TAIWANESE JOURNAL OF MATHEMATICS
Vol. 22, No. 1, pp. 95–124, February 2018
DOI: 10.11650/tjm/8161

Dynamics of a Stochastic Fractional Reaction-Diffusion Equation

Linfang Liu and Xianlong Fu*

Abstract. In this paper, we study the dynamics of a stochastic fractional reaction-diffusion equation with multiplicative noise in three spatial dimension. By proving the well-posedness and conducting a priori estimates for the solutions of the considered equation we obtain simultaneously the existence and the regularity of random attractors of the random dynamical systems for the equation. The main approach here is to establish an abstract result on existence of bi-spatial random attractors for random dynamical systems. Moreover, we also give estimate of the Hausdorff dimension in L^2 space for the obtained random attractor.

1. Introduction

In this paper we investigate the well-posedness and dynamics of the following stochastic fractional reaction-diffusion equation:

$$\frac{\partial u}{\partial t} + \lambda u + (-\Delta)^\alpha u = f(u) + g(x) + \beta u \circ \frac{dW}{dt}, \quad x \in \mathbb{R}^3,\ t > 0, \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0, \quad x \in \mathbb{R}^3 \tag{1.2}$$

and the periodic boundary condition

$$u(x + 2\pi e_i, t) = u(x,t), \quad x \in \mathbb{R}^3,\ t > 0,\ i = 1,2,3, \tag{1.3}$$

where e_i $(i = 1,2,3)$ is an orthonormal basis of $\mathbb{R}^3$. λ, α and β are positive constants with $\alpha \in (1/2, 1)$, $g \in L^2(\mathbb{R}^3)$ is a given function, W is a two-sided real-valued Wiener process

Received March 18, 2017; Accepted June 21, 2017.
Communicated by Cheng-Hsiung Hsu.
2010 *Mathematics Subject Classification.* 35B40, 35R11, 35R60, 37L55, 60H15.
Key words and phrases. stochastic fractional reaction-diffusion equation, well-posedness, bi-spatial random attractor, regularity, Hausdorff dimension.
This work is supported by NSF of China (Nos. 11671142 and 11371087), Science and Technology Commission of Shanghai Municipality (STCSM, grant No. 13dz2260400) and Shanghai Leading Academic Discipline Project (No. B407).
*Corresponding author.

on a probability space to be specified later, and f is a nonlinear function satisfying the following conditions:

$$f(s)s \le -\beta_1|s|^p + \gamma_1, \tag{1.4}$$

$$|f(s)| \le \beta_2|s|^{p-1} + \gamma_2, \tag{1.5}$$

$$|f'(s)| \le \beta_3, \tag{1.6}$$

where $p > 1$, β_i $(i = 1, 2, 3)$ and γ_j $(j = 1, 2)$ are positive numbers.

As pointed out in [2], some research puts forward that classical diffusion equations are inadequate to model many real situations. For instance, a particle plume spreads faster than that predicted by the classical model, and may exhibit significant self-organization phenomena or asymmetry, see details in [21]. In this case, these situations are called anomalous diffusion. One popular model for anomalous diffusion is the fractional diffusion equation, where the usual second derivative operator in space, i.e., the Laplacian operator $-\Delta$, is replaced by a fractional derivative operator of order $0 < \alpha < 2$ as $(-\Delta)^\alpha$. In fact, the fractions of the Laplacian are the infinitesimal generators of Lèvy stable diffusion processes and appear not only in anomalous diffusions in plasma, but also in some flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics. Meanwhile, some stochastic perturbations should be included to obtain a more realistic model and to better understand the dynamical behavior of the phenomena being studied. In many cases one may represent the micro effects by random perturbations in the dynamics of the macro observable through multiplicative or additive noise in the governing equation. In these cases, one may reformulate the equation as a random dynamical system. A key step to study a stochastic partial differential equation is to examine the asymptotic behavior of the random dynamical systems generated by its solutions. Some fundamental advances in this direction can be found, for example, in papers [9,10] by Crauel et al. where they developed the theory of random attractors in a manner which closely parallels the deterministic case [23]. Also, Debussche [11] proved that the Hausdorff dimension of the random attractor could be estimated by using global Lyapunov exponents.

Particularly, in [13] Guo and Zhou have investigated a stochastic fractional reaction-diffusion equation with Wiener noise on a bounded domain. They proved there the well-posedness, existence and uniqueness of an invariant measure as well as the strong law of large numbers and the convergence to equilibrium. While in [15] Lu et al. have studied the dynamics of the 3D-fractional Gizburg-Landau equation, and obtained a global attractor with finite Hausdorff dimension. In addition, Lu et al. have established in [16] the existence result of random attractor in $L^2(\mathbb{R}^n)$ of a stochastic fractional power dissipative equations with additive noise. We would like to mention the work in $[3, 5, 6, 17, 24, 25]$ on general stochastic reaction-diffusion equations for our references.

In the present note we are interested in the dynamics of solutions to the stochastic fractional reaction-diffusion equation (1.1)–(1.3) with multiplicative noise, since, to our best knowledge, there is little work on the topic for this kind of systems. We shall discuss the existence and uniqueness of random attractors for the associated dynamical system not only in $L^2(D)$ but also in more regular spaces $H^1(D)$ and $H^{2\alpha}(D)$ with $1/2 < \alpha < 1$. We stress that, by employing the bi-spacial attractors technique as in [14] we may achieve the existence of the bi-spatial random attractors in various spaces at the same time. Moreover, by using the method introduced by [11] we will investigate the Hausdorff dimension of the obtained attractor and give an estimate for its Hausdorff dimension. Clearly, our work here extends somewhat that in Ref. [16] and it can also be regarded as a development of those for classical stochastic reaction-diffusion equations mentioned above.

The paper is organized as follows. In Section 2, we introduce some preliminaries, concepts and basic theory for random dynamical systems as well as some notations related to fractional derivative equations and fractional Sobolev spaces. In Section 3, we transfer the stochastic equation into a deterministic one only with random parameters and discuss the well-posedness of solutions to the considered fractional equation. Then a priori estimates of solutions is conducted in Section 4. Following that, in Section 5, we explore the existence and regularity of bi-spatial random attractor via establishing an abstract results about the existence of bi-spatial random attractor for general random dynamical systems. Finally, in Section 6, we estimate the Hausdorff dimension of the random attractor in L^2 space.

2. Preliminaries

In this section, we introduce some basic concepts related to (bi-spatial) random attractors of stochastic dynamical systems, for more details about these concepts we refer to [1, 7, 9, 10, 14, 22, 23].

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with the Borel σ-algebra $\mathcal{B}(X)$, and $\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}$ be a family of measure preserving transformations of a probability space $(\Omega, \mathcal{F}, P)$.

Definition 2.1. $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta\colon \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$, and $P(\theta_t(\cdot)) = P(\cdot)$ for all $t \in \mathbb{R}$.

Definition 2.2. A random dynamical system (RDS) on X over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a mapping

$$\varphi\colon \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega)x,$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies for P-a.e. $\omega \in \Omega$,

(i) $\varphi(0, \omega) = \mathrm{Id}_X$ on X;

(ii) $\varphi(t+s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega)$ for all $t, s \in \mathbb{R}^+$ (called cocycle property).

An RDS φ is said to be continuous if $\varphi(t, \omega) \colon X \to X$ is continuous for all $t \in \mathbb{R}^+$.

Definition 2.3. A set-valued map $B \colon \Omega \to 2^X$ is called a random set in X if the mapping $\omega \mapsto \mathrm{dist}(x, B(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for all $x \in X$. A random set $B \colon \Omega \to 2^X$ is called a random closed set if $B(\omega)$ is nonempty and closed for each $\omega \in \Omega$.

Definition 2.4. A random set $B \colon \Omega \to 2^X$ is called a bounded random set if there is a random variable $r(\omega) \in [0, \infty)$, $\omega \in \Omega$, such that

$$d(B(\omega)) := \sup\{\|x\|_X : x \in B(\omega)\} \le r(\omega) \quad \text{for all } \omega \in \Omega.$$

A bounded random set B is said to be tempered with respect to $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ if for P-a.e., $\omega \in \Omega$,

$$\lim_{t\to+\infty} e^{-\mu t} d(B(\theta_{-t}\omega)) = 0 \quad \text{for all } \mu > 0.$$

Definition 2.5. Let $\mathcal{D}$ be the collection of random sets in X. Then a random set $B \in \mathcal{D}$ is called a $\mathcal{D}$-random absorbing set for an RDS φ if for any random set $D \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists a $T_D(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega) D(\theta_{-t}\omega) \subset B(\omega) \quad \text{for all } t \ge T_D(\omega).$$

Definition 2.6. Let $\mathcal{D}$ be the collection of random sets in X. Then φ is said to be $\mathcal{D}$-asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\{\varphi(t_n, \theta_{-t_n}\omega)x_n\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to +\infty$, and $x_n \in D(\theta_{-t_n}\omega)$ with $\{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

In addition, a collection $\mathcal{D}$ of random sets in X is called inclusion closed if whenever E is a random set, and F is in $\mathcal{D}$ with $E(\omega) \subset F(\omega)$ for all $\omega \in \Omega$, then E must belong to $\mathcal{D}$. A collection $\mathcal{D}$ of random sets in X is said to be universe if it is inclusion-closed.

Next we turn to introduce the concepts of bi-spatial random attractors. Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two sparable Banach spaces. In what follows, one may think that X (called an initial space) contains all initial data and Y (called an terminate space) contains all values of solutions for a stochastic PDE. Both X and Y may not be embedded in any direction. We assume that (X, Y) are limit-identical in the following sense:

$$x_n \in X \cap Y, \quad \|x_n - x_0\|_X \to 0 \quad \text{and} \quad \|x_n - y_0\|_Y \to 0 \quad \text{imply} \quad x_0 = y_0 \in X \cap Y.$$

Define $\|x\|_{X\cap Y} = \|x\|_X + \|x\|_Y$ for any $x \in X \cap Y$, then it is easy to prove that $(X \cap Y, \|\cdot\|_{X\cap Y})$ is a normed linear space (but it may not be complete). In fact, it was proved in [14] that

Lemma 2.7. *Let X and Y be two Banach spaces. Then $(X \cap Y, \|\cdot\|_{X\cap Y})$ is a Banach space if and only if (X, Y) is a limit-identical pair. Moreover, suppose (X, Y) is limit-identical and $A \subset X \cap Y$, then A is compact in $X \cap Y$ if and only if A is compact in X and Y respectively.*

In this paper, a RDS φ on X is assumed to take its values (except the values of $\varphi(0, \omega)$) in the terminate space Y, that is,

$$\varphi(t, \omega)X \subset Y \quad \text{for all } t > 0,\ P\text{-a.e., } \omega \in \Omega. \tag{2.1}$$

We now introduce the definition of bi-spatial random attractors. Assume that $\mathcal{D}$ be a universe of bounded set-mappings in X.

Definition 2.8. Let φ be a RDS on X taking its values in Y. Then a set mapping $\mathcal{A}\colon \Omega \to 2^{X\cap Y}$ is said to be an ($\mathcal{D}$-)(X, Y)-random attractor for φ if

(i) $\mathcal{A}$ is a random set in X and $\mathcal{A}(\omega) \subset \mathcal{K}(\omega)$ for some $\mathcal{K} \in \mathcal{D}$.

(ii) $\mathcal{A}(\omega)$ is compact in Y.

(iii) $\mathcal{A}(\omega)$ is invariant under the system φ, i.e.,

$$\varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega) \quad \text{for all } t \geq 0.$$

(iv) $\mathcal{A}$ is an attracting set in Y in the sense that, for every $B \in \mathcal{D}$,

$$\lim_{t\to+\infty} \text{dist}_Y \left(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), \mathcal{A}(\omega)\right) = 0,$$

where $\text{dist}_Y(\,\cdot\,,\cdot\,)$ denotes the Hausdorff semi-distance under the norm of Y defined as, for two nonempty sets $A, B \subset Y$,

$$\text{dist}_Y(A, B) := \sup_{a\in A} \text{dist}_Y(a, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_Y.$$

Remark 2.9. As it was pointed out in [8], if a RDS φ possesses a random attractor in a universe $\mathcal{D}$, then the random attractor is unique in $\mathcal{D}$.

We shall establish an abstract result on existence of bi-spacial random attractors for a RDS in Subsection 5.1, by which we may investigate the existence and regularity of the RDS associated to the considered equation. Another aim of this paper is to study the Hausdorff dimension of obtained random attractor by applying the following proposition (see [11]).

Proposition 2.10. *Let $\mathcal{A}(\omega)$ be a compact measurable set which is invariant under a random map $\varphi(\omega)$, $\omega \in \Omega$, for some ergodic metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. Assume that the following conditions are satisfied.*

(i) *$\varphi(\omega)$ is a.s. uniformly differentiable on $\mathcal{A}(\omega)$, that is, for every $u, u+h \in \mathcal{A}(\omega)$ there exists $D\varphi(\omega, u)$ in $\mathscr{L}(X)$, the space of the bounded linear operators from X to X, such that*

$$\|\varphi(\omega)(u+h) - \varphi(\omega)u - D\varphi(\omega, u)h\| \leq \overline{k}(\omega)\|h\|^{1+\delta},$$

where $\delta > 0$ and $\overline{k}(\omega)$ is a random variable satisfying $\overline{k}(\omega) \geq 1$ and the expectation $E(\ln \overline{k}) < \infty$.

(ii) *For some $d \in \mathbb{N}$, there is a random variable $\overline{\omega}_d(\omega)$ with $E(\ln(\overline{\omega}_d)) < 0$ such that $\omega_d(D\varphi(\omega, u)) \leq \overline{\omega}_d(\omega)$ for all $u \in \mathcal{A}(\omega)$, where*

$$\omega_d(D\varphi(\omega, u)) = \alpha_1(D\varphi(\omega, u)) \cdots \alpha_d(D\varphi(\omega, u))$$

and

$$\alpha_d(D\varphi(\omega, u)) = \sup_{\substack{G \subset X \\ \dim G \leq d}} \inf_{\substack{v \in G \\ \|v\|_X = 1}} \|D\varphi(\omega, u)v\|.$$

(iii) *There is a random variable $\overline{\alpha_1}(\omega) \geq 1$ with $E(\ln \overline{\alpha_1}) < \infty$ such that $\alpha_1(D\varphi(\omega, u)) \leq \overline{\alpha_1}(\omega)$ for any $u \in \mathcal{A}(\omega)$.*

Then the Hausdorff dimension $\dim_H(\mathcal{A}(\omega))$ of $\mathcal{A}(\omega)$ is a.s. less than d.

Finally, we recollect here some notations related to fractional derivative equations and fractional Sobolev spaces. First, we present the definition and the property of $(-\triangle)^\alpha$ through the Fourier series (see [20]). If u is a periodic function, it can be expressed by a Fourier series $u = \sum_{k \in \mathbb{Z}^3} u_k e^{ikx}$ with u_k the Fourier coefficients. It then follows that $u_{x_i} = \sum_{k \in \mathbb{Z}^3} ik_i u_k e^{ikx}$ $(i = 1, 2, 3)$, and $(-\triangle)^\alpha$ is defined by

$$(-\triangle)^\alpha u = \sum_{k \in \mathbb{Z}^3} |k|^{2\alpha} u_k e^{ikx},$$

where $\triangle = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$.

Let $D = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi] \subset \mathbb{R}^3$ and let $H^\alpha = H^\alpha(D)$ denote the Sobolev space of order α with the norm:

$$\|u\|_{H^\alpha(D)} = \left(\sum_{k \in \mathbb{Z}^3} |k|^{2\alpha} |u_k|^2 + \sum_{k \in \mathbb{Z}^3} |u_k|^2 \right)^{1/2} = \left(\|(-\triangle)^\alpha u\|^2 + \|u\|^2 \right)^{1/2}.$$

We denote by H_p^α those functions that are 2π-periodic in all the coordinate variables and, when restricted to D, lie in $H^\alpha(D)$. Throughout the whole paper, we denote by $(\,\cdot\,,\cdot\,)$ the usual inner product of $L^2(D)$, by $\|\cdot\|_{H^m}$ the norm of the Sobolev space $H^m(D)$, and $\|\cdot\|_m = \|\cdot\|_{L^m(D)}$ $(m = 1, 2, \ldots, \infty)$.

By virtue of the definition of $(-\triangle)^\alpha$, we have the following formula for integration by parts.

Lemma 2.11. [12] *If $f, g \in H_p^{2\alpha}(D)$, then there holds*

$$\int_D (-\triangle)^{\alpha} f \cdot g \, dx = \int_D (-\triangle)^{\alpha_1} f \cdot (-\triangle)^{\alpha_2} g \, dx,$$

where α_1, α_2 are nonnegative constants satisfying $\alpha_1 + \alpha_2 = \alpha$.

In addition, the following Gagliardo-Nirenberg inequality (see [18]) will be used later.

Lemma 2.12. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Let $u \in L^q(\Omega)$ and its derivatives of the order m, $D^m u$ belong to $L^r(\Omega)$, where $1 \le q, r \le \infty$. Then for the derivatives $D^j u$, $0 \le j < m$, there holds*

$$\|D^j u\|_{L^p} \le c \left(\|D^m u\|_{L^r} + \|u\|_{L^q} \right)^{\theta} \|u\|_{L^q}^{1-\theta}, \tag{2.2}$$

where

$$\frac{1}{p} = \frac{j}{n} + \theta \left(\frac{1}{r} - \frac{m}{n} \right) + (1-\theta)\frac{1}{q}$$

for all θ in the interval

$$\frac{j}{m} \le \theta \le 1.$$

Here the constant $c > 0$ depends only on n, m, j, q, r and θ, with the two exceptional cases:

(i) *If $j = 0$, $rm < n$ and $q = \infty$, then we make the additional assumption that either u tends to zero at infinity or $u \in L^{\widetilde{q}}$ for some $\widetilde{q} > 0$.*

(ii) *If $1 < r < \infty$ and $m - j - n/r$ is a nonnegative integer, then inequality* (2.2) *holds only for $j/m \le \theta < 1$.*

As we can see that (1.1) is a stochastic PDE which should be understood in the sense of Stravonich. To study this equation we need to transfer the stochastic fractional reaction-diffusion equation (1.1) into a deterministic one with random parameters. Thanks to the special linear multiplicative noise, the equation (1.1) can be reduced to an equation with random coefficients by Ornstein-Ulenbeck transformation.

We consider the probability space $(\Omega, \mathcal{F}, P)$ where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Defined a shift on Ω by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \ t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is an ergodic metric dynamical system. We identify $W(t)$ with $\omega(t)$, i.e., $\omega(t) = W(t,\omega)$, $t \in \mathbb{R}$.

We now introduce the stationary process

$$z(t,\omega) = z(\theta_t\omega) = -\int_{-\infty}^{0} e^{\tau}(\theta_t\omega)(\tau)\,d\tau, \quad t \in \mathbb{R},$$

which satisfies the stochastic differential equation:

$$dz + z\,dt = dW(t).$$

Moreover, for any $t, s \in \mathbb{R}$,

$$z(t, \theta_s\omega) = z(t+s, \omega), \quad P\text{-a.s.}$$

Here the exceptional set may be a priori depending on t and s. In fact, we suppose that z has a continuous modification. Once this modification is chosen, the exceptional set is independent of t. It is known that the random variable $z(\omega)$ is tempered (see [1]), there exists a θ_t-invariant set $\widetilde{\Omega} \subset \Omega$ of full P measure such that for every $\omega \in \widetilde{\Omega}$, $z(\theta_t\omega)$ is continuous in t, and

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0 \quad \text{for all } \omega \in \widetilde{\Omega},$$

and

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_t\omega)\,dt = 0 \quad \text{for all } \omega \in \widetilde{\Omega}.$$

In the sequel, we will write $\widetilde{\Omega}$ as Ω for convenience.

Then, by setting the unknown $v(t)$ as $v(t) = e^{-\beta z(\theta_t\omega)}u(t)$, we may transfer (1.1)–(1.3) to obtain the following random differential equation

$$\frac{\partial v}{\partial t} + (-\Delta)^{\alpha} v + \lambda v = e^{-\beta z(\theta_t\omega)} f(u) + e^{-\beta z(\theta_t\omega)} g(x) + \beta z(\theta_t\omega) v, \quad x \in \mathbb{R}^3,\ t > 0, \tag{2.3}$$

with the initial data

$$v(x,0) = v_0, \quad x \in \mathbb{R}^3 \tag{2.4}$$

and the periodic boundary condition

$$v(x + 2\pi e_i, t) = v(x,t), \quad x \in \mathbb{R}^3,\ t > 0,\ i = 1,2,3. \tag{2.5}$$

Thus, from now on, we shall carry on our discussion mainly for the system (2.3)–(2.5) which is clearly equivalent for (1.1)–(1.3).

3. Existence and uniqueness of solutions

As the initial task, in this section, we establish the existence and uniqueness of weak solutions to the problem (2.3)–(2.5) by applying the classical Faedo-Galerkin method. Throughout this section $T > 0$ is a fixed time. We start by stating two lemmas to be used below.

Lemma 3.1. [23] *Let X_0, X and X_1 be three Banach spaces with $X_0 \subset X \subset X_1$ and X_i $(i = 0, 1)$ are reflexive. Assume that X_0 is compactly embedded in X. Let*

$$W = \left\{ u \in L^{p_0}(0, T; X_0) : u' = \frac{du}{dt} \in L^{p_1}(0, T; X_1) \right\},$$

where $1 < p_i < \infty$, $i = 0, 1$. Then with the norm

$$\|u\|_{L^{p_0}(0,T;X_0)} + \|u'\|_{L^{p_1}(0,T;X_1)},$$

W is compactly embedded in $L^{p_0}(0, T; X)$.

Lemma 3.2. [19] *Let V, H be Hilbert spaces, V^* is the dual of V with $V \subset\subset H = H^* \subset V^*$. Suppose that*

$$u \in L^2(0, T; V) \quad \text{and} \quad \frac{du}{dt} \in L^2(0, T, V^*).$$

Then $u \in C([0, T]; H)$.

Now, we present the result of existence and uniqueness of solution to (2.3)–(2.5).

Theorem 3.3. *Suppose that $v_0 \in L^2(D)$, and f satisfies (1.4)–(1.6). Then there exists a unique global solution $v = v(x, t)$ to (2.3) such that*

$$v \in L^\infty(0, T; L^2(D) \cap H^\alpha(D)), \quad v' \in L^\infty(0, T; H^{-\alpha}(D)) \quad \text{and} \quad v \in C([0, T]; L^2(D)).$$

In particular, if $v_0 \in H^\alpha(D)$, then

$$v \in L^\infty(0, T; H^\alpha(D) \cap H^{\alpha+1}(D)), \quad v' \in L^\infty(0, T; H^{-\alpha}(D)) \quad \text{and} \quad v \in C([0, T]; H^\alpha(D)).$$

Proof. We first prove the existence of solutions making use of the Faedo-Galerkin method.

Let $\{e_k, k = 1, 2, \ldots\}$ be the orthonormal basis of $L^2(D)$. Given a positive integer m, one can find a function $v_m = v_m(\cdot, t)$ of the form

$$v_m = \sum_{k=1}^{m} d_m^k(t) e_k, \tag{3.1}$$

where the coefficients $d_m^k(t)$, for $0 \le t \le T$, $k = 1, 2, \ldots, m$, are chosen so that

$$\begin{aligned} &\left(\frac{dv_m}{dt}, e_k\right) + ((-\Delta)^\alpha v_m, e_k) + \lambda(v_m, e_k) \\ &= e^{-\beta z}(f(u_m), e_k) + e^{-\beta z}(g, e_k) + \beta z(v_m, e_k), \end{aligned} \tag{3.2}$$

and

$$v_m(0) = v_{0m} \in X_m = \text{span}\{e_k\}_{k=1}^m, \quad v_{0m} \to v_0 \quad \text{in } L^2(D) \text{ as } m \to \infty. \tag{3.3}$$

Note that (3.2) is a system of nonlinear ODE subject to the initial condition (3.3). From (1.6), we know that the nonlinear term is Lipschtiz in X_m. According to standard existence theory for nonlinear differential equations, there exists a unique solution of (3.2)–(3.3) for $0 \le t \le t_m$. Next we shall show that we can take $t_m = T$ independent of m. To this end, we first prove that v_m is uniformly bounded in $L^\infty(0, T; L^2(D))$. Multiply (3.2) by $d_m^k(t)$, sum for k from 1 to m, and recall (3.1) to get

$$\left(\frac{dv_m}{dt}, v_m\right) + ((-\Delta)^\alpha v_m, v_m) + \lambda(v_m, v_m) = e^{-\beta z}(f(u_m), v_m) + e^{-\beta z}(g, v_m) + \beta z(v_m, v_m).$$

Then integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_m\|^2 + \|(-\Delta)^{\alpha/2} v_m\|^2 + \lambda\|v_m\|^2 \\ &= \beta z\|v_m\|^2 + e^{-\beta z}\int_D f(u_m)v_m\,dx + e^{-\beta z}\int_D g v_m\,dx. \end{aligned} \tag{3.4}$$

Applying (1.4), we deduce that

$$e^{-\beta z}\int_D f(u_m)v_m\,dx \le -\beta_1 e^{-2\beta z}\|u_m\|_p^p + \gamma_1|D|e^{-2\beta z} \tag{3.5}$$

and by Young's inequality, we have

$$e^{-\beta z}\int_D g v_m\,dx \le \frac{\lambda}{4}\|v_m\|^2 + \frac{\|g\|^2}{\lambda}e^{-2\beta z}. \tag{3.6}$$

Thus it follows from (3.4)–(3.6) that

$$\begin{aligned} &\frac{d}{dt}\|v_m\|^2 + 2\|(-\Delta)^{\alpha/2} v_m\|^2 + \frac{3\lambda}{2}\|v_m\|^2 + 2\beta_1 e^{-2\beta z}\|u_m\|_p^p \\ &\le 2\beta z\|v_m\|^2 + 2\gamma_1|D|e^{-\beta z} + \frac{2\|g\|^2}{\lambda}e^{-2\beta z}. \end{aligned}$$

By Grownall lemma, we obtain that

$$\begin{aligned} &\|v_m(t,\omega,v_{0m}(\omega))\|^2 + 2\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_r\omega)\,dr}\|(-\Delta)^{\alpha/2} v_m(s,\omega,v_{0m}(\omega))\|^2\,ds \\ &+ \frac{\lambda}{2}\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_r\omega)\,dr}\|v_m(s,\omega,v_{0m}(\omega))\|^2\,ds \\ &+ 2\beta_1\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_r\omega)\,dr+(p-2)\beta z(\theta_s\omega)}\|v_m(s,\omega,v_{0m}(\omega))\|_p^p\,ds \\ \le\, & e^{-\lambda t+2\beta\int_0^t z(\theta_r\omega)\,dr}\|v_0(\omega)\|^2 \\ &+ 2\left(\gamma_1|D| + \frac{1}{\lambda}\|g\|^2\right)\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_r\omega)\,dr-2\beta z(\theta_s\omega)}\,ds. \end{aligned} \tag{3.7}$$

Note that $z(\theta_t\omega)$ is continuous in t for fixed ω. Therefore, for every $\omega \in \Omega$ and $T > 0$, we obtain from (3.7) that

$$\{v_m\}_{m=1}^{\infty} \text{ is bounded in } L^\infty(0,T;L^2(D)) \cap L^2(0,T;H^\alpha(D)) \cap L^p(0,T;L^p(D)). \tag{3.8}$$

Following exactly the same arguments as in the proofs of Lemma 4.4 (see details there), we can also get that

$$\|(-\Delta)^\alpha v_m\|^2 \le C(\omega). \tag{3.9}$$

So, from Sobolev embedding theorem, (3.9) implies

$$\{v_m\}_{m=1}^{\infty} \text{ is bounded in } L^\infty(0,T;H^\alpha(D)). \tag{3.10}$$

In addition, by (1.5), we have

$$\int_0^T \int_D |f(e^{\beta z} v_m)|^q \, dxdt \le c \int_0^T \int_D |v_m|^p \, dxdt + \widetilde{c},$$

with $1/q + 1/p = 1$, which along with (3.8) shows that

$$\{f(e^{-\beta z} v_m)\}_{m=1}^{\infty} \text{ is bounded in } L^q(0,T;L^q(D)).$$

Now we prove that $v_m' \in L^\infty(0,T;H^{-\alpha}(D))$. Indeed, for all $\varphi \in H^\alpha(D)$, we have

$$(v_m', \varphi) = -((-\Delta)^\alpha v_m, \varphi) - \lambda(v_m, \varphi) + \beta z(v_m, \varphi) + e^{-\beta z}(f(u_m), \varphi) + e^{-\beta z}(g, \varphi). \tag{3.11}$$

From (1.5), together with Hölder inequality and Gagliardo-Nirenberg inequality, (3.11) implies that

$$\begin{aligned} |(v_m', \varphi)| &\le \|(-\Delta)^{\alpha/2} v_m\| \|(-\Delta)^{\alpha/2}\varphi\| + \lambda \|v_m\|\|\varphi\| + \beta|z|\|v_m\|\|\varphi\| \\ &\quad + \beta_2 e^{(p-2)\beta z} \|v_m\|_{L^p}^{p-1} \|\varphi\|_{L^p} + \gamma_2 e^{-\beta z}\|\varphi\| + e^{-\beta z}\|g\|\|\varphi\| \\ &\le c(\|(-\Delta)^{\alpha/2}\varphi\| + \|\varphi\|_{L^p} + \|\varphi\|), \end{aligned} \tag{3.12}$$

where we have used the fact that $z(\theta_t\omega)$ is continuous in t, so that $z(\theta_t\omega)$ is bounded on $[0, T]$ for any fixed T. By Sobolev embedding theorem, we see that

$$\|\varphi\|_{L^p} \le c\|(-\Delta)^{\alpha/2}\varphi\| \quad \text{and} \quad \|\varphi\| \le \widetilde{c}\|(-\Delta)^{\alpha/2}\varphi\|.$$

Thus, substituting it into (3.12) gives that

$$|(v_m', \varphi)| \le c\|(-\Delta)^{\alpha/2}\varphi\|, \quad \forall\, \varphi \in H^\alpha(D),$$

which implies

$$v_m' \in L^\infty(0,T;H^{-\alpha}(D)). \tag{3.13}$$

Therefore, by (3.9) and (3.13), we deduce that there exists a subsequence of v_{m_l} of v_m, such that

$$v_{m_l} \xrightarrow{w^*} v \text{ in } L^\infty(0,T;H^{2\alpha}(D)), \tag{3.14}$$

and

$$v'_{m_l} \xrightarrow{w^*} v' \text{ in } L^\infty(0,T;H^{-\alpha}(D)), \tag{3.15}$$

for some $v \in L^\infty(0,T;H^{-\alpha}(D))$.

On the other hand, from (3.10) and (3.13), we infer that

$$v_m \text{ is bounded in } L^2(0,T;H^\alpha(D)) \text{ and } v'_m \text{ is bounded in } L^2(0,T;H^{-\alpha}(D)). \tag{3.16}$$

Let $W = \{v : v \in L^2(0,T;H^\alpha(D)), v' \in L^2(0,T;H^{-\alpha}(D))\}$. Since $H^\alpha(D)$ is compactly embedded in $L^2(D)$, W is compactly embedded in $L^2(0,T;L^2(D))$ due to Lemma 3.1. Then, without loss of generality, we have that v_{m_l} satisfies

$$v_{m_l} \to v \text{ strongly in } L^2(0,T;L^2(D)) \text{ a.e.} \tag{3.17}$$

So by (3.17) and the continuity of f, we can verify that

$$f(e^{-\beta z}v_{m_l}) \to f(e^{-\beta z}v) \text{ weakly in } L^q(0,T;L^q(D)). \tag{3.18}$$

By (3.2) we deduce that

$$\begin{aligned}(v'_{m_l}, e_k) = &-((-\Delta)^\alpha v_{m_l}, e_k) - \lambda(v_{m_l}, e_k) + \beta z(v_{m_l}, e_k)\\ &+ e^{-\beta z}(f(u_{m_l}), e_k) + e^{-\beta z}(g, e_k).\end{aligned} \tag{3.19}$$

Now applying (3.14), (3.15) and (3.18) to (3.19) yields that

$$(v_t, e_k) = -((-\Delta)^\alpha v, e_k) - \lambda(v, e_k) + \beta z(v, e_k) + e^{-\beta z}(f(u), e_k) + e^{-\beta z}(g, e_k).$$

This equality holds for any fixed k. Hence, by the density of the basis e_k, we get

$$(v_t, \varphi) + ((-\Delta)^\alpha v, \varphi) + \lambda(v, \varphi) = \beta z(v, \varphi) + e^{-\beta z}(f(u), \varphi) + e^{-\beta z}(g, \varphi), \quad \forall\, \varphi \in H^\alpha(D).$$

By (3.8), (3.16) and Lemma 3.2, we obtain that $v_{m_l} \in C([0,T];L^2(D))$. Then,

$$v_{m_l}(0) \to v(0) \text{ weakly in } L^2(D).$$

But from (3.3), we have

$$v_{m_l}(0) \to v_0 \text{ weakly in } L^2(D).$$

Therefore, $v(0) = v_0$.

Next we show the uniqueness of the solutions.

Assume that there are solutions v_1 and v_2 to the problem (2.3)–(2.5). Let $\widetilde{w} = v_1 - v_2$. Then, we get

$$\frac{\partial \widetilde{w}}{\partial t} + (-\Delta)^\alpha \widetilde{w} + \lambda \widetilde{w} = \beta z \widetilde{w} + e^{-\beta z} \left(f(e^{\beta z} v_1) - f(e^{\beta z} v_2) \right) \tag{3.20}$$

with $\widetilde{w}(0) = 0$. Take the inner product of (3.20) with $\widetilde{w}$ in $L^2(D)$ to find

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{w}\|^2 + \|(-\Delta)^{\alpha/2}\widetilde{w}\|^2 + \lambda\|\widetilde{w}\|^2 = \beta z\|\widetilde{w}\|^2 + e^{-\beta z}\left(f(e^{\beta z} v_1) - f(e^{\beta z} v_2), \widetilde{w} \right). \tag{3.21}$$

From condition (1.6), we have

$$e^{-\beta z}\left(f(e^{\beta z} v_1) - f(e^{\beta z} v_2), \widetilde{w} \right) \le \beta_3 \|\widetilde{w}\|^2.$$

Then (3.21) implies that

$$\frac{d}{dt}\|\widetilde{w}\|^2 + 2\|(-\Delta)^{\alpha/2}\widetilde{w}\|^2 + 2\lambda\|\widetilde{w}\|^2 \le 2(\beta z + \beta_3)\|\widetilde{w}\|^2.$$

By Gronwall inequality, for all $t \in [0,T]$, we get that

$$\|\widetilde{w}(t)\|^2 \le e^{2\int_0^t \beta_3 + \beta z(\theta_s\omega)\, ds}\|\widetilde{w}(0)\|^2 = 0.$$

So $\widetilde{w} = 0$. The proof is completed. □

From Theorem 3.3, we know that (2.3) has a unique solution $v(\,\cdot\,,\omega,v_0) \in C([0,\infty), L^2(D))$ with $v(0,\omega,v_0) = v_0(\omega)$. Moreover, we have that $v(t,\omega,v_0)$ is unique and continuous with respect to v_0 in $L^2(D)$ for all $t \ge 0$. We define a mapping $\varphi\colon \mathbb{R}^+ \times \Omega \times L^2(D) \to L^2(D)$ by

$$\varphi(t,\omega)v_0 = v(t,\omega,v_0) \quad \text{for all } v_0 \in L^2(D),\ t \ge 0 \text{ and } \omega \in \Omega. \tag{3.22}$$

Then φ is a continuous random dynamical system associated with the problem (2.3)–(2.5) in $L^2(D)$.

Let $u(t,\omega,u_0) = e^{\beta z(\theta_t\omega)} v(t,\omega,v_0)$. Then the process u is the solution of problem (1.1)–(1.3). We now define a mapping $\phi\colon \mathbb{R}^+ \times \Omega \times L^2(D) \to L^2(D)$ by

$$\phi(t,\omega)u_0 = u(t,\omega,u_0) = e^{\beta z(\theta_t\omega)} v(t,\omega, e^{-\beta z(\omega)}u_0),$$

for $u_0 \in L^2(D)$, $t \ge 0$, and for all $\omega \in \Omega$. It is easy to check that ϕ satisfies the two conditions in Definition 2.2. Therefore, ϕ is a continuous random dynamical system associated with the problem (1.1)–(1.3) on $L^2(D)$.

Observe that, the two random dynamical systems φ and ϕ are equivalent, so ϕ has a random attractor if and only if φ possesses one. Hence, in the sequel, we only need to study the dynamics for the random dynamical system φ.

4. A priori estimates

To study the dynamics for the random dynamical system φ we have to conduct a priori estimates of the solutions to problem (2.3)–(2.5), this is exactly the goal of this section. The estimates established here are necessary for proving the asymptotical compactness of the corresponding random dynamical system φ. In what follows, we always assume that $\mathcal{D}$ is the collection of all tempered random subsets in $L^2(D)$ and $g \in H^1(\mathbb{R}^3)$. And for brevity, we write $\int_D f\,dx$ as $\int f$ and use c and c_i $(i = 1, 2, \ldots)$ to denote different positive constants which may change their values from line to line or even in the same lines.

Lemma 4.1. *Assume that (1.4)–(1.6) hold. Let $B = \{B(\omega)\} \in \mathcal{D}$, $v_0(\omega) \in B(\omega)$ and $R_0 > 0$ be fixed. Then for P-a.e. $\omega \in \Omega$, there exists $T_{0B}(\omega) > 0$ such that for any $t \geq T_{0B}(\omega)$, one has*

$$\|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 \leq R_0^2.$$

Proof. Taking the inner product of (2.3) with v in $L^2(D)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|(-\Delta)^{\alpha/2}v\|^2 + \lambda\|v\|^2 = \beta z\|v\|^2 + e^{-\beta z}(f, v) + e^{-\beta z}(g, v). \tag{4.1}$$

By (1.4), we have

$$e^{-\beta z}(f, v) \leq -\beta_1 e^{-2\beta z}\|u\|_p^p + \gamma_1|D|e^{-2\beta z}, \tag{4.2}$$

and using Young's inequality, we find

$$e^{-\beta z}(g, v) \leq \frac{\lambda}{4}\|v\|^2 + \frac{e^{-2\beta z}}{\lambda}\|g\|^2. \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1) to get

$$\begin{aligned} &\frac{d}{dt}\|v\|^2 + 2\|(-\Delta)^{\alpha/2}v\|^2 + \frac{3\lambda}{2}\|v\|^2 + 2\beta_1 e^{-2\beta z}\|u\|_p^p \\ &\leq 2\beta z\|v\|^2 + 2\gamma_1|D|e^{-2\beta z} + \frac{2e^{-2\beta z}}{\lambda}\|g\|^2. \end{aligned} \tag{4.4}$$

Multiplying (4.4) by $e^{\lambda t - 2\beta\int_0^t z(\theta_r\omega)\,dr}$ and integrating over $[0, t]$ yield that

$$\begin{aligned} &\|v(t, \omega, v_0(\omega))\|^2 + 2\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_r\omega)\,dr}\|(-\Delta)^{\alpha/2}v(s, \omega, v_0(\omega))\|^2\,ds \\ &\quad + 2\beta_1\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_r\omega)\,dr - 2\beta z(\theta_s\omega)}\|u(s, \omega, u_0(\omega))\|_p^p\,ds \\ &\leq e^{-\lambda t + 2\beta\int_0^t z(\theta_r\omega)\,dr}\|v_0(\omega)\|^2 \\ &\quad + 2\left(\gamma_1|D| + \frac{1}{\lambda}\|g\|^2\right)\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_r\omega)\,dr - 2\beta z(\theta_s\omega)}\,ds. \end{aligned} \tag{4.5}$$

Replacing ω by $\theta_{-t}\omega$, then we deduce from (4.5),

$$\begin{aligned}
&\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\\
&+2\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_{r-t}\omega)\,dr}\|(-\Delta)^{\alpha/2}v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\,ds\\
&+2\beta_1\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_{r-t}\omega)\,dr-2\beta z(\theta_{s-t}\omega)}\|u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|_p^p\,ds\\
&\le e^{-\lambda t+2\beta\int_0^t z(\theta_{r-t}\omega)\,dr}\|v_0(\theta_{-t}\omega)\|^2\\
&+2\left(\gamma_1|D|+\frac{1}{\lambda}\|g\|^2\right)\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_{r-t}\omega)\,dr-2\beta z(\theta_{s-t}\omega)}\,ds\\
&= e^{-\lambda t+2\beta\int_{-t}^0 z(\theta_r\omega)\,dr}\|v_0(\theta_{-t}\omega)\|^2\\
&+2\left(\gamma_1|D|+\frac{1}{\lambda}\|g\|^2\right)\int_{-t}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr-2\beta z(\theta_s\omega)}\,ds.
\end{aligned} \tag{4.6}$$

Since $\{B(\omega)\}\in\mathcal{D}$ is tempered, for any $v_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$, we see that

$$\lim_{t\to+\infty} e^{-\lambda t+2\beta\int_{-t}^0 z(\theta_r\omega)\,dr}\|v_0(\theta_{-t}\omega)\|^2=\lim_{t\to+\infty} e^{-\lambda t+2\beta\int_{-t}^0 z(\theta_r\omega)\,dr-2\beta z(\theta_t\omega)}=0.$$

Therefore, there exist $R_0>0$ and $T_{0B}(\omega)>0$ such that for any $t\ge T_{0B}(\omega)$,

$$\begin{aligned}
&\lim_{t\to+\infty} e^{-\lambda t+2\beta\int_{-t}^0 z(\theta_r\omega)\,dr}\|v_0(\theta_{-t}\omega)\|^2\\
&+2\left(\gamma_1|D|+\frac{1}{\lambda}\|g\|^2\right)\int_{-t}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr-2\beta z(\theta_s\omega)}\,ds\\
&\le R_0^2,
\end{aligned}$$

which along with (4.6) shows that, for any $t\ge T_{0B}(\omega)$,

$$\begin{aligned}
&\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2+2\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_{r-t}\omega)\,dr}\|(-\Delta)^{\alpha/2}v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\,ds\\
&+2\beta_1\int_0^t e^{\lambda(s-t)+2\beta\int_s^t z(\theta_{r-t}\omega)\,dr-2\beta z(\theta_{s-t}\omega)}\|u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|_p^p\,ds\\
&\le R_0^2.
\end{aligned}$$

The proof is completed. □

Lemma 4.2. *Assume that* (1.4)–(1.6) *hold. Then for P-a.e. $\omega\in\Omega$, there exists $T_{1B}(\omega)>T_{0B}(\omega)>0$ such that for any $t\ge T_{1B}(\omega)$, we have*

$$\|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2\le R_1^2(\omega),$$

where $R_1^2(\omega)=c_1'R_0^2\int_{-1}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr}\,ds+\frac{2\|\nabla g\|^2}{\lambda}\int_{-1}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr-2\beta z(\theta_s\omega)}\,ds$ for some $c_1'>0$.

Proof. Taking the inner product of (2.3) with $-\Delta v$ in $L^2(D)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2+\|(-\Delta)^{(1+\alpha)/2}v\|^2+\lambda\|\nabla v\|^2\\&=\beta z\|\nabla v\|^2+e^{-\beta z}(f(u),-\Delta v)+e^{-\beta z}(g,-\Delta v).\end{aligned}\tag{4.7}$$

By condition (1.6), it can be seen that

$$e^{-\beta z}(f(u),-\Delta v)=e^{-\beta z}\int f'(u)\nabla u\cdot\nabla v\le\beta_3\|\nabla v\|^2.\tag{4.8}$$

And from Young's inequality, it is easy to get

$$e^{-\beta z}(g,-\Delta v)\le\frac{\lambda}{4}\|\nabla v\|^2+\frac{e^{-2\beta z}}{\lambda}\|\nabla g\|^2.\tag{4.9}$$

So, it follows from (4.7)–(4.9) that

$$\frac{d}{dt}\|\nabla v\|^2+2\|(-\Delta)^{(1+\alpha)/2}v\|^2+\frac{3\lambda}{2}\|\nabla v\|^2\le 2\beta z\|\nabla v\|^2+2\beta_3\|\nabla v\|^2+\frac{2e^{-2\beta z}}{\lambda}\|\nabla g\|^2.\tag{4.10}$$

Take $t\ge T_{0B}(\omega)$, multiply (4.10) by $e^{\lambda t-2\beta\int_0^t z(\theta_s\omega)\,ds}$ and integrate over $[s,t+1]$ with $s\in[t,t+1]$, then integrate the resulting inequality over $[t,t+1]$ with respect to s, we then have

$$\begin{aligned}&\|\nabla v(t+1,\omega,v_0(\omega))\|^2\\&\quad+2\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_s^{t+1}z(\theta_r\omega)\,dr}\|(-\Delta)^{(1+\alpha)/2}v(s,\omega,v_0(\omega))\|^2\,ds\\&\le(2\beta_3+1)\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_s^{t+1}z(\theta_r\omega)\,dr}\|\nabla v(s,\omega,v_0(\omega))\|^2\,ds\\&\quad+\frac{2\|\nabla g\|^2}{\lambda}\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_s^{t+1}z(\theta_r\omega)\,dr-2\beta z(\theta_s\omega)}\,ds.\end{aligned}\tag{4.11}$$

Using the Gagliardo-Nirenberg inequality in Lemma 2.12, we find that

$$\begin{aligned}(2\beta_3+1)\|\nabla v\|^2&\le c_1(2\beta_3+1)\left(\|(-\Delta)^{(1+\alpha)/2}v\|+\|v\|\right)^{2/(1+\alpha)}\|v\|^{2\alpha/(1+\alpha)}\\&\le\|(-\Delta)^{(1+\alpha)/2}v\|^2+c_1'\|v\|^2,\end{aligned}$$

where $c_1'=(c_1(2\beta_3+1))^{(1+\alpha)/\alpha}\left(\frac{2}{1+\alpha}\right)^{1/\alpha}\cdot\frac{\alpha}{1+\alpha}+1$.

Substituting ω by $\theta_{-t-1}\omega$ in (4.11) and combining above inequality, we get

$$\begin{aligned}&\|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2\\&\le c_1'\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_s^{t+1}z(\theta_{r-t-1}\omega)\,dr}\|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2\,ds\\&\quad+\frac{2\|\nabla g\|^2}{\lambda}\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_s^{t+1}z(\theta_{r-t-1}\omega)\,dr-2\beta z(\theta_{s-t-1}\omega)}\,ds.\end{aligned}\tag{4.12}$$

Since $t \geq T_{0B}(\omega)$, by Lemma 4.1, one has

$$\begin{aligned}
&c_1' \int_t^{t+1} e^{\lambda(s-t-1)+2\beta\int_s^{t+1} z(\theta_{r-t-1}\omega)\,dr}\|v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2\,ds \\
&\leq c_1' R_0^2 \int_t^{t+1} e^{\lambda(s-t-1)+2\beta\int_s^{t+1} z(\theta_{r-t-1}\omega)\,dr}\,ds \\
&= c_1' R_0^2 \int_{-1}^0 e^{\lambda s+2\beta\int_{s+t+1}^{t+1} z(\theta_{r-t-1}\omega)\,dr}\,ds \\
&= c_1' R_0^2 \int_{-1}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr}\,ds.
\end{aligned} \tag{4.13}$$

In addition,

$$\begin{aligned}
&\frac{2\|\nabla g\|^2}{\lambda}\int_t^{t+1} e^{\lambda(s-t-1)+2\beta\int_s^{t+1} z(\theta_{r-t-1}\omega)\,dr-2\beta z(\theta_{s-t-1}\omega)}\,ds \\
&\leq \frac{2\|\nabla g\|^2}{\lambda}\int_{-1}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr-2\beta z(\theta_s\omega)}\,ds.
\end{aligned} \tag{4.14}$$

Then, from (4.12)–(4.14), we obtain

$$\|\nabla v(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 \leq R_1^2(\omega),$$

where $R_1^2(\omega) = c_1' R_0^2 \int_{-1}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr}\,ds + \frac{2\|\nabla g\|^2}{\lambda}\int_{-1}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr-2\beta z(\theta_s\omega)}\,ds$, which completes the proof. □

Lemma 4.3. *Assume that* (1.4)–(1.6) *hold. Then for P-a.e. $\omega \in \Omega$, there exists $T_{2B}(\omega) > T_{1B}(\omega) > 0$ such that for any $t \geq T_{2B}(\omega)$, we have*

$$\|(-\Delta)^{(1+\alpha)/2} v(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 \leq R_2^2(\omega),$$

where $R_2^2(\omega) = R_0^2(c_2'+c_3')\int_{-1}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr}\,ds + 2\|\nabla g\|^2\int_{-1}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr-2\beta z(\theta_s\omega)}\,ds$ with constants $c_2', c_3' > 0$.

Proof. Take the inner product of (2.3) with $(-\Delta)^{1+\alpha}v$ in $L^2(D)$ to find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{(1+\alpha)/2}v\|^2 + \|(-\Delta)^{1/2+\alpha}\|^2 + \lambda\|(-\Delta)^{(1+\alpha)/2}v\|^2 \\
&= \beta z\|(-\Delta)^{(1+\alpha)/2}v\|^2 + e^{-\beta z}(f(u),(-\Delta)^{1+\alpha}v) + e^{-\beta z}(g,(-\Delta)^{1+\alpha}v).
\end{aligned} \tag{4.15}$$

By (1.6) as well as Young's inequality, we obtain

$$e^{-\beta z}(f(u),(-\Delta)^{1+\alpha}v) \leq \beta_3\|\nabla v\|\cdot\|(-\Delta)^{1/2+\alpha}v\| \leq \frac{1}{4}\|(-\Delta)^{1/2+\alpha}v\|^2 + \beta_3^2\|\nabla v\|^2 \tag{4.16}$$

and

$$e^{-\beta z}(g,(-\Delta)^{1+\alpha}v) \leq \frac{1}{4}\|(-\Delta)^{1/2+\alpha}v\|^2 + e^{-2\beta z}\|\nabla g\|^2. \tag{4.17}$$

Then it follows from (4.15)–(4.17) that

$$\begin{aligned}&\frac{d}{dt}\|(-\Delta)^{(1+\alpha)/2}v\|^2+\|(-\Delta)^{1/2+\alpha}v\|^2+2\lambda\|(-\Delta)^{(1+\alpha)/2}v\|^2\\&\le 2\beta z\|(-\Delta)^{(1+\alpha)/2}v\|^2+2\beta_3^2\|\nabla v\|^2+2e^{-2\beta z}\|\nabla g\|^2.\end{aligned}\tag{4.18}$$

Take $t>T_{2B}(\omega)>T_{1B}(\omega)$, multiply (4.18) by $e^{\lambda t-2\beta\int_0^t z(\theta_s\omega)\,ds}$ and integrate over $[s,t+1]$ with $s\in[t,t+1]$, then integrate the resulting inequality over $[t,t+1]$ with respect to s, we get

$$\begin{aligned}&\|(-\Delta)^{(1+\alpha)/2}v(t+1,\omega,v_0(\omega))\|^2\\&\quad+\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_s^{t+1}z(\theta_r\omega)\,dr}\|(-\Delta)^{1/2+\alpha}v(s,\omega,v_0(\omega))\|^2\,ds\\&\le\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_s^{t+1}z(\theta_r\omega)\,dr}\|(-\Delta)^{(1+\alpha)/2}v(s,\omega,v_0(\omega))\|^2\,ds\\&\quad+2\beta_3^2\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_s^{t+1}z(\theta_r\omega)\,dr}\|\nabla v(s,\omega,v_0(\omega))\|^2\,ds\\&\quad+2\|\nabla g\|^2\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_s^{t+1}z(\theta_r\omega)\,dr-2\beta z(\theta_s\omega)}\,ds.\end{aligned}\tag{4.19}$$

On the other hand, by the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned}\|(-\Delta)^{(1+\alpha)/2}v\|^2&\le c_2\left(\|(-\Delta)^{1/2+\alpha}v\|+\|v\|\right)^{(2+2\alpha)/(1+2\alpha)}\cdot\|v\|^{2\alpha/(1+2\alpha)}\\&\le\frac{1}{2}\|(-\Delta)^{1/2+\alpha}v\|^2+c_2'\|v\|^2,\end{aligned}\tag{4.20}$$

where $c_2'=\frac{\alpha}{1+2\alpha}\left(\frac{4(1+\alpha)}{1+2\alpha}\right)^{(1+\alpha)/\alpha}c_2^{(1+2\alpha)/\alpha}+\frac{1}{2}$. Similarly, we have

$$\begin{aligned}2\beta_3^2\|\nabla v\|^2&\le 2\beta_3^2c_3\left(\|(-\Delta)^{1/2+\alpha}v\|+\|v\|\right)^{2/(1+2\alpha)}\cdot\|v\|^{4\alpha/(1+2\alpha)}\\&\le\frac{1}{2}\|(-\Delta)^{1/2+\alpha}v\|^2+c_3'\|v\|^2,\end{aligned}\tag{4.21}$$

where $c_3'=\frac{2\alpha}{1+2\alpha}\left(\frac{4}{1+2\alpha}\right)^{1/(2\alpha)}(2\beta_3^2c_3)^{(1+2\alpha)/(2\alpha)}+\frac{1}{2}$.

Substituting (4.20) and (4.21) into (4.19), and replacing ω by $\theta_{-t-1}\omega$, we get

$$\begin{aligned}&\|(-\Delta)^{(1+\alpha)/2}v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2\\&\le(c_2'+c_3')\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_t^{t+1}z(\theta_{r-t-1}\omega)\,dr}\|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2\,ds\\&\quad+2\|\nabla g\|^2\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_t^{t+1}z(\theta_{r-t-1}\omega)\,dr-2\beta z(\theta_{s-t-1}\omega)}\,ds\\&\le R_0^2(c_2'+c_3')\int_t^{t+1}e^{\lambda(s-t-1)+2\beta\int_s^{t+1}z(\theta_{r-t-1}\omega)\,dr}\,ds\end{aligned}$$

$$
\begin{aligned}
&+ 2\|\nabla g\|^2 \int_t^{t+1} e^{\lambda(s-t-1)+2\beta \int_t^{t+1} z(\theta_{r-t-1}\omega)\,dr - 2\beta z(\theta_{s-t-1}\omega)}\,ds \\
&\le (c_2' + c_3') R_0^2 \int_{-1}^0 e^{\lambda s + 2\beta \int_s^0 z(\theta_r\omega)\,dr}\,ds + 2\|\nabla g\|^2 \int_{-1}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr - 2\beta z(\theta_s\omega)}\,ds \\
&:= R_2^2(\omega).
\end{aligned}
$$

Then the proof is completed. □

Lemma 4.4. *Assume that* (1.4)–(1.6) *hold. Then for* P*-a.e.* $\omega \in \Omega$*, there exists* $T_{3B}(\omega) > T_{2B}(\omega) > 0$ *such that for any* $t \ge T_{3B}(\omega)$*, we have*

$$
\|(-\Delta)^\alpha v(t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 \le R_3^2(\omega),
$$

where $R_3^2(\omega) = cR_0^2 \int_{-1}^0 e^{\lambda s+2\beta\int_s^0 z(\theta_r\omega)\,dr}\,ds + 2\|\nabla g\|^2 \int_{-1}^0 e^{\lambda s + 2\beta\int_s^0 z(\theta_r\omega)\,dr - 2\beta z(\theta_s\omega)}\,ds.$

Proof. Multiply (2.3) by $(-\Delta)^{2\alpha} v$ and integrate in $L^2(D)$ to obtain

$$
\begin{aligned}
&\frac{d}{dt}\|(-\Delta)^\alpha v\|^2 + 2\|(-\Delta)^{3\alpha/2} v\|^2 + 2\lambda \|(-\Delta)^\alpha v\|^2 \\
&= 2e^{-\beta z}(f(u), (-\Delta)^{2\alpha} v) + 2e^{-\beta z}(g, (-\Delta)^{2\alpha} v) + 2\beta z \|(-\Delta)^\alpha v\|^2.
\end{aligned} \tag{4.22}
$$

Using condition (1.6), Gigliardo-Nirenberg inequality and Young's inequality again, we see

$$
\begin{aligned}
&2e^{-\beta z}(f(u), (-\Delta)^{2\alpha} v) \\
&= 2e^{-\beta z}(f'(u)\nabla u, (-\Delta)^{2\alpha-1/2} v) = 2(f'(u)\nabla v, (-\Delta)^{2\alpha-1/2} v) \\
&\le 2\beta_3 \|\nabla v\| \|(-\Delta)^{2\alpha-1/2} v\| \le \beta_3^2 \|\nabla v\|^2 + \|(-\Delta)^{2\alpha-1/2} v\|^2 \\
&\le c_4 \left(\|(-\Delta)^{3\alpha/2} v\| + \|v\| \right)^{2/(3\alpha)} \|v\|^{2(3\alpha-1)/(3\alpha)} \\
&\quad + c_5 \left(\|(-\Delta)^{3\alpha/2} v\| + \|v\| \right)^{2(4\alpha-1)/(3\alpha)} \|v\|^{2(1-\alpha)/(3\alpha)} \\
&\le \frac{1}{2}\|(-\Delta)^{3\alpha/2} v\|^2 + c\|v\|^2,
\end{aligned} \tag{4.23}
$$

and

$$
\begin{aligned}
2e^{-\beta z}(g, (-\Delta)^{2\alpha} v) &\le 2e^{-\beta z}\|\nabla g\| \|(-\Delta)^{2\alpha-1/2} v\| \\
&\le e^{-2\beta z}\|\nabla g\|^2 + \frac{1}{2}\|(-\Delta)^{3\alpha/2} v\|^2 + c\|v\|^2.
\end{aligned} \tag{4.24}
$$

So it follows from (4.22)–(4.24) that

$$
\begin{aligned}
&\frac{d}{dt}\|(-\Delta)^\alpha v\|^2 + \|(-\Delta)^{3\alpha/2} v\|^2 + 2\lambda\|(-\Delta)^\alpha v\|^2 \\
&\le 2\beta z \|(-\Delta)^\alpha v\|^2 + c\|v\|^2 + 2e^{-2\beta z}\|\nabla g\|^2.
\end{aligned}
$$

By a similar process as performed in Lemma 4.3, we obtain that there exists $T_{3B}(\omega) > T_{2B}(\omega)$ such that for any $t \geq T_{3B}(\omega)$, we have

$$\|(-\Delta)^{\alpha} v(t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 \leq R_3^2(\omega),$$

where $R_3^2(\omega) = cR_0^2 \int_{-1}^{0} e^{\lambda s + 2\beta \int_s^0 z(\theta_r \omega)\, dr}\, ds + 2\|\nabla g\|^2 \int_{-1}^{0} e^{\lambda s + 2\beta \int_s^0 z(\theta_r\omega)\, dr - 2\beta z(\theta_s \omega)}\, ds$. This is the desired assertion. □

5. Existence of random attractors

In this section we study the existence and uniqueness of random attractor in space $L^2(D)$ for the associated RDS φ (defined by (3.22)) and its regularity as well, that is, we will show at the same time that the obtained random attractor in $L^2(D)$ is also actually in the spaces $H^1(D)$ and $H^{2\alpha}(D)$. For this purpose, as we mentioned in Section 1, we first need to establish an abstract result on existence of the so-called bi-spacial random attractors.

5.1. An abstract result

Theorem 5.1. *Let (X, Y) be a limit-identical pair of Banach spaces and $\mathcal{D}$ be a universe in X. Assume that φ is a continuous random dynamical system on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ taking its values in Y in the sense of (2.1). Moreover,*

(i) *φ has a closed and random absorbing set K ($K \in \mathcal{D}$) in X.*

(ii) *φ is $\mathcal{D}$-asymptotically compact in X.*

(iii) *φ is $\mathcal{D}$-asymptotically compact in Y.*

Then φ has a unique (X, Y)-random attractor $\mathcal{A}(\omega)$ given by

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}\omega)\left(K(\theta_{-t}\omega) \cap Y\right)}^{Y}. \tag{5.1}$$

Proof. The proof is similar to that of Theorem 3.1 in [14], so we only sketch it here.

Let $\mathcal{K}_1(\omega) = K(\omega) \cap Y$. Since φ is assumed to take its values in Y, $\mathcal{K}_1(\omega)$ is also a $\mathcal{D}$-absorbing set (but it may not be a closed set in X). By virtue of assumption (iii) and the definition of $\mathcal{A}(\omega)$, it is easy to prove that $\mathcal{A}(\omega)$ is compact in Y. So, to prove the existence of (X, Y)-random attractor for φ, we only need to verify the conditions (i), (iii) and (iv) in Definition 2.8.

First, we check the conditions (i) and (iii) by showing $\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_X$, where $\mathcal{A}_X$ (resp. $\mathcal{A}_1$) is the omega-limit set of K (resp. $\mathcal{K}_1$) in X. Indeed, if $x \in \mathcal{A}_1(\omega)$, then there exist $t_n \to +\infty$ and a sequence $\{x_n\}$ with $x_n \in \mathcal{K}_1(\theta_{-t_n}\omega)$ such that

$$\lim_{n\to\infty} \|\varphi(t_n, \theta_{-t_n}\omega)x_n - x\|_X = 0. \tag{5.2}$$

Since φ takes its values in Y in the sense of (2.1), and φ is $\mathcal{D}$-asymptotically compact in Y by assumption (iii), there exist a subsequence $\varphi(t_{n_j}, \theta_{-t_{n_j}}\omega)x_{n_j}$ of $\varphi(t_n, \theta_{-t_n}\omega)x_n$ and a $y \in Y$ such that

$$\lim_{n\to\infty} \|\varphi(t_{n_j}, \theta_{-t_{n_j}}\omega)x_{n_j} - y\|_Y = 0. \tag{5.3}$$

Hence, $y \in \mathcal{A}$, the omega-limit set of $\mathcal{K}_1$ in Y. Noting that (X, Y)-is assumed to be a limit-identical pair, we see from (5.2) and (5.3) that $x = y \in \mathcal{A}(\omega)$ and thus $\mathcal{A}_1(\omega) \subset \mathcal{A}(\omega)$. Conversely, by the assumption (ii), that is, φ is $\mathcal{D}$-asymptotically compact in X, we can check similarly that $\mathcal{A}_1(\omega) \supset \mathcal{A}(\omega)$ also holds. Hence, $\mathcal{A} = \mathcal{A}_1$.

Next we prove $\mathcal{A}_X = \mathcal{A}_1$. Obviously, the omega-limits set $\mathcal{A}_X$ of K in X is a random attractor of φ in X, and the $\mathcal{F}$-measurability of $\mathcal{A}_X$ follows from [4]. Let now $x \in \mathcal{A}_X(\omega)$. Then by the invariance of $\mathcal{A}_X$ we can write $x = \varphi(t_n, \theta_{-t_n}\omega)x_n$ for some $t_n \to +\infty$ and $x_n \in \mathcal{A}_X(\theta_{-t_n}\omega) \subset K(\theta_{-t_n}\omega)$. Since the absorbing set $\mathcal{K}_1$ absorbs K (here we use $K \in \mathcal{D}$), there is a large time T such that

$$y_n := \varphi(T, \theta_{-T}\theta_{T-t_n}\omega)x_n \in \mathcal{K}_1(\theta_{T-t_n}\omega).$$

By the cocycle property of φ, we have

$$\varphi(t_n - T, \theta_{T-t_n}\omega)y_n = \varphi(t_n, \theta_{-t_n}\omega)x_n = x.$$

Since $t_n - T \to +\infty$, we have $x \in \mathcal{A}_1(\omega)$, which indicates that $\mathcal{A}_X(\omega) \subset \mathcal{A}_1(\omega)$. On the other hand, it is obvious that $\mathcal{A}_X(\omega) \supset \mathcal{A}_1(\omega)$ since $K(\omega) \supset \mathcal{K}_1(\omega)$. So $\mathcal{A}_1 = \mathcal{A}_X$ and thus $\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_X$. In particular, $\mathcal{A}$ is invariant and $\mathcal{F}$-measurable in X such that $\mathcal{A} \subset K$ with $K \in \mathcal{D}$.

Then we turn to check the condition (iv) in Definition 2.8. Precisely, we prove the attracting of $\mathcal{A}$ in Y, namely, $\mathcal{A}$ attracts every element $D \in \mathcal{D}$ under the topology of Y. If it is not true, then there exist $\delta > 0$ and $z_n \in D(\theta_{-t_n}\omega)$ with $t_n \to +\infty$ such that

$$d_Y\left(\varphi(t_n, \theta_{-t_n}\omega)z_n, \mathcal{A}(\omega)\right) \geq \delta \quad \text{for all } n \in \mathbb{N}. \tag{5.4}$$

Note that φ is $\mathcal{D}$-asymptotically compact in Y, there is a $z \in Y$ such that up to a subsequence (relabeled by $\varphi(t_n, \theta_{-t_n}\omega)z_n$)

$$\lim_{n\to\infty} \|\varphi(t_n, \theta_{-t_n}\omega)z_n - z\|_Y = 0. \tag{5.5}$$

Because $\mathcal{K}_1$ is an absorbing set, there is a T_0 large enough such that

$$z_n' := \varphi(T_0, \theta_{-t_n}\omega)z_n \in \mathcal{K}_1(\theta_{-(t_n-T_0)}\omega). \tag{5.6}$$

Then, by the cocycle property of φ, we see from (5.5) and (5.6) that

$$\varphi(t_n - T_0, \theta_{-(t_n-T_0)}\omega)z_n' = \varphi(t_n, \theta_{-t_n}\omega)z_n \to z \quad \text{in } Y, \text{ as } n \to \infty$$

which implies $z \in \mathcal{A}(\omega)$. This contradicts (5.4), thus $\mathcal{A}$ attracts D.

Finally, we show the uniqueness of bi-spatial random attractor. Suppose that there is another (X,Y)-random attractor $\widetilde{\mathcal{A}}$. From Definition 2.8, there is a $B \in \mathcal{D}$ such that $\widetilde{\mathcal{A}}(\omega) \subset B$. Then $\mathcal{A}$ attracts $\widetilde{\mathcal{A}}$, by the invariance of $\widetilde{\mathcal{A}}$, we deduce that

$$\mathrm{dist}_Y\Big(\widetilde{\mathcal{A}}(\omega), \mathcal{A}(\omega)\Big) = \mathrm{dist}_Y\Big(\varphi(t,\theta_{-t}\omega)\widetilde{\mathcal{A}}(\theta_{-t}\omega), \mathcal{A}(\omega)\Big) \to 0 \quad \text{as } t \to +\infty.$$

Therefore, $\mathrm{dist}_Y\Big(\widetilde{\mathcal{A}}(\omega), \mathcal{A}(\omega)\Big) = 0$, which implies that $\widetilde{\mathcal{A}}(\omega) \subset \mathcal{A}(\omega)$, and the inverse inclusion can be proved by a similarly process. Hence, $\widetilde{\mathcal{A}} = \mathcal{A}$. □

Remark 5.2. (i) As pointed out in [14], the absorption is assumed under the topology of the initial space X only. We do not need that the absorbing set is included in the terminate space. Therefore it is unnecessary to introduce a similar class as $\mathcal{D}$ in the terminate space. Besides, this work stresses that the existence of a bi-spatial random attractors is independent of continuity of RDS φ in the terminate space Y.

(ii) From the proofs of Theorem 5.1 and Lemma 2.7, it is easy to see that, for the RDS φ, the (X,Y)-random attractor (given by (5.1)) and the (X,X)-random attractor for a RDS φ are the same.

It is easy to see that the following result on existence of random attractors for a continuous RDS in space X, which can be found in [6], follows immediately from Theorem 5.1.

Corollary 5.3. *Let φ be a continuous random dynamical system on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. Suppose that $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is a closed random absorbing set $K(\omega) \in \mathcal{D}$ of φ and φ is $\mathcal{D}$-asymptotically compact in X. Then the random dynamical system φ has a unique random attractor $\mathcal{A}_X$ in X which is given by*

$$\mathcal{A}_X(\omega) = \bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}\varphi(t,\theta_{-t}\omega)K(\theta_{-t}\omega)}.$$

5.2. Existence and regularity of bi-spatial random attractor

We now study in this subsection the existence and uniqueness of bi-spatial random attractor of the random dynamical system associated with (1.1)–(1.3) by making use of Theorem 5.1 and the estimates obtained in Section 4. We will obtain for RDS φ the existence of (L^2, L^2)-random attractor, (L^2, H^1)-random attractor, $(L^2, H^{2\alpha})$-random attractor, respectively.

Theorem 5.4. *Suppose that* (1.4)–(1.6) *hold, then the random dynamical system ϕ generated by the problem* (1.1)–(1.3) *has a unique $(L^2(D), L^2(D))$-random attractor.*

Proof. Set $B(\omega) = \left\{v_0 \in L^2(D) : \|v_0\|^2_{L^2(D)} \leq R\right\}$, by Lemma 4.1 we have

$$\|\varphi(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 = \|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 \leq R_0^2.$$

Therefore, the set $\mathcal{K}(\omega) = \left\{v \in L^2(D) : \|v\|^2_{L^2(D)} \leq R_0^2\right\} \in \mathcal{D}$ is a closed random absorbing set for the random dynamical system φ.

On the other hand, using Lemma 4.3 and the Sobolev compact embedding $H^{1+\alpha}(D) \hookrightarrow L^2(D)$, we deduce that φ is $\mathcal{D}$-asymptotically compact in $L^2(D)$, then from Corollary 5.3 it follows immediately that there exists a unique $(L^2(D), L^2(D))$-random attractor for φ. Thus, by the equivalence between ϕ and φ, we obtain that the random dynamical system ϕ generated by (1.1) has a unique $(L^2(D), L^2(D))$-random attractor. □

Theorem 5.5. *Under the condition of Theorem* 5.4*, the random dynamical system* ϕ *has a unique* $(L^2(D), H^1(D))$*-random attractor.*

Proof. From Theorem 5.4, we see that φ has a closed random absorbing set $\mathcal{K}(\omega)$ ($\in \mathcal{D}$) in $L^2(D)$, and φ is $\mathcal{D}$-asymptotically compact in X. Meanwhile, by Lemmas 4.2, 4.3 and the Sobolev compact embedding $H^{1+\alpha}(D) \hookrightarrow H^1(D)$, we know that the random dynamical system φ is also $\mathcal{D}$-asymptotically compact in $H^1(D)$. Thus, by Theorem 5.1, we obtain that φ has a unique $(L^2(D), H^1(D))$-random attractor. So, ϕ has a unique $(L^2(D), H^1(D))$-random attractor as well. □

Theorem 5.6. *Under the condition of Theorem* 5.4*, the random dynamical system* ϕ *has a unique* $(L^2(D), H^{2\alpha}(D))$*-random attractor.*

Proof. From Theorem 5.4, φ has a closed random absorbing set $\mathcal{K}(\omega)$ ($\in \mathcal{D}$) in $L^2(D)$, and φ is $\mathcal{D}$-asymptotically compact in X. Moreover, by Lemma 4.4 and the Sobolev compact embedding $H^{1+\alpha}(D) \hookrightarrow H^{2\alpha}(D)$, we see that random dynamical system φ is $\mathcal{D}$-asymptotically compact in $H^{2\alpha}(D)$. Thus, by Theorem 5.1 we infer that φ has a $(L^2(D), H^{2\alpha}(D))$-random attractor. Therefore, ϕ has also a unique $(L^2(D), H^{2\alpha}(D))$-random attractor. □

Remark 5.7. Clearly, by Remark 5.2(ii), we see that the bi-spatial random attractors obtained in Theorems 5.4–5.6 are the same. We denote the random attractor by $\mathcal{A}(\omega)$ below.

6. Hausdorff dimension of the random attractor $\mathcal{A}(\omega)$

In this section, we consider the problem of the Hausdorff dimension of the random attractor $\mathcal{A}(\omega)$ obtained in Section 5, more precisely, we will show that the Hausdorff dimension of $\mathcal{A}(\omega)$ is finite. For this we impose the following additional condition on the nonlinear term f:

$$|f''(u)| \leq \beta_4 \tag{6.1}$$

for some $\beta_4 > 0$.

We first rewrite (2.3) in the abstract form:

$$\frac{dv}{dt} = L(v), \tag{6.2}$$

where

$$L(v) = -(-\Delta)^\alpha v - \lambda v + e^{-\beta z} f(e^{\beta z} v) + e^{-\beta z} g + \beta z v.$$

Let $v(t) = \varphi(t,\omega)v_0$ be the solution to problem (2.3)–(2.5) with $v_0 \in \mathcal{A}(\omega)$ and put

$$\varphi(\omega) := \varphi(1,\omega), \quad \phi(\omega) := \phi(1,\omega).$$

As one can see that, if $\varphi(\omega)$ is almost surely uniformly differentiable on $\mathcal{A}(\omega)$ with the Fréchet derivative $D\varphi(\omega, v)$ with $v \in \mathcal{A}(\omega)$, then so is $\phi(\omega)$ and $D\phi(\omega, u) = e^{\beta z(\omega)} D\varphi(\omega, v)$. Thus, instead of ϕ, we shall verify the differentiability properties for φ.

Lemma 6.1. *Assume that* (1.6) *and* (6.1) *hold. Let $\mathcal{A}(\omega)$ be the random attractor of φ. Then the mapping φ is almost surely uniformly differentiable on $\mathcal{A}(\omega)$, that is, there exists a linear operator $D\varphi(\omega, v)$ in $\mathscr{L}(L^2(D))$, such that if v and $v + h$ are in $\mathcal{A}(\omega)$, there holds*

$$\|\varphi(\omega)(v+h) - \varphi(\omega)v - D\varphi(\omega, v)h\| \leq K(\omega)\|h\|^{1+\delta},$$

where $\delta > 0$ and $K(\omega)$ is a random variable satisfying

$$K(\omega) \geq 1, \quad E(\ln K) < \infty, \quad \omega \in \Omega.$$

Moreover, for any $v_0 \in \mathcal{A}(\omega)$, $D\varphi(\omega, v_0)h = V(1)$, where V is the solution of

$$\frac{dV}{dt} = L'(v)V, \quad V(0) = h, \tag{6.3}$$

with $L'(v)V = -(-\Delta)^\alpha V - \lambda V + f'(e^{\beta z} v)V + \beta z V$ and $v(t) = e^{-\beta z(\theta_t \omega)} \phi(t,\omega) u_0$.

Proof. Let $w(t) = v_1(t) - v_2(t) - V(t)$, where $v_1(t)$, $v_2(t)$ are two solutions of equation (6.2) with $v_j(0) = v_j^0$, $j = 1, 2$, $V(t)$ satisfies system (6.3) with $h = v_1^0 - v_2^0$. Then $w(t)$ satisfies the equation

$$\frac{dw}{dt} = -(-\Delta)^\alpha w - \lambda w + \beta z w + \psi_1(t) + \psi_2(t), \quad w(0) = 0, \tag{6.4}$$

where $\psi_1(t) = e^{-\beta z}(f(u_1) - f(u_2) - f'(u)(u_1 - u_2))$ and $\psi_2(t) = f'(u)w$. Applying Taylor's formula for the function f (which has bounded second derivatives due to (6.1)) at the point u, we deduce that

$$|\psi_1| \le c_1 e^{-\beta z(\theta_t\omega)}|u_1 - u_2|^2 \le c_1 e^{\beta z(\theta_t\omega)}|v_1 - v_2|^2. \tag{6.5}$$

Now take the inner product of (6.4) with w in $L^2(D)$ to obtain

$$\frac{d}{dt}\|w\|^2 + 2\|(-\Delta)^{\alpha/2}w\|^2 + 2\lambda\|w\|^2 = 2\beta z\|w\|^2 + 2(\psi_1, w) + 2(\psi_2, w).$$

From (6.5), we infer that

$$2(\psi_1, w) \le \frac{4c_1^2 e^{2\beta z}}{\lambda}\|v_1 - v_2\|^4 + \lambda\|w\|^2.$$

And (1.6) implies that

$$2(\psi_2, w) = 2(f'(u)w, w) \le 2\beta_3\|w\|^2.$$

So,

$$\frac{d}{dt}\|w\|^2 + 2\|(-\Delta)^{\alpha/2}w\|^2 \le 2\beta z\|w\|^2 + 2\beta_3\|w\|^2 + \frac{4c_1^2 e^{2\beta z}}{\lambda}\|v_1 - v_2\|^4.$$

Then by Gronwall lemma, we get that

$$\|w(1)\|^2 \le C_1(\omega)\left(\int_0^1 e^{2\beta z(\theta_s\omega)}\|v_1(s) - v_2(s)\|^4\, ds\right), \tag{6.6}$$

where $C_1(\omega) = \frac{4c_1^2}{\lambda}e^{2\beta_3 + 2\beta M}$ with $M = \max_{0\le t\le 1} z(\theta_t\omega)$.

On the other hand, we have

$$\frac{d}{dt}(v_1 - v_2) + (-\Delta)^\alpha(v_1 - v_2) + \lambda(v_1 - v_2) = e^{-\beta z}(f(u_1) - f(u_2)) + \beta z(v_1 - v_2).$$

Taking the inner product of above equality with $v_1 - v_2$ in $L^2(D)$, we infer that

$$\begin{aligned}&\frac{d}{dt}\|v_1 - v_2\|^2 + 2\|(-\Delta)^{\alpha/2}(v_1 - v_2)\|^2 + 2\lambda\|v_1 - v_2\|^2\\ &= 2e^{-2\beta z}(f(u_1) - f(u_2), u_1 - u_2) + 2\beta z\|v_1 - v_2\|^2.\end{aligned} \tag{6.7}$$

By virtue of (1.6),

$$2e^{-2\beta z}(f(u_1) - f(u_2), u_1 - u_2) \le 2\beta_3\|v_1 - v_2\|^2. \tag{6.8}$$

Thus it follows from (6.7) and (6.8) that

$$\frac{d}{dt}\|v_1 - v_2\|^2 \le 2(\beta_3 + \beta z)\|v_1 - v_2\|^2.$$

Using Gronwall lemma again, we then obtain

$$\begin{aligned}\|v_1(t)-v_2(t)\|^2 &\le e^{2\int_0^t(\beta_3+\beta z(\theta_s\omega))\,ds}\|v_1(0)-v_2(0)\|^2\\ &\le e^{2\beta_3+2\beta M}\|v_1(0)-v_2(0)\|^2,\quad \forall\, t\in[0,1],\end{aligned}$$

which combing with (6.6) gives that

$$\|w(1)\|\le K_1(\omega)\|v_1(0)-v_2(0)\|^{1+\delta},$$

where $K_1(\omega)=\frac{2c_1}{\sqrt{\lambda}}e^{3\beta_3+4M\beta}$, $M=\max_{0\le t\le 1}z(\theta_t\omega)$ and $\delta=1$. Choose $K(\omega)=\max\{K_1(\omega),1\}$. Hence, we have $E(\ln K(\omega))<\infty$.

Therefore, φ is uniformly differentiable on $\mathcal{A}$. Furthermore, the differential of φ in $L^2(D)$ at $v_0\in\mathcal{A}$ is $D\varphi(\omega,v_0)\colon L^2(D)\to L^2(D)$ given by $D\varphi(\omega,v_0)h=V(1)$. So, ϕ is also uniformly differentiable. And, the differential in $L^2(D)$ of ϕ at u is $D\phi(\omega,u)=e^{\beta z(\omega)}D\varphi(\omega,v)$. The proof is completed. □

To prove that the Hausdorff dimension of $\mathcal{A}$ is finite, we are only remained to verify the conditions (ii) and (iii) of Proposition 2.10. In fact, from (6.3), we have

$$\begin{aligned}&\frac{d}{dt}\|V\|^2+2\|(-\Delta)^{\alpha/2}V\|^2+2\lambda\|V\|^2\\ &=2\beta z\|V\|^2+2(f'(e^{\beta z}v)V,V)\le 2\beta z\|V\|^2+2\beta_3\|V\|^2,\end{aligned}$$

which implies that

$$\|V(t)\|^2\le\|V(0)\|^2e^{2\beta_3t+2\beta\int_0^t z(\theta_s\omega)\,ds}.$$

Since $\alpha_1(D\varphi(\omega,v))$ is equal to the norm of $D\varphi(\omega,v)\in\mathscr{L}(X)$, we choose

$$\overline{\alpha_1}(\omega)=\max\left\{e^{\beta z(\omega)+\beta_3+\beta M},1\right\}.$$

Then one has

$$\alpha_1(D\phi(\omega,u))\le\overline{\alpha_1}(\omega)\quad\text{and}\quad E(\ln\overline{\alpha}_1)<\infty.$$

From [11], we see that

$$\omega_d(D\phi(\omega,u))=\sup_{\substack{h_i\in L^2(D),\|h_i\|=1\\ i=1,2,\dots,d}}\exp\left\{\beta z(\omega)+\int_0^1\operatorname{Tr}\left(L'(v(s))\circ Q_d(s)\right)\,ds\right\},$$

where $Q_d(s)$ is the orthogonal projector in $L^2(D)$ onto the space spanned by $V_1(s),V_2(s),\dots,V_d(s)$, and $V_j(s)$ is the solution of equation (6.3) with $V_j(0)=h_j$, respectively.

Let $\eta_j(s)$ $(j\in\mathbb{N})$ be an orthonormal basis of $L^2(D)$ such that

$$Q_d(s)L^2=\operatorname{span}\{\eta_1(s),\eta_2(s),\dots,\eta_d(s)\}.$$

Then, we get

$$\begin{aligned}\operatorname{Tr}\big(L'(v(s))\circ Q_d(s)\big) &= \sum_{j=1}^{\infty}\big(L'(v(s))\circ Q_d(s)\eta_j(s),\eta_j(s)\big)\\ &= \sum_{j=1}^{d}\big(L'(v(s))\eta_j(s),\eta_j(s)\big)\\ &= -\lambda\sum_{j=1}^{d}\|\eta_j(s)\|^2-\sum_{j=1}^{d}\|(-\Delta)^{\alpha/2}\eta_j(s)\|^2\\ &\quad+\beta z\sum_{j=1}^{d}\|\eta_j(s)\|^2+f'(u)\sum_{j=1}^{d}\|\eta_j(s)\|^2\\ &\le(\lambda+M\beta+\beta_3)\sum_{j=1}^{d}\|\eta_j(s)\|^2-\sum_{j=1}^{d}\|(-\Delta)^{\alpha/2}\eta_j(s)\|^2.\end{aligned}$$

By virtue of Sobolev-Lieb-Thirring inequality, it follows that

$$\sum_{j=1}^{d}\|(-\Delta)^{\alpha/2}\eta_j(s)\|^2\ge k|D|^{\alpha}d^{1+\alpha}-d,$$

where the constant $k>0$ is independent of $\{\eta_j\}_{j=1}^{d}$ and all the parameters of in the equation (2.3). So we infer that

$$\operatorname{Tr}\big(L'(v(s))\circ Q_d(s)\big)\le(\lambda+M\beta+\beta_3)d+d-k|D|^{\alpha}d^{1+\alpha}=k_1d-k_2d^{1+\alpha},$$

where $k_1=1+\lambda+M\beta+\beta_3$ and $k_2=k|D|^{\alpha}$. Denote

$$\overline{\omega}_d(\omega)=\exp\left\{\beta z(\omega)+k_1d-k_2d^{1+\alpha}\right\}$$

and choose

$$d=\left[\left(\frac{k_1}{k_2}\right)^{1/\alpha}\right]+1. \tag{6.9}$$

Note that $E(z(\omega))=0$, we have $\omega_d(D\phi(\omega,u))\le\overline{\omega}_d(\omega)$ and $E(\ln(\overline{\omega}_d))<0$.

Consequently, by Proposition 2.10, we conclude the main result in this section as follows:

Theorem 6.2. *Assume that* (1.4)–(1.6) *and* (6.1) *hold, and let $\mathcal{A}'$ be the random attractor of random dynamical system ϕ associated with the problem* (1.1)–(1.3)*. Then the Hausdorff dimension of $\mathcal{A}'$ is less than or equal to d with d given by* (6.9)*.*

Proof. Lemma 6.1 and Proposition 2.10 imply the assertion immediately. □

Acknowledgments

We would like to thank the referees greatly for the very careful review and the valuable comments to this manuscript.

References

[1] L. Arnold, *Random Dynamical Systems*, Springer Monographs in Mathematics, Spring-Verlag, Berlin, 1998.

[2] B. Baeumer, M. Kovács and M. M. Meerschaert, *Numerical solutions for fractional reaction-diffusion equations*, Comput. Math. Appl. **55** (2008), no. 10, 2212–2226.

[3] P. W. Bates, K. Lu and B. Wang, *Random attractors for stochastic reaction-diffusion equations on unbounded domains*, J. Differential Equations **246** (2009), no. 2, 845–869.

[4] Z. Brzeźniak, T. Caraballo, J. A. Langa, Y. Li, G. Łukaszewicz and J. Real, *Random attractors for stochastic 2D-Navier-Stokes equations in some unbounded domains*, J. Differential Equations **255** (2013), no. 11, 3897–3919.

[5] D. Cao, C. Sun and M. Yang, *Dynamics for a stochastic reaction-diffusion equation with additive noise*, J. Differential Equations **259** (2015), no. 3, 838–872.

[6] T. Caraballo, J. Langa and J. C. Robinson, *Stability and random attractors for a reaction-diffusion equation with multiplicative noise*, Discrete Contin. Dynam. Systems **6** (2000), no. 4, 875–892.

[7] J. W. Cholewa and T. Dłotko, *Bi-spaces global attractors in abstract parabolic equations*, in *Evolution Equations* (Warsaw, 2001) 13–26, Banach Center Publi. **60**, Polish Acad. Sci. Inst. Math., Warsaw, 2003.

[8] I. Chueshov, *Monotone Random Systems Theory and Applications*, Lecture Notes in Mathematics **1779**, Springer-Verlag, Berlin, 2002.

[9] H. Crauel, A. Debussche and F. Flandoli, *Random attractors*, J. Dynam. Differential Equations **9** (1997), no. 2, 307–341.

[10] H. Crauel and F. Flandoli, *Attractors for random dynamical systems*, Probab. Theory Related Fields **100** (1994), no. 3, 365–393.

[11] A. Debussche, *Hausdorff dimension of a random invariant set*, J. Math. Pures Appl. (9) **77** (1998), no. 10, 967–988.

[12] B. Guo and M. Zeng, *Solutions for the fractional Landau-Lifshitz equation*, J. Math. Anal. Appl. **361** (2010), no. 1, 131–138.

[13] B. Guo and G. Zhou, *Ergodicity of the stochastic fractional reaction-diffusion equation*, Nonlinear Anal. **109** (2014), 1–22.

[14] Y. Li, A. Gu and J. Li, *Existence and continuity of bi-spatial random attractors and application to stochastic semilinear Laplacian equations*, J. Differential Equations **258** (2015), no. 2, 504–534.

[15] H. Lu, P. W. Bates, S. Lü and M. Zhang, *Dynamics of the 3-D fractional complex Ginzburg-Landau equation*, J. Differential Equations **259** (2015), no. 10, 5276–5301.

[16] H. Lu, P. W. Bates, J. Xie and M. Zhang, *Asymptotic behavior of stochastic fractional power dissipative equations on* $\mathbb{R}^n$, Nonlinear Anal. **128** (2015), 176–198.

[17] C. Marinelli and M. Rökner, *Well-posedness and asymptotic behavior for stochastic reaction-diffusion equations with multiplicative Poisson noise*, Electron. J. Probab. **15** (2010), no. 49, 1528–1555.

[18] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 115–162.

[19] J. C. Robinson, *Infinite-dimensional Dynamical Systems: An introduction to dissipative parabolic PDEs and the theory of global attractors*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.

[20] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and applications*, Gordon and Breach Science, New York, 1987.

[21] I. M. Sokolov and J. Klafter, *From diffusion to anomalous diffusion: A century after Einstein's Brownian motion*, Chaos **15** (2005), no. 2, 026103, 7 pp.

[22] C. Sun and C. Zhong, *Attractors for the semilinear reaction-diffusion equation with distribution derivatives in unbounded domains*, Nonlinear Anal. **63** (2005), no. 1, 49–65.

[23] R. Temman, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Second edition, Applied Mathematical Sciences **68**, Springer-Verlag, New York, 1997.

[24] B. Wang, *Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems*, J. Differential Equations **253** (2012), no. 5, 1544–1583.

[25] W. Zhao, H^1*-random attractors for stochastic reaction-diffusion equations with additive noise*, Nonlinear Anal. **84** (2013), 61–72.

Linfang Liu and Xianlong Fu
Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, P. R. China
E-mail address: liulinfang1988@163.com, xlfu@math.ecnu.edu.cn