An Extending Result on Spectral Radius of Bipartite Graphs

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Abstract. In this paper, we study the spectral radius of bipartite graphs. Let G be a bipartite graph with e edges without isolated vertices. It was known that the spectral radius of G is at most the square root of e, and the upper bound is attained if and only if G is a complete bipartite graph. Suppose that G is not a complete bipartite graph and $(e-1, e+1)$ is not a pair of twin primes. We describe the maximal spectral radius of G. As a byproduct of our study, we obtain a spectral characterization of a pair $(e-1, e+1)$ of integers to be a pair of twin primes.

1. Introduction

Let G denote a bipartite graph with e edges without isolated vertices. The *spectral radius* of G is the largest eigenvalue of the adjacency matrix of G. It was shown in [1, Proposition 2.1] that the spectral radius $\rho(G)$ of G satisfies $\rho(G) \leq \sqrt{e}$, with equality if and only if G is a complete bipartite graph. There are several extending results of the above result, which aim to solve an analog of the Brualdi-Hoffman conjecture for nonbipartite graphs [3], proposed in [1]. These extending results are scattered in [1, 4, 11]. To illustrate another extending result, we need some notations. For $2 \leq s \leq t$, let $K_{s,t}^-$ denote the graph obtained from the complete bipartite graph $K_{s,t}$ of bipartition orders s and t by deleting an edge, and $K_{s,t}^+$ denote the graph obtained from $K_{s,t}$ by adding a new edge xy, where x is a new vertex and y is a vertex in the part of order s. Note that $K_{2,t+1}^- = K_{2,t}^+$, and $K_{s,t}^-$ and $K_{s,t}^+$ are not complete bipartite graphs. For $e \geq 2$, let $\rho(e)$ denote the maximal value $\rho(G)$ of a bipartite graph G with e edges which is not a union of a complete bipartite graph and some isolated vertices. For the case that $(e-1, e+1)$ is not a pair of *twin primes*, i.e., a pair of primes with difference two, we will describe the bipartite graph G with e edges such that $\rho(G) = \rho(e)$. Indeed we will show in Theorem 5.1 that if $e \geq 3$ and $\rho(G) = \rho(e)$ then $G \in \{K_{s',t'}^-, K_{s'',t''}^+\}$, where s' and t' (resp. s'' and t'') are chosen to minimize s subject to $2 \leq s \leq t$ and $e = st-1$ (resp. $e = st+1$). The case that $(e-1, e+1)$ is a pair of twin primes is not completely solved. Nevertheless, we find that the values of $\rho(e)$ in this case tend to be smaller than others. Indeed, this property characterizes a pair

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of twin primes. See Theorem 5.2 for the detailed description. Our results are the main tools in [12] to determine if $K_{s,t}^-$ and $K_{s,t}^+$ are determined by their eigenvalues.

2. Preliminaries

Let $D = (d_1, d_2, \ldots, d_p)$ be a sequence of nonincreasing positive integers of length p. Let G_D denote the bipartite graph with bipartition $X \cup Y$, where $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ $(q = d_1)$, and $x_i y_j$ is an edge if and only if $j \leq d_i$. Note that D is the degree sequence of the part X in the bipartition $X \cup Y$ of G_D. As $e = d_1 + d_2 + \cdots + d_p$, D is a *partition* of the number e of edges in G_D. The degree sequence $D^* = (d_1^*, d_2^*, \ldots, d_q^*)$ of the other part Y forms the *conjugate partition* of e, where $e = d_1^* + d_2^* + \cdots + d_q^*$ and $d_j^* = |\{i \mid d_i \geq j\}|$. See [2, Section 8.3] for details. The sequence D will define a *Ferrers diagram* of 1's that has p rows with d_i 1's in row i for $1 \leq i \leq p$. For example, the Ferrers diagram $F(D)$ of the sequence $D = (4, 2, 2, 1, 1)$ is illustrated in Figure 2.1. One can check that $D^* = (5, 3, 1, 1)$ in the above example.

$$
F(D) = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & & \\ 1 & 1 & & \\ 1 & & & \\ 1 & & & \end{array}
$$

Figure 2.1: The Ferrers diagram $F(D)$ of $D = (4, 2, 2, 1, 1)$.

The graph G_D is important in the study of the spectral radius of bipartite graphs with prescribed degree sequence of one part of the bipartition.

Lemma 2.1. [1, Theorem 3.1] *Let G be a bipartite graph without isolated vertices such that one part in the bipartition of G has degree sequence $D = (d_1, \ldots, d_p)$. Then $\rho(G) \leq \rho(G_D)$ with equality if and only if $G = G_D$ (up to isomorphism).*

The following lemma is used in the proof of Lemma 2.1 which may be traced back to [13].

Lemma 2.2. *Let G be a bipartite graph and $(u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)$ be a positive Perron eigenvector of the adjacency matrix of G according to the bipartition $X \cup Y$, where vertices in the part Y of G are ordered to ensure $v_1 \geq v_2 \geq \cdots \geq v_q$. For $1 \leq i < j \leq q$, if $x_k y_j$ is an edge and $x_k y_i$ is not an edge in G for some $x_k \in X$, then the new bipartite*

graph G' with the same vertex set as G obtained by deleting the edge $x_k y_j$ and adding a new edge $x_k y_i$ has spectral radius $\rho(G') \geq \rho(G)$.

A bipartite graph G is *biregular* if the degrees of vertices in the same part of its bipartition are the same constant. Let H, H' be two bipartite graphs with given ordered bipartitions $VH = X \cup Y$ and $VH' = X' \cup Y'$, where $VH \cap VH' = \phi$. The *bipartite sum* $H + H'$ of H and H' (with respect to the given ordered bipartitions) is the graph obtained from H and H' by adding an edge between x and y for each pair $(x, y) \in X \times Y' \cup X' \times Y$. Chia-an Liu and the third author [11] found upper bounds of $\rho(G)$ expressed by degree sequences of two parts of the bipartition of G.

Lemma 2.3. [11] *Let G be a bipartite graph with bipartition $X \cup Y$ of orders p and q respectively such that the part X has degree sequence $D = (d_1, \ldots, d_p)$, and the other part Y has degree sequence $D' = (d'_1, d'_2, \ldots, d'_q)$, both in nonincreasing order. For $1 \leq s \leq p$ and $1 \leq t \leq q$, let $X_{s,t} = d_s d'_t + \sum_{i=1}^{s-1}(d_i - d_s) + \sum_{j=1}^{t-1}(d'_j - d'_t)$, $Y_{s,t} = \sum_{i=1}^{s-1}(d_i - d_s) \cdot \sum_{j=1}^{t-1}(d'_j - d'_t)$. Then*

$$\rho(G) \leq \phi_{s,t} := \sqrt{\frac{X_{s,t} + \sqrt{X_{s,t}^2 - 4Y_{s,t}}}{2}}.$$

Furthermore, if G is connected then the above equality holds if and only if there exist nonnegative integers $s' < s$ and $t' < t$, and a biregular graph H of bipartition orders $p - s'$ and $q - t'$ respectively such that $G = K_{s',t'} + H$.

The idea of the proof in Lemma 2.3 is to apply Perron-Frobenius Theorem for the spectral radius to matrices that are similar to the adjacency matrix of G by diagonal matrices with variables on diagonals. Results using this powerful method are also in [5–10, 14, 15].

3. Graphs closed to $K_{p,q}$

Applying Lemma 2.3 to the graph $G = G_D$ for a given sequence $D = (d_1, d_2, \ldots, d_p)$ of nonincreasing positive integers of length p, one immediately finds that $d'_j = d^*_j$ and

$$\sum_{j=1}^{t-1}(d'_j - d'_t) = \sum_{i=d'_t+1}^{p} d_i.$$

Moreover, if s is chosen such that $d_s < d_{s-1}$ and $t = d_s + 1$, then $d'_t = s - 1$ and the corresponding Ferrers diagram $F(D)$ has a blank in the (s, t) position, so

$$X_{s,t} = d_s(s - 1) + \sum_{i=1}^{s-1}(d_i - d_s) + \sum_{i=s}^{p} d_i = e$$

and

$$(3.1) \qquad Y_{s,t} = \sum_{i=1}^{s-1}(d_i - d_s) \cdot \sum_{i=s}^{p} d_i,$$

completely expressed by D. Hence we have the following simpler form of Lemma 2.3.

Lemma 3.1. *Assume that s is chosen satisfying $d_s < d_{s-1}$ in the sequence $D = (d_1, d_2, \ldots, d_p)$ of positive integers and $e = d_1 + d_2 + \cdots + d_p$. Then*

$$\rho(G_D) \leq \sqrt{\frac{e + \sqrt{e^2 - 4\sum_{i=1}^{s-1}(d_i - d_s) \cdot \sum_{i=s}^{p} d_i}}{2}},$$

with equality if and only if D contains exactly two different values.

The following are a few special cases of G_D that satisfy the equality in Lemma 3.1.

Example 3.2. [11] Suppose that $2 \leq p \leq q$ and $K_{p,q}^e$ (resp. $^eK_{p,q}$) is the graph obtained from $K_{p,q}$ by deleting $k := pq - e$ edges incident on a common vertex in the part of order q (resp. p). Then

$$\rho(K_{p,q}^e) = \sqrt{\frac{e + \sqrt{e^2 - 4k(q-1)(p-k)}}{2}}, \quad k = pq - e < p,$$

$$\rho(^eK_{p,q}) = \sqrt{\frac{e + \sqrt{e^2 - 4k(p-1)(q-k)}}{2}}, \quad k = pq - e < q.$$

Applying Example 3.2 to the graph $K_{p,q}^- = K_{p,q}^{pq-1} = {}^{pq-1}K_{p,q}$, one immediate finds that

$$\rho(K_{p,q}^-) = \sqrt{\frac{e + \sqrt{e^2 - 4(e - (p+q) + 2)}}{2}},$$

which obtains maximum (resp. minimum) when p is minimum (resp. p is maximum) subject to the fixed number $e = pq - 1$ of edges and $2 \leq p \leq q$. Note that

$$e - (p+q) + 2 \leq e - 2\sqrt{pq} + 2 = e - 2\sqrt{e+1} + 2 < e - 1 - \sqrt{e-1}, \quad e \geq 6.$$

Hence

$$\rho(K_{p,q}^-) > \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}, \quad q \geq p \geq 3.$$

As $K_{2,2}^-$ has 3 edges, one can check that

$$(3.2) \qquad \rho(K_{2,2}^-) = \sqrt{\frac{3 + \sqrt{5}}{2}} < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}.$$

Similarly $K_{p,q}^+ = K_{p,q+1}^{pq+1}$ has spectral radius

$$\rho(K_{p,q}^+) = \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - q)}}{2}},$$

which obtains maximum (resp. minimum) when p is minimum (resp. p is maximum) subject to the fixed number $e = pq + 1$ and $2 \le p \le q$. Note that $e - 1 - q \le e - 1 - \sqrt{e - 1}$ in this case. Hence

$$\rho(K_{p,q}^+) \ge \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e - 1})}}{2}}$$

with equality if and only if $p = q = \sqrt{e - 1}$. This proves the following lemma.

Lemma 3.3. *The following* (i)–(iii) *hold.*

(i) *For all positive integers $2 \le p' \le q'$, $(p', q') \ne (2, 2)$, $2 \le p'' \le q''$ satisfying $e = p'q' - 1 = p''q'' + 1$, we have*

$$\rho(K_{p',q'}^-),\ \rho(K_{p'',q''}^+) \ge \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e - 1})}}{2}}.$$

Moreover the above equality does not hold for $\rho(K_{p',q'}^-)$, and holds for $\rho(K_{p'',q''}^+)$ if and only if $p'' = q''$.

(ii) *If $e + 1$ is not a prime and $p' \ge 2$ is the least integer such that p' divides $e + 1$ and $q' := (e + 1)/p'$ so that $e = p'q' - 1$, then for any positive integers $2 \le p \le q$ with $e = pq - 1$, we have $\rho(K_{p,q}^-) \le \rho(K_{p',q'}^-)$, with equality if and only if $(p, q) = (p', q')$.*

(iii) *If $e - 1$ is not a prime, and $p'' \ge 2$ is the least integer such that p'' divides $e - 1$ and $q'' := (e - 1)/p''$ so that $e = p''q'' + 1$, then for positive integers $2 \le p \le q$ with $e = pq + 1$, we have $\rho(K_{p,q}^+) \le \rho(K_{p'',q''}^+)$, with equality if and only if $(p, q) = (p'', q'')$.*

Note that the condition $2 \le p' \le q'$, $(p', q') \ne (2, 2)$ in (i) is from the previous condition $3 \le p' \le q'$ and $K_{2,q}^- = K_{2,q-1}^+$ for $q \ge 3$.

4. Graphs with at least two edges different from $K_{p,q}$

In this section, we consider bipartite graphs which are not complete bipartite and are not considered in Lemma 3.3(i). The following lemma is for the special case that the graph has the form $G = G_D$.

Lemma 4.1. *Let $D = (d_1, d_2, \ldots, d_p)$ be a partition of e. Suppose that G_D is not a complete bipartite graph and is not one of the graphs $K^-_{p',q'}$ or $K^+_{p'',q''}$ for any $2 \le p' \le q'$, $(p', q') \ne (2, 2)$, $2 \le p'' \le q''$ such that $e = p'q' - 1 = p''q'' + 1$. Then*

$$\rho(G_D) < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}.$$

Proof. When $e \le 3$, $G_D = K^-_{2,2}$ is the only graph satisfies the assumption above and the inequality holds by (3.2). We assume that $e \ge 4$. The assumption implies that $q = d_1 \ge 2$ and $4 \le e \le pq - 2$. Using D^* to replace D if necessary, we might assume that $2 \le p \le q$ and $q \ge 3$. Since G_D is not complete, we choose s such that $1 \le s \le p$ and $d_{s-1} > d_s$. Set $t = d_s + 1$. According to the partition $(s-1, 1, p-s)$ of rows and the partition $(t-1, 1, q-t)$ of columns, the Ferrers diagram $F(D)$ is divided into 9 blocks and the number b_{ij} of 1's in the block (i, j) for $1 \le i, j \le 3$ is shown as

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} (s-1)d_s & s-1 & \sum_{i=1}^{s-1}(d_i - d_s - 1) \\ d_s & 0 & 0 \\ \sum_{i=s+1}^{p} d_i & 0 & 0 \end{bmatrix}.$$

Note that $b_{11} = b_{12}b_{21}$ and $b_{11} + b_{12} + b_{13} + b_{21} + b_{31} = e$. Referring to Lemma 3.1 and (3.1), it suffices to show that $Y_{s,t} > e - 1 - \sqrt{e-1}$. Note that

$$Y_{s,t} = \sum_{i=1}^{s-1}(d_i - d_s) \cdot \sum_{i=s}^{p} d_i = \left(s - 1 + \sum_{i=1}^{s-1}(d_i - d_s - 1)\right)\left(d_s + \sum_{i=s+1}^{p} d_i\right)$$
$$= (b_{12} + b_{13})(b_{21} + b_{31}) = b_{11} + b_{12}b_{31} + b_{21}b_{13} + b_{13}b_{31}.$$

Note that $b_{12}b_{21} \ne 0$, and that $G \ne K^-_{p',q'}$ implies that $b_{13} \ne 0$ or $b_{31} \ne 0$. If both parts b_{13} and b_{31} are not zero then $b_{12}b_{31} \ge b_{12} + b_{31} - 1$, $b_{21}b_{13} \ge b_{21} + b_{13} - 1$, and $b_{13}b_{31} \ge 1$, so $Y_{s,t} \ge b_{11} + (b_{12} + b_{31} - 1) + (b_{21} + b_{13} - 1) + 1 = e - 1 > e - 1 - \sqrt{e-1}$. The proof is completed. The above proof holds for any s with $d_{s-1} < d_s$. We choose the least one with such property, and might assume one of the following two cases (i)–(ii).

Case (i): $b_{31} = 0$ and $b_{13} \ne 0$. Then $s = p = b_{12} + 1 \ge 2$, and $G = {}^e K_{p,q}$, where $e = pq - (q - d_p) \ge (p-1)q + 1 > (p-1)^2 + 1$. Thus

$$Y_{s,t} = b_{11} + b_{21}b_{13} \ge e - 1 - b_{12} = e - p > e - 1 - \sqrt{e-1}.$$

Case (ii): $b_{13} = 0$ and $b_{31} \ne 0$. The condition $b_{31} \ne 0$ implies that $q \ge p \ge 3$. The condition $b_{13} = 0$ implies that $t = q$ and $b_{21} = q - 1 \ge 2$. The proof is further divided into the following two cases (iia) and (iib).

Case (iia): $1 \leq b_{31} < b_{21}$. If $s < p - 1$, let $s' = s + 1$ and $t' = d_{s'} + 1$. Then $d_{s'-1} > d_{s'}$ and $d_{s'+1} \neq 0$. Let b'_{ij} be the b_{ij} corresponding to the new choice of s' and t'. Then $b'_{13}b'_{31} \neq 0$ and the proof is completed as in the beginning. Note that $s \neq p$ since $b_{31} \neq 0$. Then we may assume $s = p - 1$. This implies that $b_{31} = d_p < q - 1$ and $e = pq - 1 - q + d_p \geq p^2 - p > (p-1)^2 + 1$. Let $s' = p$ and $t' = d_p + 1$, and then

$$Y_{s',t'} = b'_{21}(b'_{12} + b'_{13}) \geq e - 1 - b'_{12} = e - p > e - 1 - \sqrt{e-1}.$$

Case (iib): $b_{31} \geq b_{21}$. If $b_{12} = 1$ then by the assumption $G \neq K^+_{p'',q''}$, there exists another $s'' > s$ such that $d_{s''} < d_{s''-1}$. Apply the above proof on $(s,t) = (s'',t'')$. Since $b''_{13} \geq 1$, we might assume $b''_{31} = 0$. Then $s'' = p$ and $e = (p-1)(q-1)+d_p+1 > (p-1)^2+1$. Hence

$$Y_{s'',t''} = b''_{21}(b''_{12} + b''_{13}) \geq e - 1 - b''_{12} = e - p > e - 1 - \sqrt{e-1}.$$

We now assume in the last situation that $b_{12} > 1$. Then

$$Y_{s,t} = b_{11} + (b_{12} - 1)b_{31} + b_{31} \geq b_{11} + b_{12} + 2b_{31} - 2 \geq e - 2 > e - 1 - \sqrt{e-1}. \qquad \square$$

We now study the general case.

Proposition 4.2. *Let G be a bipartite graph without isolated vertices which is neither a complete bipartite graph nor one of the graphs $K^-_{p',q'}$, $K^+_{p'',q''}$ for any $2 \leq p' \leq q'$, $(p',q') \neq (2,2)$, $2 \leq p'' \leq q''$, such that $e = p'q' - 1 = p''q'' + 1$ is the number of edges in G. Then*

$$\rho(G) < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}.$$

Proof. If G is not connected, then

$$\rho(G) \leq \sqrt{e-1} < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}.$$

We assume G is connected. Let G_D be the graph obtained from a degree sequence D of any part, say X, in the bipartition $X \cup Y$ of G. Then $\rho(G) \leq \rho(G_D)$ by Lemma 2.1. The proof is finished if G_D satisfies the assumption of Lemma 4.1. Let D' be the degree sequence of the other part Y in the bipartition of G. Then we might assume that $G \neq G_D$, $G \neq G_{D'}$, and G_D and $G_{D'}$ are graphs of the forms $K_{p,q}$, $K^-_{p',q'}$, or $K^+_{p'',q''}$.

For $y_i \in Y$, let $N(y_i)$ be the set of neighbors of y_i in G. Suppose for this moment that $|N(y_i)| = |N(y_j)|$ and $N(y_i) \neq N(y_j)$ for some $y_i, y_j \in Y$. Assume that y_i is before y_j in the order that makes the entries in the latter part of the positive Perron eigenvector nonincreasing. Let G'' be the bipartite graph obtained from G by moving an edge incident

on y_j but not on y_i to incident on y_i, keeping the other endpoint of this edge unchanged. Let D'' be the new degree sequence on the part Y of the new bipartite graph G''. Then $\rho(G) \leq \rho(G'') \leq \rho(G_{D''})$, where the first inequality is obtained from Lemma 2.2. We will show that $G_{D''}$ is not of the form $K_{p,q}$, $K^-_{p',q'}$, or $K^+_{p'',q''}$. Thus the proof follows from Lemma 4.1. Suppose $G_{D''}$ is of the form $K_{p,q}$, $K^-_{p',q'}$, and $K^+_{p'',q''}$. Note that the elements in the degree sequence of any part of $K_{p,q}$, $K^-_{p',q'}$, or $K^+_{p'',q''}$ is of the form $k, \ldots, k, \ell$, where ℓ could be $1, k-1, k, k+1$, for some positive integer k. Noticing that D'' is obtained from D' by replacing two given equal values a by $a-1$ and $a+1$. If $a-1 > 1$, then the difference between $a+1$ and $a-1$ is two, a contradiction. If $a-1 = 1$, then $G_{D''}$ must be $K^+_{3,q-1}$ and $D' = (3, \ldots, 3, 2, 2)$. So $G_{D'}$ is not a graph of the form $K_{p,q}$, $K^-_{p',q'}$, or $K^+_{p'',q''}$, a contradiction. Hence we might assume that if $|N(y_i)| = |N(y_j)|$ then $N(y_i) = N(y_j)$ for all $y_i, y_j \in Y$. Reordering the vertices in Y such that the former has larger degree and then doing the same thing for X, we find indeed $G = G_D = G_{D'}$ since G is connected, a contradiction. $\qquad\square$

We provide two applications of Proposition 4.2.

Corollary 4.3. *Let G be a bipartite graph with e edges without isolated vertices. Suppose that G is not a complete bipartite graph, and $(e-1, e+1)$ is a pair of twin primes. Then*

$$\rho(G) < \sqrt{\frac{e + \sqrt{e^2 - 4(e-1-\sqrt{e-1})}}{2}}.$$

Proof. If $(e-1, e+1)$ is a pair of primes then there is no way to express G as a graph of the forms $K^-_{p',q'}$ or $K^+_{p'',q''}$. The proof follows from Proposition 4.2. $\qquad\square$

Corollary 4.4. *Let G be a bipartite graph without isolated vertices which is not one of the graphs $K_{p,q}$, $K^-_{p',q'}$, $K^+_{p'',q''}$ for any $1 \leq p \leq q$, $2 \leq p' \leq q'$, $2 \leq p'' \leq q''$ such that $e = pq = p'q' - 1 = p''q'' + 1$ is the number of edges in G. Assume that $e = st + 1$ (resp. $e = st - 1$) for $2 \leq s \leq t$. Then*

$$\rho(G) < \rho(K^+_{s,t}) \qquad (\text{resp. } \rho(G) < \rho(K^-_{s,t})).$$

Proof. If $s = t = 2$ and $e = st - 1 = 3$ then either $G = 3K_2$ the disjoint union of three edges or $G = K_{1,2} \cup K_2$ the disjoint of a path of order 3 and an edge. One can easily check that $\rho(G) < \rho(K^-_{2,2})$. The remaining cases are from Proposition 4.2 and Lemma 3.3(i) and noticing that $K^-_{2,t+1} = K^+_{2,t}$ for $t \geq 2$. $\qquad\square$

It is worth mentioning that the result $\rho(G) < \rho(K^-_{s,t})$ in Corollary 4.4 had also been proven in [1, Theorem 8.1] under more assumptions.

5. Main theorems

For $e \geq 2$, recall that $\rho(e)$ is the maximal value $\rho(G)$ of a bipartite graph G with e edges which is not a union of a complete bipartite graph and some isolated vertices. Note that

$$\rho(2) = \rho(2K_2) = 1 \quad \text{and} \quad \rho(3) = \rho(K_{2,2}^-) = \sqrt{\frac{3 + \sqrt{5}}{2}}.$$

Two theorems about $\rho(e)$ are given in this section.

Theorem 5.1. *Let G be a bipartite graph with $e \geq 3$ edges without isolated vertices such that $\rho(G) = \rho(e)$. Then the following (i)–(iv) hold.*

  (i) *If e is odd then $G = K_{2,q}^-$, where $q = (e+1)/2$.*

  (ii) *If e is even, $e-1$ is a prime and $e+1$ is not a prime, then $G = K_{p',q'}^-$, where $p' \geq 3$ is the least integer that divides $e+1$ and $q' = (e+1)/p'$.*

  (iii) *If e is even, $e-1$ is not a prime and $e+1$ is a prime, then $G = K_{p'',q''}^+$, where $p'' \geq 3$ is the least integer that divides $e-1$ and $q'' = (e-1)/p''$.*

  (iv) *If e is even and neither $e-1$ nor $e+1$ is a prime, then $G \in \{K_{p',q'}^-, K_{p'',q''}^+\}$, where p', q' are as in (ii) and p'', q'' are as in (iii).*

Proof. By the definition of $\rho(e)$, G is not a complete graph. From Lemma 3.3(i) and Proposition 4.2, we only need to compare the spectral radii $\rho(K_{p,q}^-)$ and $\rho(K_{p,q}^+)$ for all possible positive integers $2 \leq p \leq q$ that keep the graphs having e edges. This has been done in Lemma 3.3(ii)–(iii). $\square$

Theorem 5.2. *Let $e \geq 4$ be an integer. Then $(e-1, e+1)$ is a pair of twin primes if and only if*

$$\rho(e) < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}.$$

Proof. The necessity is by Corollary 4.3. The sufficiency is from Theorem 5.1 and Lemma 3.3(i). $\square$

    Due to Yitang Zhang's recent result [16], the conjecture if there are infinite pairs of twin primes obtains much attention. Theorem 5.2 provides a spectral description of the pairs of twin primes.

6. Numerical comparisons

In the case (iv) of Theorem 5.1, the two graphs $K^-_{p',q'}$ and $K^+_{p'',q''}$ are candidates to be extremal graph. For even $e \leq 100$ and neither $e-1$ nor $e+1$ is a prime, we shall determine which graph has larger spectral radius. The symbol $-$ in the last column of the following table means that $K^-_{p',q'}$ wins, i.e., $\rho(K^-_{p',q'}) > \rho(K^+_{p'',q''})$ and $+$ otherwise.

e	$\rho(K^-_{p',q'})$	$\rho(K^+_{p'',q''})$	winner
26	$\sqrt{13 + 3\sqrt{17}}$	$\sqrt{13 + \sqrt{149}}$	$-$
34	$\sqrt{17 + \sqrt{265}}$	$\sqrt{17 + \sqrt{267}}$	$+$
50	$\sqrt{25 + \sqrt{593}}$	$\sqrt{25 + \sqrt{583}}$	$-$
56	$\sqrt{28 + \sqrt{748}}$	$\sqrt{28 + \sqrt{740}}$	$-$
64	$\sqrt{32 + \sqrt{976}}$	$\sqrt{32 + \sqrt{982}}$	$+$
76	$\sqrt{38 + \sqrt{1384}}$	$\sqrt{38 + \sqrt{1394}}$	$+$
86	$\sqrt{43 + \sqrt{1813}}$	$\sqrt{43 + \sqrt{1781}}$	$-$
92	$\sqrt{46 + \sqrt{2096}}$	$\sqrt{46 + \sqrt{2078}}$	$-$
94	$\sqrt{47 + \sqrt{2137}}$	$\sqrt{47 + \sqrt{2147}}$	$+$

Table 6.1: Comparisons of $\rho(K^-_{p',q'})$ and $\rho(K^+_{p'',q''})$ in case (iv) of Theorem 5.1 for $e \leq 100$.

Acknowledgments

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