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Fixed Point Theorems via MNC in Ordered Banach Space with Application to Fractional Integro-differential Evolution Equations

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Abstract. In this paper, we propose fixed point results through the notion of measure of noncompactness (MNC) in partially ordered Banach spaces. We also prove some new coupled fixed point results via MNC for more general class of function. To achieve this result, we relaxed the conditions of boundedness, closedness and convexity of the set at the expense that the operator is monotone and bounded. Further, we apply the obtained fixed point theorems to prove the existence of mild solutions for fractional integro-differential evolution equations with nonlocal conditions. At the end, an example is given to illustrate the rationality of the abstract results for fractional parabolic equations.

1. Introduction and preliminaries

It is well known that the fixed point theorems are very important for proving the existence of solutions for some nonlinear differential and integral equations, see [1] and the references therein. The mixed arguments from different branches of mathematics are used in the research of fixed point theory. The first hybrid fixed point theorems in partially ordered sets is obtained by Ran and Reurings [21], where they extended the Banach contraction principle to partially ordered sets with some applications to linear and nonlinear matrix equations. Subsequently, Nieto and Rodríguez-López [17, 18] extended the results in [20] to the monotone mappings in partially ordered metric spaces using the mixed arguments from algebra, analysis and geometry and applied the abstract results to study the unique solution for a first order ordinary differential equation with periodic boundary conditions. Further improvements of the above mentioned results in partially ordered linear spaces can be found in [13, 16] and the references therein.

In 1930, Kuratowskii [14] opened up a new direction of research with the introduction of measure of noncompactness (MNC). The MNC [14] combine with some algebraic arguments are useful for studying the mathematical formulations, particularly for solving

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the existence of solutions of some nonlinear problems under certain conditions. The Kuratowskii and Hausdorff MNC [4–6, 12] in a metric space are well known in the literature. However, as far as we know that the applications of MNC in partially ordered normed linear spaces are seldom [7–9, 19].

Throughout this paper, we assume that $(\mathcal{E}, \|\cdot\|)$ is an infinite dimensional Banach space. Let $\mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$. If D is a subset of $\mathcal{E}$, we denote by $\mathrm{conv}(D)$ and $\overline{\mathrm{conv}}(D)$ the closure and the convex closure of D, respectively. Moreover, we denote by $\mathfrak{M}_{\mathcal{E}}$ the family of nonempty bounded subset of $\mathcal{E}$ and by $\mathfrak{N}_{\mathcal{E}}$ the family of all relatively compact subset of $\mathcal{E}$. We use the following definition of MNC given in [6].

Definition 1.1. A mapping $\beta\colon \mathfrak{M}_{\mathcal{E}} \to \mathbb{R}^+$ is said to be an MNC in $\mathcal{E}$ if it satisfies

(1) The family $\operatorname{Ker}\beta = \{D \in \mathfrak{M}_{\mathcal{E}} : \beta(D) = 0\}$ is nonempty and $\operatorname{Ker}\beta \subset \mathfrak{N}_{\mathcal{E}}$;

(2) $\beta(C) \leq \beta(D)$ for any nonempty subsets $C, D \in \mathfrak{M}_{\mathcal{E}}$ with $C \subset D$;

(3) $\beta(\mathrm{conv}(D)) = \beta(D)$ for any nonempty subset $D \in \mathfrak{M}_{\mathcal{E}}$;

(4) $\beta(\overline{\mathrm{conv}}(D)) = \beta(D)$ for any nonempty subset $D \in \mathfrak{M}_{\mathcal{E}}$;

(5) $\beta(\lambda C + (1-\lambda)D) \leq \lambda\beta(C) + (1-\lambda)\beta(D)$ for any nonempty subsets $C, D \in \mathfrak{M}_{\mathcal{E}}$ and $\lambda \in [0, 1]$;

(6) If $\{D_n\}$ is a sequence of closed sets from $\mathfrak{M}_{\mathcal{E}}$ such that $D_{n+1} \subset D_n$, $n \geq 1$ and if $\lim_{n\to\infty} \beta(D_n) = 0$, the intersection set $D_\infty = \bigcap_{n=1}^{\infty} D_n$ is nonempty.

It follows from Definition 1.1(6) that D_∞ is a member of the family $\operatorname{Ker}\beta$. Since $\beta(D_\infty) \leq \beta(D_n)$ for any n, we can deduce that $\beta(D_\infty) = 0$. This implies that $D_\infty \in \operatorname{Ker}\beta$.

Definition 1.2. [10] Let Ψ be a set of functions $\chi\colon \mathbb{R}^+ \to [0, 1)$ satisfying

$$\chi(t_n) \to 1 \quad \Longrightarrow \quad t_n \to 0.$$

Here are the examples of the function $\chi \in \Psi$:

$$\chi(t) = \begin{cases} e^{-2t} & \text{if } t > 0, \\ 1 & \text{if } t = 0, \end{cases} \qquad \chi(t) = \begin{cases} \frac{1}{1+t} & \text{if } t > 0, \\ 1 & \text{if } t = 0. \end{cases}$$

Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem and Banach's fixed point theorem. The following fixed point theorem of Darbo type was proved by Banaś and Goebel in [6].

Lemma 1.3. *Let D be a nonempty, bounded, closed and convex subset of a Banach space E and let $Q\colon D \to D$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that*

$$\beta(Q(S)) \leq k\beta(S)$$

for any nonempty subset $S \subset D$. Then Q has at least one fixed point in D.

Recently, Lemma 1.3 has been extended by Aghajani et al. in [3]. They obtained the following two fixed point theorems.

Lemma 1.4. *Let D be a nonempty, bounded, closed and convex subset of a Banach space E and let $\chi \in \Psi$, $Q\colon D \to D$ be a continuous mapping satisfying*

$$\beta(Q(S)) \leq \chi(\beta(S))\beta(S)$$

for any nonempty subset $S \subset D$. Then Q has at least one fixed point in D.

Lemma 1.5. *Let D be a nonempty, bounded, closed and convex subset of a Banach space E and let $Q\colon D \to D$ be a continuous mapping satisfying*

$$\beta(Q(S)) \leq \phi(\beta(S))$$

for any nonempty subset $S \subset D$, where $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and upper semi-continuous function such that $\phi(t) < t$ for all $t > 0$. Then Q has at least one fixed point in D.

Clearly, if we take $\chi(t) \equiv k \in [0,1)$ for any $t \in \mathbb{R}^+$ in Lemma 1.4, or take $\phi(t) = kt$ for any $t \in \mathbb{R}^+$ and $k \in [0,1)$ in Lemma 1.5, Lemmas 1.4 and 1.5 degenerate into Lemma 1.3.

In the present paper, we will extend the results in Lemmas 1.4 and 1.5 into partially ordered Banach spaces. By doing this, we also improve and generalize the works mentioned in [7–9, 19]. For this, we first define a notion of MNC in partially ordered Banach spaces. We use this notion to prove some fixed point theorems for contraction condition (2.1) in partially ordered Banach spaces whose positive cone K is normal, and then to prove some coupled fixed point theorems in partially ordered Banach spaces. To achieve this result, we relaxed the conditions of boundedness, closedness and convexity of the set at the expense that the operator is monotone and bounded. We also supply some new coupled fixed point results via MNC for more general class of function than Ψ. Further, we apply the obtained fixed point theorems to prove the existence of mild solutions for fractional integro-differential evolution equations with nonlocal conditions. At the end, an example is given to illustrate the rationality of the abstract results for fractional parabolic equations.

2. Fixed point theorems

Let $\mathcal{E}$ be a Banach space with the norm $\|\cdot\|$ whose positive cone is defined by $K = \{x \in \mathcal{E} : x \geq 0\}$. Then $(\mathcal{E}, \|\cdot\|)$ is now a partially ordered Banach space with the order relation $\sqsubseteq$ induced by cone K.

Now, we establish the fixed point theorems via MNC in partially ordered Banach spaces.

Theorem 2.1. *Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space, whose positive cone K is normal. Suppose that $\mathcal{Q}\colon \mathcal{E} \to \mathcal{E}$ is a continuous, nondecreasing and bounded mapping satisfying the following contraction:*

$$\beta(\mathcal{Q}(\mathcal{C})) \leq \chi(\beta(\mathcal{C}))\beta(\mathcal{C}) \tag{2.1}$$

for all bounded subset $\mathcal{C}$ in $\mathcal{E}$, where β denotes the MNC in $\mathcal{E}^2$ and $\chi \in \Psi$.

If there exists an element $u_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{Q}u_0$, then $\mathcal{Q}$ has a fixed point u^* and the sequence $\{\mathcal{Q}^n x_0\}$ of successive iterations converges monotonically to u^*.

Proof. Starting from the given $u_0 \in \mathcal{E}$, we define a sequence $\{u_n\}$ of points in $\mathcal{E}$ by

$$u_{n+1} = \mathcal{Q}u_n, \quad n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}. \tag{2.2}$$

Since $\mathcal{Q}$ is nondecreasing and $u_0 \sqsubseteq \mathcal{Q}u_0$, we have

$$u_0 \sqsubseteq u_1 \sqsubseteq u_2 \sqsubseteq \cdots \sqsubseteq u_n \sqsubseteq \cdots . \tag{2.3}$$

Denote $\mathcal{C}_n = \overline{\operatorname{conv}}\{u_n, u_{n+1}, \ldots\}$ for $n \in \mathbb{N}^*$. By (2.2) and (2.3), each $\mathcal{C}_n$ is a bounded and closed subset in $\mathcal{E}$ and

$$\mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n \supset \cdots .$$

Therefore, by (2.1), we obtain

$$\beta(\mathcal{Q}(\mathcal{C}_n)) \leq \chi(\beta(\mathcal{C}_n))\beta(\mathcal{C}_n) \leq \beta(\mathcal{C}_n). \tag{2.4}$$

This implies that $\beta(\mathcal{C}_n)$ is a positive decreasing sequence of real numbers. Thus, there exists $r \geq 0$ such that $\beta(\mathcal{C}_n) \to r$ as $n \to \infty$. We show that $r = 0$. Suppose, to the contrary, that $r \neq 0$. Therefore from (2.4) we have

$$\frac{\beta(\mathcal{C}_{n+1})}{\beta(\mathcal{C}_n)} \leq \chi(\beta(\mathcal{C}_n)) < 1.$$

This yields

$$\chi(\beta(\mathcal{C}_n)) \to 1 \quad \text{as } n \to \infty.$$

Since $\chi \in \Psi$, we get $r = 0$, and hence

$$\beta(\mathcal{C}_n) \to 0 \quad \text{as } n \to \infty.$$

Since $\mathcal{C}_n \subset \mathcal{C}_{n-1}$, we have

$$\overline{\mathcal{C}}_\infty = \bigcap_{n=1}^{\infty} \mathcal{C}_n \neq \emptyset \quad \text{and} \quad \mathcal{C}_\infty \in \operatorname{Ker}\beta.$$

Hence, for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\beta(\mathcal{C}_n) < \epsilon, \quad \forall\, n \geq n_0.$$

This concludes that $\overline{\mathcal{C}}_{n_0}$ and consequently $\mathcal{C}_0$ is a compact chain in $\mathcal{E}$. Hence, $\{u_n\}$ has a convergent subsequence. Applying the monotone property of $\mathcal{Q}$ and the normality of cone K, the whole sequence $\{u_n\} = \{\mathcal{Q}^n u_0\}$ converges monotonically to a point, say $u^* \in \mathcal{C}_0$. Finally, from the continuity of $\mathcal{Q}$, we get

$$\mathcal{Q}u^* = \mathcal{Q}\left(\lim_{n\to\infty} u_n\right) = \lim_{n\to\infty} \mathcal{Q}u_n = \lim_{n\to\infty} u_{n+1} = u^*. \qquad \square$$

If we set

$$\psi(t) = \begin{cases} \varphi(t) & \text{for } 0 \leq t \leq \beta(\mathcal{E}), \\ \varphi(\beta(\mathcal{E})) & \text{for } t > \beta(\mathcal{E}) \end{cases} \tag{2.5}$$

and $\chi(t) = \psi(t)/t$ for $t > 0$ and $\chi(0) = 1/2$, then we have

Theorem 2.2. *Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space whose positive cone K is normal. Suppose that $\mathcal{Q}\colon \mathcal{E} \to \mathcal{E}$ is a continuous, nondecreasing and bounded mapping satisfying the following conditions:*

$$\beta(\mathcal{Q}(\mathcal{C})) \leq \varphi(\beta(\mathcal{C}))$$

for any bounded subset $\mathcal{C}$ in $\mathcal{E}$, where β denotes the MNC in $\mathcal{E}^2$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and upper semi-continuous function such that $\varphi(t) < t$ for all $t > 0$.

If there exists an element $u_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{Q}u_0$, then $\mathcal{Q}$ has a fixed point u^* and the sequence $\{\mathcal{Q}^n x_0\}$ of successive iterations converges monotonically to u^*.

Proof. Following the line of the proof of Theorem 2.1 and Corollary 2.2 in [2], with the setting of φ given in (2.5), we have the conclusion. $\square$

If we take $\chi(t) = k$ where $k \in [0, 1)$ in Theorem 2.1, then we have following consequence.

Corollary 2.3. *Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space whose positive cone K is normal. Suppose that $\mathcal{Q}\colon \mathcal{E} \to \mathcal{E}$ is a continuous, nondecreasing and bounded mapping satisfying the following condition:*

$$\beta(\mathcal{Q}(\mathcal{C})) \leq k\beta(\mathcal{C})$$

for all bounded subset $\mathcal{C}$ in $\mathcal{E}$, where β denotes the MNC in $\mathcal{E}^2$ and $k \in [0,1)$.

If there exists an element $u_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{Q}u_0$, then $\mathcal{Q}$ has a fixed point u^* and the sequence $\{\mathcal{Q}^n x_0\}$ of successive iterations converges monotonically to u^*.

Remark 2.4. There are many examples for the function $\varphi(t)$, such as $\varphi(t) = t/(1+t)$, $\varphi(t) = \ln(1+t)$. If we take different type of $\varphi(t)$ in Theorem 2.2, we can obtain some different fixed point theorems.

3. Coupled fixed point theorems

In this section, we prove some coupled fixed point theorems. We begin our discussion by recalling some definitions and notions.

Definition 3.1. [11] An element $(u^*, v^*) \in \mathcal{E}^2$ is called a coupled fixed point of a mapping $\mathcal{G}\colon \mathcal{E}^2 \to \mathcal{E}$ if $\mathcal{G}(u^*, v^*) = u^*$ and $\mathcal{G}(v^*, u^*) = v^*$.

Definition 3.2. Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space and let $\mathcal{G}\colon \mathcal{E}^2 \to \mathcal{E}$ be a mapping. The mapping $\mathcal{G}$ is said to have the monotone property if $\mathcal{G}(u, v)$ is monotone nondecreasing in both variables u and v, that is, for any $u, v \in \mathcal{E}$,

$$u_1, u_2 \in \mathcal{E},\ u_1 \sqsubseteq u_2 \quad \Longrightarrow \quad \mathcal{G}(u_1, v) \sqsubseteq \mathcal{G}(u_2, v)$$

and

$$v_1, v_2 \in \mathcal{E},\ v_1 \sqsubseteq v_2 \quad \Longrightarrow \quad \mathcal{G}(u, v_1) \sqsubseteq \mathcal{G}(u, v_2).$$

Lemma 3.3. [4] *Suppose that $\beta_1, \beta_2, \ldots, \beta_n$ are the MNC in Banach spaces $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$, respectively. Moreover assume that the function $F\colon [0,\infty)^n \to [0,\infty)$ is convex and $F(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \ldots, n$. Then*

$$\beta(\mathcal{C}) = F(\beta_1(\mathcal{C}_1), \beta_2(\mathcal{C}_2), \ldots, \beta_n(\mathcal{C}_n))$$

defines a MNC in $\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n$ where $\mathcal{C}_i$ denotes the natural projection of $\mathcal{C}$ into $\mathcal{E}_i$ for $i = 1, 2, \ldots, n$.

Theorem 3.4. *Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space whose positive cone K is normal. Suppose that $\mathcal{G}\colon \mathcal{E}^2 \to \mathcal{E}$ is a continuous and bounded mapping, having monotone property and satisfying*

$$\beta(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2)) \leq \frac{1}{2}\chi(\beta(\mathcal{C}_1) + \beta(\mathcal{C}_2))[\beta(\mathcal{C}_1) + \beta(\mathcal{C}_2)] \tag{3.1}$$

for all bounded subsets $\mathcal{C}_1$, $\mathcal{C}_2$ in $\mathcal{E}$, where β denotes the MNC in $\mathcal{E}$ and $\chi \in \Psi$.

If there exist elements $u_0, v_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{G}(u_0, v)$ for any $v \in \mathcal{E}$ and $v_0 \sqsubseteq \mathcal{G}(v_0, u)$ for any $u \in \mathcal{E}$, then $\mathcal{G}$ has at least one coupled fixed point (u^*, v^*).

Proof. Consider the map $\widetilde{\mathcal{G}}\colon \mathcal{E}^2 \to \mathcal{E}^2$ defined by the formula

$$\widetilde{\mathcal{G}}(u,v) = (\mathcal{G}(u,v), \mathcal{G}(v,u)).$$

Since $\mathcal{G}$ is a continuous and bounded mapping, having monotone property, it follows that $\widetilde{\mathcal{G}}$ is also a continuous and bounded mapping, having monotone property.

Following Lemma 3.3, for $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$, we define a new MNC in the space $\mathcal{E}^2$ as

$$\widetilde{\beta}(\mathcal{C}) = \beta(\mathcal{C}_1) + \beta(\mathcal{C}_2),$$

where $\mathcal{C}_i$, $i = 1, 2$, denote the natural projections of $\mathcal{C}$. Now let $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \subset \mathcal{E}^2$ be a nonempty bounded subset. Due to (3.1) we conclude that

$$\begin{aligned}
\widetilde{\beta}((\widetilde{\mathcal{G}}(\mathcal{C})) &\leq \widetilde{\beta}(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2) \times \mathcal{G}(\mathcal{C}_2 \times \mathcal{C}_1)) \\
&= \beta(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2)) + \beta(\mathcal{G}(\mathcal{C}_2 \times \mathcal{C}_1)) \\
&\leq \frac{1}{2}\chi(\beta(\mathcal{C}_1) + \beta(\mathcal{C}_2))[\beta(\mathcal{C}_1) + \beta(\mathcal{C}_2)] + \frac{1}{2}\chi(\beta(\mathcal{C}_2) + \beta(\mathcal{C}_1))[\beta(\mathcal{C}_2) + \beta(\mathcal{C}_1)] \\
&= \chi(\beta(\mathcal{C}_1) + \beta(\mathcal{C}_2))[\beta(\mathcal{C}_1) + \beta(\mathcal{C}_2)] \\
&= \chi(\widetilde{\beta}(\mathcal{C}))\widetilde{\beta}(\mathcal{C}).
\end{aligned}$$

That is,

$$\widetilde{\beta}((\widetilde{\mathcal{G}}(\mathcal{C}))) \leq \chi(\widetilde{\beta}(\mathcal{C}))\widetilde{\beta}(\mathcal{C}).$$

Next, we show that there is a $\widetilde{u}_0 \in \mathcal{C}$ such that $\widetilde{u}_0 \sqsubseteq \widetilde{\mathcal{G}}(\widetilde{u}_0)$. Indeed, since there exist two elements $u_0, v_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{G}(u_0, v)$ for any $v \in \mathcal{E}$ and $v_0 \sqsubseteq \mathcal{G}(v_0, u)$ for any $u \in \mathcal{E}$, set $\widetilde{u}_0 = (u_0, v_0)$. Then by the definition of $\widetilde{\mathcal{G}}$, we have

$$\widetilde{u}_0 = (u_0, v_0) \sqsubseteq (\mathcal{G}(u_0, v_0), \mathcal{G}(v_0, u_0)) = \widetilde{\mathcal{G}}(u_0, v_0) = \widetilde{\mathcal{G}}(\widetilde{u}_0).$$

Thus, following from Theorem 2.1, we conclude that $\widetilde{\mathcal{G}}$ has a fixed point, and hence $\mathcal{G}$ has a coupled fixed point. □

Theorem 3.5. *Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space whose positive cone K is normal. Suppose that $\mathcal{G}\colon \mathcal{E}^2 \to \mathcal{E}$ is a continuous and bounded mapping, having monotone property and satisfying*

$$\beta(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2)) \leq \chi(\max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\}) \max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\}$$

for all bounded subsets $\mathcal{C}_1$, $\mathcal{C}_2$ in $\mathcal{E}$, where β denotes the MNC in $\mathcal{E}$ and $\chi \in \Psi$.

If there exist elements $u_0, v_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{G}(u_0, v)$ for any $v \in \mathcal{E}$ and $v_0 \sqsubseteq \mathcal{G}(v_0, u)$ for any $u \in \mathcal{E}$, then $\mathcal{G}$ has at least a coupled fixed point (u^*, v^*).

Proof. Consider the map $\mathcal{G}\colon \mathcal{E}^2 \to \mathcal{E}^2$ defined by the formula

$$\widetilde{\mathcal{G}}(u,v) = (\mathcal{G}(u,v), \mathcal{G}(v,u)).$$

Then $\widetilde{\mathcal{G}}$ is a continuous and bounded mapping, having monotone property.

For any $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$, we define a new MNC in the space $\mathcal{E}^2$ as

$$\widetilde{\beta}(\mathcal{C}) = \max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\},$$

where $\mathcal{C}_i$, $i = 1, 2$, denote the natural projections of $\mathcal{C}$. Now let $\mathcal{C} \subset \mathcal{E}^2$ with $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ be a nonempty bounded subset. We can conclude

$$\begin{aligned}
\widetilde{\beta}((\widetilde{\mathcal{G}}(\mathcal{C})) &\leq \widetilde{\beta}(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2) \times \mathcal{G}(\mathcal{C}_2 \times \mathcal{C}_1)) \\
&= \max\{\beta(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2)), \beta(\mathcal{G}(\mathcal{C}_2 \times \mathcal{C}_1))\} \\
&\leq \max\left\{\begin{array}{l} \chi(\max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\}) \max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\}, \\ \chi(\max\{\beta(\mathcal{C}_2), \beta(\mathcal{C}_1)\}) \max\{\beta(\mathcal{C}_2), \beta(\mathcal{C}_1)\} \end{array}\right\} \\
&= \chi(\max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\}) \max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\} \\
&= \chi(\widetilde{\beta}(\mathcal{C}))\widetilde{\beta}(\mathcal{C}).
\end{aligned}$$

That is,

$$\widetilde{\beta}((\widetilde{\mathcal{G}}(\mathcal{C})) \leq \chi(\widetilde{\beta}(\mathcal{C}))\widetilde{\beta}(\mathcal{C}).$$

Next, we show that there is a $\widetilde{u}_0 \in \mathcal{C}$ such that $\widetilde{u}_0 \sqsubseteq \widetilde{\mathcal{G}}(\widetilde{u}_0)$. Since there exist elements $u_0, v_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{G}(u_0, v)$ for any $v \in \mathcal{E}$ and $v_0 \sqsubseteq \mathcal{G}(v_0, u)$ for any $u \in \mathcal{E}$, set $\widetilde{u}_0 = (u_0, v_0)$. Then by definition of $\widetilde{\mathcal{G}}$, we have

$$\widetilde{u}_0 = (u_0, v_0) \sqsubseteq (\mathcal{G}(u_0, v_0), \mathcal{G}(v_0, u_0)) = \widetilde{\mathcal{G}}(u_0, v_0) = \widetilde{\mathcal{G}}(\widetilde{u}_0).$$

Following Theorem 2.1, $\widetilde{\mathcal{G}}$ has a fixed point, and hence $\mathcal{G}$ has a coupled fixed point. Thus we conclude the result. $\square$

In the following, we are going to derive the coupled fixed point results for more general class of functions Θ than Ψ.

Definition 3.6. [13] Let Θ denote the class of all functions $\theta\colon [0,\infty) \times [0,\infty) \to [0,1)$ which satisfy the following conditions:

(1) $\theta(s,t) = \theta(t,s)$ for all $s, t \in [0,\infty)$;

(2) for any two sequences $\{s_n\}$ and $\{t_n\}$ of nonnegative real numbers,

$$\theta(s_n, t_n) \to 1 \quad \Longrightarrow \quad s_n, t_n \to 0.$$

Theorem 3.7. *Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space whose positive cone K is normal. Suppose that $\mathcal{G}\colon \mathcal{E}^2 \to \mathcal{E}$ is a continuous and bounded mapping, having monotone property and satisfying*

$$\beta(\mathcal{G}(\mathcal{C}, \mathcal{D})) \leq \theta(\beta(\mathcal{C}), \beta(\mathcal{D})) \max\{\beta(\mathcal{C}), \beta(\mathcal{D})\}$$

for all bounded subsets $\mathcal{C}$, $\mathcal{D}$ in $\mathcal{E}$, where β denotes the MNC in $\mathcal{E}$ and $\theta \in \Theta$.

If there exist elements $u_0, v_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{G}(u_0, v)$ for any $v \in \mathcal{E}$ and $v_0 \sqsubseteq \mathcal{G}(v_0, u)$ for any $u \in \mathcal{E}$, then $\mathcal{G}$ has at least a coupled fixed point (u^*, v^*).

Proof. Starting from the given $u_0, v_0 \in \mathcal{E}$, we can construct two sequences $\{u_n\}$ and $\{v_n\}$ of points in $\mathcal{E}$ by

$$u_{n+1} = \mathcal{G}(u_n, v_n) \quad \text{and} \quad v_{n+1} = \mathcal{G}(v_n, u_n), \quad n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}. \tag{3.2}$$

Since $u_0 \sqsubseteq \mathcal{G}(u_0, v_0)$, by the monotone property of $\mathcal{G}$, we have from (3.2) that

$$u_0 \sqsubseteq \mathcal{G}(u_0, v_0) = u_1 \quad \text{and} \quad v_0 \sqsubseteq \mathcal{G}(v_0, u_0) = v_1.$$

This implies that

$$u_1 = \mathcal{G}(u_0, v_0) \sqsubseteq \mathcal{G}(u_1, v_1) = u_2 \quad \text{and} \quad v_1 = \mathcal{G}(v_0, u_0) \sqsubseteq \mathcal{G}(v_1, u_1) = v_2.$$

Consequently, we have

$$u_0 \sqsubseteq u_1 \sqsubseteq u_2 \sqsubseteq \cdots \sqsubseteq u_n \sqsubseteq \cdots \tag{3.3}$$

and

$$v_0 \sqsubseteq v_1 \sqsubseteq v_2 \sqsubseteq \cdots \sqsubseteq v_n \sqsubseteq \cdots . \tag{3.4}$$

Denote $\widehat{A}_n = \overline{\operatorname{conv}}\{u_n, u_{n+1}, \ldots\}$ and $\widehat{B}_n = \overline{\operatorname{conv}}\{v_n, v_{n+1}, \ldots\}$ for $n \in \mathbb{N}^*$. By (3.2), (3.3) and (3.4), $\widehat{A}_n$ and $\widehat{B}_n$ are bounded and closed subsets in $\mathcal{E}$ satisfying

$$\widehat{A}_0 \supset \widehat{A}_1 \supset \cdots \supset \widehat{A}_n \supset \cdots \quad \text{and} \quad \widehat{B}_0 \supset \widehat{B}_1 \supset \cdots \supset \widehat{B}_n \supset \cdots .$$

Therefore, by property of θ we obtain

$$\beta(\mathcal{G}(\widehat{A}_n, \widehat{B}_n)) \leq \theta(\beta(\widehat{A}_n), \beta(\widehat{B}_n)) \max\{\beta(\widehat{A}_n), \beta(\widehat{B}_n)\} \tag{3.5}$$

and

$$\begin{aligned} \beta(\mathcal{G}(\widehat{B}_n, \widehat{A}_n)) &\leq \theta(\beta(\widehat{B}_n), \beta(\widehat{A}_n)) \max\{\beta(\widehat{B}_n), \beta(\widehat{A}_n)\} \\ &= \theta(\beta(\widehat{A}_n), \beta(\widehat{B}_n)) \max\{\beta(\widehat{A}_n), \beta(\widehat{B}_n)\} \end{aligned} \tag{3.6}$$

for all n. From (3.5) and (3.6), we get

$$\begin{aligned}\max\{\beta(\mathcal{G}(\widehat{A}_n,\widehat{B}_n)),\beta(\mathcal{G}(\widehat{B}_n,\widehat{A}_n))\} &\le \theta(\beta(\widehat{A}_n),\beta(\widehat{B}_n))\max\{\beta(\widehat{A}_n),\beta(\widehat{B}_n)\} \\ &\le \max\{\beta(\widehat{A}_n),\beta(\widehat{B}_n)\}\end{aligned} \tag{3.7}$$

for all n. This implies that $\max\{\beta(\widehat{A}_n),\beta(\widehat{B}_n)\}$ is a positive decreasing sequence of real numbers. Thus, there is $r \ge 0$ such that $\max\{\beta(\widehat{A}_n),\beta(\widehat{B}_n)\} \to r$ as $n \to \infty$. We show that $r = 0$. Suppose, to the contrary, that $r \neq 0$. Then from (3.7) we have

$$\frac{\max\{\beta(\widehat{A}_{n+1}),\beta(\widehat{B}_{n+1})\}}{\max\{\beta(\widehat{A}_n),\beta(\widehat{B}_n)\}} \le \theta(\beta(\widehat{A}_n),\beta(\widehat{B}_n)) < 1.$$

This yields

$$\theta(\beta(\widehat{A}_n),\beta(\widehat{B}_n)) \to 1 \quad \text{as } n \to \infty.$$

Since $\theta \in \Theta$, we get

$$\beta(\widehat{A}_n) \to 0 \quad \text{and} \quad \beta(\widehat{B}_n) \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

Following the line of Theorem 2.1 and using (3.8), we can show that $\widehat{A}_0$ and consequently $\widehat{A}_0$ is a compact chain in $\mathcal{E}$. Similarly, $\widehat{B}_0$ is a compact chain in $\mathcal{E}$. Hence, $\{u_n\}$ and $\{v_n\}$ have convergent subsequences, respectively. By the monotone property of $\mathcal{G}$ and the normality of cone K, the whole sequence $\{u_n\}$ (and $\{v_n\}$) is convergent and converges monotonically to a point, say $u^* \in \widehat{A}_0$ (respectively $v^* \in \widehat{B}_0$). Finally, from the continuity of $\mathcal{G}$, we get

$$\mathcal{G}(u^*,v^*) = \mathcal{G}\left(\lim_{n\to\infty}(u_n,v_n)\right) = \lim_{n\to\infty}\mathcal{G}(u_n,v_n) = \lim_{n\to\infty}u_{n+1} = u^*$$

and

$$\mathcal{G}(v^*,u^*) = \mathcal{G}\left(\lim_{n\to\infty}(v_n,u_n)\right) = \lim_{n\to\infty}\mathcal{G}(v_n,u_n) = \lim_{n\to\infty}v_{n+1} = v^*.$$

Thus $\mathcal{G}$ has a coupled fixed point (u^*,v^*). This completes the proof. □

Corollary 3.8. *Let $(\mathcal{E},\|\cdot\|,\sqsubseteq)$ be a partially ordered Banach space whose positive cone K is normal. Suppose that $\mathcal{G}\colon \mathcal{E}^2 \to \mathcal{E}$ is a continuous and bounded mapping, having monotone property and satisfying*

$$\beta(\mathcal{G}(\mathcal{C},\mathcal{D})) \le k\max\{\beta(\mathcal{C}),\beta(\mathcal{D})\}$$

for all bounded subset $\mathcal{C}$, $\mathcal{D}$ in $\mathcal{E}$, where $\widetilde{\beta}$ denotes the MNC in $\mathcal{E}^2$ and $k \in [0,1)$.

If there exist elements $u_0, v_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{G}(u_0,v)$ for any $v \in \mathcal{E}$ and $v_0 \sqsubseteq \mathcal{G}(v_0,u)$ for any $u \in \mathcal{E}$, then $\mathcal{G}$ has at least a coupled fixed point (u^*,v^*).

Proof. Putting $\theta(t_1,t_2) = k$ with $k \in [0,1)$ for all $t_1,t_2 \in [0,\infty)$ in Theorem 3.7, we conclude the result. □

Corollary 3.9. *Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space whose positive cone K is normal. Suppose that $\mathcal{G}\colon \mathcal{E}^2 \to \mathcal{E}$ is a continuous and bounded mapping, having monotone property and satisfying*

$$\beta(\mathcal{G}(\mathcal{C}\mathcal{D})) \leq \ell_1\beta(\mathcal{C}) + \ell_2\beta(\mathcal{D})$$

for all bounded chain $\mathcal{C}$, $\mathcal{D}$ in $\mathcal{E}$, where $\widetilde{\beta}$ denotes the MNC in $\mathcal{E}^2$ and $\ell_1, \ell_2 \geq 0$ such that $\ell_1 + \ell_2 \leq 1$.

If there exist elements $u_0, v_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{G}(u_0, v)$ for any $v \in \mathcal{E}$ and $v_0 \sqsubseteq \mathcal{G}(v_0, u)$ for any $u \in \mathcal{E}$, then $\mathcal{G}$ has at least a coupled fixed point (u^*, v^*).

Proof. Since

$$\ell_1\beta(\mathcal{C}) + \ell_2\beta(\mathcal{D}) \leq \max\{\beta(\mathcal{C}), \beta(\mathcal{D})\},$$

we conclude the result from Corollary 3.8 for some $\ell_1, \ell_2 \geq 0$ such that $\ell_1 + \ell_2 < 1$. □

4. Existence of mild solutions for fractional evolution equations

Let X be a partially ordered Banach space with the norm $\|\cdot\|$ and the partial order $\leq$ whose positive cone K is normal. Consider the existence of mild solutions for the initial value problem of fractional integro-differential evolution equation of the form

$$\begin{cases} {}^C D_t^\sigma x(t) + Ax(t) = f\left(t, x(t), \int_0^t k(t,s,x(s))\, ds\right) & t \in J := [0,b], \\ x(0) = x_0 \in X, \end{cases} \tag{4.1}$$

where ${}^C D_t^\sigma$ denotes the Caputo fractional derivative of order $\sigma \in (0,1)$, $-A\colon D(A) \subset X \to X$ generates a positive C_0-semigroup $T(t)$ $(t \geq 0)$ of uniformly bounded linear operator in X, $b > 0$ is a constant, f and k are given functions.

Definition 4.1. A C_0-semigroup $T(t)$ $(t \geq 0)$ is called a positive C_0-semigroup if $T(t)x \geq 0$ for all $x \geq 0$.

We denote by $\mathfrak{B}$ a set of functions $\psi\colon [0,\infty) \to [0,\infty)$ satisfying

(a1) ψ is nondecreasing,

(a2) $\psi(x) < x$ for any $x > 0$,

(a3) $\chi(x) = \psi(x)/x \in \Psi$.

Examples of functions in $\mathfrak{B}$ are $\psi(x) = \mu x$, $0 \leq \mu < 1$, $\psi(x) = x/(1+x)$, $\psi(x) = \ln(1+x)$.

Define two operator families $\{\mathcal{U}_\sigma(t)\}_{t\geq 0}$ and $\{\mathcal{V}_\sigma(t)\}_{t\geq 0}$ as

$$\begin{aligned}\mathcal{U}_\sigma(t)x &= \int_0^\infty \zeta_\sigma(\tau)S(t^\sigma\tau)x\,d\tau,\\ \mathcal{V}_\sigma(t)x &= \sigma\int_0^\infty \tau\zeta_\sigma(\tau)S(t^\sigma\tau)x\,d\tau, \quad 0<\sigma<1,\end{aligned}$$

where

$$\begin{aligned}\zeta_\sigma(\tau) &= \frac{1}{\sigma}\tau^{-1-1/\sigma}\varrho_\sigma(\tau^{-1/\sigma}),\\ \varrho_\sigma(\tau) &= \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\tau^{-\sigma n-1}\frac{\Gamma(n\sigma+1)}{n!}\sin(n\pi\sigma), \quad \tau\in(0,\infty).\end{aligned}$$

The following lemma can be found in [15, 22].

Lemma 4.2. (i) *For any $x\in X$ and fixed $t\geq 0$, one has*

$$\|\mathcal{U}_\sigma(t)x\|\leq M\|x\|, \quad \|\mathcal{V}_\sigma(t)x\|\leq \frac{M}{\Gamma(\sigma)}\|x\|.$$

(ii) *If $T(t)$ $(t\geq 0)$ is an equi-continuous semigroup, $\mathcal{U}_\sigma(t)$ and $\mathcal{V}_\sigma(t)$ are equi-continuous in X for $t>0$.*

(iii) *If $T(t)$ $(t\geq 0)$ is a positive C_0-semigroup, $\mathcal{U}_\sigma(t)$ and $\mathcal{V}_\sigma(t)$ are positive operators for all $t\geq 0$.*

Definition 4.3. A function $x\in C(J,X)$ is called a mild solution of the initial value problem of fractional integro-differential evolution equation (4.1) if it satisfies the following integral equation

$$x(t)=\mathcal{U}_\sigma(t)x_0+\int_0^t (t-s)^{\sigma-1}\mathcal{V}_\sigma(t-s)f\left(s,x(s)\int_0^s k(s,\tau,x(\tau))\,d\tau\right)ds, \quad t\in J.$$

To prove our main results, we list the following assumptions:

(H1) $T(t)$ $(t\geq 0)$ is a positive and equi-continuous C_0-semigroup, and there is a constant $M>0$ such that $\|T(t)\|\leq M$ for all $t\geq 0$.

(H2) $f(t,x,y)\colon J\times X\times X\to X$ is a nondecreasing function with respect to $x,y\in X$ and satisfies the following conditions:

(i) For any $t\in J$, the function $f(t,\cdot,\cdot)\colon X^2\to X$ is continuous and for each $(x,y)\in X^2$, the function $f(\cdot,x,y)\colon J\to X$ is measurable.

(ii) There exists a function $\gamma_1\in L^1(J,\mathbb{R}^+)$ such that

$$\|f(t,x,y)\|\leq\gamma_1(t)(\|x\|+\|y\|)$$

for any $t\in J$ and $(x,y)\in X^2$.

(iii) There exist a function $\rho_1 \in L^1(J, \mathbb{R}^+)$ and a function $\varphi \in \mathfrak{B}$ such that

$$\beta(f(t, D_1, D_2)) \leq \rho_1(t)\varphi(\beta(D_1) + \beta(D_2))$$

for any $t \in J$ and any nonempty bounded subsets $D_1, D_2 \subset X$.

(H3) $k \colon \Delta \times X \to X$, $\Delta := \{(t,s) \in J \times J : 0 \leq s \leq t \leq b\}$ is a nondecreasing function with respect to $x \in X$ and satisfies the following conditions:

(i) For any $(t,s) \in \Delta$, the function $f(t,s,\cdot) \colon X \to X$ is continuous and for each $x \in X$, the function $f(\cdot,\cdot,x) \colon \Delta \to X$ is measurable.

(ii) There exists a function $\gamma_2 \in L^1(\Delta, \mathbb{R}^+)$ such that

$$\|k(t,s,x)\| \leq \gamma_2(t,s)\|x\|$$

for any $(t,s) \in \Delta$ and $x \in X$.

(iii) There exists a function $\rho_2 \in L^1(\Delta, \mathbb{R}^+)$ such that

$$\beta(k(t,s,D)) \leq \rho_2(t,s)\beta(D)$$

for any $(t,s) \in \Delta$ and any nonempty bounded subset $D \subset X$.

(H4) There exists an element $u_0 \in C(J, X)$ satisfying

$$\begin{cases} {}^C D_t^\sigma u_0(t) + Au_0(t) \leq f\left(t, u_0(t), \int_0^t k(t,s,u_0(s))\,ds\right) & t \in J, \\ u_0(0) \leq x_0. \end{cases}$$

(H5) There exists $r > 0$ such that

$$M\|x_0\| + \frac{Mb^\sigma r}{\Gamma(\sigma+1)}(1 + L_1)\|\gamma_1\|_{L^1} \leq r. \tag{4.2}$$

For the sake of brevity, we introduce the following notions:

$$L_1 = \max_{t \in J} \int_0^t \gamma_2(t,s)\,ds, \quad L_2 = \max_{t \in J} \int_0^t \rho_2(t,s)\,ds. \tag{4.3}$$

Theorem 4.4. *Assume that the conditions* (H1)–(H5) *hold true. Then the initial value problem of fractional integro-differential evolution equation* (4.1) *has at least one mild solution provided that*

$$\frac{4Mb^\sigma}{\Gamma(\sigma+1)}(1 + 2L_2)\|\rho_1\|_{L^1} \leq 1. \tag{4.4}$$

Proof. Define an operator $\mathcal{Q}\colon PC(J,X) \to PC(J,X)$ by the formula:

$$(4.5)\quad (\mathcal{Q}x)(t) = \mathcal{U}_\sigma(t)x_0 + \int_0^t (t-s)^{\sigma-1}\mathcal{V}_\sigma(t-s) f\left(s, x(s)\int_0^s k(s,\tau,x(\tau))\,d\tau\right) ds, \quad t \in J.$$

It is easy to prove from the continuity of f and k that $\mathcal{Q}\colon PC(J,X) \to PC(J,X)$ is continuous. From the nondecreasing property of the functions f and k, we can see that $\mathcal{Q}\colon PC(J,X) \to PC(J,X)$ is nondecreasing. The remaining proof will be given in four steps.

Step 1. Let $r > 0$ be a solution of (4.2), we prove $\mathcal{Q}B_r \subset B_r$, where $B_r = \{x \in C(J,X) : \|x\| \le r\}$.

It follows from (H2)(ii), (H3)(ii), (H5) and (4.3)–(4.5) that, for any $t \in J$ and $x \in B_r$, we have

$$\begin{aligned}
\|(\mathcal{Q}x)(t)\| &\le \|\mathcal{U}_\sigma(t)x_0\| + \left\| \int_0^t (t-s)^{\sigma-1}\mathcal{V}_\sigma(t-s) f\left(s, x(s)\int_0^s k(s,\tau,x(\tau))\,d\tau\right) ds \right\| \\
&\le M\|x_0\| + \frac{M}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \left\| f\left(s, x(s)\int_0^s k(s,\tau,x(\tau))\,d\tau\right)\right\| ds \\
&\le M\|x_0\| + \frac{M}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1}\gamma_1(s) \left(\|x(s)\| + \left\|\int_0^s k(s,\tau,x(\tau))\,d\tau\right\|\right) ds \\
&\le M\|x_0\| + \frac{M}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1}\gamma_1(s) \left(\|x(s)\| + \int_0^s \gamma_2(s,\tau)\|x(\tau)\|\,d\tau\right) ds \\
&\le M\|x_0\| + \frac{Mr}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1}\gamma_1(s) \left(1 + \int_0^s \gamma_2(s,\tau)\,d\tau\right) ds \\
&\le M\|x_0\| + \frac{Mb^\sigma r}{\Gamma(\sigma+1)}(1+L_1)\|\gamma_1\|_{L^1}.
\end{aligned}$$

This implies that

$$\|\mathcal{Q}x\|_C = \sup_{t\in J} \|(\mathcal{Q}x)(t)\| \le M\|x_0\| + \frac{Mb^\sigma r}{\Gamma(\sigma+1)}(1+L_1)\|\gamma_1\|_{L^1} \le r.$$

Step 2. $\mathcal{Q}(B_r)$ is equi-continuous, where B_r is defined in Step 1.

It is a standard proof by using (H1), please see [15]. So, we omit the details here.

Step 3. $\beta(\mathcal{Q}(B_r)) \le \chi(\beta(B_r))\beta(B_r)$, where B_r is defined in Step 1.

Since $\mathcal{Q}(B_r)$ is bounded, there exists a countable subset $B_r^n = \{x_n\} \subset B_r$ such that $\mathcal{Q}(B_r^n)$ is bounded and

$$\beta(\mathcal{Q}(B_r)) \le 2\beta(\mathcal{Q}(B_r^n)).$$

For any $x \in B_r$ and $t \in J$, by (H2)(iii), (H3)(iii), (4.4) and (4.5), we have

$$\begin{aligned}
&\beta(\mathcal{Q}B_r^n)(t) \\
&\le \beta\left(\left\{\int_0^t (t-s)^{\sigma-1}\mathcal{V}_\sigma(t-s) f\left(s, x_n(s)\int_0^s k(s,\tau,x_n(\tau))\,d\tau\right) ds : x_n \in B_r^n\right\}\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{2M}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\rho_1(s)\varphi\left(\beta(B_r^n(s))+\beta\left(\left\{\int_0^s k(s,\tau,x_n(\tau))\,d\tau : x_n\in B_r^n\right\}\right)\right)ds\\
&\le \frac{2M}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\rho_1(s)\varphi\left(\beta(B_r^n(s))+2\int_0^s \rho_2(s,\tau)\beta(B_r^n(\tau))\,d\tau\right)ds\\
&\le \frac{2M}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\rho_1(s)\varphi((1+2L_2)\beta(B_r))\,ds\\
&\le \frac{2Mb^\sigma}{\Gamma(\sigma+1)}(1+2L_2)\|\rho_1\|_{L^1}\varphi(\beta(B_r))\\
&\le \frac{1}{2}\varphi(\beta(B_r))\\
&= \frac{\varphi(\beta(B_r))}{2\beta(B_r)}\beta(B_r).
\end{aligned}
$$

Since $\mathcal{Q}B_r$ is bounded and equi-continuous, we can obtain

$$\beta(\mathcal{Q}B_r)\le 2\beta(\mathcal{Q}B_r^n)=2\max_{t\in J}\beta((\mathcal{Q}B_r^n)(t))\le\chi(\beta(B_r))\beta(B_r),$$

where $\chi(\beta(B_r))=\varphi(\beta(B_r))/\beta(B_r)$ satisfying $\chi\in\Psi$.

Step 4. We prove that there is $u_0\in B_r$ satisfying $u_0\le \mathcal{Q}u_0$.

Let $h(t) = {}^C D_t^\sigma u_0(t)+Au_0(t)$ and $u_0(0)=u_0(0)$. Then by Definition 4.3, we can obtain

$$u_0(t)=\mathcal{U}_\sigma(t)u_0(0)+\int_0^t (t-s)^{\sigma-1}\mathcal{V}_\sigma(t-s)h(s)\,ds,\quad t\in J.$$

Combining this fact with (H4) and (4.5), we have

$$u_0(t)\le\mathcal{U}_\sigma(t)x_0+\int_0^t (t-s)^{\sigma-1}\mathcal{V}_\sigma(t-s)f\left(t,u_0(t),\int_0^t k(t,s,u_0(s))\,ds\right)ds=(\mathcal{Q}u_0)(t)$$

for all $t\in J$.

Hence, all the conditions of Theorem 2.1 are satisfied. By Theorem 2.1, the initial value problem of fractional integro-differential evolution equation (4.1) has at least one mild solution. □

5. An example

In this section, we apply Theorem 4.4 to consider the existence of solutions for the following fractional integro-differential equation

$$
(5.1)\qquad
\begin{cases}
\frac{\partial^{1/2}}{\partial t^{1/2}}x(t,z)+\frac{\partial}{\partial z}x(t,z)=\frac{t}{32}\left(x(t,z)+\int_0^t e^{t-s}x(s,z)\,ds\right) & t\in J,\ z\in(0,1),\\
x(t,0)=x(t,1)=0 & t\in J,\\
x(0,z)=x_0(z) & z\in(0,1),
\end{cases}
$$

where $J=[0,1]$.

Let $X = C([0,1])$ and let

$$D(A) = \{u \in X : u' \in X, u(0) = u(1) = 0\},$$
$$Au = u', \quad u \in D(A).$$

Then $A\colon D(A) \subset X \to X$ generates an equi-continuous semigroup $T(t)$ $(t \geq 0)$ in X which is given by

$$T(t)u(s) = u(t+s), \quad \forall\, u \in D(A).$$

Then $T(t)$ $(t \geq 0)$ is a positive C_0-semigroup and $\sup_{t\in[0,1]} \|T(t)\| \leq 1$.

Let $x(t)(z) = x(t,z)$, $f(t,x(t),y(t))(z) = \frac{t}{32}(x(t,z)+y(t,z))$, $y(t,z) = \int_0^t e^{t-s}x(s,z)\,ds$, $k(t,s,x(s))(z) = e^{t-s}x(s,z)$. Then the fractional integro-differential equation (5.1) can be rewritten into (4.1). It is clear that $f(t,x,y)$ is a nondecreasing function with respect to $x, y \in X$ and the assumption (H2) holds with $\gamma_1(t) = t/32$, $\rho_1(t) = t/16$ and $\varphi(r) = r/2$. Similarly, $k(t,s,x)$ is a nondecreasing function and the assumption (H3) holds with $\gamma_2(t,s) = \rho_2(t,s) = e^{t-s}$. Let $r \geq 32\sqrt{\pi}/(32\sqrt{\pi} - e) > 0$. Then the assumption (H5) holds. If there exists a function $\widetilde{v} \in C(J,X)$ such that

$$\begin{cases} \frac{\partial^{1/2}}{\partial t^{1/2}}\widetilde{v}(t,z) + \frac{\partial}{\partial z}\widetilde{v}(t,z) \leq \frac{t}{32}\left(\widetilde{v}(t,z) + \int_0^t e^{t-s}\widetilde{v}(s,z)\,ds\right) & t \in J,\ z \in (0,1), \\ \widetilde{v}(t,0) = \widetilde{v}(t,1) = 0 & t \in J, \\ \widetilde{v}(0,z) \leq \widetilde{v}_0(z) & z \in (0,1), \end{cases}$$

then $\widetilde{v}$ satisfies the assumption (H4). Therefore, all the conditions of Theorem 4.4 are satisfied. By Theorem 4.4, the fractional integro-differential equation (5.1) has at least one solution.

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