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COMPLEX POWERS OF C-SECTORIAL OPERATORS. PART I

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Abstract. The main theme of this paper is the construction of complex powers of C-sectorial operators in the setting of sequentially complete locally convex spaces. We consider the constructed powers as the integral generators of equicontinuous analytic C-regularized resolvent families, and incorporate the obtained results in the study of incomplete higher order Cauchy problems.

1. Introduction and Preliminaries

Since the study of fractional powers of operators has an extensive and long history, it would be really difficult to mention here all relevant references on this subject. For a fairly general information, the reader may consult [1, 8-13, 17, 21, 31-32, 34, 37] and [42]. On the other hand, considerable interest in the theory of fractional differential equations has been stimulated by the applications in many fields of science and technology, including physics and chemistry. The main purpose of this paper is to develop the basic theory of complex powers of C-sectorial operators in sequentially complete locally convex spaces, and to apply the obtained results to various types of abstract fractional differential equations. In a series of our follow-up researches, we will construct the complex powers of almost C-sectorial operators [32], and examine the possibilites of applications to abstract parabolic problems [42].

This paper is organized as follows. In the first section we collect the notations and material needed later on. In the second section we introduce various types of operators of C-regularized type and clarify their basic structural properties (Proposition 2.4). The

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main objective in Lemma 2.5-Lemma 2.7 is to slightly generalize several well known results from the theory of sectorial operators. In Theorem 2.8, we analyze the continuity properties, additivity and spectral mapping theorem for introduced complex powers with nonzero imaginary part. In the continuation of the second section, we consider purely imaginary powers of C-sectorial operators. The main purpose of Remark 2.12 is to make a link to recent results concerning operators with polynomially bounded resolvent ([21]) and to show that the notion of C-sectoriality is not essential in our analysis. The remaining part of the second section is almost completely devoted to the study of moment inequality (Theorem 2.16, Example 2.18). The third section of the paper starts with the analysis of generation of equicontinuous analytic C-regularized resolvent families by fractional powers (Theorem 3.1, Remark 3.2). The obtained results are applied in Example 3.3 to a class of abstract space-time fractional PDEs in ultradistribution spaces. In Theorem 3.5 and Theorem 3.7, we consider the incomplete higher order Cauchy problems, in general with Liouville right-sided time-fractional derivatives.

We use the standard notation. By E is denoted a Hausdorff sequentially complete locally convex space, SCLCS for short; the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of E, and by $L(E)$ is denoted the space which consists of all continuous linear mappings from E into E. The domain and resolvent set of a closed linear operator A on E are denoted by $D(A)$ and $\rho(A)$, respectively. Suppose F is a linear subspace of E. Then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$. Let $L(E) \ni C$ be injective. Then the C-resolvent set of A, $\rho_C(A)$ in short, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A$ is injective and $(\lambda - A)^{-1}C \in L(E)\}$. We assume that $CA \subseteq AC$. The space $D_\infty(A) := \bigcap_{n\in\mathbb{N}} D(A^n)$, topologized by the following system of seminorms $p_n(x) := \sum_{j=0}^{n} p(A^j x)$ ($p \in \circledast$, $n \in \mathbb{N}$), becomes a SCLCS. In the case that E is a Banach space, we denote by $[D(A)]$ the Banach space $D(A)$ equipped with the graph norm.

Given $s \in \mathbb{R}$ in advance, set $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$ and $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $0^\alpha := 0$, $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ ($\alpha > 0$, $t > 0$) and $\Sigma_0 := (0, \infty)$. If $\delta \in (0, \pi]$ and $d \in (0, 1]$, then we define $\Sigma_\delta := \{\lambda \in \mathbb{C} : \lambda \neq 0, \ |\arg \lambda| < \delta\}$, $B_d := \{z \in \mathbb{C} : |z| \leq d\}$ and $\Sigma(\delta, d) := \overline{\Sigma_\delta \cup B_d}$. Denote by $\mathcal{L}$ and $\mathcal{L}^{-1}$ the Laplace transform and its inverse transform, respectively. We refer the reader to [31, pp. 99–102] for the basic material concerning integration in SCLCSs.

The following definition is taken from [18, 19].

Definition 1.1.

(i) Let $\alpha > 0$. A strongly continuous operator family $(S_\alpha(t))_{t\geq 0}$ is called a (g_α, C)-regularized resolvent family having A as a subgenerator iff the following holds:

(i.1) $S_\alpha(t)A \subseteq AS_\alpha(t), \ t \geq 0$ and $S_\alpha(0) = C$, (i.2) $S_\alpha(t)C = CS_\alpha(t),\ t \geq 0$ and

(i.3) $S_\alpha(t)x = Cx + \int_0^t g_\alpha(t-s)AS_\alpha(s)x\,ds,\ t \geq 0,\ x \in D(A)$;

$(S_\alpha(t))_{t\geq 0}$ is said to be locally equicontinuous if, for every $t > 0$, the family $\{S_\alpha(s) : s \in [0,t]\}$ is equicontinuous. Furthermore, $(S_\alpha(t))_{t\geq 0}$ is said to be exponentially equicontinuous (equicontinuous) if there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t}S_\alpha(t) : t \geq 0\}$ is equicontinuous.

(ii) Let $\alpha > 0$, let $\beta \in (0,\pi]$ and let $(S_\alpha(t))_{t\geq 0}$ be a (g_α, C)-regularized resolvent family. Then it is said that $(S_\alpha(t))_{t\geq 0}$ is an analytic (g_α, C)-regularized resolvent family of angle β, if there exists a function $\mathbf{S}_\alpha : \Sigma_\beta \to L(E)$ satisfying that, for every $x \in E$, the mapping $z \mapsto \mathbf{S}_\alpha(z)x,\ z \in \Sigma_\beta$ is analytic as well as that:

(ii.1) $\mathbf{S}_\alpha(t) = S_\alpha(t),\ t > 0$ and

(ii.2) $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{S}_\alpha(z)x = Cx$ for all $\gamma \in (0,\beta)$ and $x \in E$.

It is said that $(S_\alpha(t))_{t\geq 0}$ is an exponentially equicontinuous, analytic (g_α, C)-regularized resolvent family, resp. equicontinuous analytic (g_α, C)-regularized resolvent family of angle β, if for every $\gamma \in (0,\beta)$, there exists $\omega_\gamma \geq 0$, resp. $\omega_\gamma = 0$, such that the set $\{e^{-\omega_\gamma |z|}\mathbf{S}_\alpha(z) : z \in \Sigma_\gamma\}$ is equicontinuous. Since there is no risk for confusion, we will identify in the sequel $S_\alpha(\cdot)$ and $\mathbf{S}_\alpha(\cdot)$.

The integral generator $\hat{A}$ of $(S_\alpha(t))_{t\geq 0}$ is defined by setting

$$(1)\qquad \hat{A} := \Big\{(x,y) \in E \times E : S_\alpha(t)x - Cx = \int_0^t g_\alpha(t-s)S_\alpha(s)y\,ds,\ t \geq 0\Big\}.$$

The integral generator $\hat{A}$ of $(S_\alpha(t))_{t\geq 0}$ is a linear operator in E which extends any subgenerator of $(S_\alpha(t))_{t\geq 0}$ and satisfies $C^{-1}\hat{A}C = \hat{A}$. The local equicontinuity of $(S_\alpha(t))_{t\geq 0}$ guarantees that $\hat{A}$ is a closed linear operator in E; if, additionally,

$$(2)\qquad A\int_0^t g_\alpha(t-s)S_\alpha(s)x\,ds = S_\alpha(t)x - Cx,\ t \geq 0,\ x \in E,$$

then $S_\alpha(t)S_\alpha(s) = S_\alpha(s)S_\alpha(t)$ for all $t,\ s \geq 0$ and $\hat{A} = C^{-1}AC$ is a subgenerator of $(S_\alpha(t))_{t\geq 0}$. For more details on subgenerators of (g_α, C)-regularized resolvent families, the reader may consult [19]-[20].

It is also worth noting that the class of (a,k)-regularized C-resolvent families can be introduced following the approach employed in [4].

Definition 1.2. Let $\alpha > 0$. A strongly continuous operator family $(S_\alpha(t))_{t\geq 0}$ is called a (g_α, C)-regularized resolvent family iff the following conditions are fulfilled:

(i) $S_\alpha(0) = C$, $S_\alpha(t)S_\alpha(s) = S_\alpha(s)S_\alpha(t),\ t,\ s \geq 0$ and

(ii) the functional equality

$$
\begin{aligned}
&S_\alpha(s)\big(g_\alpha * S_\alpha\big)(t)x - \big(g_\alpha * S_\alpha\big)(s)S_\alpha(t)x \\
&= \big(g_\alpha * S_\alpha\big)(t)Cx - \big(g_\alpha * S_\alpha\big)(s)Cx
\end{aligned} \tag{3}
$$

holds for any $t,\ s \geq 0$ and $x \in E$.

The notions of (local, exponential) equicontinuity, analyticity and the integral generator $\hat{A}$ of $(S_\alpha(t))_{t\geq 0}$ are understood in the sense of Definition 1.1.

In this paragraph, we would like to note some basic facts concerning the relationship between Definition 1.1 and Definition 1.2. Suppose $\alpha > 0,\ A$ is a subgenerator of a locally equicontinuous (g_α, C)-regularized resolvent family $(S_\alpha(t))_{t\geq 0}$ and (2) holds (cf. Definition 1.1). Then $(S_\alpha(t))_{t\geq 0}$ is a global (g_α, C)-regularized resolvent family in the sense of Definition 1.2 and the integral generator of $(S_\alpha(t))_{t\geq 0}$ is $C^{-1}AC$. Suppose, conversely, that $\hat{A}$ is the integral generator of a locally equicontinuous (g_α, C)-regularized resolvent family $(S_\alpha(t))_{t\geq 0}$ in the sense of Definition 1.2. Then $\hat{A}$ coincides with the infinitesimal generator of $(S_\alpha(t))_{t\geq 0},\ R(C) \subseteq \overline{D(A)}$ (cf. [4] for the notion and further properties), $\hat{A}$ is a subgenerator of a global (g_α, C)-regularized resolvent family $(S_\alpha(t))_{t\geq 0}$ in the sense of Definition 1.1 ($\hat{A} = C^{-1}\hat{A}C$ is, in fact, the integral generator of $(S_\alpha(t))_{t\geq 0}$), and (2) holds with A replaced by $\hat{A}$ therein. Henceforth we will always use Definition 1.1.

Let $\alpha > 0$, let $\beta \in \mathbb{R}$ and let the Mittag-Leffler function $E_{\alpha,\beta}(z)$ be defined by $E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} z^n/\Gamma(\alpha n+\beta)$, $z \in \mathbb{C}$. In this place, we assume that $1/\Gamma(\alpha n+\beta) = 0$ if $\alpha n + \beta \in -\mathbb{N}_0$. Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$. The Wright function $\Phi_\gamma(t)$ is defined by $\Phi_\gamma(t) := \mathcal{L}^{-1}(E_\gamma(-\lambda))(t)$, $t \geq 0$. As is well-known, for every $\alpha > 0$, there exists $c_\alpha > 0$ such that:

$$
E_\alpha(t) \leq c_\alpha \exp\big(t^{1/\alpha}\big),\ t \geq 0. \tag{4}
$$

Henceforth $\mathbf{D}_t^\alpha$ denotes the Caputo fractional derivative of order α ([2]).

The asymptotic expansion of the entire function $E_{\alpha,\beta}(z)$ is given in the following lemma (cf. [39, Theorem 1.1]):

Lemma 1.3. *Let* $0 < \sigma < \frac{1}{2}\pi$. *Then, for every* $z \in \mathbb{C} \setminus \{0\}$ *and* $m \in \mathbb{N} \setminus \{1\}$:

$$
E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_s Z_s^{1-\beta} e^{Z_s} - \sum_{j=1}^{m-1} \frac{z^{-j}}{\Gamma(\beta - \alpha j)} + O(|z|^{-m}),\ |z| \to \infty,
$$

where Z_s *is defined by* $Z_s := z^{1/\alpha} e^{2\pi i s/\alpha}$ *and the first summation is taken over all those integers* s *satisfying* $|\arg z + 2\pi s| < \alpha(\frac{\pi}{2} + \sigma)$.

For further information concerning Mittag-Leffler and Wright functions, we refer the reader to [2, Section 1.3] and references cited there.

2. Complex Powers of Operators of C-Regularized Type

Our intention in this section is to give, under the condition (H) stated below, a simplified construction of complex powers of operators of C-regularized type in SCLCSs (cf. also [10, 11]).

Definition 2.1. Let $0 \leq \omega < \pi$. Then a closed linear operator A on E is called C-sectorial of angle ω, in short $A \in \mathit{Sect}_C(\omega)$, if $\mathbb{C} \setminus \overline{\Sigma_\omega} \subseteq \rho_C(A)$ and the family

$$\Big\{\lambda\big(\lambda - A\big)^{-1}C : \lambda \notin \Sigma_{\omega'}\Big\}$$

is equicontinuous for every $\omega < \omega' < \pi$; if this is the case, then the C-spectral angle of A is defined by $\omega_C(A) := \inf\{\omega \in [0, \pi) : A \in \mathit{Sect}_C(\omega)\}$.

In the following definition, we shall introduce a more general class of operators of C-regularized type.

Definition 2.2 A closed linear operator A on E is called C-nonnegative if $(-\infty, 0) \subseteq \rho_C(A)$ and the family

$$\Big\{\lambda\big(\lambda + A\big)^{-1}C : \lambda > 0\Big\}$$

is equicontinuous; moreover, a C-nonnegative operator A is called C-positive if, in addition, $0 \in \rho_C(A)$.

Remark 2.3. It is evident that a C-sectorial operator A has to be C-nonnegative; even in the case that E is a Fréchet space and that $C = 1$, (the identity operator on E), the converse statement is not true, in general (cf. [31, Subsection 1.4.1]). Notice also that a C-positive operator A on a Banach space E need not be C-sectorial unless $C = 1$. In order to illustrate this, consider the operator $A := \xi\Delta^2 - i\varrho\Delta + \varsigma$ ($\xi > 0$, $\varrho \in \mathbb{R} \setminus \{0\}$, $\varsigma < 0$), acting on $E := L^2(\mathbb{R}^n)$ with its maximal distributional domain. Then it is not difficult to prove that $-A$ is A^{-1}-positive and that $-A$ is not A^{-1}-sectorial (cf. [20, Example 3.5.30(ii)]). The construction of fractional powers of C-nonnegative operators that are not C-sectorial is outside the scope of this paper; it is also worthwhile to mention here that the assumption on C-sectoriality, used in the construction of fractional powers of operators established in this paper, can be slightly weakened (cf. [20], [31, 32] and Remark 2.12 below for further information in this direction).

Some preliminary properties of C-nonnegative operators are collected in the following proposition. The proof is standard and therefore omitted (see [31, Chapter 1] for the case that $C = 1$ and [19, Remark 2.2]).

Proposition 2.4.

(i) *If* $0 \in \rho(C)$*, then* A *is* C*-nonnegative iff* A *is nonnegative.*

(ii) *If* A *is* C*-positive, then the family* $\{(\lambda + C)(\lambda + A)^{-1}C : \lambda > 0\}$ *is equicontinuous. Conversely, if the last family is equicontinuous and*

(ii.1) C *is nonnegative, then* A *is* C*-nonnegative;*

(ii.2) C *is positive, then* A *is* C*-positive.*

(iii) *Let* A *be* C*-nonnegative. Then the following assertions hold.*

(iii.1) *The family* $\{A(\lambda + A)^{-1}C : \lambda > 0\}$ *is equicontinuous.*

(iii.2) *If* A *is injective, then* $\lambda(\lambda + A^{-1})^{-1}C = A(\lambda^{-1} + A)^{-1}C$ *for all* $\lambda > 0$*. Hence,* A^{-1} *is* C*-nonnegative.*

(iii.3) $\lim_{\lambda\to\infty} A^n(\lambda + A)^{-n}Cx = 0 \Leftrightarrow \lim_{\lambda\to\infty} \lambda^n(\lambda + A)^{-n}Cx = Cx.$

(iii.4) $\lim_{\lambda\to 0} \lambda^n(\lambda + A)^{-n}Cx = 0 \Leftrightarrow \lim_{\lambda\to 0} A^n(\lambda + A)^{-n}Cx = Cx.$

(iii.5) *Let* E *be barreled and let* E^**, the dual space of* E*, be endowed with the strong topology, i.e., the topology of uniform convergence on bounded sets of* E*. Then the adjoint of* A*, denoted by* A^**, is* C^**-nonnegative in* E^**, provided additionally that* $D(A)$ *and* $R(C)$ *are dense in* E*.*

Let $0 \le \omega < \varphi \le \pi$ and $0 < d \le 1$. In the remaining part of this section, we will always assume that a closed linear operator A satisfies the following condition.

(H): A is C-sectorial of angle ω, $B_{d_1} \subseteq \rho_C(-A)$ and the family $\{(z - A)^{-1}C : z \in B_{d_1}\}$ is equicontinuous for all $d_1 \in (0, d)$, and the mapping $z \mapsto (z - A)^{-1}Cx$ is continuous on $\Lambda_{\omega,d} := ((\mathbb{C} \setminus \Sigma_\omega) \cup B_d)^\circ$ for every $x \in E$.

Before proceeding further, we would like to note the following fact: If $D(A)$ and $R(C)$ are dense in E, and the space E is barreled, then the operator A^* satisfies the condition (H) with C^*. For a closed linear operator A satisfying (H), one can introduce the H^∞-functional calculus $f(A)$ for appropriate holomorphic functions f. Denote by $H(\Sigma_\varphi)$ the space of all holomorphic functions on the sector Σ_φ and by $H^\infty(\Sigma_\varphi)$ the space which consists of those functions $f \in H(\Sigma_\varphi)$ such that

$$|f(z)| \le M|z|^{-s} \quad (z \in \Sigma_\varphi)$$

for some constants $M,\ s > 0$. Notice that [19, Proposition 2.16(iii)] implies that the mapping $z \mapsto (z - A)^{-1}Cx$ is analytic in $\Lambda_{\omega,d}$ as well as that $(z - A)^{-n}C \in L(E)$ and

$$\begin{aligned}(5)\qquad &\frac{d^{n-1}}{dz^{n-1}}\big(z - A\big)^{-1}Cx \\ &= (-1)^{n-1}(n-1)!\big(z - A\big)^{-n}Cx, \quad x \in E,\ z \in \Lambda_{\omega,d},\ n \in \mathbb{N}.\end{aligned}$$

Now we are in a position to define the H^∞-functional calculus $f_C(A)$ for the operator A as follows

$$f_C(A)x := \frac{1}{2\pi i}\int_{\Gamma_{\omega',d'}} f(z)\big(z-A\big)^{-1}Cx\,dz, \quad x \in E, \tag{6}$$

where $\Gamma_{\omega',d'} = \partial(\Sigma_{\omega'} \setminus B_{d'})$, the boundary of $\Sigma_{\omega'} \setminus B_{d'}$ oriented in such a way that $\Im z$ increases along $\Gamma_{\omega,d}$, with $\omega' \in (\omega,\varphi)$ and $d' \in (0,d)$ arbitrary. Then an application of Cauchy's theorem shows that the above definition does not depend on the choice of ω' and d'. Furthermore, it is routine to show that the mapping $f \mapsto f_C(A)$ is a homomorphism from $H^\infty(\Sigma_\varphi)$ into $L(E)$ in the following sense:

$$f_C(A)g_C(A) = \big(fg\big)_C(A)C, \quad f,\ g,\ fg \in H^\infty(\Sigma_\varphi). \tag{7}$$

It immediately follows from (7) that

$$\begin{aligned}\big(z^{-b}\big)_C(A)\left(\frac{1}{\lambda+z^b}\right)_C(A) =& \left(\frac{z^{-b}}{\lambda+z^b}\right)_C(A)C\\ =& \frac{1}{2\pi i}\int_{\Gamma_{\omega',d'}} \frac{z^{-b}}{\lambda+z^b}\big(z-A\big)^{-1}C^2\,dz,\end{aligned} \tag{8}$$

provided $0 < b < \pi/\varphi$ and $\lambda > 0$. Given $b \in \mathbb{C}$ with $\Re b > 0$, set $A_C^{-b} := (z^{-b})_C(A)$ and $A_C^{-0} := C$. Obviously, $A_C^{-n} = A^{-n}C$ ($n \in \mathbb{N}$), $A_C^{-b}C = CA_C^{-b}$ ($\Re b > 0$), the mapping $b \mapsto A_C^{-b}x$, $\Re b > 0$ is analytic for every fixed $x \in E$, and the following holds:

$$\frac{d}{db}A_C^{-b}x = -\frac{1}{2\pi i}\int_{\Gamma_{\omega',d'}} (\ln z)z^{-b}\big(z-A\big)^{-1}Cx\,dz, \quad x \in E,\ \Re b > 0.$$

Furthermore, the equality (7) implies

$$A_C^{-b_1}A_C^{-b_2} = A_C^{-(b_1+b_2)}C, \quad \Re b_1,\ \Re b_2 > 0. \tag{9}$$

Notice also that the mapping $z \mapsto z^{-b}$, $z \in \Sigma_\pi$ is analytic, which implies that, for every $b \in \mathbb{C}$ with $0 < \Re b < 1$, we can take $\omega' = \pi$ in the integration appearing in (6). In such a way, we obtain that:

$$\begin{aligned}A_C^{-b}x =& \lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{\Gamma_{\pi,\varepsilon}} z^{-b}\big(z-A\big)^{-1}Cx\,dz\\ =& -\frac{\sin \pi b}{\pi}\int_0^\infty \lambda^{-b}\big(\lambda+A\big)^{-1}Cx\,d\lambda, \quad 0 < \Re b < 1, \quad x \in E.\end{aligned} \tag{10}$$

Now we will prove that the family $\{A_C^{-b} : 0 < b < 1\}$ is equicontinuous. Towards this end, notice that the family $\{(1+\lambda)(\lambda+A)^{-1}C : \lambda > 0\}$ is equicontinuous, i.e., for every $p \in \circledast$, there exist a seminorm $q_p \in \circledast$ and a constant $M_p > 0$ such that:

$$p\Big((1+\lambda)\big(\lambda+A\big)^{-1}Cx\Big) \le M_p q_p(x), \quad \lambda > 0,\ x \in E.$$

The last estimate in combination with (10) indicates that:

$$\begin{aligned}
p\big(A_C^{-b}x\big) \leq & \left|\frac{\sin \pi b}{\pi}\right| \int_0^\infty \frac{\lambda^{-b}}{1+\lambda} p\Big((1+\lambda)\big(\lambda+A\big)^{-1}Cx\Big)\, d\lambda \\
= & \left|\frac{\sin \pi b}{\pi}\right| M_p q_p(x) \Big(\int_0^1 \lambda^{-b}\, d\lambda + \int_1^\infty \lambda^{-b-1}\, d\lambda \Big) \\
= & M_p q_p(x) \left(\left| \frac{\sin \pi(1-b)}{\pi(1-b)} \right| + \left| \frac{\sin \pi b}{\pi b} \right| \right) \\
\leq & 2M_p q_p(x), \quad 0 < b < 1,\ x \in E.
\end{aligned}$$

Lemma 2.5. *The family* $\{A_C^{-b} : 0 < b < 1\}$ *is equicontinuous. Furthermore, if* $D(A)$ *and* $R(C)$ *are dense in* E*, then we have:*

$$\lim_{b \to 0} A_C^{-b}x = Cx, \quad x \in E. \tag{11}$$

Proof. First of all, notice that Lemma 2.7 below implies that the set $E_0 := \bigcup_{n \in \mathbb{N}} R((-n-A)^{-1}C)$ is dense in E. By the equicontinuity of $\{A_C^{-b} : 0 < b < 1\}$, it suffices to show that, for every $n \in \mathbb{N}$ and $x \in E$,

$$\lim_{b \to 0} A_C^{-b}\big(-n-A\big)^{-1}Cx = \big(-n-A\big)^{-1}C^2x.$$

This is a direct consequence of the residue theorem and the dominated convergence theorem. Indeed,

$$\begin{aligned}
A_C^{-b}\big(-n-A\big)^{-1}Cx = & \frac{1}{2\pi i} \int_{\Gamma_{\omega',d}} z^{-b}\big(z-A\big)^{-1}\big(-n-A\big)^{-1}C^2x\, dz \\
= & \frac{1}{2\pi i} \int_{\Gamma_{\omega',d}} z^{-b} \frac{\big(-n-A\big)^{-1} - \big(z-A\big)^{-1}}{z+n} C^2x\, dz \\
= & \frac{(-1)}{2\pi i} \int_{\Gamma_{\omega',d}} \frac{z^{-b}}{z+n}\big(z-A\big)^{-1}C^2x\, dz \\
\to & \frac{(-1)}{2\pi i} \int_{\Gamma_{\omega',d}} \frac{1}{z+n}\big(z-A\big)^{-1}C^2x\, dz = \big(-n-A\big)^{-1}C^2x
\end{aligned}$$

as $b \to 0$ ($x \in E$). ∎

Lemma 2.6. *The operator* A_C^{-b} *is injective for every* $b \in \mathbb{C}$ *with* $\Re b > 0$.

Proof. Suppose $\Re b > 0$ and $A_C^{-b}x = 0$. Then we obtain from (9) that $A_C^{-n}Cx = A_C^{-n+b}A_C^{-b}x = 0$, provided $n \in \mathbb{N}$ and $n > \Re b$. Since $A_C^{-n} = A^{-n}C$ is injective for every $n \in \mathbb{N}$, it follows that $x = 0$ and that the operator A_C^{-b} is injective. ∎

Note that the semigroup property (9) and Lemma 2.5 together imply that $(A_C^{-b})_{b\geq 0}$ is a C-regularized semigroup on E, provided that $D(A)$ and $R(C)$ are dense in E. Define now the powers with negative imaginary part by

$$A_{-b} := C^{-1}A_C^{-b}, \quad \Re b > 0. \tag{12}$$

In particular, $A_{-n} = C^{-1}A^{-n}C$ for every $n \in \mathbb{N}$. It is evident that A_{-b} is closed and injective, provided that $\Re b \geq 0$. Also, by Lemma 2.6 we can define the powers with positive imaginary part by

$$A_b := \big(A_{-b}\big)^{-1} = \big(A_C^{-b}\big)^{-1}C, \quad \Re b > 0. \tag{13}$$

Clearly, $A_n = C^{-1}A^nC$ for every $n \in \mathbb{N}$, and A_b is closed (injective) due to the closedness (injectiveness) of A_{-b} ($\Re b > 0$).

Lemma 2.7. *Let $k \in \mathbb{N}$. Then the families $\{\lambda^k(\lambda + A)^{-k}C : \lambda \geq 0\}$ and $\{A^k(\lambda + A)^{-k}C : \lambda \geq 0\}$ are equicontinuous. If, additionally, $D(A)$ and $R(C)$ are dense in E, then $\lim_{\lambda\to+\infty} \lambda^k(\lambda + A)^{-k}Cx = Cx$.*

Proof. The equicontinuity of family $\{\lambda^k(\lambda + A)^{-k}C : \lambda \geq 0\}$ is a consequence of (5) and the Cauchy integral formula [19, (1)], while the equicontinuity of the family $\{A^k(\lambda + A)^{-k}C : \lambda \geq 0\}$ can be proved similarly. If $D(A)$ and $R(C)$ are dense in E, then the equality $\lim_{\lambda\to+\infty} \lambda^k(\lambda + A)^{-k}Cx = Cx$ follows from the equicontinuity of the family $\{\lambda^k(\lambda + A)^{-k}C : \lambda \geq 0\}$, along with the identity

$$\lambda^k\big(\lambda + A\big)^{-k}Cx = Cx + \sum_{j=1}^{k}(-1)^j\binom{k}{j}\big(\lambda + A\big)^{-j}A^jCx, \quad x \in D(A^k),$$

and the denseness of $D(A^k)$ in E (cf. also [10, Lemma 2.13] for the Banach space case). ■

In the following theorem, we shall collect the most important properties of introduced powers.

Theorem 2.8.

(i) *Let $\Re b > 0$. Then $A_C^{-b} \in L(E)$ and $C^{-1}A_C^{-b}C = A_C^{-b}$. Furthermore, the operators $A_{\pm b}$ are closed, injective and $C^{-1}A_{\pm b}C = A_{\pm b}$.*

(ii) *Suppose $b_1,\ b_2 \in \mathbb{C}$, $\Re b_1 \neq 0$, $\Re b_2 \neq 0$, $k \in \mathbb{N}_0$ and:*

 ($*$) *$k \geq \Re b_2$, provided $\Re b_1 < 0$ and $\Re b_2 > 0$, and $k > \Re b_2$, provided $\Re b_1 < 0$ and $\Re b_2 \in \mathbb{N}$,*

 ($**$) *$k \geq \Re b_1 + \Re b_2$, if $\Re b_1 + \Re b_2 \notin \mathbb{N}$ and $k > \Re b_1 + \Re b_2$, otherwise.*

Then the following holds:

(14) $\Lambda_{k,\omega,d} := C\big(D\big(A^k\big)\big) \,\cup \bigcup_{\lambda\in\Lambda_{\omega,d}} R\big(\big(\lambda - A\big)^{-k}C\big) \subseteq D\big(A_{b_1}A_{b_2}\big) \,\cap\, D\big(A_{b_1+b_2}\big),$

$$A_{b_1}A_{b_2}x = A_{b_1+b_2}x,\ \ x \in \Lambda_{k,\omega,d}, \tag{15}$$

and

$$A_{b_1+b_2} = \big(\big(\lambda - A\big)^{-k}C\big)^{-1}A_{b_1}A_{b_2}\big(\lambda - A\big)^{-k}C, \quad \lambda \in \Lambda_{\omega,d}. \tag{16}$$

Furthermore, $A_{b_1}A_{b_2}$ is closable and $C^{-1}\overline{A_{b_1}A_{b_2}}C \subseteq A_{b_1+b_2}$, with the equality in case that $\Re b_1 < 0$ and $\Re b_2 < 0$, or that $D(A)$ and $R(C)$ are dense in E.

(iii)(iii.1) *Let $\Re b > 0$. Then $\lim_{b'\to b} A_{b'}x = A_b x$ for all $x \in C(D(A^{1+\lfloor \Re b\rfloor}))$.*

(iii.2) *Let $\Re b < 0$ and $\theta \in (0, \frac{\pi}{2})$. Then $\lim_{b'\to b} A_{b'}x = A_b x$ for all $x \in R(C)$. Moreover, $\lim_{b'\to 0, b'\in -\Sigma_\theta} A_{b'}x = x$ for all $x \in R(C)$, provided that $D(A)$ and $R(C)$ are dense in E.*

(iv) *Suppose $0 < b < \pi/\omega$. Then A_b satisfies (H) with ω and d replaced respectively by $b\omega$ and bd. Furthermore,*

$$\big(A_b\big)_c = A_{bc}, \quad c \in \mathbb{R}. \tag{17}$$

Proof.

(i) We will only prove that $C^{-1}A_bC = A_b$. The assumption $(x, y) \in A_b$, i.e., $(x, y) \in (A_C^{-b})^{-1}C$ (cf. (13)) simply implies $(A_C^{-b})^{-1}Cx = y$, $Cy = C(A_C^{-b})^{-1}Cx = (A_C^{-b})^{-1}CCx$ and $(x, y) \in C^{-1}[(A_C^{-b})^{-1}C]C = C^{-1}A_bC$. Suppose, conversely, $(x, y) \in C^{-1}A_bC = C^{-1}[(A_C^{-b})^{-1}C]C$. Then $C^2x = A_C^{-b}Cy = CA_C^{-b}y$, $Cx = A_C^{-b}y$ and $(x, y) \in (A_C^{-b})^{-1}C = A_b$.

(ii) It is clear that there exist four possible cases:

(ii.1) $\Re b_1 < 0$ and $\Re b_2 < 0$, (ii.2) $\Re b_1 < 0$ and $\Re b_2 > 0$,

(ii.3) $\Re b_1 > 0$ and $\Re b_2 < 0$, (ii.4) $\Re b_1 > 0$ and $\Re b_2 > 0$.

Consider first (ii.1). Then we easily infer from (9) that $R(C) \subseteq D(A_{b_1}A_{b_2}) \,\cap\, D(A_{b_1+b_2})$ and that (15) holds. Suppose $k \in \mathbb{N}$ and $x = (\lambda - A)^{-k}Cy$ for some $y \in E$ and $\lambda \in \Lambda_{\omega,d}$. Let $\omega' \in (\omega, \pi)$ and $d' \in (0, d)$ be such that λ lies on the left of $\Gamma_{\omega',d'}$, and let $\Gamma_{\omega'',d''}$ lie on the left of $\Gamma_{\omega',d'}$. Then we obtain inductively that, for every $z \in \rho_C(A) \setminus \{\lambda\}$:

$$\begin{aligned} &\big(z - A\big)^{-1}C\big(\lambda - A\big)^{-k}Cx \\ &= \frac{(-1)^k}{(z-\lambda)^k}\big(z - A\big)^{-1}C^2x + \sum_{i=1}^{k} \frac{(-1)^{k-i}\big(\lambda - A\big)^{-i}C^2x}{\big(z-\lambda\big)^{k+1-i}}, \end{aligned} \tag{18}$$

which implies with the help of the residue theorem and the Fubini theorem that:

$$\begin{aligned}
&A_C^{b_1+b_2}(\lambda-A)^{-k}Cx\\
&=\frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}} z^{b_1+b_2}\Big[\frac{(-1)^k}{(z-\lambda)^k}(z-A)^{-1}C^2x\\
&+\sum_{i=1}^{k}\frac{(-1)^{k-i}}{(z-\lambda)^{k+1-i}}(\lambda-A)^{-i}C^2x\Big]\,dz\\
&=\frac{(-1)^k}{2\pi i}\int\limits_{\Gamma_{\omega',d'}} z^{b_1+b_2}\frac{(z-A)^{-1}C^2x}{(z-\lambda)^k}\,dz \qquad (19)\\
&=\frac{(-1)^k}{(2\pi i)^2}\int\limits_{\Gamma_{\omega'',d''}}\int\limits_{\Gamma_{\omega',d'}} \mu^{b_1}z^{b_2}\frac{(z-A)^{-1}C^2x-(\mu-A)^{-1}C^2x}{(z-\lambda)^k(\mu-z)}\,dz\,d\mu\\
&=\frac{(-1)^k}{(2\pi i)^2}\int\limits_{\Gamma_{\omega'',d''}} \mu^{b_1}(\mu-A)^{-1}C\int\limits_{\Gamma_{\omega',d'}} z^{b_2}\frac{(z-A)^{-1}Cx}{(z-\lambda)^k}\,dz\,d\mu\\
&=A_C^{b_1}A_{b_2}(\lambda-A)^{-k}Cx, \qquad (20)
\end{aligned}$$

where (20) follows from the formula obtained in (19). This, in turn, implies (15). Suppose now that (ii.4) holds and $x=Cy$ for some $y\in D(A^k)$. Observe that the residue theorem in combination with (5) implies that, for every $r\in\mathbb{C}$ with $\Re r\in(0,\Re b_1+\Re b_2]$:

$$C^2y=\frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}}\frac{(z-A)^{-1}C^2A^{\lfloor\Re r+1\rfloor}y}{z^{\lfloor\Re r+1\rfloor}}\,dz.$$

Using this equality, it can be easily seen that, for such a number r, we have the following:

$$A_rCy=\frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}} z^{r-\lfloor\Re r\rfloor-1}(z-A)^{-1}CA^{\lfloor\Re r+1\rfloor}y\,dz. \tag{21}$$

Notice also that, for every $n\in\mathbb{N}_0,\ z\in\rho_C(A)\setminus\{0\}$ and $x\in D(A^{n+1})$:

$$z^{-(n+1)}(z-A)^{-1}CA^{n+1}x=(z-A)^{-1}Cx-\sum_{i=0}^{n}z^{-(i+1)}A^iCx. \tag{22}$$

Keeping in mind (21)-(22), the method used in the proof of [20, Proposition 1.4.4], and the already given argumentation, one gets $CA_{b_2}Cy=A_c^{-b_1}A_{b_1+b_2}Cy$ and (15).

Suppose now $x=(\lambda-A)^{-k}Cy$, where $y\in E$ and $\lambda\in\Lambda_{\omega,d}$. Then $Cx=C(\lambda-A)^{-k}Cy\in C(D(A^k))$ and, by the first part of assertion, $Cx\in D(A_r)$ with $A_rCx=\frac{1}{2\pi i}\int_{\Gamma_{\omega',d'}}\mu^{r-\lfloor\Re r\rfloor-1}(\mu-A)^{-1}CA^{\lfloor\Re r+1\rfloor}(\lambda-A)^{-k}Cy\,d\mu$. In combination with (22) and (18), the above implies:

$$
\begin{aligned}
A_rCx&=\frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}}\mu^{r-\lfloor\Re r\rfloor-1}\sum_{j=0}^{\lfloor\Re r\rfloor+1}(-1)^j\lambda^{\lfloor\Re r+1-j\rfloor}\\
&\quad\times\binom{\lfloor\Re r\rfloor+1}{j}\big(\mu-A\big)^{-1}C\big(\lambda-A\big)^{j-k}Cy\,d\mu\\
&=\frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}}\mu^{r-\lfloor\Re r\rfloor-1}\Big\{\sum_{j=0}^{\lfloor\Re r\rfloor+1}(-1)^j\lambda^{\lfloor\Re r+1-j\rfloor}\\
&\quad\times\binom{\lfloor\Re r\rfloor+1}{j}\Big[\frac{(-1)^{k-j}}{(\mu-\lambda)^{k-j}}\big(\mu-A\big)^{-1}C^2y\\
&\quad+\sum_{l=1}^{k-j}\frac{(-1)^{k-j-l}}{\big(\mu-\lambda\big)^{k-j-l+1}}\big(\lambda-A\big)^{-l}C^2y\Big]\Big\}\,d\mu
\end{aligned}
$$

and $x\in D(C^{-1}A_rC)=D(A_r)$. It is checked at once that:

$$A_C^{-b}A_r\subseteq A_rA_C^{-b},\quad b>0,\ r\in\mathbb{C},\ \Re r\neq 0. \tag{23}$$

Now we obtain from (9), (13) and (23) that $CA_{b_2}x=A_C^{-b_1}A_{b_1+b_2}x$ and that (15) holds. Now the proof of (16), and the proof of (15) in case that (ii.2) or (ii.3) holds, become standard and therefore omitted. The remnant of proof of (ii) will be given provided that $\Re b_1>0$ and $\Re b_2>0$. Suppose $A_{b_1}A_{b_2}x=y$. Then $A_{b_1}A_{b_2}(\lambda-A)^{-k}Cx=(\lambda-A)^{-k}Cy$, $(\lambda-A)^{-k}CA_{b_1}A_{b_2}x=(\lambda-A)^{-k}Cy$ and $((\lambda-A)^{-k}C)^{-1}A_{b_1}A_{b_2}(\lambda-A)^{-k}Cx=y$. By (16), one yields $A_{b_1+b_2}x=y$ and $A_{b_1}A_{b_2}\subseteq A_{b_1+b_2}$, which implies the closedness of $A_{b_1}A_{b_2}$ and

$$C^{-1}\overline{A_{b_1}A_{b_2}}C\subseteq C^{-1}A_{b_1+b_2}C=A_{b_1+b_2}.$$

Keeping in mind Lemma 2.7, the proof of inclusion $A_{b_1+b_2}\subseteq C^{-1}\overline{A_{b_1}A_{b_2}}C$ follows similarly as in the proof of [10, Theorem 4.1(5)]. If $\Re b\notin\mathbb{N}$, then (iii.1) follows from (21) and the dominated convergence theorem. Suppose now $\Re b\in\mathbb{N}$. Then it is not difficult to prove that $\lim_{b'\to b,\Re b'\geq\Re b}A_{b'}x=A_bx$. Using the equality

$$C^2y=\frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}}\frac{\big(z-A\big)^{-1}C^2A^{\Re b+1}y}{z^{\Re b+1}}\,dz,$$

we easily infer that, for every $b'\in\mathbb{C}$ with $\Re b'\in(\Re b-\varepsilon,\Re b)$, $\varepsilon>0$ sufficiently small:

$$(24)\qquad A_{b'}Cy = \frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}} z^{b'-\Re b-1}(z-A)^{-1}CA^{\Re b+1}y\,dz, \quad y\in D(A^{\Re b+1}).$$

In combination with the dominated convergence theorem and the residue theorem, the above implies $\lim_{b'\to b,\ \Re b'<\Re b} A_{b'}x = A_b x$. The equality $\lim_{b'\to b} A_{b'}x = A_b x$, $x\in R(C)$, $\Re b<0$ follows immediately from the definition of powers, while the second assertion in (iii.2) is a consequence of Lemma 2.5 and the proof of [42, Theorem 2.21, p. 93]. Now we will prove (iv). Put, for every $\lambda\in\Lambda_{b\omega,bd}$,

$$R_b(\lambda)x := \frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}} \frac{(z-A)^{-1}Cx}{\lambda - z^b}\,dz, \quad x\in E,$$

where $\omega'\in(\omega,\pi)$ is chosen so that λ lies on the left of $\Gamma_{d\omega',db}$. Then it can be simply proved that, for every $x\in E$, the mapping $\lambda\mapsto R_b(\lambda)x$, $\lambda\in\Lambda_{b\omega,bd}$ is analytic as well as that the mapping $\lambda\mapsto R_b(\lambda)\in L(E)$, $\lambda\in\Lambda_{b\omega,bd}$ is a non-degenerate C-pseudoresolvent in the sense of [26, Definition 3.1]. Also, the injectiveness of $R_b(\lambda)$ for $\lambda\in\Lambda_{b\omega,bd}$ is trivially verified. By [26, Theorem 3.4(i)-(ii)], we get that there exists a closed linear operator B_b on E such that the following holds: $D(B_b) = \{x\in E : R(R_b(\lambda))\}$ ($D(B_b)$ is independent of $\lambda\in\Lambda_{b\omega,bd}$), $B_b x = (\lambda - R_b(\lambda)^{-1}C)x$, $x\in D(B_b)$, $\lambda - B_b$ is injective and $R_b(\lambda)(\lambda - B_b)\subseteq(\lambda - B_b)R_b(\lambda) = C$ ($\lambda\in\Lambda_{b\omega,bd}$). Plugging $\lambda=0$, we obtain $B_b\subseteq A_b$. The assumption $x\in D(A_b)$ implies by (13) that $Cx\in R(A_c^{-b}) = R(R_b(0))$, so that $A_b = B_b$, $\Lambda_{b\omega,bd}\subseteq\rho_C(A_b)$ and $R_b(\lambda) = (\lambda - A_b)^{-1}C$, $\lambda\in\Lambda_{b\omega,bd}$. Direct computation shows that the family $\{(\lambda - A_b)^{-1}C : \lambda\in B_{bd_1}\}$ is equicontinuous for all $d_1\in(0,d)$. It remains to be proved that, for every $d_1\in(0,d)$ and $\omega_1\in(\omega,\pi)$, the family $\{R_b(\lambda) : \lambda\in\mathbb{C}\setminus\overline{\Sigma_{b\omega_1}},\ |\lambda|\geq bd_1\}$ is equicontinuous and that (17) holds. If $b\in(0,1)$, then the first assertion follows from an insignificant modification of the proof of [42, Theorem 2.23, pp. 95–97], while the proof of (17) may be left to the reader as an easy exercise. Suppose now $1<b<\pi/\omega$, $n\in\mathbb{N}$, $b=b_1 n$ for some $0<b_1<1$, and $b_1 n\omega_1 > n\omega' > b_1\omega$. Without loss of generality, we may assume that $b<\pi/\omega_1$. Denote, for every $\lambda\in\mathbb{C}\setminus\overline{\Sigma_{b\omega_1}}$ with $|\lambda|\geq b_1 n d_1$, by $\lambda_1,\cdots,\lambda_n$ the n-th roots of λ. Then $\lambda_j\in\Lambda_{\omega',d_1}$ $(1\leq j\leq n)$ and, by the foregoing, we have that:

$$\begin{aligned}
(\lambda - A_{b_1 n})^{-1}Cx &= \Bigl(\lambda - \bigl(A_{b_1}\bigr)_n\Bigr)^{-1}Cx = \frac{(-1)}{2\pi i}\int\limits_{\Gamma_{\omega',d'}} \frac{(z-A_{b_1})^{-1}Cx}{\lambda - z^n}\,dz \\
&= \frac{(-1)}{2\pi i}\int\limits_{\Gamma_{\omega',d'}} \Bigl[\frac{1}{(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_{n-1})(z-\lambda_1)} \\
&\quad + \cdots + \frac{1}{(\lambda_n-\lambda_1)\cdots(\lambda_n-\lambda_{n-1})(z-\lambda_n)}\Bigr](z-A_{b_1})^{-1}Cx\,dz \\
&= \frac{(\lambda_1 - A_{b_1})^{-1}Cx}{(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_{n-1})} + \cdots + \frac{(\lambda_n - A_{b_1})^{-1}Cx}{(\lambda_n-\lambda_1)\cdots(\lambda_n-\lambda_{n-1})},
\end{aligned}$$

which completes the proof by a routine argument. ■

Before proceeding further, we would like to observe the following fact. Suppose $b_1,\ b_2 \in \mathbb{R}$, $k \in \mathbb{N}_0$ and:

($*'$) $k \geq b_2$, provided $b_1 \leq 0$ and $b_2 \geq 0$,

($*'*'$) $k \geq b_1 + b_2$, otherwise.

Then (14)-(16) continue to hold.

Following [31, Definition 7.1.2], we introduce the purely imaginary powers of A as follows.

Definition 2.9. Let $\tau \in \mathbb{R} \setminus \{0\}$. Then the power $A_{i\tau}$ is defined by

(25) $$A_{i\tau} := C^{-2}\big(A+1\big)_2 A_{-1}A_{1+i\tau}\big(A+1\big)_{-2}C^2.$$

It is clear from the definition that $A_{i\tau}$ is a linear operator. Now we will prove that $A_{i\tau}$ is closed. Taking into account the equalities (18), (21) and the residue theorem, one gets that $(A+1)A_{-1}A_{1+i\tau}(A+1)_{-2}C^2 \in L(E)$ and that, for every $x \in E$,

(26) $$(A+1)A_{-1}A_{1+i\tau}\big(A+1\big)_{-2}C^2 x = \frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}} z^{-1+i\tau}\frac{z}{z+1}\big(z-A\big)^{-1}C^2 x\,dz.$$

Keeping in mind that $C^{-1}(A+1)_1 C = (A+1)_1 = C^{-1}(A+1)C$, it readily follows that $x \in D(A_{i\tau})$ iff $\frac{1}{2\pi i}\int_{\Gamma_{\omega',d'}} z^{-1+i\tau}\frac{z}{z+1}(z-A)^{-1}Cx\,dz \in D(C^{-2}(A+1)C)$. If this is the case, then we have the following equality:

(27) $$A_{i\tau}x = C^{-2}(A+1)C\frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}} z^{-1+i\tau}\frac{z}{z+1}\big(z-A\big)^{-1}Cx\,dz.$$

The closedness of $A_{i\tau}$ now follows from (27), along with the closedness of the operator $A+1$ and the dominated convergence theorem. Notice that the operator $A_{i\tau}$ can be introduced equivalently by

$$A_{i\tau} = C^{-j}\big(A+\lambda\big)_q A_{-p}A_{p+i\tau}\big(A+\lambda\big)_{-q}C^j,$$

where $p,\ q,\ j \in \mathbb{N}$, $q > p$ and $\lambda > 0$.

Theorem 2.10. *Let $\tau,\ \tau_1,\ \tau_2 \in \mathbb{R}$ and $k \in \mathbb{N}$. Then the following holds:*

(i) $C(D(A^k)) \cup \bigcup_{\lambda \in \Lambda_{\omega,d}} R((\lambda - A)^{-k}C) \subseteq D(A_{i\tau})$.

(ii) $C^{-1}A_{i\tau}C = A_{i\tau}$.

(iii) $A_{i\tau}$ *is injective and* $A_{i\tau} = (A_{-i\tau})^{-1}$.

(iv) $A_{i\tau_1}A_{i\tau_2} \subseteq A_{i(\tau_1+\tau_2)}$, $C(D(A^k)) \cup \bigcup_{\lambda\in\wedge_{\omega,d}} R((\lambda-A)^{-k}C) \subseteq D(A_{i\tau_1}A_{i\tau_2})$,

$$A_{i(\tau_1+\tau_2)} = \big((\lambda-A)^{-k}C\big)^{-1}A_{i\tau_1}A_{i\tau_2}\big(\lambda-A\big)^{-k}C, \tag{28}$$

$A_{i\tau_1}A_{i\tau_2}$ is closable, $C^{-1}\overline{A_{i\tau_1}A_{i\tau_2}}C \subseteq A_{i(\tau_1+\tau_2)}$, with the equality in the case that $D(A)$ and $R(C)$ are dense in E.

(v) *Let $\Re b < 0$ and $\tau \in \mathbb{R}$. Then the following holds:*

$$A_{i\tau}A_b \subseteq A_{b+i\tau}, \tag{29}$$

$$A_{i\tau}A_b \subseteq A_{i\tau+b}, \tag{30}$$

$$A_{b+i\tau} = \big(\big(\lambda-A\big)^{-k}C)\big)^{-1}A_bA_{i\tau}\big(\lambda-A\big)^{-k}C, \quad k\in\mathbb{N},\ \lambda\in\wedge_{\omega,d}, \tag{31}$$

$$A_{b+i\tau} = \big(\big(\lambda-A\big)^{-k}C)\big)^{-1}A_{i\tau}A_b\big(\lambda-A\big)^{-k}C, \quad k\in\mathbb{N},\ \lambda\in\wedge_{\omega,d}, \tag{32}$$

the operators $A_{i\tau}A_b$ and $A_bA_{i\tau}$ are closable, $C^{-1}\overline{A_{i\tau}A_b}C \subseteq A_{b+i\tau}$ and $C^{-1}\overline{A_bA_{i\tau}}C \subseteq A_{b+i\tau}$. If $D(A)$ and $R(C)$ are dense in E, then we also have the converse inclusions.

(vi) *Let $\Re b > 0$ and $\tau \in \mathbb{R}$. Then (29)-(30) hold. In the case that $k \geq \lceil \Re b\rceil$, we have (31)-(32). Furthermore, the operators $A_{i\tau}A_b$ and $A_bA_{i\tau}$ are closable, $C^{-1}\overline{A_{i\tau}A_b}C \subseteq A_{b+i\tau}$ and $C^{-1}\overline{A_bA_{i\tau}}C \subseteq A_{b+i\tau}$. If $D(A)$ and $R(C)$ are dense in E, then we also have the converse inclusions.*

(vii) *Suppose $\tau \in \mathbb{R}$, $x \in E$ and $\lambda \in \wedge_{\omega,d}$. Then the following equality holds: $\lim_{b\to i\tau,\Re b\geq 0} A_b(\lambda-A)^{-1}Cx = A_{i\tau}(\lambda-A)^{-1}Cx$ and $\lim_{b\to i\tau,\Re b<0} A_C^b(\lambda-A)^{-1}Cx = A_{i\tau}(\lambda-A)^{-1}C^2x$.*

Proof. The assertion (ii) is an immediate consequence of the equalities (27) and $C^{-2}(A+1)C^2 = C^{-1}(A+1)C$. Suppose now $x \in C(D(A))$ and $x = Cy$ for some $y \in D(A)$. Then (27) immediately implies

$$A_{i\tau}x = \frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}} z^{-1+i\tau}\frac{z}{z+1}\big(z-A\big)^{-1}C(A+1)y\,dz.$$

Using this equality, the assertion (ii) and the proof of Theorem 2.8, we easily infer that $C(D(A^k)) \cup \bigcup_{\lambda\in\wedge_{\omega,d}} R((\lambda-A)^{-k}C) \subseteq D(A_{i\tau})$. This proves (i). The injectivity of $A_{i\tau}$ follows from the injectivity of each single operator appearing in the representation (25). Let $x \in D(A_{i\tau})$ be fixed. To complete the proof of (iii), it is enough to show that $A_{-i\tau}A_{i\tau}x = x$. Using Definition 2.9, the above is equivalent with

$$C^{-2}\big(A+1\big)_2A_{-1}A_{1-i\tau}A_{-1}A_{1+i\tau}\big(A+1\big)_{-2}C^2x = x. \tag{33}$$

Since $C(D(A)^2) \subseteq D(A_{1+i\tau})$ and $L(E) \ni A_{-1}C$ commutes with $A_{1+i\tau}$, we have the equivalence of (33) with the following equality

$$C^{-2}(A+1)_2 A_{-1} A_{1-i\tau} A_{1+i\tau} A_{-1} (A+1)_{-2} C^2 x = x,$$

which follows from the semigroup property $A_{1-i\tau}A_{1+i\tau}y = A_2 y$, $y \in C(D(A^3))$ and a straightforward computation. The inclusion $A_{i\tau_1}A_{i\tau_2} \subseteq A_{i(\tau_1+\tau_2)}$ can be proved in almost the same way, which implies that the operator $A_{i\tau_1}A_{i\tau_2}$ is closed as well as $C^{-1}\overline{A_{i\tau_1}A_{i\tau_2}}C \subseteq A_{i(\tau_1+\tau_2)}$. Further on, the inclusion $C(D(A)) \subseteq D(A_{i\tau_1}A_{i\tau_2})$ simply follows from the semigroup property for the powers and the fact that, for every $y \in D(A)$, one has $A_{i(\tau_1+\tau_2)}(A+1)_{-2}C^3 y \in D(C^{-2}(A+1)_2)$. Using now the equality (18), the proof of Theorem 2.8 and $C(D(A^k)) \subseteq D(A_{i\tau_1}A_{i\tau_2})$, it is checked at once that $\bigcup_{\lambda\in\wedge_{\omega,d}} R((\lambda-A)^{-k}C) \subseteq D(A_{i\tau_1}A_{i\tau_2})$. The proof of equality (28) is standard and as such will not be given. In the case that $D(A)$ and $R(C)$ are dense in E, the equality $C^{-1}\overline{A_{i\tau_1}A_{i\tau_2}}C = A_{i(\tau_1+\tau_2)}$ follows from the commutation of $A_{i\tau}$ with the bounded linear operator $(\lambda - A)^{-k}C$ ($\lambda \in \wedge_{\omega,d}$), and the corresponding proof of [10, Theorem 4.1(5)]. This completes the proof of (iv). Suppose now $\Re b < 0$ and $\tau \in \mathbb{R} \setminus \{0\}$. By (27) and an elementary argumentation, we get that

$$A_C^b A_{i\tau} x = A_C^{b+i\tau} x, \quad x \in D(A_{i\tau}). \tag{34}$$

Then it is not difficult to show that (29)-(30) holds. It is also simple to prove that $A_{i\tau}A_C^b \in L(E)$ and that (31)-(32) hold. Hence, the operators $A_{i\tau}A_b$ and $A_b A_{i\tau}$ are closable, $C^{-1}\overline{A_{i\tau}A_b}C \subseteq A_{b+i\tau}$ and $C^{-1}\overline{A_b A_{i\tau}}C \subseteq A_{b+i\tau}$. The proof of [10, Theorem 4.1(5)] implies that the converse inclusions hold in the last two equalities, provided that $D(A)$ and $R(C)$ are dense in E. Let $\Re b > 0$ and $\tau \in \mathbb{R}\setminus\{0\}$. Then $A_C^{-b}A_{-i\tau}x = A_C^{-b-i\tau}x$, $x \in D(A_{-i\tau})$. Therefore, $A_{-i\tau}Cx = (A_C^{-b})^{-1}CA_C^{-b-i\tau}x$, $x \in D(A_{-i\tau})$ and $A_b A_C^{-b-i\tau}x = A_{-i\tau}Cx$, $x \in D(A_{-i\tau})$. For any $x \in D(A_b) \cap D(A_{-i\tau})$, the above implies $A_C^{-b-i\tau}A_b x = CA_{-i\tau}x$ and $A_b x = A_{b+i\tau}A_{-i\tau}x$. Plugging $y = A_{-i\tau}x$ for such an element $x \in E$, we get that $A_b A_{i\tau} y = A_{b+i\tau} y$ and that (29) holds. One obtains similarly that $A_C^{-b-i\tau}[A_{i\tau}A_b x] = A_C^{-b}A_b x = Cx$, $x \in D(A_{i\tau}A_b)$ and that (30) holds. The equality (31) can be shown as before and, for the remaining part of the proof, we will only prove the equality

$$A_{i\tau}A_b(\lambda - A)^{-k}Cx = A_{b+i\tau}(\lambda - A)^{-k}Cx,$$

for $k \geq \lceil \Re b \rceil$, $\lambda \in \wedge_{\omega,d}$ and $x \in E$ given in advance (cf. (32)). By the definition of the power $A_{i\tau}$ and the semigroup property established in Theorem 2.8, we easily infer that $A_b(\lambda - A)^{-k}Cx \in D(A_{i\tau})$ and

$$\begin{aligned} &A_{i\tau}A_b(\lambda - A)^{-k}Cx \\ &\quad = C^{-2}(A+1)_2 A_{-1}A_{1+i\tau}(A+1)_{-2}C^2 A_b(\lambda - A)^{-k}Cx \end{aligned}$$

$$= C^{-2}(A+1)_2 A_{-1} A_{1+b+i\tau}(A+1)_{-2} C^2 (\lambda - A)^{-k} Cx$$
$$= C^{-2}(A+1)_2 A_{b+i\tau}(A+1)_{-2} C^2 (\lambda - A)^{-k} Cx$$
$$= C^{-2}(A+1)_2 (A+1)_{-2} C^2 A_{b+i\tau} (\lambda - A)^{-k} Cx$$
$$= A_{b+i\tau}(\lambda - A)^{-k} Cx.$$

Finally, we will prove (vii). By the foregoing arguments, it can be simply justified that

$$A_{i\tau}(\lambda - A)^{-1} Cx = \frac{(-1)}{2\pi i} \int\limits_{\Gamma_{\omega',d'}} \frac{z^{i\tau}}{z-\lambda}(z-A)^{-1} Cx\,dz, \tag{35}$$

where λ lies on the left of the contour $\Gamma_{\omega',d'}$. The dominated convergence theorem yields $\lim_{\tau'\to\tau} A_{i\tau'}(\lambda - A)^{-1}Cx = A_{i\tau}(\lambda - A)^{-1}Cx$. Suppose now $b \in \mathbb{C}$ and $\Re b \in (0,1)$. Then we have the following obvious equalities

$$A_b Cx = \frac{1}{2\pi i} \int\limits_{\Gamma_{\omega',d'}} z^{-1+ib} \frac{z}{z+1}(z-A)^{-1} Cx\,dz$$

and

$$A_b(\lambda - A)^{-1} Cx = \frac{(-1)}{2\pi i} \int\limits_{\Gamma_{\omega',d'}} \frac{z^{b}}{z-\lambda}(z-A)^{-1} Cx\,dz. \tag{36}$$

By (36) and the dominated convergence theorem, we get

$$\lim_{b\to i\tau,\ \Re b>0} A_b(\lambda - A)^{-1}Cx = A_{i\tau}(\lambda - A)^{-1}Cx.$$

Further on, the proof of Lemma 2.5 implies

$$\lim_{b\to i\tau,\ \Re b<0} A_C^b(\lambda - A)^{-1}Cx = \frac{(-1)}{2\pi i} \int\limits_{\Gamma_{\omega',d'}} \frac{z^{i\tau}}{z-\lambda}(z-A)^{-1} C^2x\,dz.$$

We have also the following equality

$$A_{i\tau}(\lambda - A)^{-1} Cx = \frac{(-1)}{2\pi i} \int\limits_{\Gamma_{\omega',d'}} \frac{z^{i\tau}}{z-\lambda}(z-A)^{-1} Cx\,dz,$$

which thereby completes the proof of theorem. ■

Remark 2.11. Suppose $\alpha,\ \beta \in \mathbb{C}$. Keeping in mind Theorem 2.8(ii) and Theorem 2.10(iv)-(vi), we obtain the additivity property of powers $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$. Using the equalities (16) and (31), it can be also simply proved that $D(A_\beta) \cap D(A_{\alpha+\beta}) \subseteq$

$D(A_\alpha A_\beta)$ with $A_\alpha A_\beta x = A_{\alpha+\beta}x,\ x \in D(A_\beta)\ \cap\ D(A_{\alpha+\beta})$. If $D(A)$ and $R(C)$ are dense in E, then the following holds $C^{-1}\overline{A_\alpha A_\beta}C = A_{\alpha+\beta},\ \alpha,\ \beta \in \mathbb{C}$ (cf. also [31, Theorem 7.1.1]).

It would take too long to consider some other properties and applications of purely imaginary powers of C-sectorial operators. For further information in this direction, the reader may consult, among many other papers and monographs, [31, Sections 7-10], [42, pp. 105-116] and the references cited there.

Remark 2.12.

(i) Given $\beta \geq -1,\ \varepsilon \in (0,1]$ and $c \in (0,1)$, put $P_{\beta,\varepsilon,c} := \{\xi + i\eta : \xi \geq \varepsilon,\ \eta \in \mathbb{R},\ \ |\eta| \leq c(1+\xi)^{-\beta}\}$. Assume $(E, ||\cdot||)$ is a Banach space, $\alpha \geq -1$ and A is a closed linear operator on E with the following property:

$$(0,\infty) \subseteq \rho(A) \text{ and } \sup_{\lambda>0}\big(1+|\lambda|\big)^{-\alpha}\big\|\big(\lambda - A\big)^{-1}\big\| < \infty.$$

By the usual series argument, we have that there exist $d \in (0,1]$, $c \in (0,1)$ and $\varepsilon \in (0,1]$ such that $(\varepsilon, c(1+\varepsilon)^{-\alpha}) \in \partial B_d$,

$$P_{\alpha,\varepsilon,c} \cup B_d \subseteq \rho(A) \text{ and } \sup_{\lambda \in P_{\alpha,\varepsilon,c}\cup B_d}\big(1+|\lambda|\big)^{-\alpha}\big\|\big(\lambda - A\big)^{-1}\big\| < \infty.$$

Put $n_\alpha := \lfloor\alpha\rfloor + 2$ if $\alpha \notin \mathbb{Z}$, and $n_\alpha := \alpha + 1$, otherwise. Denote by $(-A)^b$ ($b \in \mathbb{C}$) the complex power defined in [20, Section 1.4]. We would like to notice here that the method developed in this paper, with $C = (-A)^{-n_\alpha}$, gives the definition of power $(-A)_b$. It is not difficult to prove that $(-A)^b \subseteq (-A)_b$ for all $b \in \mathbb{C}$. Moreover, the set appearing in [21, Remark 4.1, p. 61, l. -7], resp. [21, Remark 4.1, p. 61, l. -6], coincides with $D((-A_{\omega+\sigma})_{\alpha+\varepsilon})$, resp. $D((-A_\sigma)_{\alpha+\varepsilon})$, and the equality $(-A)_b = (-A)^{k+n_\alpha}(-A)^b(-A)^{-(k+n_\alpha)}$ holds provided that $\Re b \geq 0$ and $k \in \mathbb{N}$.

(ii) It is also worth noting that the method described above can be employed in a more general situation. Let $\alpha \geq -1$, $\varepsilon \in (0,1]$, $c \in (0,1)$, $d \in (0,1]$ and n_α be as in the previous part of this remark, and let $\Omega_{\alpha,\varepsilon,c,d}$ be an open neighborhood of the region $P_{\alpha,\varepsilon,c}\ \cup\ B_d$. Suppose that the following condition holds:

(H1): $\Omega_{\alpha,\varepsilon,c,d} \subseteq \rho_C(-A)$, the family $\{(1+|z|)^{-\alpha}(z+A)^{-1}C : z \in \Omega_{\alpha,\varepsilon,c,d}\}$ is equicontinuous, and the mapping $z \mapsto (z+A)^{-1}Cx,\ z \in \Omega_{\alpha,\varepsilon,c,d}$ is continuous for every $x \in E$.

Then there exists a sufficiently small number $\kappa > 0$ such that the operator $\mathcal{C} := (d+\kappa-A)^{-n_\alpha}C \in L(E)$ is injective and commutes with A (cf. (5)). Making use of (18) and the inclusion $R(C) \subseteq R((z+A)^n)$, $n \in \mathbb{N}$, $z \in \Omega_{\alpha,\varepsilon,c,d}$, it can be easily seen that, for every $z \in P_{\alpha,\varepsilon,c} \cup B_d$,

$$\big(z+A\big)^{-1}\mathcal{C}x = \frac{\big(z+A\big)^{-1}Cx}{\big(d+\kappa+z\big)^{n_\alpha}} + \sum_{i=1}^{n_\alpha} \frac{\big(d+\kappa-A\big)^{-i}Cx}{\big(d+\kappa+z\big)^{n_\alpha+1-i}},$$

and that the family $\{z(z+A)^{-1}\mathcal{C} : z \in P_{\alpha,\varepsilon,c} \cup B_d\}$ is equicontinuous. Therefore, we are in a position to construct the power A_b ($b \in \mathbb{C}$). Notice that such a construction does not depend on the choice of numbers α, ε, c, d, κ and n_α, and that the assertion of Theorem 3.1 below can be reformulated in the context of this remark (with some obvious additional difficulties in the case $\alpha \notin \mathbb{Z}$). The analysis of existence and growth of mild solutions of the abstract Cauchy problems governed by fractionally integrated C-semigroups and cosine functions in locally convex spaces falls out from the framework of this paper (cf. [14, 33, 21] for further information in this direction).

Now we focus our attention towards proving the well-known moment inequality for fractional powers.

Lemma 2.13. *Let* $\alpha,\ \gamma \in \mathbb{C}$, *let* $-\infty < \Re\alpha < \Re\gamma < +\infty$ *and* $x \in D(A_\gamma)$. *Then* $Cx \in D(A_\alpha)$ *and* $A_\alpha Cx = A_C^{\alpha-\gamma}A_\gamma x$.

Proof. We will prove the assertion of lemma only in the case $\Re\gamma = 0$ and $\alpha = i\tau$ for some $\tau \in \mathbb{R}\setminus\{0\}$. The proof in other cases is simple and as such will not be given. Notice that the equality (27), the definition of $A_C^{i\tau-\gamma}$ and the standard argumentation shows that $A_{-i\tau}A_C^{i\tau-\gamma}A_\gamma x = Cx$. Since $A_{i\tau} = (A_{-i\tau})^{-1}$, the above implies $Cx \in D(A_{i\tau})$ and $A_{i\tau}Cx = A_C^{i\tau-\gamma}A_\gamma x$. ∎

Lemma 2.14. *Let* $n \in \mathbb{N}_0$, *let* $b \in \mathbb{C}$ *and let* $\Re b \in (0, n+1)\setminus\mathbb{N}$. *Then, for every* $x \in E$,

$$A_C^{-b}x = \frac{(-1)^n n!}{(1-b)(2-b)\cdots(n-b)} \frac{\sin\pi(n-b)}{\pi} \int_0^\infty t^{n-b}\big(t+A\big)^{-(n+1)}Cx\,dt, \tag{37}$$

where $(1-b)(2-b)\cdots(n-b) := 1$ *for* $n=0$.

Proof. This lemma can be proved following the lines of the proof of [12, Theorem 5.27, p. 138]. As a matter of fact, the proof of cited theorem combined with (5) implies that, for every $x \in E$,

$$C^n A_C^{-b}x = \frac{(-1)^n n!}{(1-b)(2-b)\cdots(n-b)} \frac{\sin\pi(n-b)}{\pi} \int_0^\infty t^{n-b}\big(t+A\big)^{-(n+1)}C^{n+1}x\,dt.$$

This completes the proof by applying the operator C^{-n} on both sides of the above equality.

Lemma 2.15. *Suppose* $\alpha_0,\ \beta_0 \in \mathbb{C},\ \Re\alpha_0 > \Re\beta_0 > 0,\ n \in \mathbb{N}_0$ *and* $\Re\alpha_0 \in (n, n+1]$. *Then, for every* $p \in \circledast$, *there exist* $c_{p,\alpha_0,\beta_0} > 0$ *and* $q_p \in \circledast$ *such that:*

$$p\big(CA_C^{-\beta_0}x\big) \le c_{p,\alpha_0,\beta_0} q_p\big(A_C^{-\alpha_0}x\big)^{\frac{\Re\beta_0}{\Re\alpha_0}} q_p\big(Cx\big)^{\frac{\Re\alpha_0-\Re\beta_0}{\Re\beta_0}}, \quad x \in E, \tag{38}$$

and

$$p\big(A_C^{-\beta_0}x\big) \le c_{p,\alpha_0,\beta_0} q_p\big(A_{-\alpha_0}x\big)^{\frac{\Re\beta_0}{\Re\alpha_0}} q_p\big(x\big)^{\frac{\Re\alpha_0-\Re\beta_0}{\Re\beta_0}}, \quad x \in D(A_{-\alpha_0}). \tag{39}$$

Proof. Without loss of generality, we may assume that $\Re\beta_0 \notin \mathbb{N}$. Our intention is to prove that, for every $p \in \circledast$, there exist $c_p' > 0$ and $q_p' \in \circledast$ such that:

$$p\big(s^{n+1-\alpha_0}C\big(s+A\big)^{-(n+1)}Cx\big) \le c_p' q_p'\big(A_C^{-\alpha_0}x\big), \quad s > 0,\ x \in E. \tag{40}$$

Suppose first $\Re\alpha_0 \in (n, n+1)$. Due to (15), we have $A_{\alpha_0}C^{-1}A_C^{-\alpha_0}Cx = Cx$, $x \in E$, which clearly implies $C^2x = CA_{\alpha_0}A_C^{-\alpha_0}x$, $x \in E$. Fix, for the time being, a number $s > 0$ and an element $x \in E$. Making use of (14) and the inclusion $((t+A)^{-(n+1)}C)A_{\alpha_0} \subseteq A_{\alpha_0}((t+A)^{-(n+1)}C)$, $t > 0$, we get that:

$$s^{n+1-\alpha_0}C\big(s+A\big)^{-(n+1)}Cx = s^{n+1-\alpha_0}A_{\alpha_0}\big(s+A\big)^{-(n+1)}C\big(A_C^{-\alpha_0}x\big).$$

Let $\omega' \in (\omega, \pi)$ and $d' \in (0, d)$. Taking into account the short computation preceding the formula (23), the above equality implies that:

$$\begin{aligned}&s^{n+1-\alpha_0}C\big(s+A\big)^{-(n+1)}Cx\\ &= \sum_{j=0}^{n+1}\big(-1\big)^{n+1-j}\binom{n+1}{j}\int\limits_{\Gamma_{\omega',d'}} \frac{z^{\alpha_0-(n+1)}s^{2n+2-j-\alpha_0}}{2\pi i\big(z+s\big)^{n+1-j}}\big(z+A\big)^{-1}C\big(A_C^{-\alpha_0}x\big)\,dz.\end{aligned}$$

By the binomial formula, we obtain that:

$$\begin{aligned}&s^{n+1-\alpha_0}C\big(s+A\big)^{-(n+1)}Cx\\ &= \frac{1}{2\pi i}\int\limits_{\Gamma_{\omega',d'}} \frac{z^{\alpha_0-1}s^{n+1-\alpha_0}}{\big(z+s\big)^{n+1}}z\big(z+A\big)^{-1}C\big(A_C^{-\alpha_0}x\big)\,dz.\end{aligned} \tag{41}$$

It is checked at once that the p-value of the above integral, taken over the curve $\{d'e^{i\theta} : \theta \in [-\omega', \omega']\}$, can be majorized by $c_p'q_p'(A_C^{-\alpha_0}x)$, for some $c_p' > 0$ and $q_p' \in \circledast$, independent of $s > 0$ and $x \in E$. The same conclusion holds for the above integral along the curves $\Gamma_{\omega',d',\pm} := \{re^{\pm i\omega'} : r \ge d'\}$, and we will prove this provided $s \ge 2d'$. If this is the case, then the integral

$$\int\limits_{d'}^{\infty} \frac{s^{n+1-\alpha_0}\big(re^{i\omega'}\big)^{\alpha_0-1}}{\big(re^{i\omega'}+s\big)^{n+1}}re^{i\omega'}\big(re^{i\omega'}+A\big)^{-1}C\big(A_C^{-\alpha_0}x\big)\,dr$$

can be written as the sum of corresponding integrals taken over the intervals $[d', 2s]$ and $[2s, \infty)$, and for the estimation of the first (second) of these integrals, the inequality $\sup_{s>0, z\in\Gamma_{\omega',d',\pm}} |s/(z+s)| < \infty$ $((r/r-s)^{\alpha_0-1} \leq 2^{\alpha_0-1}$, $r \geq 2s)$ can be employed. Therefore, we have proved (40). Arguing similarly, we obtain that for each $p \in \circledast$ there exist $c_p'' > 0$ and $q_p'' \in \circledast$ such that:

$$p\big(s^{n+1-\alpha_0}\big(s+A\big)^{-(n+1)}Cx\big) \leq c_p'' q_p''\big(A_{-\alpha_0}x\big), \quad s>0,\ x \in D\big(A_{-\alpha_0}\big).$$

Further on, Lemma 2.7 implies that, for every $p \in \circledast$, there exist $c_p''' > 0$ and $q_p''' \in \circledast$ such that:

$$p\big(s^{n-\beta_0}\big(s+A\big)^{-(n+1)}Cx\big) \leq c_p''' s^{-\Re\beta_0-1} q_p'''(x), \quad s>0,\ x \in E.$$

Let $c_p \geq \max(c_p', c_p'', c_p''')$ and let $q_p \in \circledast$ satisfy $q_p \geq \max(q_p', q_p'', q_p''')$. Put $c_{p,\alpha_0,\beta_0} := c_p/(\alpha_0-\beta_0)+c_p/\beta_0$ and notice that the equality $q_p(A_C^{-\alpha_0}x)q_p(Cx) = 0$ $(q_p(A_{-\alpha_0}x)q_p$ $(x) = 0)$ for some $x \in E$ $(x \in D(A_{-\alpha_0}))$ implies $p(C(s+A)^{-(n+1)}Cx) = 0$, $s > 0$ $(p((s+A)^{-(n+1)}Cx) = 0,\ s > 0)$, and by Lemma 2.14, $p(CA_C^{-\beta_0}x) = 0$ $(p(A_C^{-\beta_0}x) = 0)$. The remaining part of the proof in the case $\Re\alpha_0 < n+1$ follows from a slight modification of the corresponding parts of the proof of [12, Theorem 5.34, pp. 141-142]. In the case $\Re\alpha_0 = n+1$, then the whole procedure still works by replacing the number n with $n+1$; for example, in the estimation of the term $p(A_C^{-\beta_0}x)$, appearing in the proof of cited theorem, one has to start from the formula (37) with n replaced by $n+1$ therein. ■

Now we are able to prove the moment inequality for C-sectorial operators.

Theorem 2.16. *Suppose* $\alpha,\ \beta,\ \gamma \in \mathbb{C}$, $-\infty < \Re\alpha < \Re\beta < \Re\gamma < +\infty$. *Then, for every* $p \in \circledast$, *there exist* $c_{p,\alpha,\beta,\gamma} > 0$ *and* $q_p \in \circledast$ *such that:*

$$p\big(CA_\beta Cx\big) \leq c_{p,\alpha,\beta,\gamma} q_p\big(A_\alpha Cx\big)^{\frac{\Re\gamma-\Re\beta}{\Re\gamma-\Re\alpha}} q_p\big(A_\gamma Cx\big)^{\frac{\Re\beta-\Re\alpha}{\Re\gamma-\Re\alpha}}, \quad x \in D(A_\gamma), \tag{42}$$

and

$$p\big(A_\beta Cx\big) \leq c_{p,\alpha,\beta,\gamma} q_p\big(A_\alpha x\big)^{\frac{\Re\gamma-\Re\beta}{\Re\gamma-\Re\alpha}} q_p\big(A_\gamma x\big)^{\frac{\Re\beta-\Re\alpha}{\Re\gamma-\Re\alpha}}, \quad x \in D(A_{\alpha-\gamma}A_\gamma). \tag{43}$$

Proof. Keeping in mind Lemma 2.13 and the obvious equality $A_{\alpha-\gamma}A_\gamma x = A_\alpha x$ $(x \in D(A_{\alpha-\gamma}A_\gamma))$, the result immediately follows by plugging $\alpha_0 = \gamma-\alpha$ and $\beta_0 = \gamma-\beta$ in Lemma 2.15. ■

The following lemma of independent interest has not been used in the proof of moment inequality (cf. also [12, Theorem 5.34, pp. 141-142] for the case $C = 1$).

Lemma 2.17. *Suppose* $b \in (0,1)$. *Then the family* $\{C^{-1}\lambda^b A_C^{-b} A(\lambda+A)^{-1}C :$ $\lambda > 0\}$ *is equicontinuous in* $L(E)$.

Proof. Let $\varepsilon \in (0,1)$ be arbitrarily chosen. Then it is not difficult to prove, with the help of (10), that the following equality holds:

$$
\begin{aligned}
&C^{-1}\lambda^b A_C^{-b} A(\lambda+A)^{-1}Cx \\
(44)\qquad &= -\frac{\sin b\pi}{\pi}\int_0^\infty s^{-b}\Big[\frac{\lambda\big(\lambda+A\big)^{-1}Cx}{1-s} - \frac{s\lambda\big(s\lambda+A\big)^{-1}Cx}{1-s}\Big]\,ds, \quad x \in E.
\end{aligned}
$$

Since the family $\{\lambda(\lambda+A)^{-1}C : \lambda > 0\}$ is equicontinuous in $L(E)$, we immediately obtain that, for every $p \in ⊛$, there exist $c_p > 0$ and $q_p \in ⊛$ such that, for every $x \in E$:

$$
(45)\qquad \Big(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty}\Big) s^{-b}\Big[\frac{\lambda\big(\lambda+A\big)^{-1}Cx}{1-s} - \frac{s\lambda\big(s\lambda+A\big)^{-1}Cx}{1-s}\Big]\,ds \le c_p q_p(x).
$$

Using the obvious equality

$$
\lambda\big(\lambda+A\big)^{-1}Cx - s\lambda\big(s\lambda+A\big)^{-1}Cx = \int_\lambda^{s\lambda}\Big[\big(\xi+A\big)^{-1}Cx - \xi\big(\xi+A\big)^{-2}Cx\Big]\,d\xi, \quad x \in E,
$$

the inequality of the same type holds for the integral appearing in (45), taken over the interval $[1-\varepsilon, 1+\varepsilon]$. This completes the proof of lemma. ■

The following application of moment inequality is for illustration purposes only.

Example 2.18. Let E be one of the spaces $L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$ and let $0 \le l \le n$. Put $\mathbb{N}_0^l := \{\alpha \in \mathbb{N}_0^n : \alpha_{l+1} = \cdots = \alpha_n = 0\}$ and recall that the space E_l $(0 \le l \le n)$ is defined by $E_l := \{f \in E : f^{(\alpha)} \in E \text{ for all } \alpha \in \mathbb{N}_0^l\}$. The calibration $(q_\alpha(f) := ||f^{(\alpha)}||_E,\ f \in E_l;\ \alpha \in \mathbb{N}_0^l)$ induces a Fréchet topology on E_l. Let $\mathbf{T_l}(\cdot)$ and $\mathbf{C_{r,l}}$ possess the same meaning as in [41], let $m \in \mathbb{N}$, $a_\alpha \in \mathbb{C}$, $0 \le |\alpha| \le m$, and let $P(D)f = \sum_{|\alpha| \le m} a_\alpha f^{(\alpha)}$ act with its maximal distributional domain. Set $P(x) := \sum_{|\alpha| \le m} a_\alpha i^{|\alpha|} x^\alpha$, $x \in \mathbb{R}^n$, and assume that $\sup_{x \in \mathbb{R}^n} \Re P(x) < 0$. Suppose $-\infty < \varsigma < \tau < \upsilon < +\infty$. By [41, Theorem 2.2], the operator $-P(D)$ is $\mathbf{C_{r,l}}$-sectorial and, since the condition (H) holds, we can construct the powers of $-P(D)$. Then the moment inequality and the arguments used in its proof show that, for every $\alpha \in \mathbb{N}_0^l$, there exists a constant $M_\alpha < \infty$ such that the following differential inequality holds for each $f \in D((-P(D))_{\varsigma-\upsilon}(-P(D))_\upsilon)$:

$$
q_\alpha\big(\big(-P(D)\big)_\tau \mathbf{C_{r,l}} f\big) \le M_\alpha q_\alpha\big(\big(-P(D)\big)_\varsigma f\big)^{\frac{\upsilon-\tau}{\upsilon-\varsigma}} q_\alpha\big(\big(-P(D)\big)_\upsilon f\big)^{\frac{\tau-\varsigma}{\upsilon-\varsigma}}.
$$

In (13) we define A_b in an indirect way. We finally give an explicit formula for $A_b x$ to end this section.

Proposition 2.19. *Let $n \in \mathbb{N}$, let $b \in \mathbb{C}$ and let $n-1 < \Re b < n$. Then the following holds:*

$$A_b x = (-1)^n \frac{\sin \pi b}{\pi} C^{-1} \int_0^\infty \lambda^{b-n} A^n (\lambda + A)^{-1} Cx \, d\lambda, \quad x \in C(D(A^n)). \tag{46}$$

Proof. Suppose first $0 < \Re b < 1$ and $x \in C(D(A))$. Then we immediately obtain from (10) and (13) that

$$A_{b-1}x = -\frac{\sin \pi b}{\pi} C^{-1} \int_0^\infty \lambda^{b-1} (\lambda + A)^{-1} Cx \, d\lambda.$$

Therefore, the fact that $A_b x = A_{b-1}Ax$ and the closedness of A_b together imply

$$A_b x = -\frac{\sin \pi b}{\pi} C^{-1} \int_0^\infty \lambda^{b-1} A(\lambda + A)^{-1} Cx \, d\lambda. \tag{47}$$

In the general case $n - 1 < \Re b < n$, note that $0 < \Re b - n + 1 < 1$ and that Theorem 2.8(ii) yields $A_b x = A_{b-n+1} A_{n-1} x = A_{b-n+1} A^{n-1} x$, $x \in C(D(A^n))$. By the closedness of A_b, we obtain (46) from (47), immediately. ■

Since $C(D(A^n)) \subseteq R((\lambda - A)^{-n}C)$, $\lambda \in \wedge_{\omega,d}$, the following representation formula can be also proved, for any $\lambda \in \wedge_{\omega,d}$ and $x \in R((\lambda - A)^{-n}C)$:

$$\begin{aligned} A_b x = (-1)^n \frac{\sin \pi b}{\pi} \big((\lambda - A)^{-n} C\big)^{-1} C^{-1} \\ \times \int_0^\infty \lambda^{b-n} A^n (\lambda + A)^{-1} C (\lambda - A)^{-n} Cx \, d\lambda. \end{aligned}$$

3. Fractional Powers as Generators of C-regularized Fractional Resolvent Families

We start this section by stating the following result which shows that the operators $-A_b$ generate equicontinuous C-regularized fractional resolvent families for suitable indices b. To do this, we follow the approach similar to that established in [29, Theorem 3.1(b)/(c)].

Theorem 3.1.

(i) *Suppose $D(A)$ and $R(C)$ are dense in E, $0 < \alpha < 2$, $d \in (0,1]$, $\Sigma(\alpha\pi/2, d) \subseteq \rho_C(-A)$, $0 < \gamma < 2$, $b \in (0, (2-\gamma)/(2-\alpha))$ and $\omega \in (\pi - (\alpha\pi)/2, \min(\pi, (\pi - (\pi\gamma)/2)/b)]$. Let $\Gamma_{\omega,d} = \partial(\Sigma_\omega \setminus B_d)$ be oriented in such a way that $\Im\lambda$ increases along $\Gamma_{\omega,d}$ and let the family $\{(1 + |\lambda|)(\lambda + A)^{-1}C : \lambda \subseteq \Sigma(\alpha'\pi/2, d)\}$ be equicontinuous for every $\alpha' \in (0, \alpha)$. Put $S^b_\gamma(0) := C$ and*

$$S^b_\gamma(t)x := \frac{1}{2\pi i} \int\limits_{\Gamma_{\omega,d}} E_\gamma(-\lambda^b t^\gamma)(\lambda - A)^{-1} Cx \, d\lambda, \ t > 0, \ x \in E. \tag{48}$$

Then $-A_b$ *is the densely defined generator of an equicontinuous analytic* (g_γ, C)*-regularized resolvent family* $(S^b_\gamma(t))_{t\geq 0}$ *of angle* $\theta := \min(\pi, (\pi(1-b)/\gamma) + \pi((\alpha b/\gamma)-1)/2)$.

(ii) *Suppose* $D(A)$ *and* $R(C)$ *are dense in* E, $0<\alpha<2$, $0<\gamma<2$, $b\in(0,(2-\gamma)/(2-\alpha))$, $\omega\in(\pi-(\pi\alpha)/2, \min(\pi,(\pi-(\pi\gamma)/2)/b)]$, *and* $-A$ *is a subgenerator of an equicontinuous* (g_α, C)*-regularized resolvent family* $(S_\alpha(t))_{t\geq 0}$. *Define*

$$f^b_{\gamma,\alpha}(t,s) := \frac{1}{2\pi i}\int\limits_{\Gamma_\omega} E_\gamma\big(-\lambda^b t^\gamma\big)\big(-\lambda\big)^{\frac{1}{\alpha}-1} e^{-(-\lambda)^{\frac{1}{\alpha}} s}\, d\lambda, \tag{49}$$

where the contour Γ_ω *is oriented in such a way that* $\Im\lambda$ *increases along* Γ_ω. *Put*

$$S^b_\gamma(t)x := \int\limits_0^\infty f^b_{\gamma,\alpha}(t,s)S_\alpha(s)x\, ds,\ t>0,\ x\in E \text{ and } S^b_\gamma(0) := C. \tag{50}$$

Assume, additionally, that there exists $d\in(0,1]$ *such that* $B_d\subseteq\rho_C(-A)$ *and that the family* $\{(\lambda+A)^{-1}C : \lambda\in B_d\}$ *is equicontinuous. Then* $-A_b$ *is the densely defined generator of the equicontinuous analytic* (g_γ, C)*-regularized resolvent family* $(S^b_\gamma(t))_{t\geq 0}$ *of angle* θ.

Proof. (i): Suppose $\omega_1\in(\pi-(\pi\alpha)/2, \min(\pi,(\pi-(\pi\gamma)/2)/b)]$, $\omega_1<\omega$, $d_1\in(0,1]$ and $\Gamma(\omega,d)$ lies on the right of $\Gamma(\omega_1,d_1)$. Using Lemma 1.3, [19, Proposition 2.16(i)] and the Cauchy theorem (cf. also [2, (1.28)]), it is checked at once that one can interchange the path of integration $\Gamma(\omega,d)$, appearing in (48), with $\Gamma(\omega_1,d_1)$. The equicontinuity of operator family $(S^b_\gamma(t))_{t\geq 0}\subseteq L(E)$ is a consequence of Lemma 1.3 and the choice of ω, whereas the strong continuity of $(S^b_\gamma(t))_{t\geq 0}$ at $t=0$ can be proved as follows. By Lemma 2.7 and the equicontinuity of $(S^b_\gamma(t))_{t\geq 0}$, it suffices to show that, for every $n\in\mathbb{N}$ and $x\in E$, $\lim_{t\to 0+} S^b_\gamma(t)(-n-A)^{-1}Cx = (-n-A)^{-1}C^2x$. Towards this end, notice that the residue theorem and the dominated convergence theorem, in combination with Lemma 1.3, imply the following:

$$S^b_\gamma(t)\big(-n-A\big)^{-1}Cx - \big(-n-A\big)^{-1}C^2x =$$
$$\frac{1}{2\pi i}\int\limits_{\Gamma(\omega,d)} E_\gamma\big(-\lambda^b t^\gamma\big)\big(\lambda-A\big)^{-1}C\big(-n-A\big)^{-1}Cx\, d\lambda - \big(-n-A\big)^{-1}C^2x$$
$$= \frac{1}{2\pi i}\int\limits_{\Gamma(\omega,d)} E_\gamma\big(-\lambda^b t^\gamma\big)\frac{\big(-n-A\big)^{-1}C^2x - \big(\lambda-A\big)^{-1}C^2x}{\lambda+n}\, d\lambda - \big(-n-A\big)^{-1}C^2x$$
$$= \frac{(-1)}{2\pi i}\int\limits_{\Gamma(\omega,d)} E_\gamma\big(-\lambda^b t^\gamma\big)\frac{\big(\lambda-A\big)^{-1}C^2x}{\lambda+n}\, d\lambda - \big(-n-A\big)^{-1}C^2x$$

$$\to \frac{(-1)}{2\pi i} \int\limits_{\Gamma(\omega,d)} \frac{\big(\lambda - A\big)^{-1} C^2 x}{\lambda + n}\, d\lambda - \big(-n - A\big)^{-1} C^2 x = 0,\ t \to 0+ .$$

Now we will prove that:

$$\mathcal{L}(\lambda, x) := \int_0^\infty e^{-\lambda t} S_\gamma^b(t) x\, dt = \lambda^{\gamma-1}\big(\lambda^\gamma + A_b\big)^{-1} C x, \quad \lambda > 0,\ x \in E. \tag{51}$$

The above equality follows from the use of Laplace transform and Fubini's theorem. Indeed, observe that:

$$\begin{aligned} A_C^{-b} =& \frac{1}{2\pi i} \int_{\Gamma_{\omega',d}} z^{-b}\big(z - A\big)^{-1} C\, dz \\ =& \frac{1}{2\pi i} \int_{\Gamma_{\omega',d}} \frac{\lambda^\gamma z^{-b} + 1}{\lambda^\gamma + z^b}\big(z - A\big)^{-1} C\, dz \\ =& \lambda^\gamma \Big(\frac{z^{-b}}{\lambda^\gamma + z^b}\Big)_C (A) + \Big(\frac{1}{\lambda^\gamma + z^b}\Big)_C (A),\ \lambda > 0, \end{aligned}$$

and that (cf. [2, (1.26), (1.28)] and (4)), for every $z \in \Gamma(\omega', d)$, one has:

$$\int\limits_0^\infty e^{-\lambda t} E_\gamma\big(-z^b t^\gamma\big)\, dt = \frac{\lambda^{\gamma-1}}{\lambda^\gamma + z^b},\ \lambda > 0.$$

By this equality and (8), we have further that:

$$\begin{aligned} C\mathcal{L}(\lambda, x) =& \frac{C}{2\pi i} \int_{\Gamma_{\omega',d}} \frac{\lambda^{\gamma-1}}{\lambda^\gamma + z^b}\big(z - A\big)^{-1} C\, dz = C\Big(\frac{\lambda^{\gamma-1}}{\lambda^\gamma + z^b}\Big)_C (A) \\ =& \lambda^{\gamma-1} C A_C^{-b} - \lambda^\gamma \Big(\frac{\lambda^{\gamma-1}}{\lambda^\gamma + z^b} z^{-b}\Big)_C (A) C \\ =& \lambda^{\gamma-1} C A_C^{-b} - \lambda^\gamma \Big(\frac{\lambda^{\gamma-1}}{\lambda^\gamma + z^b}\Big)_C (A) \big(z^{-b}\big)_C (A) \\ =& \lambda^{\gamma-1} C A_C^{-b} - \lambda^\gamma \mathcal{L}(\lambda) A_C^{-b}. \end{aligned}$$

Now it readily follows that $(0, \infty) \subseteq \rho_C(-A_b)$ and that (51) holds, which implies together with the equality $C^{-1} A_b C = A_b$ that $-A_b$ is the integral generator of the equicontinuous (g_γ, C)-regularized resolvent family $(S_\gamma^b(t))_{t \geq 0}$ (cf. also [19, Theorem 2.7]). Clearly, Theorem 2.8(ii) implies that A_b is densely defined in E. Using the inequality $b < (2 - \gamma)/(2 - \alpha)$, Theorem 2.8(iv) and its proof, we have that A_b is C-sectorial of angle $b\pi(1 - (\alpha/2))$ and that the mapping $\lambda \mapsto (\lambda + A_b)^{-1} C x$, $\lambda \in \Sigma_{\pi - b\pi(1 - (\alpha/2))}$ is analytic for all $x \in E$. By [19, Theorem 3.7], we obtain that, for every $b' \in (0, b)$, the operator $-A_{b'}$ is the integral generator of an equicontinuous analytic $(g_{2(1-b'(1-(\alpha/2)))}, C)$-regularized resolvent family of angle $\frac{\pi - b'(\pi - \frac{\alpha\pi}{2})}{2(1-b'(1-(\alpha/2)))} - \frac{\pi}{2}$. Take now $b' \in (0, b)$ such that $2(1 - b'(1 - (\alpha/2))) > \gamma$. Then an application of [19, Theorem 3.9(ii)] shows that the operator $-A_{b'}$ is the integral generator of the equicontinuous analytic (g_γ, C)-regularized resolvent family $(S_\gamma^{b'}(t))_{t\geq 0}$ of angle $\min(\pi, (\pi(1-b')/\gamma)+\pi((\alpha b'/\gamma)-1)/2)$. This completes the proof by letting $b' \to b-$. The proof of (ii) follows immediately from the proof of [29, Theorem 3.1(c)] and the first part of theorem. ■

Remark 3.2.

(i) In the case that there exists $d \in (0, 1]$ such that the family $\{(\lambda - A)^{-1}C : \lambda \in B_d\}$ is equicontinuous, Theorem 3.1 extends the assertion of [29, Theorem 3.1]. Suppose now that $D(A)$ and $R(C)$ are not densely defined in E as well as that all remaining assumptions quoted in the formulation of Theorem 3.1 hold. Then it can be proved that, for every $\sigma > 0$, the operator $-A_b$ is the integral generator of an analytic $(g_\gamma, g_{\sigma+1})$-regularized C-resolvent family of angle θ and of subexponential growth (cf. [19] for the notion).

(ii) In the previous version of the paper, we have also considered the case $0 \notin \rho_C(A)$. If $D(A)$ and $R(C)$ are dense in E, $0 < b \leq 2/(\alpha+2)$ and $\gamma = \alpha b$, then the assertion of Theorem 3.1 can be proved without the use of spectral mapping theorem (the integral generator in this case is the operator $C^{-1}\overline{s - \lim_{\varepsilon\to 0+} -(A+\varepsilon)_b C}$, defined usually). The method used in the proof relies upon the recent results on generalized subordination kernels ([5]), whose value in the existing theory has not been analyzed very well so far, and an elementary argumentation from the real analysis. This method will not be employed in our follow-up researches and, because of that, we will omit details in the interest of brevity.

The following example illustrates an application of Theorem 3.1.

Example 3.3. Suppose $(M_p)_{p\in\mathbb{N}_0}$ is a sequence of positive numbers which satisfies $M_0 = 1$, (M.1), (M.2) and (M.3') (cf. [20, Sections 1.3 and 3.5-3.6] for definitions and additional information). Put

$$E := \Big\{ f \in C^\infty[0, 1] \ ; \ \|f\| := \sup_{p\in\mathbb{N}_0} \frac{\|f^{(p)}\|_\infty}{M_p} < \infty \Big\},$$

$A := -d/ds$, $D(A) =: \{f \in E : f' \in E,\ f(0) = 0\}$ and $E^{(M_p)}(A) := \{f \in D_\infty(A) : \sup_{p\in\mathbb{N}_0} \frac{h^p\|f^{(p)}\|_\infty}{M_p} < \infty$ for all $h > 0\}$. Then A generates a non-dense ultradistribution semigroup of (M_p)-class, $\rho(A) = \mathbb{C}$ and there exists an injective operator $C \in L(E)$ such that $E^{(M_p)}(A) \subseteq C(D_\infty(A))$,

$$C\big(g_\beta * f\big) = g_\beta * Cf, \quad \beta > 0,\ f \in E, \tag{52}$$

and that A generates a bounded C-regularized semigroup $(S(t))_{t\geq 0}$ on E (cf. [20, Example 3.5.15, (316), Theorem 3.6.4, Lemma 3.6.5]). Therefore, we are in a position to construct the fractional powers of $-A$. Using (10), (16) and (52), it readily follows that $(f,g) \in (-A)_b$ iff $f(t) = \int_0^t g_b(t-s)g(s)\,ds$, $t \in [0,1]$ $(b > 0)$. Suppose now $1 < \gamma < 2$, $1 < b < 2-\gamma$ and $f_0,\ f_1 \in E^{(M_p)}(A)$. By Remark 3.2(i), we obtain that for each $\sigma > 0$ the operator $-(-A)_b$ is the integral generator of an analytic $(g_\gamma, g_{\sigma+1})$-regularized C-resolvent family $(S^b_{\gamma,\sigma}(t))_{t\geq 0}$ of angle θ, with θ being defined in the formulation of Theorem 3.1 with $\alpha = 1$. Furthermore, $S^b_{\gamma,\sigma}(t)f = \int_0^t g_\sigma(t-s)S^b_\gamma(s)f\,ds$, $t\geq 0$, $f\in E$ and the mapping $t\mapsto S^b_\gamma(t)f$, $t\geq 0$ is continuous for any $f\in \overline{D(A)}$ (cf. (48)). Now it is not difficult to prove that there exists a unique function $u\in C([0,\infty):[D((-A)_b)])\ \cap C^1([0,\infty):E)$ which solves the problem:

$$
(53)\qquad \begin{aligned}
&u(t,x) + \int\limits_0^x g_b(x-s)\mathbf{D}_t^\gamma u(t,s)\,ds = 0,\\
&\qquad\qquad u(0,x) = f_0(x) \text{ and } \frac{\partial}{\partial t}u(0,x) = f_1(x), \quad t\geq 0,\ x\in[0,1].
\end{aligned}
$$

Moreover, the solution $u(t)$ is given by

$$
u(t,\cdot) = S^b_\gamma(t)C^{-1}f_0 + \int_0^t S^b_\gamma(s)C^{-1}f_1\,ds, \quad t\geq 0,
$$

and can be analytically extended to the sector Σ_θ (cf. also [24, Proposition 3.4] for some inhomogeneous fractional equations). It is worth noting that the problem (53) is a sort of backwards diffusion equation with space-time fractional derivatives (cf. [3] and [15] for some applications in describing the mechanism of anomalous diffusion in transport processes).

From the previous analysis, it is clear that the results obtained in this paper can be applied to a class of abstract differential equations considered in ultradistribution spaces. For example, it is not difficult to prove, with the help of Theorem 2.8(iv) and [19, Theorem 3.15], that there exists an injective operator $C_1\in L(E)$ such that the operator $(-A)^{1/2}$, resp. $-A$, generates a global C_1-regularized group, resp. a global C_1-regularized cosine function.

In the remaining part of the paper, we consider the constructed powers as the integral generators of C-regularized semigroups of growth order $r>0$ (cf. [8, 35, 38] and [20]-[21]).

Definition 3.4.

(i) An operator family $(T(t))_{t>0}\subseteq L(E)$ is said to be a C-regularized semigroup of growth order $r>0$ iff the following holds:

(a) $T(t+s)C = T(t)T(s), \quad t,\ s > 0,$
(b) for every $x \in E$, the mapping $t \mapsto T(t)x$, $t > 0$ is continuous,
(c) the family $\{t^r T(t) : t \in (0,1]\}$ is equicontinuous, and
(d) $T(t)x = 0$ for all $t > 0$ implies $x = 0$.

(ii) Suppose $\gamma \in (0, \frac{\pi}{2}]$, $(T(t))_{t>0}$ is a C-regularized semigroup of growth order $r > 0$, and the mapping $t \mapsto T(t)x$, $t > 0$ has an analytic extension to the sector Σ_γ, denoted by the same symbol. If there exists $\omega \in \mathbb{R}$ such that, for every $\delta \in (0,\gamma)$, the family $\{z^r e^{-\omega \Re z} T(z) : z \in \Sigma_\delta\}$ is equicontinuous, then $(T(t))_{t \in \Sigma_\gamma}$ is said to be an analytic C-regularized semigroup of growth order r.

The integral generator $\hat{G}$, resp. the infinitesimal generator G, of $(T(t))_{t>0}$ (cf. [21] and [25]), is defined by

$$\hat{G} := \Big\{(x,y) \in E \times E : T(t)x - T(s)x = \int_s^t T(r)y\, dr \text{ for all } t,\ s > 0 \text{ with } t \geq s\Big\},$$

resp,,

$$G := \Big\{(x,y) \in E \times E : \lim_{t \to 0+} \frac{T(t)x - Cx}{t} = Cy\Big\}.$$

The integral generator $\hat{G}$ is a closed linear operator which satisfies $C^{-1}\hat{G}C = \hat{G}$. Moreover, $G \subseteq \hat{G}$ and G is a closable linear operator. The closure of G, denoted by $\overline{G}$, is said to be the complete infinitesimal generator, in short, the c.i.g. of $(T(t))_{t>0}$. The integral generator $\hat{G}$ contains the c.i.g. $\overline{G}$ and satisfies $\hat{G} = \{(x,y) \in E \times E : (T(s)x, T(s)y) \in G \text{ for all } s > 0\}$. The set $\{x \in E : \lim_{t \to 0+} T(t)x = Cx\}$, resp. $\{x \in E : \lim_{z \to 0, z \in \Sigma_{\gamma'}} T_b(z)x = Cx \text{ for all } \gamma' \in (0,\gamma)\}$ is said to be the continuity set of $(T(t))_{t>0}$, resp. $(T(z))_{z \in \Sigma_\gamma}$.

The subsequent assertions correspond to [21, Theorem 3.1/3.2].

Theorem 3.5. *Suppose $b \in (0, 1/2)$ and a closed linear operator A satisfies (H1) with $\alpha > -1$. Denote by Γ the frontier of the region $-(P_{\alpha,\varepsilon,d} \cup B_d)$, oriented in such a way that $\Im\lambda$ increases along the curve $\{z \in \mathbb{C} : |z| = d,\ z \in \partial(-(P_{\alpha,\varepsilon,d} \cup B_d))\}$. Set $\gamma := \arctan(\cos(\pi b))$ and*

$$T_b(z)x := \frac{1}{2\pi i} \int\limits_\Gamma e^{-z\lambda^b} \big(\lambda - A\big)^{-1} Cx\, d\lambda, \quad x \in E,\ z \in \Sigma_\gamma. \tag{54}$$

(i) *Then $(T_b(z))_{z \in \Sigma_\gamma}$ is an analytic C-regularized semigroup of growth order $(\alpha + 1)/b$, and the integral generator of $(T_b(z))_{z \in \Sigma_\gamma}$ is the operator $\hat{G} = -A_b$. Denote by $\Omega_b(A)$, resp. $\Omega_{b,\theta}(A)$, the continuity set of $(T_b(z))_{z \in \Sigma_\gamma}$, resp. $(T_b(te^{i\theta}))_{t>0}$. Then the following holds:*

(a) *For every $\delta \in (0,\gamma)$, the family $\{z^{(\alpha+1)/b} T_b(z) : z \in \Sigma_\delta\}$ is equicontinuous.*

(b) *The mapping $z \mapsto T_b(z)x,\ z \in \Sigma_\gamma$ is analytic for every $x \in E$, $\bigcup_{z\in\Sigma_\gamma} R(T_b(z)) \subseteq D_\infty(A)$, and for every $n \in \mathbb{N}$, $x \in E$ and $z \in \Sigma_\gamma$, the following holds:*

$$\frac{d^n}{dz^n}T_b(z)x = \frac{(-1)^n}{2\pi i}\int\limits_{\Gamma} \lambda^{nb}e^{-z\lambda^b}\big(\lambda - A\big)^{-1}Cx\,d\lambda \tag{55}$$

and

$$A^nT_b(z)x = \frac{1}{2\pi i}\int\limits_{\Gamma} e^{-z\lambda^b}\lambda^n\big(\lambda - A\big)^{-1}Cx\,d\lambda. \tag{56}$$

(b') *Let $\beta > 0$, and let $|\theta| < \gamma$. Then*

$$D_-^{\beta}T_b\big(te^{i\theta}\big)x = \frac{1}{2\pi i}\int\limits_{\Gamma} \big(e^{i\theta}\lambda^b\big)^{\beta}e^{-te^{i\theta}\lambda^b}\big(\lambda - A\big)^{-1}Cx\,d\lambda \tag{57}$$

holds for all $t > 0$ and $x \in E$, where D_-^{β} denotes the Liouville right-sided fractional derivative of order β (see [16, (2.3.4)]); and

$$A_{b\beta}T_b\big(te^{i\theta}\big)x = \frac{1}{2\pi i}\int\limits_{\Gamma} \lambda^{b\beta}e^{-te^{i\theta}\lambda^b}\big(\lambda - A\big)^{-1}Cx\,d\lambda \tag{58}$$

for all $t > 0$ and $x \in R(C)$.

(c) *We have $D(A^{\lfloor b+\alpha\rfloor+1}) \subseteq \Omega_b(A)$, provided $\lfloor b+\alpha\rfloor \geq 0$.*

(d) *If $\lfloor b+\alpha\rfloor \geq 0$, $x \in D(A^{\lfloor b+\alpha\rfloor+2})$ and $\gamma' \in (0,\gamma)$, then*

$$\lim_{z\to 0, z\in\Sigma_{\gamma'}} \frac{T_b(z)x - Cx}{z} = \frac{(-1)}{2\pi i}\int\limits_{\Gamma} (-\lambda)^{b-1}\big(\lambda - A\big)^{-1}CAx\,d\lambda. \tag{59}$$

(e) *For every $z \in \Sigma_\gamma$, $T_b(z)$ is an injective operator.*

(ii) *Suppose $n \in \mathbb{N}\setminus\{1,2\}$, $|\theta| < \arctan(\cos(\pi/n))$ and $x \in \Omega_{1/n,\theta}(A)$. Then the function $u : (0,\infty) \to E$, defined by $u(t) := T_{1/n}(te^{i\theta})x,\ t > 0$, is a solution of the abstract Cauchy problem*

$$(P_n):\begin{cases} u \in C((0,\infty) : D_\infty(A)) \cap C^\infty((0,\infty) : E),\\ \frac{d^n}{dt^n}u(t) = (-1)^n e^{in\theta}Au(t),\ t > 0,\\ \lim_{t\to 0+} u(t) = Cx,\ \textit{and the set } \{u(t) : t > 0\} \textit{ is bounded.}\end{cases}$$

Moreover, $u(\cdot)$ can be analytically extended to the sector $\Sigma_{\arctan(\cos(\pi/n))-|\theta|}$ and, for every $\delta \in (0, \arctan(\cos(\pi/n)) - |\theta|)$ and $j \in \mathbb{N}_0$, we have that the set $\{z^{j+n\alpha+n}u^{(j)}(z) : z \in \Sigma_\delta\}$ is bounded. The previous conclusions hold in the case $(1/n) + \alpha \geq 0$ and $x \in D(A^{\lfloor (1/n)+\alpha\rfloor+1})$.

(ii') *Suppose* $\beta > 0$, $|\theta| < \gamma$ *and* $x \in \Omega_{b,\theta}(A) \cap R(C)$. *Then the function* $u : (0,\infty) \to E$, *defined by* $u(t) := T_b(te^{i\theta})x,\ t > 0$, *is a solution of the fractional abstract Cauchy problem*

$$(P_\beta) : \begin{cases} u \in C((0,\infty) : D_\infty(A)) \cap C^\infty((0,\infty) : E), \\ D_-^\beta u(t) = e^{i\theta\beta} A_{b\beta} u(t),\ t > 0, \\ \lim_{t\to 0+} u(t) = Cx,\ \textit{and the set}\ \{u(t) : t > 0\}\ \textit{is bounded.} \end{cases}$$

Moreover, $u(\cdot)$ *can be analytically extended to the sector* $\Sigma_{\arctan(\cos(b\pi))-|\theta|}$ *and, for every* $\delta \in (0, \arctan(\cos(b\pi)) - |\theta|)$ *and* $j \in \mathbb{N}_0$, *we have that the set* $\{z^{j+(1+\alpha)/b} u^{(j)}(z) : z \in \Sigma_\delta\}$ *is bounded. The previous conclusions hold in the case* $b + \alpha \geq 0$ *and* $x \in D(A^{\lfloor b+\alpha \rfloor + 1}) \cap R(C)$.

Outline of the proof. We will only prove the first part of theorem. In almost the same way as in the proof of [21, Theorem 3.2] (cf. also [19, Theorem 3.15/3.16], [20, Theorem 1.4.15] and [37, Section 2]), one can prove that $(T_b(z))_{z\in\Sigma_\gamma}$ is an analytic C-regularized semigroup of growth order $(\alpha+1)/b$ as well as that (c), (e) and (55)-(59) hold. Now we will prove that the operator $-A_b$ (cf. Remark 2.12(ii) for the definition and notation used below) is the integral generator of $(T_b(z))_{z\in\Sigma_\gamma}$. Define $S_b(z) := \frac{1}{2\pi i}\int_\Gamma e^{-z\lambda^b}(\lambda - A)^{-1}\mathcal{C}x\,d\lambda$, $x \in E$, $z \in \Sigma_\gamma$ and $S_{b,1}(z) := \int_0^z S_b(\sigma)x\,d\sigma$, $x \in E$, $z \in \Sigma_\gamma$. By the proof of Theorem 3.1, we easily infer that $(S_{b,1}(t))_{t\geq 0}$ is a once integrated C-semigroup which do have the operator $-A_b$ as the integral generator. Since $T_b(z)(d+\kappa-A)^{-n_\alpha}C = S_b(z)C$, $z \in \Sigma_\gamma$, we immediately obtain that $(x,y) \in \hat{G}$ iff

$$S_b(t)x - S_b(s)x = \int_s^t S_b(r)y\,dr \text{ for any } t > s > 0 \text{ with } t \geq s. \tag{60}$$

Using the fact that $\lim_{s\to 0+} S_b(s)(-n-A)^{-1}\mathcal{C}x = \mathcal{C}(-n-A)^{-1}\mathcal{C}x$, $x \in E$, $n \in \mathbb{N}$, and an elementary argumentation, one gets that (60) is equivalent with

$$\int_0^t S_b(\sigma)x\,d\sigma - t\mathcal{C}x = \int_0^t \Big(\int_0^r S_b(\sigma)y\,d\sigma\Big)\,dr, \quad t \geq 0,$$

which holds since the integral generator of $(S_{b,1}(t))_{t\geq 0}$ is $-A_b$. Now we will prove the assertion (b') in the non-trivial case $\beta \in (0,\infty)\setminus\mathbb{N}$. Since $0 < x + \arctan(\cos x) < \pi/2$, provided $0 < x < \pi/2$, we have $|b\arg\lambda + \theta| < b\pi + \arctan(\cos(b\pi)) < \pi/2$, $\lambda \in \Gamma$, which implies that $\Re(e^{i\theta}\lambda^b) = |\lambda|^b\cos(b\arg\lambda + \theta) > 0$, $\lambda \in \Gamma$ and

$$\begin{aligned} \int_t^\infty g_{\beta-\lceil\beta\rceil}(s-t)e^{-se^{i\theta}\lambda^b}\,ds &= \Big(\int_0^\infty g_{\beta-\lceil\beta\rceil}(v)e^{-ve^{i\theta}\lambda^b}\,dv\Big)e^{-te^{i\theta}\lambda^b} \\ &= \big(e^{i\theta}\lambda^b\big)^{\beta-\lceil\beta\rceil}e^{-te^{i\theta}\lambda^b}, \end{aligned}$$

for any $\lambda \in \Gamma$. By definitions of D_-^β and $T_b(\cdot)$, we obtain

$$
\begin{aligned}
&D_-^\beta T_b(te^{i\theta})x \\
=&\Big(-\frac{d}{dt}\Big)^{\lceil\beta\rceil}\int_t^\infty g_{\lceil\beta\rceil-\beta}(s-t)\Big(\frac{1}{2\pi i}\int_\Gamma e^{-se^{i\theta}\lambda^b}(\lambda-A)^{-1}Cx\,d\lambda\Big)\,ds \\
=&\Big(-\frac{d}{dt}\Big)^{\lceil\beta\rceil}\frac{1}{2\pi i}\int_\Gamma\Big(\int_t^\infty g_{\lceil\beta\rceil-\beta}(s-t)e^{-se^{i\theta}\lambda^b}\,ds\Big)(\lambda-A)^{-1}Cx\,d\lambda \\
=&\Big(-\frac{d}{dt}\Big)^{\lceil\beta\rceil}\frac{1}{2\pi i}\int_\Gamma(e^{i\theta}\lambda^b)^{\beta-\lceil\beta\rceil}e^{-te^{i\theta}\lambda^b}(\lambda-A)^{-1}Cx\,d\lambda \\
=&(-1)^{\lceil\beta\rceil}\frac{1}{2\pi i}\int_\Gamma(e^{i\theta}\lambda^b)^{\beta-\lceil\beta\rceil}(-1)^{\lceil\beta\rceil}(e^{i\theta}\lambda^b)^{\lceil\beta\rceil}e^{-te^{i\theta}\lambda^b}(\lambda-A)^{-1}Cx\,d\lambda \\
=&\frac{1}{2\pi i}\int_\Gamma(e^{i\theta}\lambda^b)^{\beta}e^{-te^{i\theta}\lambda^b}(\lambda-A)^{-1}Cx\,d\lambda,\quad t>0,\ x\in E.
\end{aligned}
$$

Suppose now $b\beta \notin \mathbb{N}$. Keeping in mind that $\bigcup_{z\in\Sigma_\gamma} R(T_b(z)) \subseteq D_\infty(A)$, it is routine to show that, with a suitable choice of the contour Γ',

$$
\begin{aligned}
&A_{b\beta}T_b(te^{i\theta})x = A_{b\beta-\lceil b\beta\rceil}A^{\lceil b\beta\rceil}T_b(te^{i\theta})x \\
=&A_{b\beta-\lceil b\beta\rceil}\frac{1}{2\pi i}\int_\Gamma e^{-te^{i\theta}\lambda^b}\lambda^{\lceil b\beta\rceil}(\lambda-A)^{-1}Cx\,d\lambda \\
=&C^{-1}\frac{1}{2\pi i}\int_{\Gamma'}\mu^{b\beta-\lceil b\beta\rceil}(\mu-A)^{-1}C\Big(\frac{1}{2\pi i}\int_\Gamma e^{-te^{i\theta}\lambda^b}\lambda^{\lceil b\beta\rceil}(\lambda-A)^{-1}Cx\,d\lambda\Big)\,d\mu \\
=&\frac{1}{2\pi i}\int_\Gamma e^{-te^{i\theta}\lambda^b}\lambda^{b\beta}(\lambda-A)^{-1}Cx\,d\lambda,\quad x\in R(C).
\end{aligned}
$$

This proves (57)-(58). ∎

Remark 3.6. Recall that $D_-^n = (-1)^n D^n$ for $n \in \mathbb{N}$, where D^n denotes the usual derivative operator of order n ([16, (2.3.5)]).

Theorem 3.7. *Suppose* $d \in (0,1]$, $\gamma \in (0,\pi/2)$, $\alpha > -1$ *and* $b \in (0, \pi/(2(\pi-\gamma)))$. *Set* $\varphi := \arctan(\cos(b(\pi-\gamma)))$ *and assume that* $\Sigma(\gamma,d) \subseteq \rho_C(-A)$ *and that the family* $\{(1+|\lambda|)^{-\alpha}(\lambda+A)^{-1}C : \lambda \in \Sigma(\gamma,d)\}$ *is equicontinuous.*

(i) *Denote by* Γ *the frontier of the region* $-\Sigma(\gamma,d)$, *oriented counterclockwise. Then* $(T_b(z))_{z\in\Sigma_\varphi}$ *(cf. (54)) is an analytic* C*-regularized semigroup of growth order* $(\alpha+1)/b$, *and the integral generator of* $(T_b(z))_{z\in\Sigma_\varphi}$ *is the operator*

$\hat{G} = -A_b$. *Denote by* $\Omega_b(A)$, *resp.* $\Omega_{b,\theta}(A)$, *the continuity set of* $(T_b(z))_{z\in\Sigma_\varphi}$, *resp.* $(T_b(te^{i\theta}))_{t>0}$. *Then the following holds:*

(a) *For every* $\delta \in (0,\varphi)$, *the family* $\{z^{(\alpha+1)/b}T_b(z) : z \in \Sigma_\delta\}$ *is equicontinuous.*

(b) *The mapping* $z \mapsto T_b(z)x,\ z \in \Sigma_\varphi$ *is analytic for every* $x \in E$, $\bigcup_{z\in\Sigma_\varphi} R(T_b(z)) \subseteq D_\infty(A)$, *and (55)-(58) hold with* γ *replaced by* φ *therein.*

(c) *We have* $D(A^{\lfloor b+\alpha\rfloor+1}) \subseteq \Omega_b(A)$, *provided* $\lfloor b+\alpha\rfloor \geq 0$.

(d) *If* $\lfloor b+\alpha\rfloor \geq 0$, $x \in D(A^{\lfloor b+\alpha\rfloor+2})$ *and* $\varphi' \in (0,\varphi)$, *then (59) holds.*

(ii) *Suppose* $n \in \mathbb{N}\backslash\{1\}$, $|\theta| < \arctan(\cos((\pi-\gamma)/n))$ *and* $x \in \Omega_{1/n,\theta}(A)$. *Then the function* $u : (0,\infty) \to E$, *defined by* $u(t) := T_{1/n}(te^{i\theta})x,\ t > 0$, *is a solution of the abstract Cauchy problem* (P_n). *Put* $a_{n,\theta} := \arctan(\cos((\pi-\gamma)/n)) - |\theta|$. *Then the solution* $u(\cdot)$ *can be analytically extended to the sector* $\Sigma_{a_{n,\theta}}$ *and, for every* $\delta \in (0, a_{n,\theta})$ *and* $i \in \mathbb{N}_0$, *we have that the set* $\{z^{i+n\alpha+n}u^{(i)}(z) : z \in \Sigma_\delta\}$ *is bounded. The previous conclusions hold in the case* $(1/n) + \alpha \geq 0$ *and* $x \in D(A^{\lfloor (1/n)+\alpha\rfloor+1})$.

(ii') *Suppose* $\beta > 0$, $|\theta| < \varphi$ *and* $x \in \Omega_{b,\theta}(A) \cap R(C)$. *Then the function* $u : (0,\infty) \to E$, *defined by* $u(t) := T_b(te^{i\theta})x,\ t > 0$, *is a solution of the abstract Cauchy problem* (P_β). *Put* $a_{b,\theta} := \arctan(\cos((\pi-\gamma)b)) - |\theta|$. *Then the solution* $u(\cdot)$ *can be analytically extended to the sector* $\Sigma_{a_{b,\theta}}$ *and, for every* $\delta \in (0, a_{b,\theta})$ *and* $i \in \mathbb{N}_0$, *we have that the set* $\{z^{i+(\alpha+1)/b}u^{(i)}(z) : z \in \Sigma_\delta\}$ *is bounded. The previous conclusions hold in the case* $b+\alpha \geq 0$ *and* $x \in D(A^{\lfloor b+\alpha\rfloor+1}) \cap R(C)$.

The assertions of Theorem 3.5 and Theorem 3.7 can be reformulated, with some obvious modifications, in the case $\alpha = -1$. We would also like to mention that the uniqueness of solutions of the problem (P_β) can be proved provided that $\beta = 2$, $n(A) \leq 1$, $C = I$ and that E is a Banach space ([21]). We leave to the interested reader problems of:

(i) finding general conditions under which the problem (P_β) has a unique solution (in this context, we also refer the reader to [35, Proposition 2, Theorem 3] for the case $\beta = 2$),

(ii) describing the c.i.g. of the semigroup $(T_b(t))_{t>0}$ appearing in Theorem 3.5 and Theorem 3.7.

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