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THREE TYPES OF VARIATIONAL RELATION PROBLEMS

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Abstract. Variational relation problems were introduced by Luc in [1] as a general model for a large class of problems in nonlinear analysis and applied mathematics. Since this manner of approach provides unified results for several mathematical problems it has been used in many recent papers (see [2-9]). In this paper we investigate the existence of solutions for three types of variational relation problems which encompass several generalized equilibrium problems, variational inequalities and variational inclusions studied in a long list of papers in the field.

1. Introduction

Let X, Y and Z be nonempty sets. A nonempty subset R of the product $X \times Y \times Z$ determines a relation $R(x, y, z)$ in a natural manner: we say that $R(x, y, z)$ holds if and only if $(x, y, z) \in R$. When Z is a parameter set, then R is called a variational relation. Variational relation problems were introduced by Luc in [1] as a general model for a large class of problems in nonlinear analysis and applied mathematics, including optimization problems, variational inequalities, variational inclusions, equilibrium problems, etc. Since this manner of approach provides unified results for several mathematical problems it has been used in many recent papers (see [2-10]). The present paper fits into this interesting group of works, establishing several existence theorems for the solutions of the following three types of variational relation problems:

Assume that X is a convex set in a topological vector space and Y and Z are two sets, endowed for each problem with an adequate topological and/or algebraic structure. Let $T : X \multimap Y$, $P : X \multimap Z$ be two set-valued mappings and $R(x, y, z)$ be a relation linking elements $x \in X$, $y \in Y$, $z \in Z$.

(VRP 1_a) Find $\bar{x} \in X$ such that $R(\bar{x}, y, z)$ holds for all $y \in T(\bar{x})$ and all $z \in P(\bar{x})$.

(VRP 1_b) Find $\bar{x} \in X$ such that for each $y \in T(\bar{x})$ there exists $z \in P(\bar{x})$ such that $R(\bar{x}, y, z)$ holds.

(VRP 2) Find $\bar{x} \in X$ and $\bar{z} \in P(\bar{x})$ such that $R(\bar{x}, y, \bar{z})$ holds for all $y \in T(\bar{x})$.

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These problems encompass several generalized equilibrium problems, variational inequalities and variational inclusions studied in a long list of papers in the field. To motivate our investigation we list below a few typical examples:

(a) Assume that V is a topological vector space ordered by a closed convex cone C with nonempty interior and $f : X \times Y \times X \to V$ is a single-valued mapping. Taking $X = Z$, $P(x) = X$ and the relation R defined by

$$R(x, y, z) \text{ holds iff } f(x, y, z) \not< 0,$$

problem (VRP 1_a) reduces to the following:

Find $\bar{x} \in X$ such that $f(\bar{x}, y, z) \not< 0$ for all $y \in T(\bar{x})$ and $z \in X$.

Here, for $v \in V$, $v \not< 0$ means $v \notin -\text{int}\, C$. This is a vector variational inequality studied in [11] and [12].

(b) Let E and F be two topological vector spaces, X be a convex subset of E, $\eta : X \times X \to E$ be a single-valued mapping and $C : X \multimap F$ a set-valued mapping with closed convex cones values. Set $Y = L(E, F)$ (the family of all continuous linear operators from E into F), $Z = X$ and $P(x) = X$. Denote by $\langle y, x \rangle$ the evaluation of $y \in L(E, F)$ at $x \in E$. Define the variational relation R as follows:

$$R(x, y, z) \text{ hods iff } \langle y, \eta(x, z) \rangle \in C(x),$$

Problem (VRP 1_a) is formulated now as follows:

Find $\bar{x} \in X$ such that $\langle T(\bar{x}), \eta(\bar{x}, z) \rangle \subseteq C(\bar{x})$ for all $z \in X$.

Here $\langle T(\bar{x}), \eta(\bar{x}, z) \rangle = \bigcup\limits_{y \in T(\bar{x})} \langle y, \eta(x, z) \rangle$. This problem is considered in [13].

Replacing the relation R by one of the relations R_1, R_2 defined below

$$R_1(x, y, z) \text{ hods iff } \langle y, \eta(x, z) \rangle \notin C(x) \setminus \{0\},$$

$$R_2(x, y, z) \text{ hods iff } \langle y, \eta(x, z) \rangle \notin \text{ int}\, C(x),$$

problem (VRP 1_b) collapses to the strong, respectively the weak generalized vector variational-like inequality investigated in [14]-[16].

(c) Let X be a nonempty convex set in a topological vector space, Z be a nonempty set in a topological vector space and V be a topological vector space. Let $C : X \multimap V$ be a set-valued mapping such that, for each $x \in X$, $C(x)$ is a closed, convex and pointed cone with nonempty interior and $F : X \times X \times Z \multimap V$ be a set-valued mapping. Set $X = Y$, $T(x) = X$ and define the relation R by

$$R(x, y, z) \text{ hods iff } F(x, y, z) \nsubseteq \text{int}\, C(x).$$

Then problems (VRP 1_b) and (VRP 2) are formulated as follows:

$$\text{Find } \bar{x} \in X \text{ such that for each } y \in X \text{ there exists } z \in P(\bar{x})$$
$$\text{such that } F(\bar{x}, y, z) \not\subseteq \operatorname{int} C(\bar{x});$$

and respectively,

$$\text{Find } \bar{x} \in X \text{ and } \bar{z} \in P(\bar{x}) \text{ such that } F(\bar{x}, y, \bar{z}) \not\subseteq \operatorname{int} C(\bar{x}) \text{ holds for all } y \in X.$$

The two problems above or particular forms of them are studied in [17]-[23].

As one of reviewers remarked, problems (VRP 1_a) and (VRP 2) can be regarded as special cases of some variational relation problems investigated in the recent literature. Using a combined variable (y, z), problem (VRP 1_a) becomes: find $\bar{x} \in X$ such that $R(\bar{x}, (y, z))$ holds for every $(y, z) \in (T \times P)(\bar{x})$. Thus (VRP 1_a) is a special case of the variational relation problem studied in [10], which in turn is a particular form of that studied in [7]. Considering now the pair (x, z) as a unique variable, problem (VRP 2) looks like: find $(\bar{x}, \bar{z}) \in X \times Z$ which is fixed point for the map $(x, z) \mapsto \{x\} \times P(x)$ and satisfies $R((\bar{x}, \bar{z}), y)$ for all $y \in T(\bar{x}, \bar{z})$. So, problem (VRP2) can be considered as a special case of the problem studied by Luc in [1] (see also [2], [5] and [9]), in which the third variable is missing. Therefore existence theorems for the solutions of problems (VRP 1_a) and (VRP 2) could be derived from the corresponding results found in the above mentioned papers. However we do not use this method. The proofs of all existence theorems (except Theorem 4.3) rely, more or less, on a variational relation model involving inclusion of two set-valued mappings.

The paper is structured as follows. In the next section we give some notations and preliminaries results. In Section 3 we study an auxiliary inclusion problem by using a generalized KKM theorem. The obtained result is then applied to establish, in a unified manner, existence criteria for problems (VRP 1_a) and (VRP 1_b). Section 4 is dedicated to the problem (VRP 2).

2. Preliminaries

Definition 2.1. Let X be a nonempty convex set in a vector space and Y be a nonempty set. A set-valued mapping $Q : Y \multimap X$ is said to be generalized KKM if for every finite set $\{y_0, y_1, ..., y_n\} \subseteq Y$, there exists $\{x_0, x_1, \ldots, x_n\} \subseteq X$ such that for each index set $I \subseteq \{0, \ldots, n\}$ one has $\operatorname{co}\{x_i : i \in I\} \subseteq \bigcup_{i \in I} Q(y_i)$.

The following KKM result is a version of Theorem 3.1 in [24] and Theorem 3.5 in [25].

Lemma 2.1. *Let X be a nonempty convex set in a topological vector space and Y be a nonempty set. If $Q : Y \multimap X$ is a generalized KKM mapping with closed values such that $Q(y')$ is compact for at least one $y' \in Y$, then $\bigcap_{y \in Y} Q(y) \neq \emptyset$.*

Proof. If we show that the family $\{Q(y) : y \in Y\}$ has the finite intersection property, by a standard topological argument the desired conclusion follows. So, let $\{y_0, ..., y_n\}$ be a finite subset of Y and $\{x_0, \ldots, x_n\} \subseteq X$ the companion set in Definition 2.1. Put $B = \text{co}\{x_0, \ldots, x_n\}$, $G_i = Q(y_i) \cap B$, for $i = 0, \ldots, n$. Obviously, each G_i is a closed subset of B. Let $\Delta_n = \text{co}\{e_0, e_1, \ldots, e_n\}$ be the standard n-dimensional simplex. The function $\varphi : \Delta_n \to B$ defined by $\varphi\big(\sum\limits_{i=0}^{n} \alpha_i e_i\big) = \sum\limits_{i=0}^{n} \alpha_i x_i$ ($\alpha_i \geq 0, \sum\limits_{i=0}^{n} \alpha_i = 1$) is continuous and satisfies

$$\varphi(\text{co}\{e_i : i \in I\}) = \text{co}\{x_i : i \in I\} \subseteq \bigcup_{i \in I} G_i,$$

for any index set $I \subseteq \{0, \ldots, n\}$. Therefore, $\text{co}\{e_i : i \in I\} \subseteq \bigcup\limits_{i \in I} \varphi^{-1}(G_i)$. Since the sets $\varphi^{-1}(G_i)$ are closed, by the KKM-principle, $\bigcap\limits_{i=0}^{n} \varphi^{-1}(G_i) \neq \emptyset$, and hence $\bigcap\limits_{i=0}^{n} \varphi^{-1}(Q(y_i)) \neq \emptyset$. If $x \in \bigcap\limits_{i=0}^{n} \varphi^{-1}(Q(y_i))$, then $\varphi(x) \in \bigcap\limits_{i=0}^{n} Q(y_i)$. This completes the proof. ■

We recall some continuity properties of set-valued mappings. Assume that X and Y are topological spaces. A set-valued mapping $T : X \multimap Y$ is said to be: (i) upper semicontinuous (respectively, lower semicontinuous) if for every $x \in X$ and for every open subset D of Y with $T(x) \subseteq D$ (respectively, $T(x) \cap D \neq \emptyset$) there is a neighborhood U of x such that $T(x') \subseteq D$ (respectively $T(x') \cap D \neq \emptyset$) for all $x' \in U$; (ii) closed if its graph (that is, the set $GrT = \{(x, y) \in X \times Y : \ y \in T(x)\}$) is a closed subset of $X \times Y$.

The following facts are known (see for instance [26]):

(i) If T has compact values, then T is upper semicontinuous if and only if for every net $\{x_t\}$ in X converging to $x \in X$ and for any net $\{y_t\}$ with $y_t \in T(x_t)$ there exist $y \in T(x)$ and a subnet $\{y_{t_\alpha}\}$ of $\{y_t\}$ converging to y.

(ii) If T is upper semicontinuous with compact values, then $T(K)$ is compact whenever $K \subseteq X$ is compact.

(iii) If Y is compact and T is closed, then T is upper semicontinuous.

Let X, Y and Z be convex sets in topological vector spaces and $R(x, y, z)$ be a relation linking elements $x \in X$, $y \in Y$ and $z \in Z$. R is said to be closed (respectively, convex) if the set $\{(x, y, z) \in X \times Y \times Z : R(x, y, z))$ holds$\}$ is closed in $X \times Y \times Z$ (respectively, convex). Similarly, R is said to be closed (respectively, convex) in two of the three variables, say x and z, if for each $y \in Y$ the set $\{(x, z) \in X \times Z : R(x, y, z))$ holds$\}$ is closed in $X \times Z$ (respectively, convex). The complementary relation of R is denoted by R^c, that is $R^c(x, y, z)$ hold if and only if $R(x, y, z)$ does not hold.

For the sake of simplicity, from now on all topological (vector) spaces are assumed to be Hausdorff.

3. Existence of Solutions to Problems (VRP 1_a) and (VRP 1_b)

Let $T : X \multimap Y$ be a set-valued mapping between two topological spaces. We say that T has open (respectively, closed) fibers if the inverse mapping $T^- : Y \multimap X$ has open (respectively, closed) values.

In order to study the solution existence of problems (VRP 1_a) and (VRP 1_b) we establish the following inclusion result:

Theorem 3.1. *Let X be a convex set in a topological vector space and Y be a nonempty set. Assume that $T, S : X \multimap Y$ are two set-valued mappings with nonempty values satisfying:*

(i) *T has open fibers and $X \setminus T^-(y)$ is compact for at least one $y \in Y$;*

(ii) *S has closed fibers;*

(iii) *the set $E = \{x \in X : x \in (S^- T)(x)\}$ is compact;*

(iv) *S^- is generalized KKM mapping.*

Then there exists $\bar{x} \in X$ such that $T(\bar{x}) \subseteq S(\bar{x})$.

Proof. Consider the map $Q : Y \multimap X$ defined by

$$Q(y) = (X \setminus T^-(y)) \cup (E \cap S^-(y)).$$

We show that Q is a generalized KKM mapping. If $\{y_0, \ldots, y_n\}$ is a finite subset of Y, by (iv), there exists a set $\{x_0, \ldots, x_n\} \subseteq X$ such that for each subset of indices $I \subseteq \{0, \ldots, n\}$,

$$co\{x_i : i \in I\} \subseteq \bigcup_{i \in I} S^-(y_i). \tag{1}$$

Let x be a point from the convex hull of $\{x_i : i \in I\}$. We prove that (1) implies

$$x \in \bigcup_{i \in I} Q(y_i). \tag{2}$$

If $x \in E$, by (1) one has

$$x \in E \cap \big(\bigcup_{i \in I} S^-(y_i)\big) = \bigcup_{i \in I} \big(E \cap S^-(y_i)\big) \subseteq \bigcup_{i \in I} Q(y_i).$$

If $x \in X \setminus E$, we claim that $x \in X \setminus T^{-}(y_i)$, for some index $i \in I$. Indeed, in contrary case $y_i \in T(x)$, for all $i \in I$. Thus, $S^{-}(y_i) \subseteq S^{-}(T(x))$. In view of (1), we have

$$x \in co\{x_i : i \in I\} \subseteq \bigcup_{i \in I} S^{-}(y_i) \subseteq S^{-}(T(x)); \text{ a contradiction.}$$

Hence, $x \in \bigcup_{i \in I}(X \setminus T^{-}(y_i)) \subseteq \bigcup_{i \in I} Q(y_i)$.

Since Q has closed values and $Q(y)$ is compact for at least one $y \in Y$, by Lemma 2.1, there exists $\bar{x} \in \bigcap_{y \in Y} Q(y)$. For each $y \in T(\bar{x})$, i.e. $\bar{x} \notin X \setminus T^{-}(y)$, since $\bar{x} \in Q(y)$, we have $\bar{x} \in S^{-}(y)$, that is $y \in S(\bar{x})$. Thus $T(\bar{x}) \subseteq S(\bar{x})$ and this means exactly the conclusion of the theorem. ∎

Remark 3.1. Let us observe that

$$E = \{x \in X : \exists y \in Y : x \in T^{-}(y) \cap S^{-}(y)\} = \left(T^{-} \cap S^{-}\right)(Y).$$

Hence condition (iii) in Theorem 3.1 means actually the compactness of the range of the set-valued map $T^{-} \cap S^{-}$.

Condition (iii) in Theorem 3.1 is essential. The next example shows that without it the result may fail.

Example 1. Let $X = Y = [0, 3)$. For each $x \in [0, 3)$, set $S(x) = [0, x]$ and

$$T(x) = \begin{cases} [0, x+2) & \text{if } x \in [0,1), \\ (x-1, 3) & \text{if } x \in [1,3), \end{cases}$$

Simple calculations show that for all $y \in [0, 3), S^{-}(y) = [y, 3)$,

$$T^{-}(y) = \begin{cases} (0, y+1) & \text{if } y \in [0,2) \\ (y-2, 3) & \text{if } y \in [2,3), \end{cases}$$

and the set $E = (S^{-} \cap T^{-})(Y) = [0, 3)$. One can readily see that all the hypotheses of Theorem 3.1, excepting condition (iii), are fulfilled and there is no $\bar{x} \in [0, 3)$ satisfying the conclusion of the theorem.

In what follows, unless other specified, X is a nonempty convex set in a topological vector space, Y, Z are nonempty sets, $T : X \multimap Y$ and $P : X \multimap Z$ are set-valued mappings with nonempty values and $R(x, y, z)$ is a relation linking elements $x \in X$, $y \in Y$, $z \in Z$.

The problems (VRP 1_a) and (VRP 1_b) can be jointly studied adopting the following notations: for subset U and point x, $\alpha_1(x, U)$ and $\alpha_2(x, U)$ means $\forall x \in U$ and respectively, $\exists x \in U$. For a suitable choice of $\alpha \in \{\alpha_1, \alpha_2\}$ the problems (VRP 1_a) and (VRP 1_b) can be formulated as follows:

(VRP 1) Find $\bar{x} \in X$ such that for each $y \in T(\bar{x})$,

$$\alpha(z, P(\bar{x})),\ R(\bar{x}, y, z) \text{ holds.}$$

The concept of generalized KKM mapping can be adapted to variational relations as follows:

Definition 3.1. The relation R is said to be generalized KKM w.r.t. (α, P) if for any finite subset $\{y_0, y_1, \ldots, y_n\}$ of Y there exists a corresponding subset $\{x_0, x_1, \ldots, x_n\}$ of X such that for any nonempty set $I \subseteq \{0, \ldots, n\}$ and any $x \in co\{x_i : i \in I\}$ one can find some index $i \in I$ such that $\alpha(z, P(x))$, $R(x, y_i, z)$ holds. The relation R is said to be KKM w.r.t. (α, P) if $X = Y$ and $y_i = x_i$ for each index i.

Usually the relation R is given by equality/inequality of real-valued functions, or by inclusions/intrsections of set-valued mappings. As it is seen in [5], in these cases the previous concept reduces to several concepts of generalized diagonal quasi-concavity present in many papers. We are now in position to obtain an existence theorem of the solution for problem (VRP 1).

Theorem 3.2. *Assume that the data of problem (VRP 1) satisfy the following conditions:*

(i) *T has open fibers and $X \setminus T^{-}(y)$ is compact for at least one $y \in Y$;*

(ii) *for each $y \in Y$, the set $\{x \in X : \alpha(z, P(x)), R(x, y, z)$ holds$\}$ is closed;*

(iii) *the set $\{x \in X : \exists y \in T(x), \alpha(z, P(x)), R(x, y, z)$ holds$\}$ is compact;*

(iv) *R is generalized KKM w.r.t. (α, P).*

Then problem (VRP 1) has solution.

Proof. Define the mapping $S : X \multimap Y$ by

$$S(x) = \{y \in Y : \alpha(z, P(x)), R(x, y, z) \text{ holds}\}.$$

A straightforward checking shows that each of conditions (ii), (iii) and (iv) implies the condition similarly noted in Theorem 3.1. The desired conclusion follows thus from Theorem 3.1. ■

Remark 3.2. Theorem 3.2 has been derived from Theorem 3.1. Actually, the two theorems are equivalent. Indeed, in order to obtain Theorem 3.1 by Theorem 3.2 take the relation R defined by $R(x, y, z)$ holds iff $y \in T(x)$.

Remark 3.3. (a) Assume that Z is a topological space. In the case $\alpha = \alpha_1$, one can easily prove (see the proof of Proposition 3.4 in [2]) that condition (ii) in Theorem 3.2 is satisfied if the set-valued map P is lower semicontinuous and the relation R is closed in the first and the third variable. Similarly, in the case $\alpha = \alpha_2$, condition (ii) in

Theorem 3.2 is satisfied if the set-valued map P is upper semicontinuous with compact values and the relation R is closed in the first and the third variable (see the proof of Theorem 14 in [8]).

(b) Assume that Y, Z are convex sets in two vector spaces. In the case $\alpha = \alpha_1$, condition (iv) in Theorem 3.2 is satisfied if there exists a relation r on $X \times Y \times Z$ such that

(i) $r \subseteq R$ (that is $R(x, y, z)$ holds whenever $r(x, y, z)$ holds);

(ii) for each $y \in Y$ there exists $x \in X$ such that $r(x, y, z)$ holds for all $z \in P(x)$;

(iii) r is convex and R^c is convex in the second and the third variable.

Proof. Let $\{y_0, y_1, \ldots, y_n\}$ be a finite subset of Y. Using (ii) we can find a corresponding subset $\{x_0, x_1, \ldots, x_n\}$ of X such that for each index i, $r(x_i, y_i, z)$ holds for all $z \in P(x_i)$. Suppose that there exist $I \subseteq \{0, 1, \ldots, n\}$ and $\widetilde{x} = \sum_{i \in I} \lambda_i x_i \in co\{x_i : i \in I\}$ such that for every index $i \in I$, $R(\widetilde{x}, y_i, z_i)$ does not hold, for some $z_i \in P(\widetilde{x})$. Set $\widetilde{y} = \sum_{i \in I} \lambda_i y_i$ and $\widetilde{z} = \sum_{i \in I} \lambda_i z_i$. Since R^c is convex in the second and the third variable, $R(\widetilde{x}, \widetilde{y}, \widetilde{z})$ does not hold. On the other side, since for each index i, $r(x_i, y_i, z_i)$ is satisfied and r is convex, it follows that $r(\widetilde{x}, \widetilde{y}, \widetilde{z})$ holds. In view of (i), $R(\widetilde{x}, \widetilde{y}, \widetilde{z})$ holds too; a contradiction. ■

Similarly, in case $\alpha = \alpha_2$, condition (iv) in Theorem 3.2 is satisfied whenever the mapping P is convex (that is, $\lambda P(x_1) + (1 - \lambda)P(x_2) \subseteq P(\lambda x_1 + (1 - \lambda)x_2)$, for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$) and there exists a relation r on $X \times Y \times Z$ such that

(i) $r \subseteq R$;

(ii) for each $y \in Y$ there exists $x \in X$ such that $r(x, y, z)$ holds for some $z \in P(x)$;

(iii) the relation r is convex and R^c is convex in the second variable.

Proof. Let $\{y_0, y_1, \ldots, y_n\}$ be a finite subset of Y. For each $i \in \{0, \ldots, n\}$ there is $x_i \in X$ and $z_i \in P(x_i)$ such that $r(x_i, y_i, z_i)$ is satisfied. Suppose that there exist $I \subseteq \{0, 1, \ldots, n\}$ and $\widetilde{x} = \sum_{i \in I} \lambda_i x_i \in co\{x_i : i \in I\}$ such that for every index $i \in I$, $R(\widetilde{x}, y_i, z)$ does not hold, for all $z \in P(\widetilde{x})$. Set $\widetilde{y} = \sum_{i \in I} \lambda_i y_i$ and $\widetilde{z} = \sum_{i \in I} \lambda_i z_i$. Since P is convex, $\widetilde{z} \in P(\widetilde{x})$. For each $i \in I$, $R(\widetilde{x}, y_i, \widetilde{z})$ does not hold, and in view of (iii), $R(\widetilde{x}, \widetilde{y}, \widetilde{z})$ does not hold too. On the other side, since r is convex, it follows that $r(\widetilde{x}, \widetilde{y}, \widetilde{z})$ holds. In view of (i), $R(\widetilde{x}, \widetilde{y}, \widetilde{z})$ holds too; a contradiction. ■

(c) If $X = Y$, for each $i \in \{1, 2\}$ one can easily prove that the relation R is KKM w.r.t. (α_i, P) whenever the following conditions are satisfied:

(i) for each $x \in X$ the set $\{y \in X : \alpha_{3-i}(z, P(x)), R(x, y, z) \text{ does not hold}\}$ is convex;

(ii) for each $x \in X$, $\alpha_i(z, P(x)), R(x, x, z)$ holds.

Example 2. Let $X = Y = Z = [\frac{1}{2}, 2]$, $T(x) = [\frac{1}{2}, 2]$, $P(x) = [x + \frac{1}{x} - \frac{3}{2},\ 6 - 2(x + \frac{1}{x})]$, for all $x \in [\frac{1}{2}, 2]$. Consider the set-valued mappings $C : [\frac{1}{2}, 2] \multimap \mathbf{R}$, $F : [\frac{1}{2}, 2] \times [\frac{1}{2}, 2] \times [\frac{1}{2}, 2] \multimap \mathbf{R}$ defined by

$$C(x) = [0, \infty) \quad \text{and } F(x, y, z) = [x(yz - 1),\ \infty)$$

and the relation R defined by $R(x, y, z)$ holds iff $F(x, y, z) \nsubseteq \operatorname{int} C(x)$.

In this case problem (VRP 1_b) becomes:

Find $\bar{x} \in [\frac{1}{2}, 2]$ such that for each $y \in [\frac{1}{2}, 2]$ there exists $z \in P(\bar{x})$ such that $F(\bar{x}, y, z) \nsubseteq \operatorname{int} C(\bar{x})$.

Observe that Theorem 3 in [18] is not applicable because two requirements of this theorem ($0 \in F(x, x, z)$ for all $x, z \in [\frac{1}{2}, 2]$ and, respectively, F is C_x-pseudomontone) are not satisfied. It is easy to see that all assumptions of Theorem 3.2, in case $\alpha = \alpha_2$, are fulfilled. For instance let us prove that the relation R is generalized KKM w.r.t. (α_2, P). Indeed, let $\{y_0, y_1, \ldots, y_n\}$ be finite subset of $[\frac{1}{2}, 2]$. Taking $x_0 = x_1 = \cdots = x_n = 1$ (then, $x \in \operatorname{co}\{x_i : i \in I\}$ means $x = 1$) and $z = \frac{1}{2} \in P(x)$, $R(x, y_, z)$ hods for each index $i \in I$.

So, by Theorem 3.2 the considered problem has solution. By direct checking one see that $\bar{x} = 1$ is the unique solution of this problem.

4. Existence of Solution for Problem $(VRP\ 2)$

The convex hull of finitely many points from a vector space is called polytope. The proof of the first result in this section relies on the following lemma:

Lemma 4.1. ([27]). *Let X be a polytope in a topological vector space an Y be a compact convex set in a topological vector space. If $F : X \multimap Y$ is a upper semicontinuous set-valued mapping with nonempty compact convex values and $p : Y \to X$ is a continuous function, then pF has a fixed point, that is there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in F(\bar{x})$ and $\bar{x} = p(\bar{y})$.*

Theorem 4.1. *Let X, Y and Z be convex sets in topological vector spaces. Assume that the following conditions are satisfied:*

(i) *T has convex values, open fibers and $X \setminus T^{-}(y)$ is compact for at least one $y \in Y$;*

(ii) *$P : X \multimap Z$ is upper semicontinuous with compact convex values;*

(iii) *the relation R is closed;*

(iv) *R is convex in the third variable and R^c is convex in the second variable;*

(v) *the set* $\{x \in X : \exists y \in T(x), \exists z \in P(x), R(x,y,z)$ *holds*$\}$ *is compact*

(vi) R *is generalized* KKM *w.r.t.* (α_2, P).

Then problem $(VRP2)$ *has solution.*

Proof. Define the mapping $S : X \multimap Y$ by

$$S(x) = \{y \in Y : \exists z \in P(x), R(x,y,z) \text{ holds}\}.$$

By (ii) and (iii), via Remark 3.3 (a), it follows that S has closed fibers in X. By (iv), S^- is generalized KKM mapping. Applying Theorem 3.1 we find a point $\bar{x} \in X$ such that

(3) for each $y \in T(\bar{x})$ there is $z \in P(\bar{x})$ for which $R(\bar{x}, y, z)$ holds.

We prove that, actually $\bar{x}$ is a solution of problem $(VRP2)$. Denote by

$$G(y) = \{z \in P(\bar{x}) : R(\bar{x}, y, z) \text{ does not hold}\}, \text{ for any } y \in T(\bar{x}).$$

By (iii), the sets $G(y)$ are open in $P(\bar{x})$. By way of contradiction, suppose that for each $z \in P(\bar{x})$ there exists $y \in T(\bar{x})$ for which $R(\bar{x}, y, z)$ does not hold. Then there exists a finite covering $\{G(y_0), \ldots G(y_n)\}$ of the compact set $P(\bar{x})$ and a partition of unity $\{\beta_0, \ldots, \beta_n\}$ corresponding to this finite partition (i.e., each β_i is a continuous function of $P(\bar{x})$ into $[0,1]$ which vanishes outside of $G(y_i)$ and $\sum_{i=0}^{n} \beta_i(z) = 1$ for all $z \in P(\bar{x})$). Set

$$K = \text{co}\{y_0, \ldots, y_n\} \text{ and } p(z) = \sum_{i=0}^{n} \beta_i(z) y_i \text{ for all } z \in P(\bar{x}).$$

Then p is a continuous mapping of $P(\bar{x})$ into K. Define a set-valued mapping $F : K \multimap P(\bar{x})$ by

$$F(y) = \{z \in P(\bar{x}) : R(\bar{x}, y, z) \text{ holds}\}, \text{ for all } y \in K.$$

Then, for every $y \in K$, $F(y)$ is nonempty (by (3)), convex (since the relation R is convex in the third variable) and compact (as closed subset of the compact K). Since $P(\bar{x})$ is compact set and the relation R is closed, F is closed mapping, hence F will be upper semicontinuous. By Lemma 4.1, there exists $(\bar{y}, \bar{z}) \in K \times P(\bar{x})$ such that $\bar{z} \in F(\bar{y})$ and $\bar{y} = p(\bar{z})$.

From $\bar{z} \in F(\bar{y})$ it follows that

(4) $R(\bar{x}, \bar{y}, \bar{z})$ holds

If $I = \{i \in \{0, 1, \ldots, n\} : \beta_i(\bar{z}) > 0\}$, then $\bar{y} = p(\bar{z}) \in \text{co}\{y_i : i \in I\}$. For each $i \in I$, $\bar{z} \in G(y_i)$, that is, $R(\bar{x}, y_i, \bar{z})$ does not hold. Since R^c is convex in the second variable, it follows that $R(\bar{x}, \bar{y}, \bar{z})$ does not hold, which contradicts (4). ∎

Remark 4.1. It is reasonable to give now an example illustrating the capacity of our theorems to cover classical results of literature.

Let us consider the case of problem $(VRP2)$ and Theorem 4.1. One of the most important particular cases of $(VRP2)$ is the classical Stampacchia variational inequality. It corresponds to the case when $X = Y$ is a compact convex subset of a Banach space E, $Z = E^*$ (the topological dual of E), $T(x) = X$ and $R(x, y, z)$ holds $\Leftrightarrow$ $\langle z, y - x\rangle \geq 0$.

Assumptions (i), (iii) and (iv) are clearly satisfied, while (iii) holds if P is assumed upper semicontinuous with compact convex values. On the other hand (v) corresponds to a coercivity condition and finally (vi) turns out to be the so-called properly quasi-monotonicity of P. So, Theorem 4.1 reduces to the old existence result from [28].

Recall that an extended real function $f : Y \to \overline{\mathbf{R}}$, where Y is a convex set in a topological vector space, is said to be lower semicontinuous (resp., quasiconvex) if the set $\{y \in Y : f(y) \leq r\}$ is closed in Y (resp., convex), for each $r \in \overline{\mathbf{R}}$. The function f is upper semicontinuous (resp., quasiconcave) if $-f$ is lower semicontinuous (resp., quasiconvex). The next theorem is a version of Theorem 4.1. Its proof uses the following Sion's minimax theorem ([29]):

Lemma 4.2. *If Y and Z are convex sets in topological vector spaces and $\varphi : Y \times Z \to \overline{\mathbf{R}}$ is a function upper semicontinuous, quasiconcave in the first variable and lower semicontinuous, quasiconvex in the second variable, then $\inf\limits_{z\in Z}\sup\limits_{y\in Y}\varphi(y, z) = \sup\limits_{y\in Y}\inf\limits_{z\in Z}\varphi(y, z)$, whenever either Y or Z is compact.*

Theorem 4.2. *The conclusion of Theorem 4.1 remains valid if condition (iii) from this theorem is replaced with the following condition:*

(iii') R is closed in the first and the third variable and R^c is closed in the second variable.

Proof. Let us fix a point $\bar{x} \in X$ satisfying the statement (3) from the previous proof. Define $\varphi : T(\bar{x}) \times P(\bar{x}) \to (-\infty, +\infty]$ by

$$\varphi(y, z) = \begin{cases} 0 & \text{if } R(\bar{x}, y, z) \text{ holds} \\ +\infty & \text{if } R(\bar{x}, y, z) \text{ does not hold.} \end{cases}$$

From (3), we have $\sup\limits_{y\in T(\bar{x})}\inf\limits_{z\in P(\bar{x})}\varphi(y, z) = 0$. By (vi) and (vii) we easily infer that the function φ is quasiconcave and upper semicontinuous in the variable y and quasiconvex and lower semicontinuous in the variable z. Thus, by Sion's minimax theorem,

$$\inf_{z\in P(\bar{x})}\sup_{y\in T(\bar{x})}\varphi(y, z) = \sup_{y\in T(\bar{x})}\inf_{z\in P(\bar{x})}\varphi(y, z) = 0.$$

Consequently, there exists $\bar{z} \in P(\bar{x})$ such that for each $y \in T(\bar{x})$, $R(\bar{x}, y, \bar{z})$ holds. ■

Finally we establish an existence theorem for the solution of problem $(VRP2)$ when $X = Y$ and $T = 1_X$. To this purpose we need recall the following concept from the KKM theory:

Definition 4.1. Let X be a nonempty convex subset of a topological vector space and let H be a set-valued mapping with nonempty values from X into a topological space Y. A set-valued mapping $G : X \multimap Y$ is said to be KKM mapping w.r.t. H if $H(coA) \subseteq \bigcup_{x \in A} G(x)$, for every nonempty finite subset A of X. The mapping H is said to have the KKM property if, for any mapping $G : X \multimap Y$ which is KKM w.r.t. H, the family $\{\overline{G(x)} : x \in X\}$ has the finite intersection property.

Lemma 4.3. *If a set-valued mapping $H : X \multimap Y$ has the KKM property then so does its restriction to any nonempty convex subset of X.*

Proof. Let C be a nonempty convex subset of X and $G : C \multimap Y$ a KKM mapping w.r.t. $H_{|C}$. Extend G to the whole set X defining $G' : X \multimap Y$ by

$$G'(x) = \begin{cases} G(x) & \text{if } x \in C, \\ Y & \text{if } x \in X \setminus C, \end{cases}$$

One can easily check that G' is a KKM mapping w.r.t. H. Since H has the KKM property, the family $\{\overline{G'(x)}; x \in X\}$ has the finite intersection property, and hence so does its subfamily $\{\overline{G(x)} : x \in C\}$. ■

Theorem 4.3. *Let X be a convex set in a topological vector space, Z be a topological space, $P : X \multimap Z$ be a set-valued mapping and R be a relation linking elements $x \in X$, $y \in X$ and $z \in Z$. Assume that:*

(i) *P is upper semicontinuous with nonempty compact values;*

(ii) *the set-valued mapping $H : X \multimap X \times Z$ defined by $H(x) = \{x\} \times P(x)$ has the KKM property;*

(iii) *the relation R is KKM w.r.t. (α_1, P) and closed in the first and the third variables;*

(iv) *there are a nonempty compact set $K_0 \subseteq X$ and a nonempty compact convex set $K_1 \subseteq X$ such that $x \in X \setminus K_0 \Longrightarrow$ for each $z \in P(x)$ there exists $y \in K_1$ for which $R(x, y, z)$ does not hold.*

Then there exists $\bar{x} \in X$ and $\bar{z} \in P(\bar{x})$ such that $R(\bar{x}, y, \bar{z})$ holds for all $y \in X$.

Proof. Denote by $\mathcal{K}$ the family of all compact convex subsets K of X satisfying $K_1 \subseteq K$. Let $K \in \mathcal{K}$, be arbitrarily fixed. Define $G : K \multimap K \times Z$ by

$$G(y) = \{(x, z) \in K \times Z : z \in P(x) \text{ and } R(x, y, z) \text{ holds}\}.$$

We prove that G is a KKM mapping w.r.t. $H_{|K}$. Let $\{y_0, \ldots, y_n\}$ be a finite subset of K and $(x, z) \in H(co\{y_0, \ldots, y_n\})$. Then $x \in co\{y_0, \ldots, y_n\}$ and $z \in P(x)$. By (iii), there exists an index $i \in \{0, \ldots, n\}$ such that $R(x, y_i, z)$ holds. This proves that $(x, z) \in G(y_i)$, hence G is a $H_{|K}$-KKM mapping.

Since the set-valued mapping H is upper semicontinuous with nonempty compact values, $H(K)$ is compact set. For any $y \in K$, $G(y) = H(K) \cap \{(x, z) \in X \times Z :$ $R(x, y, z)$ holds$\}$, hence G has compact values. In view of (ii) and Lemma 4.3, there exists $(x_K, z_K) \in \bigcap_{y \in K} G(y)$. It follows that $x_K \in K$, $z_K \in P(x_K)$ and $R(x_K, y, z_K)$ holds for all $y \in K$. Since $K_1 \subseteq K$, by (iv) we infer that $x_K \in K_0$.

It is clear that for any $K, K' \in \mathcal{K}$, $co(K \cup K') \in \mathcal{K}$. Consequently, the ordered set $(\mathcal{K}, \subseteq)$ is directed to the right. Since $\{x_K\}_{K \in \mathcal{K}}$ is a net in the compact K_0 we may assume without loss of generality that this converges to an element $\bar{x}$ of K_0. Since P is upper semicontinuous with compact values, there exist $\bar{z} \in P(\bar{x})$ and a subnet $\{z_{K_i}\}_{i \in I}$ of the net $\{z_K\}$ converging to $\bar{z}$, where the index set I is equipped with a direction $\succeq$.

We shall prove that $R(\bar{x}, y, \bar{z})$ holds for all $y \in X$. For any $y \in X$ there exists $i_0 \in I$ such that $co(K_0 \cup \{y\}) \subseteq K_i$ for all $i \succeq i_0$. It follows that for each such index i, $R(x_{K_i}, y, z_{K_i})$ holds. The relation R being closed in the variables x, z, it follows that $R(\bar{x}, y, \bar{z})$ holds. ■

Remark 4.2. Condition (ii) in Theorem 4.3 is fulfilled if the mapping P has acyclic values (recall that a topological space is acyclic if all its reduced Cech homology groups over rationals vanish). In this case for each $x \in X$, the set $H(x)$ is acyclic since it is the product of two acyclic sets (see the Kunneth formula in [30]). Thus H is an acyclic mapping (that is, upper semicontinuous with nonempty compact acyclic values), and according to [31] it has the KKM property.

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