TAIWANESE JOURNAL OF MATHEMATICS
Vol. 22, No. 2, pp. 339–348, April 2018
DOI: 10.11650/tjm/8168

Devaney's Chaos for Maps on G-spaces

Ekta Shah

Abstract. We study the notion of sensitivity on G-spaces and through examples observe that G-sensitivity neither implies nor is implied by sensitivity. Further, we obtain necessary and sufficient conditions for a map to be G-sensitive. Next, we define the notion of Devaney's chaos on G-space and show that G-sensitivity is a redundant condition in the definition.

1. Introduction

The occurrence of chaos in the several problems ranging from physics and chemistry to ecology and economics has made the theory of chaotic dynamical system as one of the challenging fields of dynamical systems. In last decade quite a few surveys have appeared in the literature only in the topological aspect of chaos theory. For details refer [1, 5, 12, 14, 15, 19, 21, 22]. We wish to note here that most of the contents in these surveys are disjoint. Apart from topological aspect, chaos theory is being studied in other settings also. For instance, recently chaos has been defined and studied for group actions [8,24,28] or on hyperspace [3,16] or for non-autonomous dynamical systems [13,29]. The theory of chaos is hence now a very well-established branch of dynamical systems.

Conventionally, the term "chaos" means "a state of confusion with no order". However in the theory of chaotic dynamical systems, the term is defined in more precise form. In 1975, Li and Yorke [20] are the first one to connect the term "chaos" with a map. In past three decades, several alternative definitions of chaos have been proposed and studied in detail. Though, there is still no definitive, universally accepted mathematical definition of chaos. In fact, as per the authors of [19], it is impossible to accept one such universal definition of chaos. Most accepted definition of chaos includes Li Yorke chaos, distributional chaos, positive topological entropy and its variants, Devaney Chaos, stronger forms of transitivity such as weakly mixing, mixing. The present paper studies the Devaney's definition of chaos for maps on G-spaces.

Received February 25, 2017; Accepted June 26, 2017.
Communicated by Cheng-Hsiung Hsu.
2010 *Mathematics Subject Classification.* 37D05, 37B20, 37C85, 54H20.
Key words and phrases. transitive maps, sensitive dependence on initial conditions, Devaney's chaos, metric G-space.

The paper is organized in the following manner: In the next Section we give necessary terminologies required throughout the paper. The notion of sensitive dependence on G-spaces was defined in [10]. In Section 3, through examples we show G-sensitivity neither implies nor is implied by sensitivity. Further, in the same section we obtain a necessary and sufficient condition for a map to be G-sensitive. The notion of periodic points for continuous self maps and transitivity on G-space were defined in [25]. In Section 4, we give a characterization for $\mathrm{Per}_G(f)$ to be dense in a G-space. Here $\mathrm{Per}_G(f)$ is the set of G-periodic points of map f. We also show that G-transitivity of map f implies transitivity of the induced map $\widehat{f}$. The notion G-chaos is defined in Section 5 and it is observed that G-sensitvity is a redundant condition in the definition of G-chaos for compact metric space. This is similar to the result proved by Banks, et al. in [4].

2. Preliminaries

By a dynamical system we mean a pair (X, f), where X is a metric space and f is a continuous self map on X. Throughout the paper maps means self maps.

By a metric G-space, X, we mean a metric space X on which a topological group G acts continuously by an action ϑ. For $g \in G$ and $x \in X$ we denote $\vartheta(g, x)$ by gx. The *G-orbit* of a point x, denoted by $G(x)$, is the set $\{gx : g \in G\}$. The set X/G of all G-orbits in X with the quotient topology induced by the quotient map $\pi\colon X \to X/G$ defined by $\pi(x) = G(x)$, is called the *orbit space* of X and the map π is called the *orbit map*. Note that the map π is an open continuous map. Map f is said to be *pseudoequivariant* if $f(G(x)) = G(f(x))$ for all $x \in X$. For details on G-space one can refer to [7, 23, 27]. It is known that if f is pseudoequivariant continuous map, then it induces a continuous map $\widehat{f}\colon X/G \to X/G$ given by $\widehat{f}(G(x)) = G(f(x))$ [9].

Definition 2.1. Let X be a metric space and $f\colon X \to X$ be a continuous map. Map f is said to be *transitive* if for any non empty open subsets U and V of X, there is an integer $n > 0$ such that $f^n(U) \cap V \neq \emptyset$.

Definition 2.2. Let X be a metric space and $f\colon X \to X$ be a continuous map. Map f is said to have *sensitive dependence on initial conditions* if there is a constant $\delta > 0$ such that for each $x \in X$ and for every neighbourhood U of x, there is a point $y \in U$ satisfying $d(f^n(x), f^n(y)) > \delta$, for some $n > 0$; δ is called *sensitive constant* for f. If f has sensitive dependence on initial conditions, then we write f is sensitive.

The notion of chaos in the sense of Devaney was first introduced by Devaney in [11].

Definition 2.3. Let X be a metric space and $f\colon X \to X$ be a continuous map then f is called *chaotic in the sense of Devaney* or *Devaney chaotic* if

(1) f is transitive,

(2) $\mathrm{Per}(f)$ is dense in X and

(3) f is sensitive.

Since its inception the notion of Devaney's chaos have been extensively studied. Its relation with other notions of chaos have also been studied. Certain references in this direction are [6, 15, 17, 18]. In 2007, the notion of P-chaos is defined in [2] and it is shown there that P-chaos implies Devaney's chaos on continuums.

Banks, et al. in [4] observed that the condition of sensitivity in the definition of Devaney's chaos is redundant if the space is a compact metric space. We recall the theorem.

Lemma 2.4. *Let X be a compact metric space. If $f\colon X \to X$ is transitive and has dense set of periodic points then f has sensitive dependence on initial conditions.*

It was further observed in [26], transivity is equivalent to chaos on the compact interval I. We recall the theorem due to Vellekoop and Berglund.

Lemma 2.5. *Let I be a compact interval and let $f\colon I \to I$ be a continuous map. If f is transitive, then* $\mathrm{Per}(f)$ *is dense in I and also f is sensitive.*

3. G-sensitive dependence on initial conditions

We recall the definition of sensitivity on G-space.

Definition 3.1. Let X be a metric G-space and $f\colon X \to X$ be a continuous map. Then f is said to have *G-sensitive dependence on initial conditions* if there is a constant $\delta > 0$ such that for each $x \in X$ and for every neighbourhood U of x, there is a point $y \in U$ with $G(x) \neq G(y)$ satisfying for some $n > 0$, $d(f^n(u), f^n(v)) > \delta$, for all $u \in G(x)$ and all $v \in G(y)$; δ is called *G-sensitive constant* for f. If f has G-sensitive dependence on initial conditions, then we write f is G-sensitive.

Note that the G-sensitivity of f depends on the action of G. In Example 3.2, f is G-sensitive with respect to one group action but is not G-sensitive with respect to other group action. Also, f is not sensitive. In Example 3.3, f is sensitive but is not G-sensitive for some group action. Thus, we have that G-sensitivity neither implies not is implied by sensitivity. Hence, the notion of G-sensivity depends on the action of group on the space. Further, if the action of G is trivial on X, then the notion of G-sensitivity and sensitivity are equivalent.

Example 3.2. For each $n \in \mathbb{N}$, let X_n denote the $(m-1)$ sphere centered at origin and of radius $1/n$. Let $G = \mathrm{SO}(m)$ act on the subspace $X = \bigcup_{n=1}^{\infty} X_n \cup \{\mathbf{0}\}$ of $\mathbb{R}^m$ by the usual action of matrix multiplication, where $\mathbf{0}$ is the origin of $\mathbb{R}^m$. Consider the map $f\colon X \to X$ defined by

$$f(z) = \begin{cases} z & \text{if } z = \mathbf{0} \text{ or } z \in X_1, \\ z' & \text{if } z \in X_n,\ n \neq 1, \end{cases}$$

where z' is the point of intersection of the sphere X_{n-1} and the line joining z and origin. Then f is $\mathrm{SO}(m)$-sensitive. But if $\mathbb{Z}_2$ acts on X by the action $1x = x$ and $-1x = -x$ for all $x \in X$, then f is not $\mathbb{Z}_2$-sensitive. Also note that f is not sensitive as it is the identity map on X_1.

Example 3.3. Consider the subspace $X = \{\pm\frac{1}{n}, \pm(1-\frac{1}{n}) : n \in \mathbb{N}\}$ of $\mathbb{R}$ with the usual metric of $\mathbb{R}$ and the homeomorphism $h\colon X \to X$ defined by

$$h(x) = \begin{cases} x & \text{if } x \in \{-1, 0, 1\}, \\ -x_+ & \text{if } 0 < x < 1, \text{ where } x_+ \text{ is the element of } X \text{ immediately right to } x, \\ -x_- & \text{if } -1 < x < 0, \text{ where } x_- \text{ is the element of } X \text{ immediately left to } x. \end{cases}$$

Suppose the group $G = \{h^n : n \in \mathbb{Z}\}$ acts on X by the usual action. If f is the left shift fixing -1, 0 and 1, then f is sensitive but not G-sensitive.

In the following theorem we obtain a necessary and sufficient condition for a map f to be G-sensitivity.

Theorem 3.4. *Let X be a compact metric G-space, with G compact. Suppose $f\colon X \to X$ is a continuous pseudoequivariant onto map. Then f is G-sensitive if and only if the induced map $\widehat{f}\colon X/G \to X/G$ is sensitive.*

Proof. Suppose f is G-sensitive. Therefore there exists a G-sensitive constant, say δ, satisfying the definition of G-sensitivity. Choose β such that $0 < \beta < \delta$. We show that β is sensitive constant for $\widehat{f}$. Let $G(x) \in X/G$ and $\widehat{U}$ be a neighbourhood of $G(X)$. Then $\pi^{-1}(\widehat{U})$ is an open set containing x. But f is G-sensitive. Therefore there exist $y \in \pi^{-1}(\widehat{U})$ with $G(x) \neq G(y)$ and an integer $n > 0$ such that $d(f^n(u), f^n(v)) > \delta$ for all $u \in G(x)$ and all $v \in G(y)$. But this implies, there is an integer $n > 0$ such that

$$\widehat{d}\big(\widehat{f}^n(G(x)), \widehat{f}^n(G(y))\big) > \beta$$

for $G(y) \in \widehat{U}$ with $G(x) \neq G(y)$, where $\widehat{d}$ is the metric on X/G induced from d. Therefore, $\widehat{f}$ is sensitive with sensitivity constant β.

Conversely, suppose $\widehat{f}$ is sensitive with sensitive constant δ. We show that any η, $0 < \eta < \delta$, is a G-sensitive constant for f. Let $x \in X$ and U be an open set containing x.

Since π is an open map, it follows that $\pi(U)$ is an open subset of X/G containing $G(x)$. Therefore by sensitivity $\widehat{f}$, there are $G(y) \in \pi(U)$ and an integer $n > 0$ such that

$$\widehat{d}\big(\widehat{f}^n(G(x)), \widehat{f}^n(G(y))\big) > \delta.$$

This further implies that there is an integer $n > 0$, with $G(x) \neq G(y)$ such that

$$d(f^n(u), f^n(v)) > \delta$$

for all $u \in G(x)$ and all $v \in G(y)$. We complete the proof by showing that $gy \in U$ for some $g \in G$. Now, $\pi(U) = \bigcup_{t \in G} tU$. Therefore for each $p \in G$, $py \in \bigcup_{t \in G} tU$. In particular, $y \in \bigcup_{t \in G} tU$. Therefore, there is $g \in G$ such that $gy \in U$. Note that $G(y) = G(gy)$. Hence the proof follows. □

4. G-periodic points and G-transitivity

The notion of G-periodic points was defined in [25]. We recall the definition.

Definition 4.1. Let X be a metric G-space and $f\colon X \to X$ be a continuous map. Then a point $x \in X$ is said to be a *G-periodic point of f* if there exist an integer $n > 0$ and a $g \in G$ such that $f^n(x) = gx$. The smallest such positive integer n is called the *period* of x. We denote the set of all G-periodic points of f by $\mathrm{Per}_G(f)$.

Obviously if x is a periodic point then x is a G-periodic point. But converse need not be true follows from Example 4.2.

Example 4.2. Let $X = S^1$, the unit circle of the plane and suppose $G = U_4$, the group of 4th roots of unity, acts on X by the usual action of complex multiplication. Define the map $f\colon X \to X$ by $f(e^{i\theta}) = e^{i\theta/2}$. Then $\mathrm{Per}_G(f) = \{e^{i0}, e^{i\pi}\}$, whereas $\mathrm{Per}(f) = \{e^{i0}\}$.

In the following theorem we obtain a characterization for $\mathrm{Per}_G(f)$ to be a dense subset of X.

Proposition 4.3. *Let X be a compact metric G-space, with G compact. Suppose $f\colon X \to X$ is a pseudoequivariant continuous map defined on X. Then $\mathrm{Per}_G(f)$ is dense in X if and only if $\mathrm{Per}(\widehat{f})$ is dense in the orbit space X/G.*

Proof. Suppose $\mathrm{Per}(\widehat{f})$ is dense in X/G. Let U be an open subset of X. We show that U contains a G-periodic point of f. Since π is an open map, it follows that $\pi(U)$ is an open subset of X/G. Therefore there is a periodic point $G(x)$ of $\widehat{f}$ such that $G(x) \in \pi(U)$. But $\pi(U) = \bigcup_{g \in G} gU$, implies

$$tx \in \bigcup_{g \in G} gU$$

for each $t \in G$. Therefore, in particular, $x \in g'U$ for some $g' \in G$. But this implies, there is $g \in G$, such that $gx \in U$. We complete the proof by showing that, gx is a G-periodic point of f. Now, $G(x)$ is a periodic point of $\widehat{f}$, say with period n. Therefore using pseudoequivariancy of f, we obtain $G(f^n(x)) = G(x)$. Thus, there are a positive integer n and an $l \in G$, such that $f^n(gx) = lx$. Hence the proof.

Conversely, suppose $\mathrm{Per}_G(f)$ is dense in X. Let $\widehat{U}$ be an open subset of X/G. We show that $\widehat{U}$ contains a periodic point of $\widehat{f}$. Using continuity of π, we have $\pi^{-1}(\widehat{U})$ is an open subset of X. Therefore, there is a G-periodic point y of f in X such that $y \in \pi^{-1}(\widehat{U})$. But π is surjective. Therefore, $\pi(y) = G(y) \in \widehat{U}$. We complete the proof by showing that $G(y)$ is a periodic point of $\widehat{f}$. Now, y is a G-periodic point of f, say of period m. Therefore, there is $g \in G$ such that $f^m(y) = gy$. But this implies $G(f^m(y)) = G(gy) = G(y)$. Thus $\widehat{f}^m(G(y)) = G(y)$. □

In [25], the notion of topological transitivity on G-space was defined and studied. We recall the definition.

Definition 4.4. Let (X, d) be a metric G-space and $f : X \to X$ be a continuous map then f is called *G-transitive* if for every pair of non-empty open subsets U and V of X, there exist $n \in \mathbb{N}$ and $g \in G$ such that $g.f^n(U) \cap V \neq \emptyset$.

It was observed that transitivity of a map implies G-transitivity of map. But converse in general is not true under the non-trivial action of G. In the following theorem we show that G-transitivity of f implies transitivity of the induced map $\widehat{f}$ on X/G.

Theorem 4.5. *Let X be a compact metric G-space, with G compact. Suppose a pseudoequivariant continuous onto map f defined on X is G-transitive. Then the induced map $\widehat{f} : X/G \to X/G$ is transitive.*

Proof. Let $\widehat{U}$ and $\widehat{V}$ be two non-empty subsets of X. Then $\pi^{-1}(\widehat{U})$ and $\pi^{-1}(\widehat{V})$ are non-empty open subsets of X. But f is a G-transitive pseudoequivariant map. Therefore there are an integer $n > 0$ and $g \in G$ such that

$$f^n(g\pi^{-1}(U)) \cap \pi^{-1}(V) \neq \emptyset.$$

This further implies

$$\pi\left(f^n(g\pi^{-1}(U)) \cap \pi^{-1}(V)\right) \neq \emptyset. \tag{4.1}$$

But $\pi \circ f = \widehat{f} \circ \pi$. Therefore, (4.1) implies

$$\widehat{f}^n(\pi(\pi^{-1}(U))) \cap \pi(\pi^{-1}(V)) \neq \emptyset.$$

Using surjectivity of π we obtain,

$$\widehat{f}^n(\widehat{U}) \cap \widehat{V} \neq \emptyset$$

for some positive integer n. Therefore $\widehat{f}$ is transitive. □

5. G-chaos

Definition 5.1. Let X be a metric G-space and $f\colon X \to X$ be a continuous map then f is called *G-chaotic* if

(1) f is G-transitive,

(2) $\mathrm{Per}_G(f)$ is dense in X and

(3) f is G-sensitive.

Since G-sensitivity is neither is implied by nor implies sensitivity, it follows that G-chaoticity of f is neither implied by nor implies chaoticity of f. Under the trivial action of G, both G-chaos and chaos are equivalent notions.

Example 5.2. Consider $I = [0, 1]$ with usual metric of real numbers. Let $G = \mathbb{Z}_2$ act on I by the action $1x = x$ and $-1x = 1 - x$ for all $x \in I$. Then the map f whose graph is given by Figure 5.1 on I is $\mathbb{Z}_2$-chaotic but is not chaotic. In fact, f is not transitive.

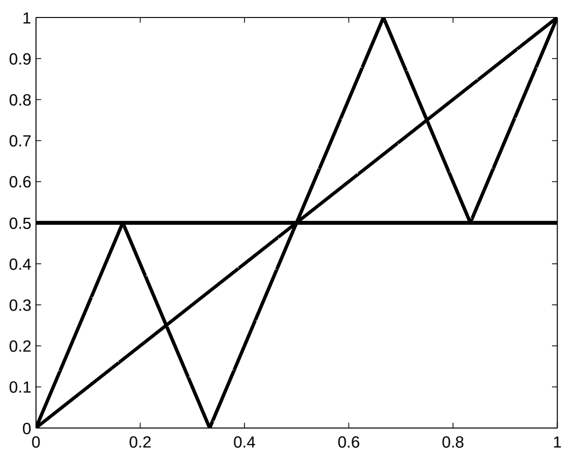

Figure 5.1: Example of map which is $\mathbb{Z}_2$-chaotic but not chaotic.

In the following theorem we obtain result similar to that of Banks et al. in [4] for G-spaces.

Theorem 5.3. *Let X be a compact metric G-space, with G compact and $f\colon X \to X$ be a pseudoequivariant continuous surjective map. If f is G-transitive and if* $\mathrm{Per}_G(f)$ *is dense in X/G. Then f is G-sensitive.*

Proof. Since f is G-transitive, by Theorem 4.5 we have $\widehat{f}$ is transitive. Further, $\mathrm{Per}_G(f)$ is dense in X. Therefore by Theorem 4.3, $\mathrm{Per}(\widehat{f})$ is dense in X/G. Hence by Lemma 2.4, $\widehat{f}$ is sensitive. But, then by Theorem 3.4, we have that f is G-sensitive. $\square$

Through example we now observe that G-sensitivity and dense set of G-periodic points need not imply G-transitivity (see Example 5.4). Also, we give an example of map which is G-sensitive and G-transitivity but the set of G-periodic points are not dense (see Example 5.5).

Example 5.4. Suppose $G_1 = \mathbb{Z}_2$ acts on S^1 by the action $1e^{i\theta} = e^{i\theta}$ and $-1e^{i\theta} = e^{-i\theta}$ and suppose $G_2 = \mathbb{Z}_2$ acts on $I = [0,1]$ by the action $1x = x$ and $-1x = 1 - x$ for all $x \in I$. Consider $X = S^1 \times I$ with taxi cab metric (i.e., maximum of distance in S^1 and I). Let $G = G_1 \times G_2$ act on X by the action $(g_1, g_2)(e^{i\theta}, t) = (g_1 e^{i\theta}, g_2 t)$. Then the map $f\colon X \to X$ defined by $f(e^{i\theta}, t) = (e^{2i\theta}, t)$ is G-sensitive. Also, $\mathrm{Per}_G(f)$ is dense in X. Note that f is not G-transitive.

Example 5.5. Consider $X = S^1 \setminus \{e^{2i\pi r} : r \in \mathbb{Q}\}$ with usual metric. Suppose $G = \mathbb{Z}_2$ acts on X by the action $1e^{2\pi i\theta} = e^{2\pi i\theta}$ and $-1e^{2\pi i\theta} = e^{-2\pi i\theta}$. Then the map $f\colon X \to X$ defined by $f(e^{2\pi i\theta}) = e^{4\pi i\theta}$ is $\mathbb{Z}_2$-transitive and $\mathbb{Z}_2$-sensitive. But $\mathrm{Per}_G(f) = \emptyset$ and hence is not dense in X.

In the following proposition we obtain a result similar to that of [26].

Proposition 5.6. *Let I denote the compact interval of the real line. Suppose a compact group G acts on I such that I/G is homeomorphic to some compact interval $J = [a, b]$. If $f\colon I \to I$ is G-transitive then f is G-chaotic.*

Proof. Given f is G-transitive. Therefore by Theorem 4.5, $\widehat{f}$ is transitive. Also, X/G is homeomorphic to the interval J. Therefore by Lemma 2.5, $\widehat{f}$ is sensitive and $\mathrm{Per}(\widehat{f})$ is dense in X/G. Therefore by Theorem 3.4, f is G-sensitive. Also, Proposition 4.3 implies $\mathrm{Per}_G(f)$ is dense in X. Hence, f is G-chaotic. □

References

[1] E. Akin and J. D. Carlson, *Conceptions of topological transitivity*, Topology Appl. **159** (2012), no. 12, 2815–2830.

[2] T. Arai and N. Chinen, *P-chaos implies distributional chaos and chaos in the sense of Devaney with positive topological entropy*, Topology Appl. **154** (2007), no. 7, 1254-1262.

[3] J. Banks, *Chaos for induced hyperspace maps*, Chaos Solitons Fractals **25** (2005), no. 3, 681–685.

[4] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, *On Devaney's definition of chaos*, Amer. Math. Monthly **99** (1992), no. 4, 332–334.

[5] F. Blanchard, *Topological chaos: what may this mean?*, J. Difference Equ. Appl. **15** (2009), no. 1, 23–46.

[6] F. Blanchard, E. Glasner, S. Kolyada and A. Maass, *On Li-Yorke pairs*, J. Reine Angew. Math. **547** (2002), 51-68.

[7] G. E. Bredon, *Introduction to Compact Transformation Groups*, Pure and Applied Mathematics **46**, Academic Press, New York, 1972.

[8] G. Cairns, A. Kolganova and A. Neilsen, *Topological transitivity and mixing notion for group actions*, Rocky Mountain J. Math. **37** (2007), no. 2, 371–397.

[9] R. Das, *Expansive self-homeomorphisms on G-spaces*, Period. Math. Hungar. **31** (1995), no. 2, 123–130.

[10] ______, *Chaos of product map on G-spaces*, Int. Math. Forum **8** (2013), no. 13-16, 647–652.

[11] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Benjamin/Cummings, Menlo Park, CA, 1986.

[12] D. Dikranjan, M. Sanchis and S. Virili, *New and old facts about entropy in uniform spaces and topological groups*, Topology Appl. **159** (2012), no. 7, 1916–1942.

[13] J. Dvořáková, *Chaos in nonautonomous discrete dynamical systems*, Commun. Nonlinear Sci. Numer. Simul. **17** (2012), no. 12, 4649–4652.

[14] F. Falniowski, M. Kulczycki, D. Kwietniak and J. Li, *Two results on entropy, chaos and independence in symbolic dynamics*, Discrete COntin. Dyn. Syst. Ser. B **20** (2015), no. 10, 3487–3505.

[15] G. L. Forti, *Various notions of chaos for discrete dynamical systems*, Aequationes Math. **70** (2005), no. 1-2, 1-13.

[16] J. L. G. Guirao, D. Kwietniak, M. Lampart, P. Oprocha and A. Peris, *Chaos on hyperspaces*, Nonlinear Anal. **71** (2009), no. 1-2, 1–8.

[17] W. Huang and X. Ye, *Devaneys chaos or* 2*-scattering implies Li-Yorkes chaos*, Topology Appl. **117** (2002), no. 3, 259-272.

[18] S. H. Li, *ω-chaos and topological entropy*, Trans. Amer. Math. Soc. **339** (1993), no. 1, 243-249.

[19] J. Li and X. D. Ye, *Recent development of chaos theory in topological dynamics*, Acta Math. Sin. (Engl. Ser.) **32** (2016), no. 1, 83–114.

[20] T. Y. Li and J. A. Yorke, *Period three implies chaos*, Amer. Math. Monthly **82** (1975), no. 10, 985–992.

[21] J. Llibre, *Brief survey on the topological entropy*, Discrete Contin. Dyn. Syst. Ser. B **20** (2015), no. 10, 3363–3374.

[22] P. Oprocha and G. Zhang, *Topological aspects of dynamics of pairs, tuples and sets*, in *Recent Progress in General Topology III*, 665–709, Atlantis Press, Paris, 2014.

[23] R. S. Palais, *The Classifications of G-spaces*, Mem. Amer. Math. Soc. **36**, 1960.

[24] P. Sankaran, *Chaotic group actions on the rationals*, Indian J. Pure Appl. Math. **40** (2009), no. 3, 221–228.

[25] E. Shah, *Dynamical Properties of Maps on Topological Spaces and G-spaces*, Ph.D. Thesis, 2005.

[26] M. Vellekoop and R. Berglund, *On intervals, transitivity = chaos*, Amer. Math. Monthly **101** (1994), no. 4, 353–355.

[27] J. de Vries, *Elements of Topological Dynamics*, Mathematics and its Applications **257**, Kluwer Academic Publishers Group, Dordrecht, 1993.

[28] Z. Wang and G. Zhang, *Chaotic behaviour of group actions*, in *Dynamics and Numbers*, 299–315, Contemp. Math. **669**, Amer. Math. Soc., Providence, RI, 2016.

[29] H. Zhu, Y. Shi and H. Shao, *Devaney chaos in nonautonomous discrete systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **26** (2016), no. 11, 1650190, 10 pp.

Ekta Shah
Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara-390002, India
E-mail address: ekta19001@gmail.com, shah.ekta-math@msubaroda.ac.in