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Pointwise Multipliers on BMO Spaces with Non-doubling Measures

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Abstract. Let μ be a non-negative Radon measure satisfying the polynomial growth condition. In this paper, the authors characterize the set of pointwise multipliers on a BMO type space $\mathrm{RBMO}(\mu)$ introduced by Tolsa.

1. Introduction

The characterization of pointwise multipliers on the space $\mathrm{BMO}_\phi(\mathbb{T}^d)$ was first studied by Janson [9], where $\mathbb{T}^d$ is the d-dimensional torus and $\mathrm{BMO}_\phi(\mathbb{T}^d)$ is the space of bounded mean oscillation defined by a positive non-decreasing function ϕ. Independently, when $d = 1$, Stegenga [21] established a characterization of pointwise multipliers on the space $\mathrm{BMO}(\mathbb{T})$ to study the boundedness of Toeplitz operators on the Hardy space $H^1(\mathbb{T})$. This characterization was further generalized to $\mathbb{R}^d$ in [19] and the spaces of homogeneous type in [20]. For the characterization of pointwise multipliers on other function spaces, such as weighted BMO spaces, we refer to [1, 11–18, 24] and their references. We remark that the pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^d)$ was used by Lerner [10] to study the class $\mathcal{P}(\mathbb{R}^d)$ of functions for which the Hardy-Littlewood maximal operator is bounded on the Lebesgue spaces $L^{p(\cdot)}$ with variable exponent, and positively solve a conjecture of Deining [3] saying that there are discontinuous functions belonging to $\mathcal{P}(\mathbb{R}^d)$.

On the other hand, let n be a real number such that $0 < n \le d$ and μ a non-negative Radon measure on $\mathbb{R}^d$ satisfying the following *polynomial growth condition* that there exists a positive real number C_0 such that

$$\mu(B(x,r)) \le C_0 r^n, \tag{1.1}$$

where $B(x,r)$ is the *ball centered at* $x \in \mathbb{R}^d$ *and of radius* $r > 0$. In [22], Tolsa introduced and studied the BMO-type space, $\mathrm{RBMO}(\mu)$ (the *space of regularized bounded mean oscillation*), showing that $\mathrm{RBMO}(\mu)$ satisfies a John-Nirenberg inequality. In [6], a John-Strömberg maximal characterization of $\mathrm{RBMO}(\mu)$ was established. For more references on $\mathrm{RBMO}(\mu)$, see, for example, [5, 7, 23, 25, 26] and their references. The purpose of

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this paper is to study the characterization of pointwise multipliers on RBMO(μ), where a μ-measurable function g is a pointwise multiplier on RBMO(μ), if fg is in RBMO(μ) for all $f \in$ RBMO(μ). To consider pointwise multipliers on RBMO(μ), we need to regard RBMO(μ) as a space of functions modulo μ-null-functions, since pointwise multipliers are not well defined on a space modulo constant functions. In this paper we introduce a norm $\|\cdot\|_{\mathrm{RBMO}^\natural(\mu)}$ on RBMO(μ) as a function space modulo μ-null-functions like the preceding researches.

We first recall some notion and notations. For a ball $B = B(x,r)$ and $\rho \in (0,\infty)$, denote $B(x,\rho r)$ by ρB. We also denote by x_B and r_B the center and the radius of B, respectively. For a subset $A \subset \mathbb{R}^d$, we denote by χ_A the characteristic function of A.

Definition 1.1. Let $\alpha, \beta \in (1,\infty)$. A ball $B \subset \mathbb{R}^d$ is (α,β)-*doubling* if

$$\mu(\alpha B) \le \beta\mu(B).$$

It was proved by Tolsa in [22] that if $\beta > \alpha^n$ with n as in (1.1), then for every ball $B \subset \mathbb{R}^d$, there exists some $j \in \mathbb{Z}_+ = \{0,1,2,\dots\}$ such that $\alpha^j B$ is (α,β)-doubling. Moreover, let $\beta > \alpha^d$. Tolsa [22] also showed that for μ-almost every $x \in \mathbb{R}^d$, there exist arbitrarily small (α,β)-doubling balls centered at x. Furthermore, the radius of these balls may be chosen to be of the form $\alpha^{-j}r$ for $j \in \mathbb{N} = \{1,2,\dots\}$ and any preassigned number $r \in (0,\infty)$. Throughout this paper, for any $\alpha \in (1,\infty)$ and ball B, $\widetilde{B}^\alpha$ denotes *the smallest* (α,β_α)-*doubling* ball of the form $\alpha^j B$ with $j \in \mathbb{Z}_+$, where $\beta_\alpha > \alpha^d$. When $\alpha = 5$, we write $\widetilde{B}^\alpha$ simply by $\widetilde{B}$.

Definition 1.2. Let $\mathcal{B}_{\text{double}}$ be the *set of all* $(5,\beta_5)$-*doubling balls centered in points of* $\operatorname{supp}(\mu)$.

The following coefficient $K_{B,S}$ was first introduced by Tolsa [22], where B and S are cubes in $\mathbb{R}^d$ therein; see also [2,8].

Definition 1.3. For any pair of two balls $B = B(x_B,r_B)$ and $S = B(x_S,r_S)$ satisfying $B \subset S$, let $N_{B,S}$ be the *smallest integer* k satisfying $2^k r_B \ge r_S$, and let

$$K_{B,S} = 1 + \sum_{k=1}^{N_{B,S}} \frac{\mu(2^k B)}{(2^k r_B)^n}.$$

We recall the definition of the space RBMO(μ) which was introduced by Tolsa in [22]; see also [7]. For a ball B and a function $f \in L^1_{\text{loc}}(\mu)$, let

$$(1.2)\quad m_B(f) = \frac{1}{\mu(B)}\int_B f(x)\,d\mu(x) \quad\text{and}\quad \mathrm{MO}(f,B) = \frac{1}{\mu(B)}\int_B |f(x) - m_B(f)|\,d\mu(x).$$

Definition 1.4. Let $\mathrm{RBMO}(\mu)$ be the set of all functions f such that $\|f\|_{\mathrm{RBMO}(\mu)} < \infty$, where

$$\|f\|_{\mathrm{RBMO}(\mu)} = \sup_{B \in \mathcal{B}_{\mathrm{double}}} \mathrm{MO}(f, B) + \sup_{\substack{B,S \in \mathcal{B}_{\mathrm{double}} \\ B \subset S}} \frac{|m_B(f) - m_S(f)|}{K_{B,S}}.$$

In this paper, we assume that there exists a point $x_0 \in \mathbb{R}^d$ such that $\mu(B(x_0, 1)) > 0$. Without loss of generality, we always *take* $x_0 = 0$*, the origin.* Moreover, by (1.1), we may further *assume that* $B(0,1)$ *is* $(5, \beta_5)$*-doubling.* In fact, we may assume that $\beta_5 > \mu(B(0,5))/\mu(B(0,1))$. Based on this assumption, we define the space $\mathrm{RBMO}^\natural(\mu)$ as follows.

Definition 1.5. For a function $f \in \mathrm{RBMO}(\mu)$, let

$$\|f\|_{\mathrm{RBMO}^\natural(\mu)} = \|f\|_{\mathrm{RBMO}(\mu)} + |m_{B(0,1)}(f)|,$$

and denote by $\mathrm{RBMO}^\natural(\mu)$ the space $\mathrm{RBMO}(\mu)$ equipped with the norm $\|\cdot\|_{\mathrm{RBMO}^\natural(\mu)}$.

Then $\mathrm{RBMO}^\natural(\mu)$ is a Banach space modulo μ-null-functions. Moreover, $\mathrm{RBMO}^\natural(\mu)$ has the following property: For a sequence $\{f_j\}$ in $\mathrm{RBMO}^\natural(\mu)$,

$$\begin{aligned} &\lim_{j \to \infty} \|f_j - f\|_{\mathrm{RBMO}^\natural(\mu)} = 0 \\ &\Longrightarrow f_j \chi_{B(0,R)} \to f \chi_{B(0,R)} \text{ in measure } \mu \text{ for each } R > 0. \end{aligned} \tag{1.3}$$

Since $\mathrm{RBMO}^\natural(\mu)$ is a Banach space, using the property (1.3) and the closed graph theorem, we see that every pointwise multiplier on $\mathrm{RBMO}^\natural(\mu)$ is a bounded operator.

To show the property (1.3), take a sequence $\{R_m\}$ of positive numbers such that $B(0, R_m) \in \mathcal{B}_{\mathrm{double}}$ and $R_m \to \infty$ as $m \to \infty$. Then, by the definition of the norm, we have

$$\begin{aligned} \left|m_{B(0,R_m)}(f)\right| &\le \left|m_{B(0,1)}(f) - m_{B(0,R_m)}(f)\right| + \left|m_{B(0,1)}(f)\right| \\ &\le K_{B(0,1),B(0,R_m)} \|f\|_{\mathrm{RBMO}(\mu)} + \left|m_{B(0,1)}(f)\right| \\ &\le K_{B(0,1),B(0,R_m)} \|f\|_{\mathrm{RBMO}^\natural(\mu)}, \end{aligned}$$

and

$$\begin{aligned} &\int_{B(0,R_m)} |f(x)| \, d\mu(x) \\ &\le \int_{B(0,R_m)} \left|f(x) - m_{B(0,R_m)}(f)\right| \, d\mu(x) + \left|m_{B(0,R_m)}(f)\right| \mu(B(0,R_m)) \\ &\le \left[1 + K_{B(0,1),B(0,R_m)}\right] \|f\|_{\mathrm{RBMO}^\natural(\mu)} \mu(B(0,R_m)). \end{aligned}$$

This shows the property (1.3).

To characterize pointwise multipliers on $\mathrm{RBMO}^\natural(\mu)$, we also define $\mathrm{RBMO}_\varphi(\mu)$ for a variable growth function $\varphi\colon \mathbb{R}^d \times (0,\infty) \to (0,\infty)$. For a ball B, we denote $\varphi(x_B, r_B)$ by $\varphi(B)$.

Definition 1.6. For $\varphi\colon \mathbb{R}^d \times (0,\infty) \to (0,\infty)$, let $\mathrm{RBMO}_\varphi(\mu)$ be the set of all $f \in L^1_{\mathrm{loc}}(\mu)$ such that $\|f\|_{\mathrm{RBMO}_\varphi(\mu)} < \infty$, where

$$\|f\|_{\mathrm{RBMO}_\varphi(\mu)} = \sup_{B\in\mathcal{B}_{\mathrm{double}}} \frac{\mathrm{MO}(f,B)}{\varphi(B)} + \sup_{\substack{B,S\in\mathcal{B}_{\mathrm{double}}\\ B\subset S}} \frac{|m_B(f) - m_S(f)|}{\varphi(B)K_{B,S}}.$$

For any $a \in \mathbb{R}^d$ and $r \in (0,\infty)$, let

$$\Phi^*(a,r) = 1 + \int_1^{\max(2,|a|,r)} \frac{\mu(B(0,t))}{t^{n+1}}\,dt \tag{1.4}$$

and

$$\Phi^{**}(a,r) = 1 + \int_r^{\max(2,|a|,r)} \frac{\mu(B(a,t))}{t^{n+1}}\,dt. \tag{1.5}$$

The main result of this paper is as follows.

Theorem 1.7. *A function g is a pointwise multiplier on* $\mathrm{RBMO}^\natural(\mu)$ *if and only if* $g \in \mathrm{RBMO}_\varphi(\mu) \cap L^\infty(\mu)$, *where* $\varphi = 1/(\Phi^* + \Phi^{**})$. *Moreover, there exists a positive constant* $C \geq 1$ *such that*

$$\|g\|_{\mathrm{OP}}/C \leq \|g\|_{\mathrm{RBMO}_\varphi(\mu)} + \|g\|_{L^\infty(\mu)} \leq C\|g\|_{\mathrm{OP}},$$

where $\|g\|_{\mathrm{OP}}$ *is the operator norm of pointwise multiplier* g *on* $\mathrm{RBMO}^\natural(\mu)$.

The organization of this paper is as follows. In Section 2 we give an example of functions in $\mathrm{RBMO}(\mu)$. We first show an estimate on the measure $\mu(B)$ for any ball B, by which we construct an example of functions in $\mathrm{RBMO}(\mu)$ (see Proposition 2.5 below). It is well known that a function f belongs to $\mathrm{RBMO}(\mu)$ if and only if for each ball B, the mean value $m_B(f)$ in the definition of $\mathrm{RBMO}(\mu)$ can be replaced by a number f_B (see, for example, [22, Lemma 2.8] or Lemma 3.4 below). In Proposition 2.5, for any $a \in \mathbb{R}^d$, we define a function f_a and show that $f_a \in \mathrm{RBMO}(\mu)$ by giving the suitable number $(f_a)_B$ for each ball B. This example is the logarithmic function in case that μ is the usual Lebesgue measure.

In Section 3 we establish three lemmas before showing Theorem 1.7 in Section 4. Among these lemmas, we apply Proposition 2.5 and show that for each ball $B(a,r)$, there exists a positive function $f \in \mathrm{RBMO}^\natural(\mu)$ which has uniform upper bound of norm $\|f\|_{\mathrm{RBMO}^\natural(\mu)}$ for a and satisfies the property that, for some positive constant C, $f(x) \geq$

$C(\Phi^*(a,r) + \Phi^{**}(a,r))$ if $r < 1$ and $x \in B(a,r)$, and that $m_{B(a,r)}(f) \geq C(\Phi^*(a,r) + \Phi^{**}(a,r))$ if $B(a,r) \in \mathcal{B}_{\text{double}}$ (see Lemma 3.4 below). Such a property is obvious on a metric measure space $(\mathcal{X}, d, \mu)$ when μ satisfies the doubling property and reverse doubling property (see, [11, Lemma 2.2] or [20, Lemma 3.3 and Remark 3.3], for example), while in the current setting where the measure doubling property and reverse doubling property is not assumed uniformly for all balls, it is more complicated to see this fact. Moreover, the conclusion of Theorem 1.7 is still unknown for general Campanato spaces.

Finally, we make some conventions on notation. Throughout the whole paper, C stands for a *positive constant* which is independent of choices of the main parameters, but it may vary from line to line. *Constants with subscripts*, such as C_0 and C_1, do not change in different occurrences. The symbol $f \lesssim g$ or $g \gtrsim f$ means that there exists a positive constant C independent of f and g satisfying that $f \leq Cg$, and $f \sim g$ means that $f \lesssim g \lesssim f$.

2. An example of RBMO(μ) functions

In this section, we give examples of functions in RBMO(μ) on $\mathbb{R}^d$ with a Radon measure μ as in (1.1). To this end, we begin with the following lemma.

Lemma 2.1. *Let μ be as in* (1.1)*. Then for any $a \in \mathbb{R}^d$ and $r \in (0,\infty)$,*

$$\int_0^r \frac{[\mu(B(a,t))]^2}{t^{n+1}}\,dt \leq \frac{2C_0}{n}\mu(B(a,r)).$$

Proof. We use the Riemann-Stieltjes integral. Take $\delta \in (0,r)$ and let $f(t) = -t^{-n}/n$ and $g(t) = \mu(B(a,t))^2$. Then f is continuous and g is increasing on $[\delta, r]$. Hence f is Riemann-Stieltjes integrable with respect to g. Since both f and g are increasing on $[\delta, r]$, by the integration by parts for the Riemann-Stieltjes integral, we have that g is Riemann-Stieltjes integrable with respect to f and

$$\int_\delta^r g(t)\,df(t) = f(r)g(r) - f(\delta)g(\delta) - \int_\delta^r f(t)\,dg(t).$$

On the other hand, since g is Riemann integrable and f is increasing and continuously differentiable on $[\delta, r]$, we have that gf' is Riemann integrable and

$$\int_\delta^r g(t)\,df(t) = \int_\delta^r g(t)f'(t)\,dt.$$

That is

$$\begin{aligned}\int_\delta^r \frac{[\mu(B(a,t))]^2}{t^{n+1}}\,dt &= -\frac{1}{n}r^{-n}\mu(B(a,r))^2 + \frac{1}{n}\delta^{-n}\mu(B(a,\delta))^2 + \frac{1}{n}\int_\delta^r t^{-n}\,d(\mu(B(a,t))^2) \\ &\leq \frac{C_0^2}{n}\delta^n + \frac{1}{n}\int_\delta^r t^{-n}\,d(\mu(B(a,t))^2).\end{aligned}$$

Next we estimate the integral $\int_\delta^r t^{-n}\, d(\mu(B(a,t))^2)$. For its Riemann-Stieltjes sum

$$\sum_{j=1}^m t_j^{-n} \left[\mu(B(a,t_j))^2 - \mu(B(a,t_{j-1}))^2\right], \quad \delta = t_0 < t_1 < \cdots < t_m = r,$$

using the estimate

$$\begin{aligned}
&\left[\mu(B(a,t_j))^2 - \mu(B(a,t_{j-1}))^2\right] \\
&= \left[\mu(B(a,t_j)) + \mu(B(a,t_{j-1}))\right]\left[\mu(B(a,t_j)) - \mu(B(a,t_{j-1}))\right] \\
&\le 2C_0 t_j^n \left[\mu(B(a,t_j)) - \mu(B(a,t_{j-1}))\right],
\end{aligned}$$

we have

$$\begin{aligned}
\sum_{j=1}^m t_j^{-n} \left[\mu(B(a,t_j))^2 - \mu(B(a,t_{j-1}))^2\right] &\le 2C_0 \sum_{j=1}^m \left[\mu(B(a,t_j)) - \mu(B(a,t_{j-1}))\right] \\
&= 2C_0 \left[\mu(B(a,r)) - \mu(B(a,\delta))\right].
\end{aligned}$$

This shows that

$$\int_\delta^r t^{-n}\, d(\mu(B(a,t))^2) \le 2C_0 \left[\mu(B(a,r)) - \mu(B(a,\delta))\right].$$

From the above calculation we have

$$\int_\delta^r \frac{[\mu(B(a,t))]^2}{t^{n+1}}\, dt \le \frac{C_0^2}{n}\delta^n + \frac{2C_0}{n}\mu(B(a,r)).$$

Letting $\delta \to 0$, we have the conclusion. □

As an easy corollary of Lemma 2.1, we obtain the following lemma.

Lemma 2.2. *For any ball $B = B(a,r)$,*

$$\frac{1}{\mu(B)} \int_B \int_{|a-x|}^r \frac{\mu(B(a,t))}{t^{n+1}}\, dt\, d\mu(x) \le \frac{2C_0}{n}.$$

Proof. Let χ be the characteristic function of the set $\{(x,t) \in \mathbb{R}^d \times \mathbb{R}_+ : |a-x| < t < r\}$. By Lemma 2.1, we have

$$\begin{aligned}
\int_B \int_{|a-x|}^r \frac{\mu(B(a,t))}{t^{n+1}}\, dt\, d\mu(x) &= \int_B \int_0^r \chi(x,t) \frac{\mu(B(a,t))}{t^{n+1}}\, dt\, d\mu(x) \\
&\le \int_0^r \frac{[\mu(B(a,t))]^2}{t^{n+1}}\, dt \le \frac{2C_0}{n}\mu(B).
\end{aligned}$$

□

We now recall an equivalent characterization of $\mathrm{RBMO}(\mu)$ in [22]

Lemma 2.3. [22] *A function f belongs to* $\mathrm{RBMO}(\mu)$ *if and only if the following statement holds: For $\rho \in (1,\infty)$, there exist a positive constant C and a collection of complex numbers $\{f_B\}_B$ (i.e., for each ball B, there exists f_B) such that*

$$\frac{1}{\mu(\rho B)}\int_B |f(y)-f_B|\,d\mu(y) \le C$$

and that, for any pair of balls B and S satisfying $B\subset S$,

$$|f_B - f_S| \le C K_{B,S}.$$

Moreover, the minimal constant C as above is comparable to $\|f\|_{\mathrm{RBMO}(\mu)}$.

Remark 2.4. By Lemma 2.3, we see that the space $\mathrm{RBMO}(\mu)$ is independent of the choice of $\rho \in (1,\infty)$. Thus, for the sake of simplicity, in what follows, we may choose $\rho = 5$ when we apply Lemma 2.3.

For any $a \in \mathbb{R}^d$ and $r \in (0,\infty)$, let

$$\Phi(a,r) = \int_r^1 \frac{\mu(B(a,t))}{t^{n+1}}\,dt. \tag{2.1}$$

Now we state our main result in this section.

Proposition 2.5. *For $a \in \mathbb{R}^d$, let*

$$f_a(x) = \Phi(a,|a-x|) = \int_{|x-a|}^1 \frac{\mu(B(a,t))}{t^{n+1}}\,dt.$$

Then $f_a \in \mathrm{RBMO}(\mu)$ and there exists a positive constant C, independent of a, such that

$$\|f_a\|_{\mathrm{RBMO}(\mu)} \le C.$$

Proof. We show Proposition 2.5 by applying Lemma 2.3 with $\rho = 5$ therein. For any ball $B = B(x_B, r_B)$, let Φ be as in (2.1) and

$$(f_a)_B = \begin{cases} \Phi(a,3r_B) & \text{if } |a-x_B| < 2r_B, \\ \Phi(a,|a-x_B|) & \text{otherwise.} \end{cases}$$

We first show that,

$$\int_B |f_a(x) - (f_a)_B|\,d\mu(x) \lesssim \mu(5B). \tag{2.2}$$

We consider the following two cases.

Case 1: $|a-x_B|<2r_B$. In this case, $B\subset B(a,3r_B)\subset 5B$. From this and Lemma 2.2, we deduce that

$$\begin{aligned}\frac{1}{\mu(5B)}\int_B|f_a(x)-(f_a)_B|\,d\mu(x)&=\frac{1}{\mu(5B)}\int_B\left|f_a(x)-\int_{3r_B}^1\frac{\mu(B(a,t))}{t^{n+1}}\,dt\right|\,d\mu(x)\\&\le\frac{1}{\mu(5B)}\int_{B(a,3r_B)}\left(\int_{|x-a|}^{3r_B}\frac{\mu(B(a,t))}{t^{n+1}}\,dt\right)\,d\mu(x)\\&\lesssim\frac{\mu(B(a,3r_B))}{\mu(5B)}\lesssim 1.\end{aligned}$$

Case 2: $|a-x_B|\ge 2r_B$. In this case, if $x\in B$, then $\frac{1}{2}|a-x_B|\le|x-a|\le 2|a-x_B|$. By this fact, the definitions of f_a and $(f_a)_B$ and (1.1), we conclude that

$$\frac{1}{\mu(5B)}\int_B|f_a(x)-(f_a)_B|\,d\mu(x)\le\frac{1}{\mu(B)}\int_B\left|\int_{|x-a|}^{|a-x_B|}\frac{\mu(B(a,t))}{t^{n+1}}\,dt\right|\,d\mu(x)\lesssim 1.$$

Combining these two cases we see that (2.2) holds.

Next we show that, for any pair of balls B and S satisfying $B\subset S$,

$$|(f_a)_B-(f_a)_S|\lesssim K_{B,S}. \tag{2.3}$$

To show (2.3), we first note that

$$\int_{2r_B}^{4r_S}\frac{\mu(B(x_B,2t))}{t^{n+1}}\,dt\lesssim K_{B,S}. \tag{2.4}$$

Actually, using (1.1), we have

$$\begin{aligned}\int_{2r_B}^{4r_S}\frac{\mu(B(x_B,2t))}{t^{n+1}}\,dt&\le\sum_{k=1}^{1+N_{B,S}}\int_{2^kr_B}^{2^{k+1}r_B}\frac{\mu(B(x_B,2t))}{t^{n+1}}\,dt\\&\le\sum_{k=1}^{1+N_{B,S}}\frac{\mu(2^{k+2}B)}{(2^kr_B)^n}\int_{2^kr_B}^{2^{k+1}r_B}\frac{dt}{t}\lesssim K_{B,S}.\end{aligned}$$

Now, we consider the following four cases.

Case (i): $|a-x_B|<2r_B$ and $|a-x_S|<2r_S$. In this case we have

$$|(f_a)_B-(f_a)_S|=|\Phi(a,3r_B)-\Phi(a,3r_S)|\le\int_{2r_B}^{3r_S}\frac{\mu(B(a,t))}{t^{n+1}}\,dt.$$

From $|a-x_B|<2r_B$ and $t\ge 2r_B$, it follows that $B(a,t)\subset B(x_B,2t)$. Then, using (2.4), we have (2.3).

Case (ii): $|a-x_B|<2r_B$ and $|a-x_S|\ge 2r_S$. In this case we have

$$2r_B\le 2r_S\le|a-x_S|\le|a-x_B|+|x_B-x_S|<3r_S$$

and

$$|(f_a)_B - (f_a)_S| = |\Phi(a, 3r_B) - \Phi(a, |a - x_S|)| \le \int_{2r_B}^{3r_S} \frac{\mu(B(a,t))}{t^{n+1}}\,dt.$$

By the same way as Case (i) we have (2.3).

Case (iii): $|a - x_B| \ge 2r_B$ and $|a - x_S| < 2r_S$. In this case, we have $|a - x_B| < 3r_S$ and

$$|(f_a)_B - (f_a)_S| = |\Phi(a, |a - x_B|) - \Phi(a, 3r_S)| = \int_{|a-x_B|}^{3r_S} \frac{\mu(B(a,t))}{t^{n+1}}\,dt.$$

From $t \ge |a - x_B|$ it follows that $B(a,t) \subset B(x_B, 2t)$. Then, using (2.4), we have

$$|(f_a)_B - (f_a)_S| \le \int_{|a-x_B|}^{3r_S} \frac{\mu(B(x_B,2t))}{t^{n+1}}\,dt \le \int_{2r_B}^{3r_S} \frac{\mu(B(x_B,2t))}{t^{n+1}}\,dt \lesssim K_{B,S}.$$

Case (iv): $|a - x_B| \ge 2r_S$ and $|a - x_S| \ge 2r_S$. If $|a - x_B| < 3r_S$, then $|a - x_S| < 4r_S$ and

$$|(f_a)_B - (f_a)_S| = |\Phi(a, |a - x_B|) - \Phi(a, |a - x_S|)| \le \int_{2r_S}^{4r_S} \frac{\mu(B(a,t))}{t^{n+1}}\,dt \lesssim 1.$$

This implies (2.3). If $|a - x_B| \ge 3r_S$, then

$$\frac{1}{2}|a - x_S| \le |a - x_B| \le 2|a - x_S|.$$

Using (1.1), we have

$$|(f_a)_B - (f_a)_S| = |\Phi(a, |a - x_B|) - \Phi(a, |a - x_S|)| \le \int_{|a-x_S|/2}^{2|a-x_S|} \frac{\mu(B(a,t))}{t^{n+1}}\,dt \lesssim 1.$$

Combining the four cases above, we see that (2.3) holds, which together with (2.2) completes the proof of Proposition 2.5. □

3. Lemmas

In this section we give several lemmas to prove Theorem 1.7. Recall that Φ^* and Φ^{**} are defined by (1.4) and (1.5), respectively.

Lemma 3.1. *There exists a positive constant C such that, for any $f \in \mathrm{RBMO}^\natural(\mu)$ and $(5, \beta_5)$-doubling ball $B(a,r)$,*

$$\left|m_{B(a,r)}(f)\right| \le C\|f\|_{\mathrm{RBMO}^\natural(\mu)} \left[\Phi^*(a,r) + \Phi^{**}(a,r)\right].$$

Proof. First note that, for any pair of balls $R = B(x_R, r_R)$ and $S = B(x_S, r_S)$ satisfying $R \subset S$, the following two inequalities hold

$$\left|m_{\widetilde{R}}(f) - m_{\widetilde{S}}(f)\right| \lesssim K_{R,S}\|f\|_{\mathrm{RBMO}(\mu)}, \tag{3.1}$$

and

$$K_{R,S} \lesssim 1 + \int_{r_R}^{r_S} \frac{\mu(B(x_R,t))}{t^{n+1}}\,dt. \tag{3.2}$$

For (3.1), see [22, p. 99] or [8, p. 18]. For (3.2), using (1.1), we have

$$K_{R,S} = 1 + \sum_{k=1}^{N_{R,S}} \frac{\mu(2^k R)}{(2^k r_R)^n} \lesssim 1 + \sum_{k=1}^{N_{R,S}} \int_{2^k r_R}^{2^{k+1} r_R} \frac{\mu(B(x_R,t))}{t^{n+1}}\,dt \sim 1 + \int_{r_R}^{r_S} \frac{\mu(B(x_R,t))}{t^{n+1}}\,dt.$$

To prove the lemma, it suffices to show that

$$\left| m_{B(a,r)}(f) - m_{B(0,1)}(f) \right| \lesssim \|f\|_{\mathrm{RBMO}(\mu)} \left[\Phi^*(a,r) + \Phi^{**}(a,r)\right]. \tag{3.3}$$

To this end, we consider the following three cases:

Case (i): $\max(r,1,|a|/2) = |a|/2$. In this case, we have

$$B(a,r) \cup B(0,1) \subset B(0,2|a|).$$

Then, using (3.1), (3.2) and (1.1), we have

$$\begin{aligned}
&\left| m_{B(a,r)}(f) - m_{B(0,1)}(f) \right| \\
&\le \left| m_{B(a,r)}(f) - m_{\widetilde{B(0,2|a|)}}(f) \right| + \left| m_{B(0,1)}(f) - m_{\widetilde{B(0,2|a|)}}(f) \right| \\
&\lesssim \left[K_{B(a,r),B(0,2|a|)} + K_{B(0,1),B(0,2|a|)} \right] \|f\|_{\mathrm{RBMO}(\mu)} \\
&\lesssim \left\{ \left[1 + \int_r^{2|a|} \frac{\mu(B(a,t))}{t^{n+1}}\,dt \right] + \left[1 + \int_1^{2|a|} \frac{\mu(B(0,t))}{t^{n+1}}\,dt \right] \right\} \|f\|_{\mathrm{RBMO}(\mu)} \\
&\lesssim \left[\Phi^{**}(a,r) + \Phi^*(a,r)\right] \|f\|_{\mathrm{RBMO}(\mu)}.
\end{aligned}$$

Case (ii): $\max(r,1,|a|/2) = r$. In this case, we have

$$B(a,r) \cup B(0,1) \subset B(0,3r).$$

Then, using (3.1), (3.2) and (1.1), we have

$$\begin{aligned}
&\left| m_{B(a,r)}(f) - m_{B(0,1)}(f) \right| \\
&\le \left| m_{B(a,r)}(f) - m_{\widetilde{B(0,3r)}}(f) \right| + \left| m_{B(0,1)}(f) - m_{\widetilde{B(0,3r)}}(f) \right| \\
&\lesssim \left[K_{B(a,r),B(0,3r)} + K_{B(0,1),B(0,3r)} \right] \|f\|_{\mathrm{RBMO}(\mu)} \\
&\lesssim \left\{ \left[1 + \int_r^{3r} \frac{\mu(B(a,t))}{t^{n+1}}\,dt \right] + \left[1 + \int_1^{3r} \frac{\mu(B(0,t))}{t^{n+1}}\,dt \right] \right\} \|f\|_{\mathrm{RBMO}(\mu)} \\
&\lesssim \Phi^*(a,r) \|f\|_{\mathrm{RBMO}(\mu)}.
\end{aligned}$$

Case (iii): $\max(r, 1, |a|/2) = 1$. In this case, we have

$$B(a, r) \cup B(0, 1) \subset B(a, 3).$$

Then, using (3.1), (3.2) and (1.1) again, we have

$$\begin{aligned}
&\left|m_{B(a,r)}(f) - m_{B(0,1)}(f)\right| \\
&\le \left|m_{B(a,r)}(f) - m_{\widetilde{B(a,3)}}(f)\right| + \left|m_{B(0,1)}(f) - m_{\widetilde{B(a,3)}}(f)\right| \\
&\lesssim \left[K_{B(a,r),B(a,3)} + K_{B(0,1),B(a,3)}\right] \|f\|_{\mathrm{RBMO}(\mu)} \\
&\lesssim \left\{\left[1 + \int_r^3 \frac{\mu(B(a,t))}{t^{n+1}}\, dt\right] + \left[1 + \int_1^3 \frac{\mu(B(0,t))}{t^{n+1}}\, dt\right]\right\} \|f\|_{\mathrm{RBMO}(\mu)} \\
&\lesssim \Phi^{**}(a, r)\|f\|_{\mathrm{RBMO}(\mu)}.
\end{aligned}$$

Thus, we finish the proof of (3.3) and hence Lemma 3.1. □

Recall that $\mathrm{MO}(f, B)$ is defined by (1.2).

Lemma 3.2. *Let $f \in \mathrm{RBMO}(\mu)$ and $g \in L^\infty(\mu)$. Then $fg \in \mathrm{RBMO}(\mu)$ if and only if*

$$F(f, g) = \sup_{B \in \mathcal{B}_{\mathrm{double}}} |m_B(f)|\, \mathrm{MO}(g, B) + \sup_{\substack{B,S \in \mathcal{B}_{\mathrm{double}} \\ B \subset S}} |m_B(f)| \frac{|m_B(g) - m_S(g)|}{K_{B,S}} < \infty.$$

In this case,

$$\left|\|fg\|_{\mathrm{RBMO}(\mu)} - F(f, g)\right| \le 5\|f\|_{\mathrm{RBMO}(\mu)}\|g\|_{L^\infty(\mu)}.$$

Proof. Assume that $f \in \mathrm{RBMO}(\mu)$ and $g \in L^\infty(\mu)$. Then for any $B \in \mathcal{B}_{\mathrm{double}}$, we have

$$\begin{aligned}
&\left|\int_B |(fg)(x) - m_B(fg)|\, d\mu(x) - |m_B(f)|\, \mathrm{MO}(g, B)\mu(B)\right| \\
&= \left|\|fg - m_B(fg)\|_{L^1(B,\mu)} - \|m_B(f)(g - m_B(g))\|_{L^1(B,\mu)}\right| \\
&\le \|fg - m_B(fg) - m_B(f)g + m_B(f)m_B(g)\|_{L^1(B,\mu)} \\
&\le \int_B |(fg)(x) - m_B(f)g(x)|\, d\mu(x) + \mu(B)|m_B(fg) - m_B(f)m_B(g)| \\
&\le 2\int_B |(fg)(x) - m_B(f)g(x)|\, d\mu(x) \\
&\le 2\int_B |f(x) - m_B(f)|\, d\mu(x)\, \|g\|_{L^\infty(\mu)} \le 2\mu(B)\|f\|_{\mathrm{RBMO}(\mu)}\|g\|_{L^\infty(\mu)}.
\end{aligned}$$

This shows

$$|\mathrm{MO}(fg, B) - |m_B(f)|\, \mathrm{MO}(g, B)| \le 2\|f\|_{\mathrm{RBMO}(\mu)}\|g\|_{L^\infty(\mu)}. \tag{3.4}$$

Moreover, for any $B, S \in \mathcal{B}_{\rm double}$ satisfying $B \subset S$,

$$\begin{aligned}
&||m_B(fg) - m_S(fg)| - |m_B(f)||m_B(g) - m_S(g)||\\
&\le |m_B(fg) - m_B(f)m_B(g)| + |m_S(f)m_S(g) - m_S(fg)| + |m_B(f) - m_S(f)||m_S(g)|\\
&\le 2\|f\|_{{\rm RBMO}(\mu)}\|g\|_{L^\infty(\mu)} + K_{B,S}\|f\|_{{\rm RBMO}(\mu)}\|g\|_{L^\infty(\mu)}.
\end{aligned}$$

This shows

$$\left|\frac{|m_B(fg) - m_S(fg)|}{K_{B,S}} - |m_B(f)|\frac{|m_B(g) - m_S(g)|}{K_{B,S}}\right| \le 3\|f\|_{{\rm RBMO}(\mu)}\|g\|_{L^\infty(\mu)}. \tag{3.5}$$

These two inequalities (3.4) and (3.5) imply that $fg \in {\rm RBMO}(\mu)$ if and only if $F(f, g) < \infty$. This finishes the proof of Lemma 3.2. □

The following lemma on the properties of the coefficient $K_{B,S}$ was established by Tolsa in [22]; see also [2, 4].

Lemma 3.3. [22]

(1) *There exists a positive constant C, such that, for all balls $B \subset R \subset S$, $K_{B,R} \le CK_{B,S}$, $K_{R,S} \le CK_{B,S}$, $K_{B,S} \le C(K_{B,R} + K_{R,S})$.*

(2) *For any $\alpha \in [1, \infty)$, there exists a positive constant C_α, such that, for all balls $B \subset S$ with $r_S \le \alpha r_B$, $K_{B,S} \le C_\alpha$.*

(3) *For any $\rho \in (1, \infty)$, there exists a positive constant C_ρ, such that, for all balls B, $K_{B,\widetilde{B}^\rho} \le C_\rho$. Moreover, letting $\alpha, \beta \in (1, \infty)$, $B \subset S$ be any two concentric balls such that there exists no (α, β)-doubling ball in the form of $\alpha^k B$, with $k \in \mathbb{N}$, satisfying $B \subset \alpha^k B \subset S$, then there exists a positive constant $C_{\alpha,\beta}$, such that $K_{B,S} \le C_{\alpha,\beta}$.*

Lemma 3.4. *Let Φ be as in (2.1). For any $a \in \mathbb{R}^d$, let*

$$\begin{aligned}
f^*(x) &= 1 + \max(-\Phi(0, 2), -\Phi(0, |x|)) = 1 + \int_1^{\max(2,|x|)} \frac{\mu(B(0, t))}{t^{n+1}}\,dt,\\
f^{**}(x) &= 1 + \max\left(0, \Phi(a, |a - x|) - \Phi\left(a, \max\left(1, \frac{|a|}{2}\right)\right)\right)\\
&= 1 + \max\left(0, \int_{|a-x|}^{\max(1,|a|/2)} \frac{\mu(B(a, t))}{t^{n+1}}\,dt\right),
\end{aligned}$$

and $f = f^* + f^{**}$. Then $f(x) \ge 1$ for all $x \in \mathbb{R}^d$ and there exists a positive constant C_1, independent of a, such that $\|f\|_{{\rm RBMO}^\natural(\mu)} \le C_1$. Moreover, there exists a positive constant C_2, independent of a, such that, for any $r \in (0, \infty)$,

(1) *if $r < 1$, then, for any $x \in B(a, r)$,*

$$f(x) \ge C_2[\Phi^*(a, r) + \Phi^{**}(a, r)]; \tag{3.6}$$

(2) *if* $B(a,r) \in \mathcal{B}_{\rm double}$, *then*

$$m_{B(a,r)}(f) \geq C_2[\Phi^*(a,r) + \Phi^{**}(a,r)]. \tag{3.7}$$

Proof. By the definition of f we have $f(x) \geq 1$ for all $x \in \mathbb{R}^d$. From Proposition 2.5, it follows that $f \in \mathrm{RBMO}(\mu)$ and $\|f\|_{\mathrm{RBMO}(\mu)} \lesssim 1$. Next we show that $m_{B(0,1)}(f) \lesssim 1$. In fact, for $|x| < 1$, from (1.1), we deduce that

$$f^*(x) = 1 + \int_1^2 \frac{\mu(B(0,t))}{t^{n+1}}\,dt \lesssim 1.$$

This shows $m_{B(0,1)}(f^*) \lesssim 1$. To estimate $m_{B(0,1)}(f^{**})$, we notice that, if $1 \leq |a|/2$ and $x \in B(0,1)$, then $|a-x| \geq |a|/2$ and

$$\max\left(0, \int_{|a-x|}^{|a|/2} \frac{\mu(B(a,t))}{t^{n+1}}\,dt\right) = 0$$

and hence $f^{**}(x) \equiv 1$, which implies that $m_{B(0,1)}(f^{**}) = 1$. If $1 > |a|/2$ and $x \in B(0,1) \setminus B(a,1)$, then

$$\max\left(0, \int_{|a-x|}^{1} \frac{\mu(B(a,t))}{t^{n+1}}\,dt\right) = 0.$$

Thus, to estimate $m_{B(0,1)}(f^{**})$, if $1 > |a|/2$, it suffices to consider $x \in B(a,1)$. From Lemma 2.2, $B(a,1) \subset B(0,3)$ and the $(5,\beta_5)$-doubling property of $B(0,1)$, we deduce that

$$\begin{aligned} m_{B(0,1)}(f^{**}) &\leq 1 + \frac{1}{\mu(B(0,1))} \int_{B(a,1)} \int_{|a-x|}^{1} \frac{\mu(B(a,t))}{t^{n+1}}\,dt\,d\mu(x) \\ &\lesssim 1 + \frac{\mu(B(a,1))}{\mu(B(0,1))} \lesssim 1 + \frac{\mu(B(0,3))}{\mu(B(0,1))} \lesssim 1. \end{aligned}$$

Combining the estimates of $m_{B(0,1)}(f^{**})$ and $m_{B(0,1)}(f^*)$ we have $m_{B(0,1)}(f) \lesssim 1$ and then $\|f\|_{\mathrm{RBMO}^\natural(\mu)} \lesssim 1$.

We now show (3.6). We first estimate f^*. Let $r < 1$ and $x \in B(a,r)$. If $|a| < 2r$, then we see that

$$\Phi^*(a,r) = 1 + \int_1^2 \frac{\mu(B(0,t))}{t^{n+1}}\,dt \leq 1 + \int_1^{\max(2,|x|)} \frac{\mu(B(0,t))}{t^{n+1}}\,dt = f^*(x).$$

If $|a| \geq 2r$, then $|x| \geq |a| - |x-a| \geq |a| - r \geq |a|/2$ and

$$\begin{aligned} \Phi^*(a,r) &= 1 + \int_1^{\max(2,|a|/2)} \frac{\mu(B(0,t))}{t^{n+1}}\,dt + \int_{\max(2,|a|/2)}^{\max(2,|a|)} \frac{\mu(B(0,t))}{t^{n+1}}\,dt \\ &\leq f^*(x) + C_0 \log 2 \leq (1 + C_0 \log 2) f^*(x). \end{aligned}$$

Next we estimate f^{**}. Let $r<1$ and $x\in B(a,r)$. Then

$$\begin{aligned}\Phi^{**}(a,r)&=1+\int_r^{\max(1,|a|/2)}\frac{\mu(B(a,t))}{t^{n+1}}\,dt+\int_{\max(1,|a|/2)}^{\max(2,|a|)}\frac{\mu(B(a,t))}{t^{n+1}}\,dt\\&\le 1+\int_{|a-x|}^{\max(1,|a|/2)}\frac{\mu(B(a,t))}{t^{n+1}}\,dt+C_0\log 2\\&=f^{**}(x)+C_0\log 2\le(1+C_0\log 2)f^{**}(x).\end{aligned}$$

This finishes the proof of (3.6).

Next we show (3.7) for $B(a,r)\in\mathcal{B}_{\mathrm{double}}$. We consider the following two cases:

Case (i): $r\ge\max(2,|a|)$. In this case,

$$\Phi^{*}(a,r)=1+\int_1^r\frac{\mu(B(0,t))}{t^{n+1}}\,dt\quad\text{and}\quad\Phi^{**}(a,r)=1.$$

Then $\Phi^{**}(a,r)\le m_{B(a,r)}(f^{**})$. To estimate $m_{B(a,r)}(f^{*})$, we further consider the following two subcases:

Subcase (i): $|a|<r/2$. In this subcase, choose $y_0\in B(a,2r)\setminus B(a,r)$ and B_{y_0} the *biggest* $(5,\beta_5)$*-doubling ball of the form* $B(y_0,5^{-j}r)$, $j\in\mathbb{N}$. Then $B_{y_0}\subset\widetilde{B(a,5r)}$. We claim that

$$K_{B_{y_0},\widetilde{B(a,5r)}}\lesssim 1.$$

In fact, if $r_{\widetilde{B(a,5r)}}\le r_{\widetilde{5B_{y_0}}}$, then $\widetilde{B(a,5r)}\subset 55\widetilde{B_{y_0}}$. Using (1)–(3) of Lemma 3.3, we deduce that

$$K_{B_{y_0},\widetilde{B(a,5r)}}\lesssim K_{B_{y_0},55\widetilde{B_{y_0}}}\lesssim K_{B_{y_0},5B_{y_0}}+K_{5B_{y_0},\widetilde{5B_{y_0}}}+K_{\widetilde{5B_{y_0}},55\widetilde{B_{y_0}}}\lesssim 1.$$

If $r_{\widetilde{5B_{y_0}}}\le r_{\widetilde{B(a,5r)}}$, then, by the fact that $r_{\widetilde{5B_{y_0}}}\ge r$, we see that

$$B(a,r)\subset 55\widetilde{B_{y_0}}\subset 25\widetilde{B(a,5r)},$$

which together with (1)–(3) of Lemma 3.3 further implies that

$$\begin{aligned}K_{B_{y_0},\widetilde{B(a,5r)}}&\lesssim K_{B_{y_0},25\widetilde{B(a,5r)}}\\&\lesssim K_{B_{y_0},55\widetilde{B_{y_0}}}+K_{55\widetilde{B_{y_0}},25\widetilde{B(a,5r)}}\\&\lesssim K_{B_{y_0},55\widetilde{B_{y_0}}}+K_{B(a,r),25\widetilde{B(a,5r)}}\lesssim 1.\end{aligned}$$

From this claim, it follows that there exists a positive constant C_3 such that

$$\begin{aligned}\left|m_{B(a,r)}(f^*)-m_{B_{y_0}}(f^*)\right|&\le\left|m_{B(a,r)}(f^*)-m_{\widetilde{B(a,5r)}}(f^*)\right|+\left|m_{B_{y_0}}(f^*)-m_{\widetilde{B(a,5r)}}(f^*)\right|\\&\le\left[K_{B(a,r),\widetilde{B(a,5r)}}+K_{B_{y_0},\widetilde{B(a,5r)}}\right]\|f^*\|_{\mathrm{RBMO}(\mu)}\le C_3.\end{aligned}$$

Moreover, if $x \in B_{y_0}$, then

$$|x| \geq |y_0| - |x - y_0| \geq |y_0 - a| - |a| - |x - y_0| > r - \frac{r}{2} - \frac{r}{5} = \frac{3}{10}r,$$

and hence

$$\begin{aligned}\Phi^*(a,r) &= 1 + \int_1^{\max(2,3r/10)} \frac{\mu(B(0,t))}{t^{n+1}}\,dt + \int_{\max(2,3r/10)}^{r} \frac{\mu(B(0,t))}{t^{n+1}}\,dt \\ &\leq f^*(x) + C_0 \log(10/3) \leq [1 + C_0 \log(10/3)] f^*(x),\end{aligned}$$

which implies that

$$\begin{aligned}\Phi^*(a,r) &\lesssim m_{B_{y_0}}(f^*) \\ &= m_{B(a,r)}(f^*) + \left[m_{B_{y_0}}(f^*) - m_{B(a,r)}(f^*)\right] \\ &\lesssim m_{B(a,r)}(f^*) + C_3 \lesssim m_{B(a,r)}(f^*).\end{aligned}$$

Subcase (ii): $r/2 \leq |a| \leq r$. In this subcase, take $x_0 \in B(a, r/20)$ and B_{x_0} the *biggest* $(5, \beta_5)$*-doubling ball of the form* $B(x_0, 5^{-j}r)$, $j \in \mathbb{N}$. Then $B_{x_0} \subset B(a, r/4) \subset B(a, r)$. As in Subcase (i), by Lemma 3.3 and Proposition 2.5, we see that

$$\left|m_{B_{x_0}}(f^*) - m_{B(a,r)}(f^*)\right| \leq K_{B_{x_0}, B(a,r)} \|f^*\|_{\mathrm{RBMO}(\mu)} \leq C_4.$$

Moreover, if $x \in B_{x_0}$, then

$$|x| \geq |a| - |x - a| \geq \frac{r}{2} - \frac{r}{4} = \frac{r}{4},$$

and hence

$$\begin{aligned}\Phi^*(a,r) &= 1 + \int_1^{\max(2,r/4)} \frac{\mu(B(0,t))}{t^{n+1}}\,dt + \int_{\max(2,r/4)}^{r} \frac{\mu(B(0,t))}{t^{n+1}}\,dt \\ &\leq f^*(x) + C_0 \log 4 \leq (1 + C_0 \log 4) f^*(x),\end{aligned}$$

which implies that

$$\begin{aligned}\Phi^*(a,r) &\lesssim m_{B_{x_0}}(f^*) \\ &= m_{B(a,r)}(f^*) + \left[m_{B_{x_0}}(f^*) - m_{B(a,r)}(f^*)\right] \\ &\lesssim m_{B(a,r)}(f^*) + C_4 \lesssim m_{B(a,r)}(f^*).\end{aligned}$$

Case (ii): $r < \max(2, |a|)$. In this case,

$$\Phi^*(a,r) = 1 + \int_1^{\max(2,|a|)} \frac{\mu(B(0,t))}{t^{n+1}}\,dt \quad \text{and} \quad \Phi^{**}(a,r) = 1 + \int_r^{\max(2,|a|)} \frac{\mu(B(a,t))}{t^{n+1}}\,dt.$$

We first estimate $m_{B(a,r)}(f^{**})$. Choose $x_0 \in B(a, 3r/10)$ and take B_{x_0} the biggest $(5, \beta_5)$-doubling ball of the form $B(x_0, 5^{-j}r)$, $j \in \mathbb{N}$. Then $B_{x_0} \subset B(a, r/2)$. As in the estimate of Case (i), by Lemma 3.3 and Proposition 2.5, we see that

$$\left|m_{B_{x_0}}(f^{**}) - m_{B(a,r)}(f^{**})\right| \le K_{B_{x_0},B(a,r)}\|f^{**}\|_{\mathrm{RBMO}(\mu)} \le C_5.$$

Moreover, if $x \in B_{x_0}$, then $|a-x| < r/2 < \max(1, |a|/2)$ and

$$\begin{aligned}\Phi^{**}(a,r) &= 1 + \int_r^{\max(1,|a|/2)} \frac{\mu(B(a,t))}{t^{n+1}}\,dt + \int_{\max(1,|a|/2)}^{\max(2,|a|)} \frac{\mu(B(a,t))}{t^{n+1}}\,dt \\ &\le f^{**}(x) + C_0 \log 2 \le (1 + C_0 \log 2) f^{**}(x),\end{aligned}$$

which implies that

$$\begin{aligned}\Phi^{**}(a,r) &\lesssim m_{B_{x_0}}(f^{**}) \\ &= m_{B(a,r)}(f^{**}) + \left[m_{B_{x_0}}(f^{**}) - m_{B(a,r)}(f^{**})\right] \\ &\lesssim m_{B(a,r)}(f^{**}) + C_5 \lesssim m_{B(a,r)}(f^{**}).\end{aligned}$$

It remains to estimate $m_{B(a,r)}(f^*)$. If $\max(2, |a|) = 2$, then, for any $x \in B(a,r)$,

$$\Phi^*(a,r) = 1 + \int_1^2 \frac{\mu(B(0,t))}{t^{n+1}}\,dt \le f^*(x).$$

This shows

$$\Phi^*(a,r) \le m_{B(a,r)}(f^*).$$

If $\max(2, |a|) = |a|$, then

$$\Phi^*(a,r) = 1 + \int_1^{|a|} \frac{\mu(B(0,t))}{t^{n+1}}\,dt.$$

In this case, take B_{x_0} the same as in Subcase (ii) of Case (i). Then we have

$$\left|m_{B_{x_0}}(f^*) - m_{B(a,r)}(f^*)\right| \lesssim C_6.$$

Moreover, if $x \in B_{x_0}$, then

$$|x| \ge |a| - |x-a| \ge |a| - \frac{r}{4} \ge |a| - \frac{|a|}{4} = \frac{3|a|}{4},$$

and hence

$$\begin{aligned}\Phi^*(a,r) &= 1 + \int_1^{\max(2,3|a|/4)} \frac{\mu(B(0,t))}{t^{n+1}}\,dt + \int_{\max(2,3|a|/4)}^{|a|} \frac{\mu(B(0,t))}{t^{n+1}}\,dt \\ &\le f^*(x) + C_0 \log(4/3) \le [1 + C_0 \log(4/3)] f^*(x),\end{aligned}$$

which implies that

$$\begin{aligned}\Phi^*(a,r) &\lesssim m_{B_{x_0}}(f^*)\\ &= m_{B(a,r)}(f^*) + \left[m_{B_{x_0}}(f^*) - m_{B(a,r)}(f^*)\right]\\ &\lesssim m_{B(a,r)}(f^*) + C_6 \lesssim m_{B(a,r)}(f^*).\end{aligned}$$

This finishes the proof of (3.7) and hence Lemma 3.4. □

4. Proof of the main result

In this section, using the lemmas in the previous section, we prove Theorem 1.7.

4.1. Sufficiency

Assume that $g \in \mathrm{RBMO}_\varphi(\mu) \cap L^\infty(\mu)$. From Lemma 3.1, for any $f \in \mathrm{RBMO}^\natural(\mu)$ and $B = B(a,r) \in \mathcal{B}_{\mathrm{double}}$, it follows that

$$|m_B(f)| \lesssim \|f\|_{\mathrm{RBMO}^\natural(\mu)}[\Phi^*(a,r) + \Phi^{**}(a,r)] = \|f\|_{\mathrm{RBMO}^\natural(\mu)} \frac{1}{\varphi(B)}.$$

Then

$$|m_B(f)|\,\mathrm{MO}(g,B) \lesssim \|f\|_{\mathrm{RBMO}^\natural(\mu)} \frac{\mathrm{MO}(g,B)}{\varphi(B)} \le \|f\|_{\mathrm{RBMO}^\natural(\mu)} \|g\|_{\mathrm{RBMO}_\varphi(\mu)},$$

and, for any pair of $B, S \in \mathcal{B}_{\mathrm{double}}$ satisfying $B \subset S$,

$$|m_B(f)| \frac{|m_B(g) - m_S(g)|}{K_{B,S}} \lesssim \|f\|_{\mathrm{RBMO}^\natural(\mu)} \frac{|m_B(g) - m_S(g)|}{\varphi(B) K_{B,S}} \le \|f\|_{\mathrm{RBMO}^\natural(\mu)} \|g\|_{\mathrm{RBMO}_\varphi(\mu)}.$$

These two inequalities, together with Lemma 3.2, show that $fg \in \mathrm{RBMO}(\mu)$ and

$$\begin{aligned}\|fg\|_{\mathrm{RBMO}(\mu)} &\lesssim \|f\|_{\mathrm{RBMO}^\natural(\mu)} \|g\|_{\mathrm{RBMO}_\varphi(\mu)} + \|f\|_{\mathrm{RBMO}(\mu)} \|g\|_{L^\infty(\mu)}\\ &\le \|f\|_{\mathrm{RBMO}^\natural(\mu)} \left(\|g\|_{\mathrm{RBMO}_\varphi(\mu)} + \|g\|_{L^\infty(\mu)}\right).\end{aligned}$$

Moreover, since

$$\begin{aligned}|m_{B(0,1)}(fg)| &\le \left|m_{B(0,1)}(fg) - m_{B(0,1)}(f) m_{B(0,1)}(g)\right| + \left|m_{B(0,1)}(f) m_{B(0,1)}(g)\right|\\ &\le \frac{1}{\mu(B(0,1))} \int_{B(0,1)} \left|f(x)g(x) - m_{B(0,1)}(f) g(x)\right| \, d\mu(x)\\ &\quad + \left|m_{B(0,1)}(f) m_{B(0,1)}(g)\right|\\ &\le \|f\|_{\mathrm{RBMO}^\natural(\mu)} \|g\|_{L^\infty(\mu)},\end{aligned}$$

we see that

$$\|fg\|_{\mathrm{RBMO}^\natural(\mu)} \lesssim \|f\|_{\mathrm{RBMO}^\natural(\mu)} \left(\|g\|_{\mathrm{RBMO}_\varphi(\mu)} + \|g\|_{L^\infty(\mu)}\right).$$

Thus, g is a pointwise multiplier on $\mathrm{RBMO}^\natural(\mu)$ and

$$\|g\|_{\mathrm{OP}} \lesssim \|g\|_{\mathrm{RBMO}_\varphi(\mu)} + \|g\|_{L^\infty(\mu)}.$$

4.2. Necessity

Conversely, assume that g is a pointwise multiplier on $\mathrm{RBMO}^\natural(\mu)$. For any $f \in \mathrm{RBMO}^\natural(\mu)$ and for any $B \in \mathcal{B}_{\mathrm{double}}$, we deduce by Lemma 3.1 that

$$\begin{aligned} \frac{1}{\mu(B)} \int_B |(fg)(x)| \, d\mu(x) &\le \frac{1}{\mu(B)} \int_B |(fg)(x) - m_B(fg)| \, d\mu(x) + |m_B(fg)| \\ &\lesssim \|fg\|_{\mathrm{RBMO}(\mu)} + [\Phi^*(a,r) + \Phi^{**}(a,r)] \, \|fg\|_{\mathrm{RBMO}^\natural(\mu)} \\ &\lesssim [\Phi^*(a,r) + \Phi^{**}(a,r)] \, \|f\|_{\mathrm{RBMO}^\natural(\mu)} \|g\|_{\mathrm{OP}}. \end{aligned} \tag{4.1}$$

For any $a \in \mathrm{supp}(\mu)$, let f be as in Lemma 3.4. Take a sequence $\{r_j\}$ of positive numbers less than 1 such that $B(a, r_j) \in \mathcal{B}_{\mathrm{double}}$ and $r_j \to 0$ as $j \to \infty$. Then for each j and $x \in B(a, r_j)$,

$$\|f\|_{\mathrm{RBMO}^\natural(\mu)} \lesssim 1 \quad \text{and} \quad f(x) \gtrsim \Phi^*(a, r_j) + \Phi^{**}(a, r_j). \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$\frac{1}{\mu(B(a,r_j))} \int_{B(a,r_j)} |g(x)| \, d\mu(x) \lesssim \|g\|_{\mathrm{OP}}.$$

Letting $j \to \infty$, by the Lebesgue differentiation theorem in [22, p. 96], we see that $|g(a)| \lesssim \|g\|_{\mathrm{OP}}$ for μ-almost every $a \in \mathbb{R}^d$. That is, $g \in L^\infty(\mu)$ and

$$\|g\|_{L^\infty(\mu)} \lesssim \|g\|_{\mathrm{OP}}.$$

Next we show $g \in \mathrm{RBMO}_\varphi(\mu)$. For any $f \in \mathrm{RBMO}^\natural(\mu)$, we deduce by Lemma 3.2 that, for any $B \in \mathcal{B}_{\mathrm{double}}$,

$$\begin{aligned} |m_B(f)| \, \mathrm{MO}(g, B) &\lesssim \|fg\|_{\mathrm{RBMO}(\mu)} + \|f\|_{\mathrm{RBMO}(\mu)} \|g\|_{L^\infty(\mu)} \\ &\lesssim \|f\|_{\mathrm{RBMO}^\natural(\mu)} \left(\|g\|_{\mathrm{OP}} + \|g\|_{L^\infty(\mu)} \right) \lesssim \|f\|_{\mathrm{RBMO}^\natural(\mu)} \|g\|_{\mathrm{OP}}, \end{aligned} \tag{4.3}$$

and, for any pair of $B, S \in \mathcal{B}_{\mathrm{double}}$ satisfying $B \subset S$,

$$\begin{aligned} |m_B(f)| \frac{|m_B(g) - m_S(g)|}{K_{B,S}} &\lesssim \|fg\|_{\mathrm{RBMO}(\mu)} + \|f\|_{\mathrm{RBMO}(\mu)} \|g\|_{L^\infty(\mu)} \\ &\lesssim \|f\|_{\mathrm{RBMO}^\natural(\mu)} \|g\|_{\mathrm{OP}}. \end{aligned} \tag{4.4}$$

For $B = B(a, r) \in \mathcal{B}_{\mathrm{double}}$, let f be as in Lemma 3.4. Then

$$\|f\|_{\mathrm{RBMO}^\natural(\mu)} \lesssim 1 \quad \text{and} \quad m_B(f) \gtrsim \Phi^*(a,r) + \Phi^{**}(a,r) = \frac{1}{\varphi(B)}. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we have

$$\frac{\mathrm{MO}(g,B)}{\varphi(B)} \lesssim \|g\|_{\mathrm{OP}} \quad \text{and} \quad \frac{|m_B(g) - m_S(g)|}{\varphi(B) K_{B,S}} \lesssim \|g\|_{\mathrm{OP}}.$$

This implies that $g \in \mathrm{RBMO}_\varphi(\mu)$ and $\|g\|_{\mathrm{RBMO}_\varphi(\mu)} \lesssim \|g\|_{\mathrm{OP}}$, which completes the proof of Theorem 1.7.

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