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LITTLEWOOD-PALEY CHARACTERIZATION OF WEIGHTED ANISOTROPIC HARDY SPACES

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Abstract. We obtain the weighted anisotropic Hardy space estimate for anisotropic singular integrals of convolution type, and apply it to derive Littlewood-Paley characterization of weighted anisotropic Hardy spaces.

1. Introduction and Statement of Main Results

The real-variable theory of Hardy spaces $H^p(\mathbb{R}^n)$ was developed by Fefferman and Stein [13] in the early 1970's. Since then these classes of function spaces play an important role in harmonic analysis, naturally continuing the scale of L^p spaces to the range $p \leq 1$. For example, when $0 < p \leq 1$, Riesz transforms on $\mathbb{R}^n$ are not bounded on $L^p(\mathbb{R}^n)$, however, they are bounded on Hardy spaces $H^p(\mathbb{R}^n)$. On the other hand, a local version of real Hardy spaces more suited to problems associated with partial differential equations, was introduced by Goldberg [17]. Indeed, quite a few results concerning L^p-boundedness ($1 < p < \infty$) of pseudodifferential operators were generalized to h^p-boundedness ($0 < p \leq 1$); see, for example, [21] and [22].

Extensions of classical real Hardy spaces were carried out in several directions. In particular, Garcia-Cuerva [15] and Strömberg and Torchinsky [26] studied weighted Hardy spaces associated with Muckenhoupt A_∞ weights. Another possible direction is to extend the classical real Hardy spaces to nonisotropic settings. This direction of study was initiated by Calderón and Torchinsky [10, 11], who studied parabolic Hardy spaces. Bownik *et al*. [2, 6, 7] developed a more general theory of anisotropic Hardy spaces (and their weighted counterparts) associated with expansive dilations. Recently, anisotropic local Hardy spaces were studied by Betancor and Damián [1].

The first aim of this paper is to obtain the weighted anisotropic Hardy space estimate for anisotropic singular integrals of convolution type. Suppose A is an expansive

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dilation, i.e., $n \times n$ real matrix all of whose eigenvalues λ satisfy $|\lambda| > 1$. Let λ_-, λ_+ be two positive numbers such that

$$1 < \lambda_- < \min\{|\lambda|, \lambda \in \sigma(A)|\} \le \max\{|\lambda|, \lambda \in \sigma(A)|\} < \lambda_+,$$

where $\sigma(A)$ denotes the set of eigenvalues of A. Set $\zeta_- := \ln(\lambda_-)/\ln|\det A|$ and $\zeta_+ := \ln(\lambda_+)/\ln|\det A|$. If $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$, we define its anisotropic dilation by

$$\psi_k(x) = |\det A|^k \psi(A^k x),\ k \in \mathbb{Z}.$$

Then the nontangential maximal function (resp. local nontangential maximal function) of $f \in \mathcal{S}'(\mathbb{R}^n)$, with respect to ψ, is given by

$$M_\psi f(x) = \sup_{k\in\mathbb{Z}} \sup_{y \in x+B_{-k}} |f * \psi_k(y)|$$

(resp. $M_\psi^{loc} f(x) = \sup_{k\in\mathbb{N}\cup\{0\}} \sup_{y\in x+B_{-k}} |f * \psi_k(y)|$), where B_k, $k \in \mathbb{Z}$, is a family of dilated balls defined in Section 2. Let $0 < p < \infty$ and let $w \in \mathcal{A}_\infty(A)$, that is, the class of Muckenhoupt weights associated with A (see Definition 2.2 below). We define the weighted anisotropic Hardy space $H_w^p(\mathbb{R}^n; A)$ (resp. weighted anisotropic local Hardy space $h_w^p(\mathbb{R}^n; A)$) as the collection of those tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $M_\psi f \in L_w^p(\mathbb{R}^n)$ (resp. $M_\psi^{loc} f \in L_w^p(\mathbb{R}^n)$) with the (quasi-)norm

$$\|f\|_{H_w^p(\mathbb{R}^n;A)} := \|M_\psi f\|_{L_w^p(\mathbb{R}^n)}$$

(resp. $\|f\|_{h_w^p(\mathbb{R}^n;A)} := \|M_\psi^{loc} f\|_{L_w^p(\mathbb{R}^n)}$). Here, for $0 < p < \infty$, $L_w^p(\mathbb{R}^n)$ denotes the space of Lebesgue measurable functions f satisfying

$$\|f\|_{L_w^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx\right)^{1/p} < \infty.$$

We shall show that (see Proposition 2.2 and 2.3 below) this definition of $H_w^p(\mathbb{R}^n; A)$ (resp. $h_w^p(\mathbb{R}^n; A)$) is independent of the choice of ψ and is equivalent to the radial and grand maximal function formulation.

Let ρ_A be the step homogeneous quasi-norm associated with A defined by (2.3) below. We define the anisotropic singular integrals of convolution type as follows.

Definition 1.1. Let m be a positive integer. An anisotropic kernel of order m is a distribution $K \in \mathcal{S}'(\mathbb{R}^n)$ which coincides with a C^m function away from the origin and satisfies the following conditions:

(i) There exists a constant $C > 0$ such that for all $\ell \in \mathbb{Z}$, all $x \in \mathbb{R}^n\backslash\{0\}$ with $\rho_A(x) = |\det A|^\ell$, and for all multi-indices α with $|\alpha| \le m$,

$$|\partial^\alpha[K(A^\ell \cdot)](A^{-\ell}x)| \le C[\rho_A(x)]^{-1}; \tag{1.1}$$

(ii) There exists a constant $C > 0$ such that $\|f * K\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}$ for all $f \in \mathcal{S}(\mathbb{R}^n)$.

The convolution operator T with anisotropic kernel of order m is called anisotropic singular integral operator of order m.

In the unweighted isotropic setting, i.e., $A = 2I_n$ and $w \equiv 1$, the H^p-boundedness of singular integrals of convolution type was first obtained by C. Fefferman and Stein (see [13, Theorem 12]). In the weighted isotropic setting, when $w \in A_1$, Lee *et al.* [19, 20] applied the weighted molecular theory and atomic decomposition to obtain the $H^p_w(\mathbb{R}^n)$-boundedness of a class of convolution singular integrals. This was recently extended to the case $w \in A_\infty$ by Ding *et al.* [12], who achieved their goal by applying the discrete version of Calderón's identity and Littlewood-Paley-Stein theory. In the unweighted anisotropic setting, Bownik obtained the $H^p_A(\mathbb{R}^n)$-boundedness of a class of anisotropic singular integrals of nonconvolution type by using molecular characterization of (unweighted) anisotropic Hardy spaces (see [2, Theorem 9.8]). The first main result of the present paper, which concerns the $H^p_w(\mathbb{R}^n; A)$-boundedness of anisotropic singular integrals of convolution type, is stated as follows.

Theorem 1.1. *Suppose A is an expansive dilation, $w \in \mathcal{A}_\infty(A)$, m is a positive integer, and T is an anisotropic singular integral operator of order m. Then T is bounded on $H^p_w(\mathbb{R}^n; A)$, for $p \in (\frac{q_w}{1+m\zeta_-}, 1]$. Here, q_w is defined in Section 2.*

Theorem 1.1 is proved in Section 3, by using atomic decomposition and radial maximal function characterization of $H^p_w(\mathbb{R}^n; A)$. Note that the classical Riesz transforms R_j, $j = 1, \cdots, n$, satisfy the conditions in Definition 1.1 with $A = 2I_n$ for all $m \in \mathbb{N}$. Hence it follows from Theorem 1.1 that $R_j's$ are bounded on $H^p_w(\mathbb{R}^n)$, for $0 < p \leq 1$ and $w \in A_\infty$. This has already been obtained by Lee *et al.* [19, 20] and Ding *et al.* [12], whose methods seem more complicated than that used in the present paper.

The second aim of this paper is to obtain the Littlewood-Paley characterization of weighted anisotropic Hardy spaces, that is, to demonstrate that weighted anisotropic Hardy spaces fit into the scales of weighted anisotropic Triebel-Lizorkin spaces studied by Bownik *et. al.* [5, 3, 4]. Let φ be a Schwartz function such that supp $\hat{\varphi}$ (where $\hat{}$ denotes the Fourier transform) is compact and does not contain the origin, and that

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}((A^*)^{-j}\xi) = 1 \text{ for all } \xi \in \mathbb{R}^n\backslash\{0\}, \tag{1.2}$$

where A^* is the transpose of A. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be given by

$$\hat{\Phi}(\xi) = \begin{cases} \sum_{j=-\infty}^{0} \hat{\varphi}((A^*)^{-j}\xi), & \xi \in \mathbb{R}^n\backslash\{0\}, \\ 1, & \xi = 0. \end{cases} \tag{1.3}$$

Given a smoothness parameter $\alpha \in \mathbb{R}$, an integrability exponent $0 < p < \infty$, and a summability exponent $0 < q \leq \infty$, the weighted anisotropic homogeneous Triebel-Lizorkin norm is defined by (see [4, p. 132])

$$\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^n,A,wdx)} := \left\| \left(\sum_{j=-\infty}^{\infty} (|\det A|^{j\alpha}|f*\varphi_j|)^q \right)^{1/q} \right\|_{L^p_w(\mathbb{R}^n)},$$

and the weighted anisotropic inhomogeneous Triebel-Lizorkin norm is defined by

$$\|f\|_{F_p^{\alpha,q}(\mathbb{R}^n,A,wdx)} := \|f*\Phi\|_{L^p_w(\mathbb{R}^n)} + \left\| \left(\sum_{j=1}^{\infty} (|\det A|^{j\alpha}|f*\varphi_j|)^q \right)^{1/q} \right\|_{L^p_w(\mathbb{R}^n)},$$

where $\varphi_j(x) := |\det A|^j \varphi(A^j x)$. It was proved in [5, 3] that the spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n, A, wdx)$ and $F_p^{\alpha,q}(\mathbb{R}^n, A, wdx)$ are independent of φ.

In the unweighted isotropic setting, i.e., $A = 2I_n$ and $w \equiv 1$, it is well-known that $H^p(\mathbb{R}^n) = \dot{F}_p^{0,2}(\mathbb{R}^n)$ (modulo polynomials) (resp. $h^p(\mathbb{R}^n) = F_p^{0,2}(\mathbb{R}^n)$) with equivalent norms; see Peetre [23] (resp. Triebel [27]). Bui [9] obtained such results in weighted isotropic setting. In the unweighted anisotropic setting, i.e., when A is an expansive dilation and $w \equiv 1$, Bownik in [3] identified $H^p(\mathbb{R}^n; A)$ with $\dot{F}_p^{0,2}(\mathbb{R}^n; A)$ by applying wavelet characterizations (see [3, Theorem 7.1]). In the present paper, we obtain the Littlewood-Paley characterization of weighted anisotropic Hardy spaces and weighted anisotropic local Hardy spaces.

Theorem 1.2. *Let A be an expansive dilation, $w \in \mathcal{A}_\infty(A)$ and $0 < p \leq 1$.*

(i) If $f \in H^p_w(\mathbb{R}^n; A)$, then $f \in \dot{F}_p^{0,2}(\mathbb{R}^n, A, wdx)$, and there exists a positive constant C, which is independent of f, such that

$$\|f\|_{\dot{F}_p^{0,2}(\mathbb{R}^n,A,wdx)} \leq C\|f\|_{H^p_w(\mathbb{R}^n;A)}.$$

Conversely, if $f \in \mathcal{S}'(\mathbb{R}^n)$ is in $\dot{F}_p^{0,2}(\mathbb{R}^n, A, wdx)$, then there exists a polynomial P such that $f - P \in H^p_w(\mathbb{R}^n; A)$; moreover, there exists a positive constant C', which is independent of f, such that

$$\|f - P\|_{H^p_w(\mathbb{R}^n;A)} \leq C'\|f\|_{\dot{F}_p^{0,2}(\mathbb{R}^n,A,wdx)}.$$

(ii) There exist positive constants C and C' such that

$$C\|f\|_{h^p_w(\mathbb{R}^n;A)} \leq \|f\|_{F_p^{0,2}(\mathbb{R}^n,A,wdx)} \leq C'\|f\|_{h^p_w(\mathbb{R}^n;A)}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.2 is proved in Section 4. We conclude this introduction by making some notation conventions. Throughout this paper, the letter C denotes a positive constant which is independent of the main parameters involved but whose value may differ from line to line. The notation $a \lesssim b$ or $b \gtrsim a$ for some variable quantities a and b means that $a \le Cb$ for some constant $C > 0$; $a \sim b$ stands for $a \lesssim b \lesssim a$. Denote by $\mathbb{N}$ the set $\{1, 2, \cdots\}$. The meaning of $|\cdot|$ depends on context: $|x|$ is the Euclidean norm for $x \in \mathbb{R}^n$, $|\alpha|$ is the ℓ^1-norm for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, while $|E|$ is the Lebesgue measure for a measurable set $E \subset \mathbb{R}^n$. Finally, we denote $\|A\| := \sup\{|Ax| : x \in \mathbb{R}^n \text{ with } |x| = 1\}$ for any $n \times n$ real matrix A.

2. Preliminaries and Notations

Definition 2.1. A real $n \times n$ matrix A is said to be an expansive dilation, if $\min_{\lambda \in \sigma(A)} |\lambda| > 1$, where $\sigma(A)$ is the set of eigenvalues of A. A homogeneous quasi-norm associated with an expansive dilation A is a measurable function $\rho : \mathbb{R}^n \to [0, \infty)$ satisfying that

(i) $\rho(x) > 0$ if and only if $x \ne 0$;

(ii) $\rho(Ax) = |\det A| \rho(x)$ for all $x \in \mathbb{R}^n$;

(iii) $\rho(x+y) \le H(\rho(x)+\rho(y))$ for $x, y \in \mathbb{R}^n$, where H is a constant no less than 1.

We follow the notations in Bownik's monograph [2] (see also [3-7]). For a given expansive dilation A, there exists an ellipsoid Δ and $r > 1$ such that $\Delta \subset r\Delta \subset A\Delta$ and $|\Delta| = 1$. Then, we can define a family of dilated balls around the origin $B_k := A^k\Delta$, $k \in \mathbb{Z}$, that satisfy $B_k \subset rB_k \subset B_{k+1}$. Let ω be the smallest integer such that $2B_0 \subset A^\omega B_0 = B_\omega$. Obviously $\omega \ge 1$. For any set $E \subset \mathbb{R}^n$, let $E^c := \mathbb{R}^n \backslash E$. Then for all $k, \ell \in \mathbb{Z}$ we have

$$B_k + B_\ell \subset B_{\max\{k,\ell\}+\omega}, \tag{2.1}$$

$$B_k + (B_{k+\omega})^c \subset (B_k)^c. \tag{2.2}$$

It is known that any two homogeneous quasi-norms associated with A are equivalent (cf. [2, Lemma 2.4]). Throughout this paper, for convenience we always use the *step homogeneous quasi-norm* ρ_A defined by

$$\rho_A(x) = \begin{cases} |\det A|^j, & \text{if } x \in B_{j+1} \backslash B_j, \\ 0, & \text{if } x = 0. \end{cases} \tag{2.3}$$

Using (2.1) and (2.2), it is straightforward to verify that ρ_A satisfies a triangle inequality up to a constant and the homogeneity condition $\rho_A(Ax) = |\det A| \rho_A(x)$, $x \in \mathbb{R}^n$.

We shall also consider a family of dilated balls $\{B_k^* : k \in \mathbb{Z}\}$ and the step homogeneous quasi-norm ρ_{A^*} associated with A^*, the transpose of A.

Let $\lambda_-, \lambda_+, \zeta_-$ and ζ_+ be defined as in the introduction. We will frequently use the following inequalities established in [2, Section 2]: There exists a positive constant C such that

$$C^{-1}[\rho_A(x)]^{\zeta_-} \le |x| \le C[\rho_A(x)]^{\zeta_+} \text{ for all } \rho_A(x) \ge 1, \tag{2.4}$$

$$C^{-1}[\rho_A(x)]^{\zeta_+} \le |x| \le C[\rho_A(x)]^{\zeta_-} \text{ for all } \rho_A(x) < 1, \tag{2.5}$$

$$C^{-1}(\lambda_-)^j|x| \le |A^j x| \le C(\lambda_+)^j|x| \text{ for all } j \ge 0, \text{ and} \tag{2.6}$$

$$C^{-1}(\lambda_+)^j|x| \le |A^j x| \le C(\lambda_-)^j|x| \text{ for all } j < 0. \tag{2.7}$$

Let us now recall from [5] and [6] the notion of the class of Muckenhoupt weights associated with an expansive dilation A.

Definition 2.2. Let A be an expansive dilation and $1 \le p < \infty$. A nonnegative locally integrable function w belongs to the Muckenhoupt weight class $\mathcal{A}_p(A)$ associated with A, say $w \in \mathcal{A}_p(A)$, if there exists a positive constant C so that

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}}\left(\frac{1}{|B_k|}\int_{x+B_k} w(y)dy\right)\left(\frac{1}{|B_k|}\int_{x+B_k} w(y)^{-\frac{1}{p-1}}dy\right)^{p-1} \le C, \quad if\ 1<p<\infty,$$

and

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}}\left(\frac{1}{|B_k|}\int_{x+B_k} w(y)dy\right)\left(\operatorname*{esssup}_{y\in x+B_k}\frac{1}{w(y)}\right) \le C, \quad if\ p=1.$$

We say that $w \in \mathcal{A}_\infty(A)$ if $w \in \mathcal{A}_p(A)$ for some $p \in [1,\infty)$.

If $w \in \mathcal{A}_\infty(A)$, we write $q_w := \inf\{p \in [1,\infty) : w \in \mathcal{A}_p(A)\}$ to denote the critical index of w, and we write $w(E) = \int_E w(x)dx$ for any measurable set $E \subset \mathbb{R}^n$.

For a locally integrable function f, the Hardy-Littlewood maximal function $M(f)$ is defined by

$$M(f)(x) := \sup_{k\in\mathbb{Z}}\sup_{y\in x+B_k}\frac{1}{|B_k|}\int_{y+B_k}|f(z)|dz,\ x\in\mathbb{R}^n.$$

Proposition 2.1. (see [6, Proposition 2.6]). *(i) Let $p \in [1,\infty)$ and $w \in \mathcal{A}_p(A)$. Then there exists a positive constant C such that for all $x \in \mathbb{R}^n$ and $k, m \in \mathbb{Z}$ with $k \le m$,*

$$C^{-1}|\det A|^{(m-k)/p} \le \frac{w(x+B_m)}{w(x+B_k)} \le C|\det A|^{(m-k)p}.$$

(ii) Let $p \in (1,\infty)$. Then the Hardy-Littlewood operator M is bounded on $L^p_w(\mathbb{R}^n)$ if and only if $w \in \mathcal{A}_p(A)$.

We introduce the radial maximal function and local radial maximal function (cf. [2] and [1]): If $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$, define

$$M^0_\psi f(x) = \sup_{k\in\mathbb{Z}} |f * \psi_k(x)| \text{ and } M^{0,loc}_\varphi f(x) = \sup_{k\in\mathbb{N}\cup\{0\}} |f * \psi_k(x)|.$$

For $N \in \mathbb{N} \cup \{0\}$, we define the grand maximal function and local grand maximal function of $f \in \mathcal{S}'$ by

$$M_N f(x) = \sup_{\psi\in\mathcal{S}_N(\mathbb{R}^n)} M_\psi f(x) \text{ and } M^{loc}_N f(x) = \sup_{\psi\in\mathcal{S}_N(\mathbb{R}^n)} M^{loc}_\psi f(x)$$

respectively, where

$$\mathcal{S}_N(\mathbb{R}^n) = \left\{ \psi \in \mathcal{S}(\mathbb{R}^n) : \|\psi\|_{\mathcal{S}_N(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \sup_{|\alpha|\leq N} |\partial^\alpha \psi(x)| \, [1+\rho_A(x)]^N \leq 1 \right\}.$$

In the introduction we defined $H^p_w(\mathbb{R}^n; A)$ via nontangential maximal function. The following proposition, which is a weighted analogue of [2, Theorem 7.1], states that this previous definition of $H^p_w(\mathbb{R}^n; A)$ is independent of ψ and is equivalent to the radial and grand maximal formulation.

Proposition 2.2. *Suppose that A is an expansive dilation, $w \in \mathcal{A}_\infty(A)$, $0 < p < \infty$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$. Then for any $f \in \mathcal{S}'(\mathbb{R}^n)$ the following are equivalent:*
(i) $M^0_\psi f \in L^p_w(\mathbb{R}^n)$;
(ii) $M_\psi f \in L^p_w(\mathbb{R}^n)$;
(iii) $M_N f \in L^p_w(\mathbb{R}^n)$ for sufficiently large N.
Moreover, if N is sufficiently large, then $\|M^0_\psi f\|_{L^p_w(\mathbb{R}^n)} \sim \|M_\psi f\|_{L^p_w(\mathbb{R}^n)} \sim \|M_N f\|_{L^p_w(\mathbb{R}^n)}$.

Proof. The proof is similar to that given for the case $w \equiv 1$ in [2, Theorem 7.1], so that we only sketch necessary modifications. If $F : \mathbb{R}^n \times \mathbb{Z} \to [0,\infty)$ is an arbitrary (possibly nonmeasurable) function, we define, for $\ell \in \mathbb{Z}$ and $K \in \mathbb{Z}$, the (truncated) maximal type function of F with *aperture* ℓ as

$$F^{*K}_\ell(x) := \sup_{k\in\mathbb{Z}, k\geq K} \sup_{y\in x+B_{-k+\ell}} F(y,k).$$

Let M_ω be the Hardy-Littlewood maximal operator with respect to the measure $w(x)dx$; that is,

$$M_w f(x) = \sup_{k\in\mathbb{Z}} \sup_{y\in x+B_k} \frac{1}{w(y+B_k)} \int_{y+B_k} |f(z)|w(z)dz, \; x \in \mathbb{R}^n.$$

Then we note that, instead of (7.6) in [2], we have

$$\int_{\mathbb{R}^n} F_\ell^{*K}(x)w(x)dx \lesssim |\det A|^{q(\ell-\ell')}\int_{\mathbb{R}^n} F_{\ell'}^{*K}(x)w(x)dx, \tag{2.8}$$

for all $\ell \geq \ell' \in \mathbb{Z}$, $q > q_w$, $K \in \mathbb{Z}$ and all functions $F : \mathbb{R}^n \times \mathbb{Z} \to [0,\infty)$. To see (2.8), we let $q \in (q_w,\infty)$ and $\Omega = \{x : F_{\ell'}^{*K}(x) > \lambda\}$. Suppose $F_\ell^{*K}(x) > \lambda$ for some $x \in \mathbb{R}^n$. Then there exists $k \geq K$ and $y \in x + B_{-k+\ell}$ such that $F(y,k) > \lambda$. Clearly $y+B_{-k+\ell'} \subset \Omega$. By (2.1), we also have $y+B_{-k+\ell'} \subset x+B_{-k+\ell}+B_{-k+\ell'} \subset x + B_{-k+\ell+\omega}$. Hence $y + B_{-k+\ell'} \subset \Omega \cap (x + B_{-k+\ell+\omega})$. From this, Proposition 2.1 (i), and (2.1), it follows that

$$\begin{aligned}
M_w(1_\Omega)(x) &\geq \frac{1}{w(x+B_{-k+\ell+\omega})}\int_{\Omega\cap(x+B_{-k+\ell+\omega})} w(x)dx \geq \frac{w(y+B_{-k+\ell'})}{w(x+B_{-k+\ell+\omega})}\\
&\geq \frac{w(y+B_{-k+\ell'})}{w(y+B_{-k+\ell}+B_{-k+\ell+\omega})} \geq \frac{w(y+B_{-k+\ell'})}{w(y+B_{-k+\ell+2\omega})}\\
&\gtrsim |\det A|^{-q(\ell-\ell'+2\omega)} \gtrsim |\det A|^{-q(\ell-\ell')}.
\end{aligned}$$

Therefore, we have seen that $\{x : F_\ell^{*K}(x) > \lambda\} \subset \Big\{x : M_w(1_\Omega)(x) \geq C|\det A|^{-q(\ell-\ell')}\Big\}$. Since $(\mathbb{R}^n, \rho_A, wdx)$ is a space of homogeneous type, M_w is bounded form $L^1(wdx)$ into $L^{1,\infty}(wdx)$. It follows that

$$\begin{aligned}
&w\left(\{x : F_\ell^{*K}(x) > \lambda\}\right) \leq w\left(\Big\{x : M_w(1_\Omega)(x) \geq C|\det A|^{-q(\ell-\ell')}\Big\}\right)\\
&\lesssim |\det A|^{q(\ell-\ell')}\|1_\Omega\|_{L^1(wdx)} = |\det A|^{q(\ell-\ell')} w\left(\{x : F_{\ell'}^{*K}(x) > \lambda\}\right).
\end{aligned} \tag{2.9}$$

Integrating both sides of (2.9) on $(0,\infty)$ with respect to λ yields (2.8).

For an integer K representing the truncation level and real number $L \geq 0$ representing the decay level, we define the following maximal type functions

$$\begin{aligned}
&T_\psi^{N(K,L)}f(x)\\
&= \sup_{k\in\mathbb{Z},k\geq K}\sup_{y\in\mathbb{R}^n} \frac{|f*\psi_k(y)|}{\max(1,\rho_A(A^k(x-y)))^N}\frac{(1+|\det A|^{k+K})^{-L}}{\max(1,\rho_A(A^K y))^L};\\
&M_N^{(K,L)}f(x)\\
&= \sup_{\psi\in\mathcal{S}_N(\mathbb{R}^n)}\sup_{\substack{k\in\mathbb{Z}\\ k\geq K}}\sup_{y\in x+B_{-k}} |f*\psi_k(y)|\max(1,\rho_A(A^K y))^{-L}(1+|\det A|^{k+K})^{-L}.
\end{aligned}$$

Using (2.8) together with an argument similar to that used in the proof of [2, Lemma 7.4], we obtain

$$\left\|T_\psi^{N(K,L)}f\right\|_{L^p_w(\mathbb{R}^n)} \leq C\left\|M_\psi^{(K,L)}f\right\|_{L^p_w(\mathbb{R}^n)} \tag{2.10}$$

for all $N > q_w/p$, $K \in \mathbb{Z}$, $L \geq 0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. With (2.10) in hand, the equivalence of (i), (ii) and (iii) follows by an argument similar to that used in the proof of [2, Theroem 7.1]. ■

We also have the following local version of Proposition 2.2, which provides the radial and grand maximal function characterizations of weighted anisotropic local Hardy spaces $h^p_w(\mathbb{R}^n; A)$. The proof is similar to that of Proposition 2.2 with only minor modifications, and hence we omit it.

Proposition 2.3. *Suppose that A is an expansive dilation, $w \in \mathcal{A}_\infty(A)$, $0 < p < \infty$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$. Then for any $f \in \mathcal{S}'(\mathbb{R}^n)$ the following are equivalent:*

(i) $M^{0,loc}_\psi f \in L^p_w(\mathbb{R}^n)$ *;*

(ii) $M^{loc}_\psi f \in L^p_w(\mathbb{R}^n)$*;*

(iii) $M^{loc}_N f \in L^p_w(\mathbb{R}^n)$ *for sufficiently large N.*

Moreover, if N is sufficiently large, then $\|M^{0,loc}_\psi f\|_{L^p_w(\mathbb{R}^n)} \sim \|M^{loc}_\psi f\|_{L^p_w(\mathbb{R}^n)} \sim \|M^{loc}_N f\|_{L^p_w(\mathbb{R}^n)}$.

In order to show the $H^p_w(\mathbb{R}^n; A)$-boundedness of anisotropic singular integrals, we will use the atomic characterization of $H^p_w(\mathbb{R}^n; A)$ obtained in [6].

Definition 2.3. (see [6]) Let A be an expansive dilation and $w \in \mathcal{A}_\infty(A)$. A triplet $(p, q, s)_w$ is said to be admissible, if $p \in (0, 1]$, $q \in (q_w, \infty]$ and $s \in \mathbb{N} \cup \{0\}$ such that $s \geq \left\lfloor (\frac{q_w}{p} - 1)\frac{\ln|\det A|}{\ln(\lambda_-)} \right\rfloor$. Given an admissible triplet $(p, q, s)_w$, we say that the measurable function a on $\mathbb{R}^n$ is a $(p, q, s)_w$-atom if

(i) supp $a \subset x_0 + B_{j_0}$ for some $x_0 \in \mathbb{R}^n$ and $j_0 \in \mathbb{Z}$;

(ii) $\|a\|_{L^q_w(\mathbb{R}^n)} \leq [w(x_0 + B_{j_0})]^{1/q-1/p}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for all multi-indices α with $|\alpha| \leq s$.

Let $H^{p,q,s}_w(\mathbb{R}^n; A)$ denote the space consisting of those tempered distributions admitting a decomposition $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $a_j's$ are $(p, q, s)_w$-atoms and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. And for every $f \in H^{p,q,s}_w(\mathbb{R}^n; A)$, we consider the (quasi-)norm defined by

$$\|f\|_{H^{p,q,s}_w(\mathbb{R}^n;A)} = \inf\left\{\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^{1/p} : f = \sum_{j=1}^\infty \lambda_j a_j \text{ in } \mathcal{S}'(\mathbb{R}^n),\ \{a_j\}_{j=1}^\infty \text{ are } (p, q, s)_w\text{–atoms}\right\}.$$

Proposition 2.4. (see [6, Theorem 5.5]) *Let A be an expansive dilation and $w \in \mathcal{A}_\infty(A)$. If $(p, q, s)_w$ is an admissible triplet, then $H^{p,q,s}_w(\mathbb{R}^n; A) = H^p_w(\mathbb{R}^n; A)$ with equivalent (quasi-)norms.*

The relation between $H^p_w(\mathbb{R}^n; A)$ and $h^p_w(\mathbb{R}^n; A)$ is as in the following lemma.

Lemma 2.1. *Let A be an expansive dilation, $w \in \mathcal{A}_\infty(A)$ and $0 < p \le 1$. Then*

(i) *$H^p_w(\mathbb{R}^n; A)$ is continuously embedded in $h^p_w(\mathbb{R}^n; A)$;*

(ii) *If Ψ is a function in $\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \Psi(x)dx = 1$ and $\int_{\mathbb{R}^n} \Psi(x)x^\alpha dx = 0$ for all $|\alpha| \le N$, where $N = N(p, w)$ is a sufficiently large positive integer, then $f - f * \Psi \in H^p_w(\mathbb{R}^n, A)$ and there exists a constant $C > 0$ such that*

$$\|f - f * \Psi\|_{H^p_w(\mathbb{R}^n,A)} \le C\|f\|_{h^p_w(\mathbb{R}^n,A)},$$

for all $f \in h^p_w(\mathbb{R}^n, A)$.

The proof of Lemma 2.1 is similar to that of [1, Lemma 2.1] with the only modification that one needs to use the radial and grand maximal function characterizations of weighted anisotropic Hardy spaces (see Proposition 2.2 and 2.3) instead of their unweighted counterparts; see also [8, Proposition 4.1]. We thus omit the details here.

3. Proof of Theorem 1.1

We begin with a simple lemma.

Lemma 3.1. *Suppose that K is a tempered distribution satisfying the condition (i) in Definition 1.1, and k_0 is a fixed positive integer. Then for all $\ell \in \mathbb{Z}$, all $x \in \mathbb{R}^n\backslash\{0\}$ with $|\det A|^{\ell-k_0} \le \rho_A(x) \le |\det A|^{\ell+k_0}$, and for all multi-indices α with $|\alpha| \le m$, we have*

$$\left|\partial^\alpha \left[K(A^\ell \cdot)\right](A^{-\ell}x)\right| \lesssim [\rho_A(x)]^{-1}.$$

Proof. Let $x \in \mathbb{R}^n$ be such that $\rho_A(x) = |\det A|^{\ell'}$ with $\ell - k_0 \le \ell' \le \ell + k_0$. Let $|\alpha| \le m$. By the chain rule, (2.6), (2.7) and (1.1), we have

$$\begin{aligned}\left|\partial^\alpha \left[K(A^\ell \cdot)\right](A^{-\ell}x)\right| &= \left|\partial^\alpha \left[K\left(A^{\ell'} A^{\ell-\ell'} \cdot\right)\right](A^{-\ell}x)\right| \\ &\lesssim \|A^{\ell-\ell'}\|^{|\alpha|} \sum_{|\beta|=|\alpha|} \left|\partial^\beta \left[K(A^{\ell'} \cdot)\right](A^{-\ell'}x)\right| \\ &\lesssim (\lambda_+)^{k_0 m} [\rho_A(x)]^{-1} \sim [\rho_A(x)]^{-1},\end{aligned}$$

which completes the proof. ∎

Lemma 3.2. *Suppose K is an anisotropic singular kernel of order m, and ψ is a Schwartz function such that* supp $\psi \subset B_1$. *Then $K * \psi_k$ satisfies the condition (i) in Definition 1.1 uniformly in $k \in \mathbb{Z}$. More precisely, there exists a constant $C > 0$ such that for all $\ell \in \mathbb{Z}$, all $x \in \mathbb{R}^n\backslash\{0\}$ with $\rho_A(x) = |\det A|^\ell$, and for all multi-indices α with $|\alpha| \le m$, we have*

$$\sup_{k\in\mathbb{Z}} \left|\partial^\alpha \left[(K * \psi_k)(A^\ell \cdot)\right](A^{-\ell}x)\right| \le C[\rho_A(x)]^{-1}.$$

Proof. Observe that (1.1) is dilation-invariant; that is, if we replace K by K_k in (1.1), where $K_k := |\det A|^k K(A^k\cdot)$, we get the same inequality with the same constant. In addition, we can write $K * \psi_k = (K_{-k} * \psi)_k$. Therefore, it suffices to show that $\widetilde{K} := K * \psi$ satisfies (1.1).

To this end, we write

$$\widetilde{K}(x) = (\hat{K}\hat{\psi})^{\vee}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\hat{\psi}(\xi)\hat{K}(\xi)d\xi.$$

Then for all multi-indices β with $|\beta| \le m$, we have

$$(3.1)\quad |\partial^\beta \widetilde{K}(x)| = \left|\int_{\mathbb{R}^n}(2\pi i\xi)^\beta e^{2\pi i x\cdot\xi}\hat{\psi}(\xi)\hat{K}(\xi)d\xi\right| \lesssim \|\hat{K}\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}|\xi^\beta\hat{\psi}(\xi)|d\xi \lesssim 1,$$

where we have used the fact that the L^2-boundedness of $f \mapsto f * K$ implies that $\hat{K} \in L^\infty(\mathbb{R}^n)$. From (3.1) we see that the family $\{\partial^\beta\widetilde{K} : |\beta| \le m\}$ is uniformly bounded.

Let $\ell \in \mathbb{Z}$ and $x \in \mathbb{R}^n\backslash\{0\}$ with $\rho_A(x) = |\det A|^\ell$, and let α be a multi-index such that $|\alpha| \le m$. To show that $\widetilde{K}$ satisfies (1.1), we consider two cases.

Case I. $\ell \le \omega + 1$. By the chain rule, (2.6), (2.7) and (3.1), we have

$$\begin{aligned}\left|\partial^\alpha\left[\widetilde{K}(A^\ell\cdot)\right](A^{-\ell}x)\right| &\lesssim \|A^\ell\|^{|\alpha|}\sum_{|\beta|=|\alpha|}\left|(\partial^\beta\widetilde{K})(x)\right|\\ &\lesssim (\lambda_+)^{(\omega+1)m} \lesssim 1 \lesssim |\det A|^{-\ell} = [\rho_A(x)]^{-1}.\end{aligned}$$

Case II. $\ell > \omega + 1$. Since supp $\psi \subset B_1$ we can write

$$(3.2)\quad \begin{aligned}\partial^\alpha\left[\widetilde{K}(A^\ell\cdot)\right](A^{-\ell}x) &= \int_{\mathbb{R}^n}\partial^\alpha\left\{K\left[A^\ell(\cdot - A^{-\ell}y)\right]\right\}(A^{-\ell}x)\psi(y)dy\\ &= \int_{B_1}\partial^\alpha\left[K(A^\ell\cdot)\right]\left(A^{-\ell}(x-y)\right)\psi(y)dy.\end{aligned}$$

By (2.1) and (2.2), we see that for $\ell > \omega + 1$, if $x \in B_{\ell+1}\backslash B_\ell$ and $y \in B_1$ then $x - y \in B_{\ell+\omega+1}\backslash B_{\ell-\omega}$, i.e., $|\det A|^{\ell-\omega} \le \rho_A(x-y) \le |\det A|^{\ell+\omega}$. Hence it follows from Lemma 3.1 that

$$\left|\partial^\alpha\left[K(A^\ell\cdot)\right](A^{-\ell}(x-y))\right| \lesssim [\rho_A(x-y)]^{-1} \le |\det A|^{-\ell+\omega} \sim [\rho_A(x)]^{-1}.$$

Inserting this into (3.2), we get

$$\left|\partial^\alpha\left[\widetilde{K}(A^\ell\cdot)\right](A^{-\ell}x)\right| \lesssim \|\psi\|_{L^1}[\rho_A(x)]^{-1} \sim [\rho_A(x)]^{-1}.$$

Combining both cases, we obtain the desired estimate and thus complete the proof of Lemma 3.2. ∎

Lemma 3.3. *Suppose $1 < q < \infty$ and $w \in \mathcal{A}_q(A)$. Then any anisotropic singular integral operator of order 1 is bounded on $L^q_w(\mathbb{R}^n)$.*

Proof. By [7, Proposition 3.6], it suffices to verify that there exist positive constants ε and C such that for all $x, y \in \mathbb{R}^n\backslash\{0\}$ with $\rho_A(x-y) \le |\det A|^{-2\omega}\rho_A(y)$,

$$|K(x) - K(y)| \le C\frac{[\rho_A(x-y)]^\varepsilon}{[\rho_A(y)]^{1+\varepsilon}}. \tag{3.3}$$

To this end, we assume without loss of generality that $\rho_A(x-y) = |\det A|^{j_0}$ and $\rho_A(y) = |\det A|^{j_0+j_1+2\omega}$ for certain $j_0 \in \mathbb{Z}$ and $j_1 \in \mathbb{N}\cup\{0\}$. Set $\widetilde{K} = K(A^{j_0+j_1+2\omega}\cdot)$. By the Mean Value Theorem, we have

$$\begin{aligned}|K(x) - K(y)| &= \left|\widetilde{K}\left(A^{-(j_0+j_1+2\omega)}x\right) - \widetilde{K}\left(A^{-(j_0+j_1+2\omega)}y\right)\right| \\ &\le \left|A^{-(j_0+j_1+2\omega)}(x-y)\right| \sup_{\xi \in B_{j_0+1}} \left|\nabla\widetilde{K}\left(A^{-(j_0+j_1+2\omega)}(y+\xi)\right)\right|.\end{aligned}$$

From (2.1) and (2.2) we see that if $\rho_A(y) = |\det A|^{j_0+j_1+2\omega}$ and $\xi \in B_{j_0+1}$ then $|\det A|^{j_0+j_1+\omega} \le \rho_A(y+\xi) \le |\det A|^{j_0+j_1+3\omega}$. Hence by Lemma 3.1 we have

$$\sup_{\xi \in B_{j_0+1}} \left|\nabla\widetilde{K}\left(A^{-(j_0+j_1+2\omega)}(y+\xi)\right)\right| \lesssim \sup_{\xi \in B_{j_0+1}} [\rho_A(y+\xi)]^{-1} \sim [\rho_A(y)]^{-1}.$$

It follows that

$$\begin{aligned}|K(x) - K(y)| &\lesssim \left|A^{-(j_0+j_1+2\omega)}(x-y)\right| [\rho_A(y)]^{-1} \\ &\lesssim \left[\rho_A\left(A^{-(j_0+j_1+2\omega)}(x-y)\right)\right]^{\zeta_-} [\rho_A(y)]^{-1} \\ &\sim \frac{[\rho_A(x-y)]^{\zeta_-}}{[\rho_A(y)]^{1+\zeta_-}},\end{aligned}$$

where, in the second inequality we have applied (2.5). Hence (3.3) holds with $\varepsilon = \zeta_-$, and the proof of Lemma 3.3 is thus complete. ■

We are now ready to prove Theroem 1.1.

Proof of Theorem 1.1. Since $\frac{q_w}{1+m\zeta_-} < p \le 1$, we can find $q \in (q_w, \infty)$ such that $p(1+m\zeta_-) - q > 0$. The latter implies that $(p,q,m)_w$ is an admissible triplet. Hence, from [6, Theorem 7.2] we know that, to obtain the $H^p_w(\mathbb{R}^n; A)$-boundedness of T, it suffices to show that for all $(p,q,m)_w$-atoms a, we have $\|Ta\|_{H^p_w(\mathbb{R}^n;A)} \lesssim 1$. Take $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi(x)dx \ne 0$ and supp $\psi \subset B_1$. In view of the radial maximal function characterization of $H^p_w(\mathbb{R}^n; A)$ (see Proposition 2.2), we have to show that

$$\left\|\sup_{k \in \mathbb{Z}} |(Ta) * \psi_k|\right\|_{L^p_w(\mathbb{R}^n)} \lesssim 1. \tag{3.4}$$

We assume that a is a $(p,q,m)_w$-atom associated to the dilated ball $x_0+B_{j_0}$, where $x_0\in\mathbb{R}^n$ and $j_0\in\mathbb{Z}$. Note that by [6, Proposition 2.11 (i)] we have

$$\sup_{k\in\mathbb{Z}}|(Ta)*\psi_k(x)|\le CM(Ta)(x). \tag{3.5}$$

To show (3.4) we first estimate the integration over the set $x_0+B_{j_0+\omega}$. Indeed, by (3.5), Holder's inequality, Proposition 2.1 and Lemma 3.3 we have

$$\begin{aligned}
&\int_{x_0+B_{j_0+\omega}}\left(\sup_{k\in\mathbb{Z}}|(Ta)*\psi_k(x)|\right)^p w(x)dx\\
&\lesssim\int_{x_0+B_{j_0+\omega}}|M(Ta)(x)|^p\,w(x)dx\\
&\lesssim\left[\int_{x_0+B_{j_0+\omega}}|M(Ta)(x)|^q\,w(x)dx\right]^{\frac{p}{q}}\left[w(x_0+B_{j_0+\omega})\right]^{1-\frac{p}{q}}\\
&\lesssim\|Ta\|^p_{L^q_w(\mathbb{R}^n)}\left[w(x_0+B_{j_0})\right]^{1-\frac{p}{q}}\\
&\lesssim\|a\|^p_{L^q_w(\mathbb{R}^n)}\left[w(x_0+B_{j_0})\right]^{1-\frac{p}{q}}\\
&\lesssim 1.
\end{aligned}\tag{3.6}$$

Next we fix an arbitrary integer $j\ge j_0+\omega$. Set $K^{(k)}:=(K*\psi_k)(A^j\cdot)$, $k\in\mathbb{Z}$. For $x\in x_0+(B_{j+1}\backslash B_j)$ and $y\in x_0+B_{j_0}$, by Taylor's inequality we have

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\left|K^{(k)}\left(A^{-j}(x-y)\right)-\sum_{|\beta|\le m-1}\frac{\left(\partial^\beta K^{(k)}\right)\left(A^{-j}(x-x_0)\right)\left(A^{-j}(x_0-y)\right)^\beta}{\beta!}\right|\\
&\lesssim\sup_{k\in\mathbb{Z}}\sup_{z\in B_{j_0}}\sup_{|\beta|=m}\left|\left(\partial^\beta K^{(k)}\right)\left(A^{-j}(x-x_0+z)\right)\right|\left|A^{-j}(x_0-y)\right|^m\\
&=\sup_{k\in\mathbb{Z}}\sup_{z\in B_{j_0}}\sup_{|\beta|=m}\left|\partial^\beta\left[(K*\psi_k)(A^j\cdot)\right]\left(A^{-j}(x-x_0+z)\right)\right|\left|A^{-j}(x_0-y)\right|^m.
\end{aligned}\tag{3.7}$$

Observe that if $j\ge j_0+\omega$, $z\in B_{j_0}$ and $x\in x_0+(B_{j+1}\backslash B_j)$, then (by (2.1) and (2.2)) we have $|\det A|^{j-\omega}\le\rho_A(x-x_0+z)\le|\det A|^{j+\omega}$. Hence, it follows from Lemma 3.2 and Lemma 3.1 that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\sup_{z\in B_{j_0}}\sup_{|\beta|=m}\left|\partial^\beta\left[(K*\psi_k)(A^j\cdot)\right]\left(A^{-j}(x-x_0+z)\right)\right|\\
&\lesssim\left[\rho_A(x-x_0+z)\right]^{-1}\sim|\det A|^{-j}.
\end{aligned}$$

Inserting this into (3.7) and by using (2.5), we obtain that, for all $x\in x_0+(B_{j+1}\backslash B_j)$

and $y \in x_0 + B_{j_0}$,

$$
\begin{aligned}
&\sup_{k\in\mathbb{Z}}\left|K^{(k)}\left(A^{-j}(x-y)\right) - \sum_{|\beta|\le m-1}\frac{\left(\partial^\beta K^{(k)}\right)\left(A^{-j}(x-x_0)\right)\left(A^{-j}(x_0-y)\right)^\beta}{\beta!}\right| \\
&\lesssim |\det A|^{-j}\left|A^{-j}\left(x_0-y\right)\right|^m \\
&\lesssim |\det A|^{-j}\left[\rho\left(A^{-j}(x_0-y)\right)\right]^{m\zeta_-} \\
&\lesssim |\det A|^{-j+(j_0-j)m\zeta_-}.
\end{aligned}
$$

Then, by using the vanishing moment conditions satisfied by a and Hölder's inequality, we have, for $x \in x_0 + (B_{j+1}\backslash B_j)$,

$$
\begin{aligned}
&\sup_{k\in\mathbb{Z}}|(Ta)*\psi_k(x)| \\
&= \sup_{k\in\mathbb{Z}}\left|\int_{x_0+B_{j_0}}(K*\psi_k)(x-y)a(y)dy\right| \\
&\le \sup_{k\in\mathbb{Z}}\int_{x_0+B_{j_0}}\left|K^{(k)}\left(A^{-j}(x-y)\right)\right. \\
&\quad\left.-\sum_{|\beta|\le m-1}\frac{\left(\partial^\beta K^{(k)}\right)\left(A^{-j}(x-x_0)\right)\left(A^{-j}(x_0-y)\right)^\beta}{\beta!}\right||a(y)|dy \\
&\lesssim |\det A|^{-j+(j_0-j)m\zeta_-}\int_{x_0+B_{j_0}}|a(y)|dy \\
&\lesssim |\det A|^{-j+(j_0-j)m\zeta_-}\|a\|_{L^q_w(\mathbb{R}^n)}\left(\int_{x_0+B_{j_0}}w(y)^{-\frac{q'}{q}}dy\right)^{1-\frac{1}{q}} \\
&\lesssim |\det A|^{-j+(j_0-j)m\zeta_-}\left[w(x_0+B_{j_0})\right]^{\frac{1}{q}-\frac{1}{p}}|x_0+B_{j_0}|^{\frac{q-1}{q}}\left(\frac{|x_0+B_{j_0}|}{w(x_0+B_{j_0})}\right)^{\frac{1}{q}} \\
&\lesssim |\det A|^{(j_0-j)(1+m\zeta_-)}\left[w(x_0+B_{j_0})\right]^{-\frac{1}{p}}.
\end{aligned}
$$

Hence, by Proposition 2.1 (i), we can estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^n\backslash(x_0+B_{j_0+\omega})}\sup_{k\in\mathbb{Z}}|(Ta)*\psi_k(x)|^p\,w(x)dx \\
&= \sum_{j=j_0+\omega}^{\infty}\int_{x_0+(B_{j+1}\backslash B_j)}\sup_{k\in\mathbb{Z}}|(Ta)*\psi_k(x)|^p\,w(x)dx \qquad (3.8)\\
&\lesssim \sum_{j=j_0+\omega}^{\infty}|\det A|^{p(j_0-j)(1+m\zeta_-)}\left[w(x_0+B_{j_0})\right]^{-1}w(x_0+B_{j+1})
\end{aligned}
$$

$$\lesssim \sum_{j=j_0+\omega}^{\infty} |\det A|^{p(j_0-j)(1+m\zeta_-)+q(j-j_0+1)}$$
$$\lesssim \sum_{j=\omega}^{\infty} |\det A|^{-j[p(1+m\zeta_-)-q]} \lesssim 1.$$

Here, in the last inequality, we have used that $p(1+m\zeta_-)-q>0$.

Combining (3.6) and (3.8), we get (3.4) and thus complete the proof of Theorem 1.1.

We remark that, as in the isotropic setting and the homogeneous group setting (cf. [25, pp. 28-29] and [14, pp. 106-107 and p. 201]), the result for anisotropic singular integrals on anisotropic Hardy spaces in Theorem 1.1 goes through for functions that take their values in Banach spaces. If $\mathcal{B}$ is a Banach spaces, we can consider the $\mathcal{B}$-valued tempered distributions on $\mathbb{R}^n$, i.e., the space of continuous linear maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{B}$. Then for an expansive dilation A, $0<p<\infty$ and $w\in\mathcal{A}_\infty(A)$, we define the $\mathcal{B}$-valued weighted anisotropic Hardy space $H^p_w(\mathbb{R}^n,\mathcal{B};A)$ as the collection of those $\mathcal{B}$-valued tempered distributions f for which

$$M_\psi f(x) := \sup_{k\in\mathbb{Z}} \sup_{y\in x+B_{-k}} \|f*\psi_k(y)\|_{\mathcal{B}}$$

belong to $L^p_w(\mathbb{R}^n)$, where $\psi\in\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\psi(x)dx\neq 0$. Note that the atomic decomposition can be carried out in the Banach space setting (see the remarks in [14, pp. 106-107]). Moreover, Proposition 2.2 can be extended to the Banach space setting, so that $H^p_w(\mathbb{R}^n,\mathcal{B};A)$ is independent of the choice of ψ and are equivalent to the radial and grand maximal function formulation.

For later use we formulate a generalization of Theorem 1.1 in the Banach space setting. In what follows, $\mathcal{B}_1$ and $\mathcal{B}_2$ is a pair of Banach spaces, $\mathfrak{L}(\mathcal{B}_1,\mathcal{B}_2)$ denotes the Banach space consisting of bounded linear operators form $\mathcal{B}_1$ to $\mathcal{B}_2$, and $L^p(\mathbb{R}^n,\mathcal{B}_i)$ $(i=1,2)$ are Bochner spaces.

Theorem 3.1. *Suppose A is an expansive dilation, $w\in\mathcal{A}_\infty(A)$ and m is a positive integer. Suppose further that $\mathcal{K}$ is an $\mathfrak{L}(\mathcal{B}_1,\mathcal{B}_2)$-valued tempered distribution satisfying the following conditions:*

(i) $\mathcal{K}$ coincides with an $\mathfrak{L}(\mathcal{B}_1,\mathcal{B}_2)$-valued C^m function away from the origin, and there exist a constant $C>0$ such that for all $\ell\in\mathbb{Z}$, all $x\in\mathbb{R}^n\backslash\{0\}$ with $\rho_A(x)=|\det A|^\ell$, and for all multi-indices α with $|\alpha|\le m$,

$$\|\partial^\alpha[\mathcal{K}(A^\ell\cdot)](A^{-\ell}x)\|_{\mathfrak{L}(\mathcal{B}_1,\mathcal{B}_2)} \le C[\rho_A(x)]^{-1}; \tag{3.9}$$

(ii) $\|f*\mathcal{K}\|_{L^2(\mathbb{R}^n,\mathcal{B}_2)}\le C\|f\|_{L^2(\mathbb{R}^n,\mathcal{B}_1)}$ for all $\mathcal{B}_1$-valued Schwartz functions f.

Then the operator $Tf=f*\mathcal{K}$ is bounded from $H^p_w(\mathbb{R}^n,\mathcal{B}_1;A)$ to $H^p_w(\mathbb{R}^n,\mathcal{B}_2;A)$, for $p\in(\frac{q_w}{1+m\zeta_-},1]$.

The proof is based on re-examining related assertions in the scalar-valued setting. We, therefore, omit the details.

4. Proof of Theorem 1.2

We begin with recalling the anisotropic Peetre's inequality obtained in [5].

Lemma 4.1. (see [5, Lemma 3.4]). *Suppose A is an expansive dilation, K is a compact subset of $\mathbb{R}^n$ and $r > 0$. Then there exist constants $C_1, C_2 > 0$, depending only on A, n, r and K such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$ with* supp $\hat{g} \subset K$,

$$\sup_{z\in\mathbb{R}^n} \frac{|\nabla g(x-z)|}{(1+\rho_A(z))^{1/r}} \le C_1 \sup_{z\in\mathbb{R}^n} \frac{|g(x-z)|}{(1+\rho_A(z))^{1/r}} \le C_2[M(|g|^r)(x)]^{1/r} \quad \text{for all } x \in \mathbb{R}^n.$$

Given $f \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $a > 0$, we define

$$(\varphi_j^* f)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|(f*\varphi_j)(x-y)|}{(1+|\det A|^j \rho_A(y))^a}, \ j \in \mathbb{Z}.$$

Lemma 4.2. *Suppose A is an expansive dilation, $w \in \mathcal{A}_\infty(A)$, $0 < p, q < \infty$ and $a > \max\{q_w/p, 1/q\}$. Suppose further that φ is a function in $\mathcal{S}(\mathbb{R}^n)$ such that* supp *$\hat{\varphi}$ is compact. Then there exists a constant $C > 0$ such that for all $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\left\| \left[\sum_{j=-\infty}^{\infty} \left|(\varphi_j^* f)_a\right|^q \right]^{1/q} \right\|_{L^p_w(\mathbb{R}^n)} \le C \left\| \left(\sum_{j=-\infty}^{\infty} |f*\varphi_j|^q \right)^{1/q} \right\|_{L^p_w(\mathbb{R}^n)}.$$

Proof. Set $K =$ supp $\hat{\varphi}$ and $g^{(j)}(x) = (f*\varphi_j)(A^{-j}x)$, $j \in \mathbb{Z}$. Then $(g^{(j)})^\wedge(\xi) = |\det A|^j \hat{\varphi}(\xi)\hat{f}((A^*)^j\xi)$, from which we see that supp$(g^{(j)})^\wedge \subset K$ for all $j \in \mathbb{Z}$. Hence, by Lemma 4.1, there exists a constant $C > 0$ such that

$$\sup_{z\in\mathbb{R}} \frac{|g^{(j)}(\zeta-z)|}{(1+\rho_A(z))^a} \le C \left[M\left(|g^{(j)}|^{1/a}\right)(\zeta)\right]^a \ \text{for all } j \in \mathbb{Z} \text{ and } \zeta \in \mathbb{R}^n. \tag{4.1}$$

But we have

$$\begin{aligned} \sup_{z\in\mathbb{R}} \frac{|g^{(j)}(\zeta-z)|}{(1+\rho_A(z))^a} &= \sup_{z\in\mathbb{R}^n} \frac{|(f*\varphi_j)(A^{-j}\zeta - A^{-j}z)|}{(1+\rho_A(z))^a} \\ &= \sup_{z\in\mathbb{R}^n} \frac{|(f*\varphi_j)(A^{-j}\zeta - z)|}{(1+|\det A|^j\rho_A(z))^a} \\ &= (\varphi_j^* f)_a(A^{-j}\zeta), \end{aligned} \tag{4.2}$$

and

$$
\begin{aligned}
M(|g^{(j)}|^{1/a})(\zeta) &= \sup_{k\in\mathbb{Z}} \sup_{y\in\zeta+B_k} \frac{1}{|B_k|}\int_{y+B_k} |(f*\varphi_j)(A^{-j}z)|^{1/a}dz \\
&\lesssim \sup_{k\in\mathbb{Z}} \frac{1}{|B_k|}\int_{\zeta+B_k} |(f*\varphi_j)(A^{-j}z)|^{1/a}dz \\
&= \sup_{k\in\mathbb{Z}} \frac{1}{|B_{k-j}|}\int_{A^{-j}\zeta+B_{k-j}} |(f*\varphi_j)(z)|^{1/a}dz \\
&\lesssim M\left(|f*\varphi_j|^{1/a}\right)(A^{-j}\zeta)
\end{aligned} \tag{4.3}
$$

where, in the first inequality we have used the fact that the "uncentered" Hardy-Littlewood maximal function of f is controlled by the "centered" one. Combining (4.1), (4.2) and (4.3), we obtain

$$
(\varphi_j^* f)_a(x) \le C\left[M\left(|f*\varphi_j|^{1/a}\right)(x)\right]^a, \tag{4.4}
$$

for all $j\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, where C is a constant independent of j. Since $w\in\mathcal{A}_{ap}(A)$ and $aq>1$, it follows from (4.4) and weighted Fefferman-Stein's vector-valued inequality (cf. [5, Theorem 2.5]) that

$$
\begin{aligned}
\left\|\left(\sum_{j=-\infty}^{\infty}|(\varphi_j^*f)_a|^q\right)^{1/q}\right\|_{L^p_w(\mathbb{R}^n)} &\lesssim \left\|\left(\sum_{j=-\infty}^{\infty}\left[M\left(|f*\varphi_j|^{1/a}\right)\right]^{aq}\right)^{1/q}\right\|_{L^p_w(\mathbb{R}^n)} \\
&\lesssim \left\|\left(\sum_{j=-\infty}^{\infty}|f*\varphi_j|^q\right)^{1/q}\right\|_{L^p_w(\mathbb{R}^n)}.
\end{aligned}
$$

This completes the proof of Lemma 4.2. ∎

Lemma 4.3. *Suppose A is an expansive dilation and $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Then, for every multi-index α there exists a constant $C_\alpha>0$ such that for all $\ell\in\mathbb{Z}$ and all $x\in\mathbb{R}^n\backslash\{0\}$ with $\rho_A(x)=|\det A|^\ell$,*

$$
\left\|\partial^\alpha\left[\left\{\varphi_j(A^\ell\cdot)\right\}_{j=-\infty}^{\infty}\right](A^{-\ell}x)\right\|_{\ell^2} \le C_\alpha[\rho_A(x)]^{-1}.
$$

Proof. Let $\ell\in\mathbb{Z}$ and let $x\in\mathbb{R}^n\backslash\{0\}$ such that $\rho_A(x)=|\det A|^\ell$. We write

$$
\begin{aligned}
\left\|\partial^\alpha\left[\{\varphi_j(A^\ell\cdot)\}_{j=-\infty}^{\infty}\right](A^{-\ell}x)\right\|_{\ell^2}^2 &= \sum_{j=-\infty}^{\infty}|\det A|^{2j}\left|\partial^\alpha\left[\varphi(A^{j+\ell}\cdot)\right](A^{-\ell}x)\right|^2 \\
&= \left(\sum_{j=-\infty}^{-\ell}+\sum_{j=-\ell+1}^{\infty}\right)|\det A|^{2j}\left|\partial^\alpha\left[\varphi(A^{j+\ell}\cdot)\right](A^{-\ell}x)\right|^2 \\
&:= I_1+I_2.
\end{aligned}
$$

First we estimate I_1. Let $-\infty < j \le -\ell$. By using the chain rule, (2.7), and that $\partial^\beta \varphi \in L^\infty(\mathbb{R}^n)$ $(\forall\beta)$, we have

$$\left|\partial^\alpha\left[\varphi(A^{j+\ell}\cdot)\right](A^{-\ell}x)\right|^2 \lesssim \|A^{j+\ell}\|^{2|\alpha|}\sum_{|\beta|=|\alpha|}\left|\left(\partial^\beta\varphi\right)(A^jx)\right|^2 \lesssim (\lambda_-)^{2(j+\ell)|\alpha|}.$$

It follows that,

$$I_1 \lesssim \sum_{j=-\infty}^{-\ell} |\det A|^{2j}(\lambda_-)^{2(j+\ell)|\alpha|} \lesssim |\det A|^{-2\ell} = [\rho_A(x)]^{-2}. \tag{4.5}$$

Next we estimate I_2. Let $j \ge -\ell+1$. Let M be a positive number such that $|\det A|^{M-1} > (\lambda_+)^{|\alpha|}$. By using the chain rule, (2.6), and that $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we can estimate as follows.

$$\begin{aligned}\left|\partial^\alpha\left[\varphi(A^{j+\ell}\cdot)\right](A^{-\ell}x)\right|^2 &\lesssim \|A^{j+\ell}\|^{2|\alpha|}\sum_{|\beta|=|\alpha|}\left|\left(\partial^\beta\varphi\right)(A^jx)\right|^2 \\ &\lesssim (\lambda_+)^{2(j+\ell)|\alpha|}\left[\rho_A(A^jx)\right]^{-2M} = \left(\frac{(\lambda_+)^{|\alpha|}}{|\det A|^M}\right)^{2(j+\ell)}.\end{aligned}$$

Hence,

$$\begin{aligned}I_2 &\lesssim \sum_{j=-\ell+1}^{\infty} |\det A|^{2j}\left(\frac{(\lambda_+)^{|\alpha|}}{|\det A|^M}\right)^{2(j+\ell)} \\ &= |\det A|^{-2\ell}\sum_{j=-\ell+1}^{\infty}\left(\frac{(\lambda_+)^{|\alpha|}}{|\det A|^{M-1}}\right)^{2(j+\ell)} \lesssim |\det A|^{-2\ell} = [\rho_A(x)]^{-2}.\end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we obtain the desired estimate. ■

In order to obtain the Littlewood-Paley characterization of weighted anisotropic Hardy spaces, it is useful to introduce the ℓ^2-valued weighted anisotropic Hardy spaces $H^p_w(\mathbb{R}^n, \ell^2; A)$. Let A be an expansive dilation, $0 < p < \infty$ and $w \in \mathcal{A}_\infty(A)$. Define

$$\begin{aligned}H^p_w(\mathbb{R}^n,\ell^2;A) := \Bigg\{ f = \{f_j\}_{j=-\infty}^{\infty} \subset \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^p_w(\mathbb{R}^n,\ell^2;A)} \\ = \left\|\sup_{k\in\mathbb{Z}}\left(\sum_{j=-\infty}^{\infty}|f_j * \psi_k|^2\right)^{1/2}\right\|_{L^p_w(\mathbb{R}^n)} < \infty\Bigg\},\end{aligned}$$

where $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\psi(x)dx \neq 0$. As we mentioned in the previous section, the definition of $H^p_w(\mathbb{R}^n, \ell^2; A)$ is independent of the choice of ψ.

Lemma 4.4. *Suppose A is an expansive dilation, $w \in \mathcal{A}_\infty(A)$ and $0 < p \le 1$. Suppose further that φ is a Schwartz function satisfying (1.2) and the property that* supp $\hat{\varphi}$ *is compact and does not contain the origin. Then we have*

$$\left\| \{f * \varphi_j\}_{j=-\infty}^{\infty} \right\|_{H^p_w(\mathbb{R}^n,\ell^2;A)} \lesssim \|f\|_{H^p_w(\mathbb{R}^n;A)}, \quad f \in \mathcal{S}'(\mathbb{R}^n). \tag{4.7}$$

Conversely, given $f \in \mathcal{S}'(\mathbb{R}^n)$, there exists a polynomial P such that

$$\|f - P\|_{H^p_w(\mathbb{R}^n;A)} \lesssim \left\| \{f * \varphi_j\}_{j=-\infty}^{\infty} \right\|_{H^p_w(\mathbb{R}^n,\ell^2;A)}. \tag{4.8}$$

Proof. Define a liner map $\mathcal{K} : \mathcal{S}(\mathbb{R}^n) \to \ell^2$ by

$$\mathcal{K}(\phi) = \left\{ \int_{\mathbb{R}^n} \phi(x)\varphi_j(x)dx \right\}_{j=-\infty}^{\infty}, \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

Note that $\mathcal{K}(\phi)$ does belong to ℓ^2. Indeed, since $\int_{\mathbb{R}^n} \varphi(x)dx = \hat{\varphi}(0) = 0$, for $j > 0$ we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \phi(x)\varphi_j(x)dx \right| &\le \int_{\mathbb{R}^n} |\phi(A^{-j}x) - \phi(0)||\varphi(x)|dx \\ &\le \|\nabla\phi\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |A^{-j}x||\varphi(x)|dx \lesssim (\lambda_-)^{-j}, \end{aligned}$$

where we have used the Mean Value Theorem and (2.7). For $j \le 0$, we have

$$\left| \int_{\mathbb{R}^n} \phi(x)\varphi_j(x)dx \right| \le |\det A|^j \int_{\mathbb{R}^n} |\phi(x)\varphi(A^j x)|dx \lesssim |\det A|^j.$$

Therefore, $\|\mathcal{K}(\phi)\|_{\ell^2} < \infty$. The above computation also shows that $\mathcal{K}$ is a continuous map from $\mathcal{S}(\mathbb{R}^n)$ to ℓ^2, so that it is an ℓ^2-valued tempered distribution.

Since $\mathcal{K}$ coincides with the ℓ^2-valued smooth function $\{\varphi_j(x)\}_{j=-\infty}^{\infty}$ away from the origin, from Lemma 4.3 we see that it satisfies the regularity conditions in Theorem 3.1 up to order m for arbitrarily large integer m. Furthermore, by the Plancherel theorem and [7, Remark 2.13], we have

$$\begin{aligned} \|f * \mathcal{K}\|^2_{L^2(\mathbb{R}^n,\ell^2)} &= \|\{f * \varphi_j\}_{j=-\infty}^{\infty}\|^2_{L^2(\mathbb{R}^n,\ell^2)} = \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^n} |f * \varphi_j(x)|^2 dx \\ &= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \left(\sum_{j=-\infty}^{\infty} |\hat{\varphi}((A^*)^{-j}\xi)|^2 \right) d\xi \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Therefore, we may apply Theorem 3.1 to the $\mathcal{L}(\mathbb{C}, \ell^2)$-valued kernel $\mathcal{K}$, to get

$$\left\| \{f * \varphi_j\}_{j=-\infty}^{\infty} \right\|_{H^p_w(\mathbb{R}^n,\ell^2;A)} = \|f * \mathcal{K}\|_{H^p_w(\mathbb{R}^n,\ell^2;A)} \lesssim \|f\|_{H^p_w(\mathbb{R}^n;A)},$$

which proves (4.7).

Now let us show (4.8). Since supp $\hat{\varphi}$ is compact and does not contain the origin, we can find a (sufficiently large) positive integer k_0 such that supp $\hat{\varphi} \subset B^*_{k_0} \backslash B^*_{-k_0}$. The latter implies that supp$\hat{\varphi}((A^*)^j \cdot) \cap$ supp$\hat{\varphi}((A^*)^{j'} \cdot) = \emptyset$ whenever $|j - j'| > 2k_0$. Hence, if we define

$$\theta(x) = \sum_{k=-2k_0}^{2k_0} \varphi_k(x), \tag{4.9}$$

then it follows from (1.2) that $\hat{\theta}(\xi) = 1$ for $\xi \in$ supp $\hat{\varphi}$, and consequently $\varphi_j * \theta_j = \varphi_j$ for all $j \in \mathbb{Z}$. Define the linear map $\widetilde{K}$ from the ℓ^2-valued Schwart class $\mathcal{S}(\mathbb{R}^n, \ell^2)$ to $\mathbb{C}$ by setting, for all $\vec{\phi} = \{\phi_j\}_{j=-\infty}^{\infty} \in \mathcal{S}(\mathbb{R}^n, \ell^2)$,

$$\widetilde{\mathcal{K}}(\vec{\phi}) = \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^n} \phi_j(x)\theta_j(x)dx.$$

Similarly to above, one can use $\int_{\mathbb{R}^n} \theta(x)dx = 0$ to show that the sum in the right-hand side converges absolutely, and that the map $\widetilde{K} : \mathcal{S}(\mathbb{R}^n, \ell^2) \to \mathbb{C}$ is continuous, so that $\widetilde{K}$ is an $\mathcal{L}(\ell^2, \mathbb{C})$-valued tempered distribution. If we identify $\mathcal{L}(\ell^2, \mathbb{C})$ with ℓ^2 (they are isomorphic as Hilbert spaces), then $\mathcal{K}$ coincides with the ℓ^2-valued smooth function $\{\theta_j\}_{j=-\infty}^{\infty}$ away from the origin. Thus, by Lemma 4.3, the $\mathcal{L}(\ell^2, \mathbb{C})$-valued kernel $\widetilde{K}$ satisfies the regularity conditions in Theorem 3.1 up to order m for arbitrarily large integer m. Furthermore, $\widetilde{K}$ satisfies the condition (ii) in Theorem 3.1. Indeed, by the Plancherel theorem, the Cauchy-Schwartz inequality, and [7, Remark 2.13], we have

$$\begin{aligned}
\left\| \left(\{f_j\}_{j=-\infty}^{\infty}\right) * \widetilde{K} \right\|_{L^2(\mathbb{R}^n)} &= \left\| \sum_{j=-\infty}^{\infty} f_j * \theta_j \right\|_{L^2(\mathbb{R}^n)} = \left\| \sum_{j=-\infty}^{\infty} \widehat{f_j}\widehat{\theta_j} \right\|_{L^2(\mathbb{R}^n)} \\
&\le \left\| \left(\sum_{j=-\infty}^{\infty} |\widehat{f_j}|^2 \right)^{1/2} \left(\sum_{j=-\infty}^{\infty} |\hat{\theta}((A^*)^{-j} \cdot)|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \left\| \left(\sum_{j=-\infty}^{\infty} |\widehat{f_j}|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^n)} = \left\| \left(\sum_{j=-\infty}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^n)} \\
&= \left\| \{f_j\}_{j=-\infty}^{\infty} \right\|_{L^2(\mathbb{R}^n, \ell^2)}.
\end{aligned}$$

Thus, we may apply Theorem 3.1 to the $\mathcal{L}(\ell^2, \mathbb{C})$-valued kernel $\widetilde{K}$, which yields

$$\left\| \sum_{j=-\infty}^{\infty} f * \varphi_j \right\|_{H^p_w(\mathbb{R}^n;A)} = \left\| \sum_{j=-\infty}^{\infty} f * \varphi_j * \theta_j \right\|_{H^p_w(\mathbb{R}^n;A)}$$
$$= \left\| (\{f * \varphi_j\}) * \widetilde{K} \right\|_{H^p_w(\mathbb{R}^n;A)} \lesssim \left\| \{\varphi_j * f\}_{j=-\infty}^{\infty} \right\|_{H^p_w(\mathbb{R}^n,\ell^2;A)}.$$

But note that $f - \sum_{j\in\mathbb{Z}} f * \varphi_j$ equals to a polynomial P, since its Fourier transform is supported at the origin. It follows that $f - P \in H^p_w(\mathbb{R}^n; A)$ and satisfies (4.8). ■

Lemma 4.5. *Suppose that A is an expansive dilation, φ is a Schwartz function satisfying (1.2) and the property that* supp $\hat{\varphi}$ *is compact and does not contain the origin, and $\Phi \in \mathcal{S}(\mathbb{R}^n)$ is given by (1.3). Then we have the following (quasi-)norm equivalence*

$$\|f\|_{h^p_w(\mathbb{R}^n;A)} \sim \|f - \Phi * f\|_{H^p_w(\mathbb{R}^n;A)} + \|\Phi * f\|_{L^p_w(\mathbb{R}^n)}, \quad f \in \mathcal{S}'(\mathbb{R}^n).$$

Proof. Since $\hat{\Phi}$ equals 1 near the origin, by Lemma 2.1 we have $\|f - \Phi * f\|_{H^p_w(\mathbb{R}^n;A)} \lesssim \|f\|_{h^p_w(\mathbb{R}^n;A)}$. Moreover, from the definition of $h^p_w(\mathbb{R}^n; A)$ we see that $\|\Phi * f\|_{L^p_w(\mathbb{R}^n;A)} \leq \|f\|_{h^p_w(\mathbb{R}^n;A)}$. Therefore,

$$\|f - \Phi * f\|_{H^p_w(\mathbb{R}^n;A)} + \|\Phi * f\|_{L^p_w(\mathbb{R}^n)} \lesssim \|f\|_{h^p_w(\mathbb{R}^n;A)}.$$

To see the inverse inequality, we write, by virtue of the radial maximal function characterization of $h^p_w(\mathbb{R}^n; A)$ (see Proposition 2.3),

$$\begin{aligned} \|f\|_{h^p_w(\mathbb{R}^n;A)} &\lesssim \|f - \Phi * f\|_{h^p_w(\mathbb{R}^n;A)} + \|\Phi * f\|_{h^p_w(\mathbb{R}^n;A)} \\ &= \|f - \Phi * f\|_{h^p_w(\mathbb{R}^n;A)} + \left\| \sup_{j\in\mathbb{N}\cup\{0\}} |\psi_j * \Phi * f| \right\|_{L^p_w(\mathbb{R}^n)}, \end{aligned} \tag{4.10}$$

where $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$. Since $\hat{\Phi}$ has compact support, we may apply (4.4), in which taking $j = 0$ yields that

$$|(\Phi * f)(x - y)| \lesssim \left[M\left(|\Phi * f|^{1/a} \right)(x) \right]^a (1 + \rho_A(y))^a, \quad \forall y \in \mathbb{R}^n,$$

where $a > 0$ can be chosen to be arbitrarily large. By the above inequality and (2.4), and using that $\psi \in \mathcal{S}(\mathbb{R}^n)$, we can estimate as follows

$$
\begin{aligned}
&\sup_{j\in\mathbb{N}\cup\{0\}} |\psi_j * \Phi * f(x)| \\
&= \sup_{j\in\mathbb{N}\cup\{0\}} \left| \int_{\mathbb{R}^n} |\det A|^j \psi(A^j y)(\Phi * f)(x-y)dy \right| \\
&\lesssim \sup_{j\in\mathbb{N}\cup\{0\}} \int_{\mathbb{R}^n} \frac{|\det A|^j}{(1+\rho_A(A^j y))^{a+(n+1)\zeta_+}} \\
&\qquad \left[M\left(|\Phi * f|^{1/a}\right)(x)\right]^a (1+\rho_A(y))^a dy \qquad (4.11) \\
&\le \left[M\left(|\Phi * f|^{1/a}\right)(x)\right]^a \int_{\mathbb{R}^n} \frac{1}{(1+\rho_A(y))^{(n+1)\zeta_+}} dy \\
&\lesssim \left[M\left(|\Phi * f|^{1/a}\right)(x)\right]^a \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}} dy \\
&\lesssim \left[M\left(|\Phi * f|^{1/a}\right)(x)\right]^a .
\end{aligned}
$$

Inserting this into (4.10), and applying the weighted maximal inequality, we get

$$
\|f\|_{h_w^p(\mathbb{R}^n;A)} \lesssim \|f - \Phi * f\|_{H_w^p(\mathbb{R}^n;A)} + \|\Phi * f\|_{L_w^p(\mathbb{R}^n)}.
$$

The proof is therefore complete. ■

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. (i) Let φ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying (1.2) and the property that supp$\hat{\varphi}$ is compact and does not contain the origin. As above we let k_0 be a positive integer such that supp$\hat{\varphi} \subset B^*_{k_0} \backslash B^*_{-k_0}$. Define the function θ by (4.9).

We first show that $\|f\|_{\dot{F}_p^{0,2}(\mathbb{R}^n,A,wdx)} \lesssim \|f\|_{H_w^p(\mathbb{R}^n;A)}$ for all $f \in \mathcal{S}'$. Indeed, by (4.7) in Lemma 4.4 we have

$$
\begin{aligned}
\|f\|_{\dot{F}_p^{0,2}(\mathbb{R}^n,A,wdx)} &= \left\| \{f * \varphi_j\}_{j=-\infty}^{\infty} \right\|_{L_w^p(\mathbb{R}^n,\ell^2)} \\
&\le \left\| \{f * \varphi_j\}_{j=-\infty}^{\infty} \right\|_{H_w^p(\mathbb{R}^n,\ell^2;A)} \lesssim \|f\|_{H_w^p(\mathbb{R}^n;A)}.
\end{aligned}
$$

To see the converse, we pick a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$ and supp$\hat{\psi} \subset B^*_{-k_0}$. Then $\hat{\psi}((A^*)^{-\ell}\cdot)\hat{\varphi}((A^*)^{-j}\cdot) \equiv 0$ whenever $\ell \le j$ $(i, j \in \mathbb{Z})$. Hence, for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $j \in \mathbb{Z}$ we have

$$
\begin{aligned}
&\sup_{\ell\in\mathbb{Z}} |\psi_\ell * \varphi_j * f(x)| \\
&= \sup_{\ell\in\mathbb{Z},\ell>j} \left| \int_{\mathbb{R}^n} \psi_\ell(y)(\varphi_j * f)(x-y)dy \right| \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{\ell\in\mathbb{Z},\ell>j}\left|\int_{\mathbb{R}^n}\psi(y)(\varphi_j*f)(x-A^{-\ell}y)dy\right| \\
&\lesssim \sup_{\ell\in\mathbb{Z},\ell>j}\sup_{z\in\mathbb{R}^n}\frac{|(\varphi_j*f)(x-A^{-\ell}z)|}{(1+\rho_A(z))^a}\int_{\mathbb{R}^n}(1+\rho_A(y))^a|\psi(y)|dy \\
&\lesssim \sup_{\ell\in\mathbb{Z},\ell>j}\sup_{z\in\mathbb{R}^n}\frac{|(\varphi_j*f)(x-z)|}{(1+|\det A|^\ell\rho_A(z))^a} \\
&\le \sup_{z\in\mathbb{R}^n}\frac{|(\varphi_j*f)(x-z)|}{(1+|\det A|^j\rho_A(z))^a}=(\varphi_j^*f)_a(x),
\end{aligned}
$$

where $a>0$ can be chosen to be arbitrarily large. Now, by using (4.8) in Lemma 4.4, (4.12), and Lemma 4.2, we can estimate as follows: Given $f\in\mathcal{S}'$, there exists a polynomial P such that

$$
\begin{aligned}
&\|f-P\|_{H^p_w(\mathbb{R}^n;A)} \\
&\lesssim \left\|\{\varphi_j*f\}_{j=-\infty}^{\infty}\right\|_{H^p_w(\mathbb{R}^n,\ell^2;A)} = \left\|\sup_{\ell\in\mathbb{Z}}\left(\sum_{j=-\infty}^{\infty}|\psi_\ell*\varphi_j*f|^2\right)^{1/2}\right\|_{L^p_w(\mathbb{R}^n)} \\
&\le \left\|\left(\sum_{j=-\infty}^{\infty}\sup_{\ell\in\mathbb{Z}}|\psi_\ell*\varphi_j*f|^2\right)^{1/2}\right\|_{L^p_w(\mathbb{R}^n)} \lesssim \left\|\left(\sum_{j=-\infty}^{\infty}|(\varphi_j^*f)_a|^2\right)^{1/2}\right\|_{L^p_w(\mathbb{R}^n)} \\
&\lesssim \left\|\left(\sum_{j=-\infty}^{\infty}|f*\varphi_j|^2\right)^{1/2}\right\|_{L^p_w(\mathbb{R}^n)} = \|f\|_{\dot F^{0,2}_p(\mathbb{R}^n,A,wdx)}.
\end{aligned}
$$

Therefore, the assertion (i) is established.

(ii) Let us now show that $\|f\|_{h^p_w(\mathbb{R}^n;A)}\sim\|f\|_{F^{0,2}_p(\mathbb{R}^n,A,wdx)}$ for $f\in\mathcal{S}'(\mathbb{R}^n)$. Indeed, let φ be the same as in the proof of (i) and define Φ by (1.3). By Lemma 4.5 and (i) we have

$$
\begin{aligned}
\|f\|_{h^p_w(\mathbb{R}^n;A)} &\sim \|\Phi*f\|_{L^p_w(\mathbb{R}^n)}+\|f-\Phi*f\|_{H^p_w(\mathbb{R}^n;A)} \\
&\sim \|\Phi*f\|_{L^p_w(\mathbb{R}^n)}+\left\|\left(\sum_{j=-\infty}^{\infty}|\varphi_j*(f-\Phi*f)|^2\right)^{1/2}\right\|_{L^p_w(\mathbb{R}^n)}.
\end{aligned}
$$

The polynomial P in the assertion (i) doesn't appear in the above inequalities. This is because the Fourier transform of $f-\Phi*f$ equals 0 in a neighborhood of the origin. Therefore, it remains to show that

$$\|\Phi * f\|_{L^p_w(\mathbb{R}^n)} + \left\| \left(\sum_{j=-\infty}^{\infty} |\varphi_j * (f - \Phi * f)|^2 \right)^{1/2} \right\|_{L^p_w(\mathbb{R}^n)} \sim \|f\|_{F^{0,2}_p(\mathbb{R}^n, A, w dx)}. \tag{4.13}$$

Since $\mathrm{supp}\hat{\varphi} \subset B^*_{k_0} \backslash B^*_{-k_0}$, we see (from (1.3)) that supp $\hat{\Phi} \subset B^*_{k_0}$ and that $\hat{\Phi}(\xi) = 1$ for $\xi \in B^*_{-k_0+1}$. Hence $\varphi_j * (f - \Phi * f) \equiv 0$ whenever $j \le -2k_0 + 1$. Consequently we can write

$$\begin{aligned}\text{lhs } (4.13) \lesssim \|\Phi * f\|_{L^p_w(\mathbb{R}^n)} &+ \left\| \left(\sum_{j=-2k_0+2}^{\infty} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^p_w(\mathbb{R}^n)} \\ &+ \left\| \left(\sum_{j=-2k_0+2}^{\infty} |\Phi * \varphi_j * f|^2 \right)^{1/2} \right\|_{L^p_w(\mathbb{R}^n)}.\end{aligned}$$

Since $\hat{\varphi}$ has compact support, by (4.4) and a similar computation as in (4.11) it is easy to obtain that $|\Phi * \varphi_j * f| \lesssim [M(|\varphi_j * f|^{1/a})(x)]^a$, where a can be chosen to be arbitrarily large. Hence, it follows from the weighted Fefferman-Stein inequality (cf. [5, Theorem 2.5]) that

$$\text{lhs } (4.13) \lesssim \|\Phi * f\|_{L^p_w(\mathbb{R}^n)} + \left\| \left(\sum_{j=-2k_0+2}^{\infty} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^p_w(\mathbb{R}^n)} \lesssim \|f\|_{F^{0,2}_p(\mathbb{R}^n, A, w dx)}.$$

To obtain the inverse inequality in (4.13), we note that by the support properties of $\hat{\varphi}$ and $\hat{\Phi}$ (which implies $\varphi_j * \Phi \equiv 0$ whenever $j \ge 2k_0$), (4.11), and the weighted Hardy-Littlewood maximal inequality,

$$\begin{aligned}&\|f\|_{F^{0,2}_p(\mathbb{R}^n, A, w dx)} \\ &= \|\Phi * f\|_{L^p_w(\mathbb{R}^n)} + \left\| \left(\sum_{j=1}^{\infty} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^p_w(\mathbb{R}^n)} \\ &\lesssim \|\Phi * f\|_{L^p_w(\mathbb{R}^n)} + \left\| \left(\sum_{j=1}^{\infty} |\varphi_j * (f - \Phi * f)|^2 \right)^{1/2} \right\|_{L^p_w(\mathbb{R}^n)} \\ &\quad + \left\| \left(\sum_{j=1}^{2k_0-1} |\varphi_j * \Phi * f|^2 \right)^{1/2} \right\|_{L^p_w(\mathbb{R}^n)}\end{aligned}$$

$$\begin{aligned}&\lesssim \|\Phi * f\|_{L^p_w(\mathbb{R}^n)} + \left\| \left(\sum_{j=1}^{\infty} |\varphi_j * (f - \Phi * f)|^2 \right)^{1/2} \right\|_{L^p_w(\mathbb{R}^n)} \\ &\quad + \left\| \left[M \left(|\Phi * f|^{1/a} \right) (x) \right]^a \right\|_{L^p_w(\mathbb{R}^n)} \\ &\lesssim \text{lhs } (4.13).\end{aligned}$$

Therefore, (4.13) is established and the proof Theorem 1.2 is complete.

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