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Recovery of the Schrödinger Operator on the Half-line from a Particular Set of Eigenvalues

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Abstract. In this work, we study the Schrödinger operator on the half-line with a selfadjoint boundary condition. We prove that a particular set of eigenvalues can uniquely determine the potential. The reconstruction algorithm for recovering the potential from the particular data is provided.

1. Introduction

We consider the Schrödinger operator $L(q,h)y = -y'' + q(x)y$ on $L^2(0,\infty)$ with the boundary condition

$$y'(0) - hy(0) = 0, \tag{1.1}$$

where $h \in \mathbb{R} \cup \{+\infty\}$ and real-valued function q is Lebesgue-measurable and belongs to $L_1^1 := \left\{q : \int_0^{+\infty} (1+x)\,|q(x)|\,dx < \infty\right\}$. In particular, if $h = +\infty$ then (1.1) means the Dirichlet boundary condition $y(0) = 0$.

It is known [5, 8] that the Schrödinger operator $L(q,h)$ has only a finite (or zero) number of eigenvalues, which are all negative, and absolutely continuous spectrum $[0,+\infty)$. Throughout this paper, we always assume that the operator $L(q,H_0)$ has N_0 eigenvalues with $N_0 \geq 1$ for some $H_0 \in \mathbb{R} \cup \{+\infty\}$.

Inverse problems for the Schrödinger operators on the half-line have been studied by many authors (see [2, 3, 5–8, 10, 12–15] and the references therein). The self-adjoint Schrödinger operator under study is connected with determining the radius of the human vocal tract from sound pressure measurements at the lips, the potential $q(x)$ denotes the relative concavity of the vocal tract and the eigenvalues of the operator $L(q,h)$ correspond to the frequency which can be measured in Hertz (see [1, 2, 4]). In general, for recovering the potential $q(x)$ one needs to specify the spectral function or the Weyl function or the scattering data. For the compactly supported potential, it can also be recovered from the resonance data (see [7, 14] and other works).

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In 1987, McLaughlin and Rundell [9] studied the Schrödinger operators $L_0(q,\beta_n)y = -y'' + q(y)y$ on $L^2(0,1)$ with the boundary conditions

$$y(0) = 0, \quad y'(1) + \beta_n y(1) = 0,$$

where $\beta_n \in \mathbb{R}$ $(n = 1, 2, \ldots)$ are all distinct, and proved that the set of eigenvalues $\{\lambda_j(q,\beta_n)\}_{n\geq 1}$, for some fixed index $j \geq 1$, of the operator $L_0(q,\beta_n)$, uniquely determine the potential $q \in L^2(0,1)$.

Motivated by McLaughlin-Rundell's work, we are interested in the Schrödinger operator on the half-line. Does the similar result hold for the Schrödinger operator on the half-line? If "does", how to reconstruct the potential from this kind of particular spectral data? In this paper, we will give the positive answers to these two questions. The uniqueness theorem is shown in Section 2, and the corresponding reconstruction algorithm is provided in Section 3.

2. Uniqueness theorem

Denote by $\{\lambda_j(q,h)\}$ the eigenvalues of the operator $L(q,h)$ with $\lambda_j(q,h) > \lambda_{j+1}(q,h)$. Aktosun and Weder present the following result.

Lemma 2.1. [3, Proposition 3.7] *Let $-\infty < H_2 < H_1 \leq +\infty$, and denote by N_i $(i = 1, 2)$ the number of eigenvalues of the operators $L(q,H_i)$, respectively. Then either $N_2 = N_1$ or $N_2 = N_1 + 1$. Moreover, in the former case,*

$$0 > \lambda_1(q,H_1) > \lambda_1(q,H_2) > \cdots > \lambda_{N_1}(q,H_1) > \lambda_{N_2}(q,H_2), \tag{2.1}$$

and in the latter case,

$$0 > \lambda_1(q,H_2) > \lambda_1(q,H_1) > \cdots > \lambda_{N_1}(q,H_2) > \lambda_{N_1}(q,H_1) > \lambda_{N_2}(q,H_2).$$

From Lemma 2.1, we see that if there exists some $H \in \mathbb{R}\cup\{+\infty\}$ such that the operator $L(q,H)$ has N eigenvalues, then for arbitrary $h \in (-\infty, H)$ the operator $L(q,h)$ has either N or $N+1$ eigenvalues. As mentioned above, we always assume that the operator $L(q,H_0)$ has N_0 (≥ 1) eigenvalues for some $H_0 \in \mathbb{R}\cup\{+\infty\}$. Select the sequence $\{h_n\}_{n\geq 1}$ satisfying

$$-\infty < M < \cdots < h_{n+1} < h_n < \cdots < h_2 < h_1 < H_0 \leq +\infty, \tag{2.2}$$

where M is some real number. Then each operator $L(q,h_n)$ $(n \geq 1)$ has at least N_0 eigenvalues.

Now we present one of the main results in this paper: uniqueness theorem. Together with the operator $L(q,h)$ we consider another operator $L(\widetilde{q},h)$ of the same form but with different potential $\widetilde{q}$. We agree that if a certain symbol δ denotes an object related to $L(q,h)$, then $\widetilde{\delta}$ will denote an analogous object related to $L(\widetilde{q},h)$.

Theorem 2.2. *Suppose that the parameters h_n ($n \geq 1$) satisfy* (2.2), *and for some fixed* $j = \overline{1, N_0}$,

$$\lambda_j(q, h_n) = \lambda_j(\widetilde{q}, h_n), \quad n \geq 1,$$

then $q(x) = \widetilde{q}(x)$ *a.e. on* $[0, \infty)$.

To prove Theorem 2.2, we need the following Lemma 2.3. Let $\lambda = k^2$ with $\operatorname{Im} k \geq 0$. Let $f(x, k)$ be the Jost solution of the equation

$$-f'' + q(x)f = k^2 f, \quad x \geq 0 \tag{2.3}$$

satisfying the asymptotic behavior

$$f^{(v)}(x, k) = (ik)^{(v)} e^{ikx}[1 + o(1)],\ x \to +\infty, \quad \operatorname{Im} k > 0,\ v = 0, 1. \tag{2.4}$$

It is known [5, 8] that under the assumption $q \in L_1^1$, Jost solution $f(x, k)$ satisfies that for every fixed $x \geq 0$, $f^{(v)}(x, k)$ are analytic for $\operatorname{Im} k > 0$ and continuous for $k \in \Omega := \{k : \operatorname{Im} k \geq 0\}$, and for each fixed $k \in \Omega$, $f(x, k)$ is continuously differentiable in $x \in [0, \infty)$.

Lemma 2.3. [10] *For arbitrary function* $p \in L_1^1$, *if*

$$\int_0^{+\infty} p(x) f(x, k) \widetilde{f}(x, k)\, dx = 0, \quad \forall\, k > 0,$$

then $p(x) = 0$ *a.e. on* $[0, +\infty)$.

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. First we show that the set $\{\lambda_j(q, h_n)\}_{n \geq 1}$ is bounded and infinite. Indeed, by Lemma 2.1, it is easy to see that no matter the number of eigenvalues remains N_0 for all n or becomes $N_0 + 1$ at some large $n = n_0$, (2.1) implies that $\{\lambda_j(q, h_n)\}_{n \geq n_0}$ is a strictly decreasing sequence bounded below by $\lambda_j(q, M)$.

Next we prove that $q(x) = \widetilde{q}(x)$ a.e. on $[0, +\infty)$ if $\lambda_j(q, h_n) = \lambda_j(\widetilde{q}, h_n)$ for all $n \geq 1$. To this end, we consider the equations

$$-f''(x, k) + q(x) f(x, k) = \lambda f(x, k) \tag{2.5}$$

and

$$-\widetilde{f}''(x, k) + \widetilde{q}(x) \widetilde{f}(x, k) = \lambda \widetilde{f}(x, k). \tag{2.6}$$

Multiplying (2.5) by $\widetilde{f}(x, k)$ and (2.6) by $f(x, k)$, subtracting, integrating from 0 to $+\infty$, and using (2.4) one gets

$$G(k) := \int_0^{+\infty} [\widetilde{q}(x) - q(x)] f(x, k) \widetilde{f}(x, k)\, dx = (f' \widetilde{f} - \widetilde{f}' f)(0, k), \quad k \in \Omega. \tag{2.7}$$

Denote $k_j(h_n) := \sqrt{\lambda_j(q,h_n)}$ $(= \sqrt{\lambda_j(\widetilde{q},h_n)})$ for all $n \geq 1$. Taking (1.1) with $h = h_n$ into account, we know that $k_j(h_n)$ $(n \geq 1)$ are zeros of the function $G(k)$ in the upper half complex plane $\mathbb{C}^+$, i.e., $G(k) = 0$ at $k = k_j(h_n)$ $(n \geq 1)$. From the argument above, we know that $\{k_j(h_n)\}_{n\geq 1}$ is a bounded and infinite set. Thus there is at least one accumulation point of the set $\{k_j(h_n)\}_{n\geq 1}$. By virtue of the properties of the Jost solutions together with (2.7), we know that the function $G(k)$ is analytic for $\operatorname{Im} k > 0$ and continuous for $k \in \Omega$. Thus, $G(k) = 0$ for all $k \in \Omega$. It follows from Lemma 2.3 that $q(x) = \widetilde{q}(x)$ a.e. on $[0, +\infty)$. □

3. Reconstruction algorithm

In this section, we provide the reconstruction algorithm for Theorem 2.2: recovering the potential from the particular set of eigenvalues $\{\lambda_j(q,h_n)\}_{n\geq 1}$.

The Gel'fand-Levitan method and the Marchenko method are the most classical methods for reconstructing the potential, which are related to the spectral function and the scattering data, respectively. It is shown in [12] that both spectral function and the scattering data can be reconstructed from the well known I-function:

$$I(k) = \frac{f'(0,k)}{f(0,k)},$$

which is a meromorphic function for $k \in \mathbb{C}^+$ with only simple poles in $\mathbb{C}^+$, and a continuous function for $k \in \overline{\mathbb{C}^+} \setminus \{0\}$, and it may also have a simple pole at $k = 0$ (see [11]). Therefore, we shall first reconstruct the I-function from the particular spectral data $\{\lambda_j(q,h_n)\}_{n\geq 1}$, and then recover the potential from the I-function. For convenience of readers, we briefly provide the basic procedures for recovering the potential from I-function by using the Gel'fand-Levitan method. One can find the detailed procedures in [12].

Let $\{\lambda_m := \lambda_m(q,\infty)\}_{m=1}^N$ be all the eigenvalues of the operator $L(q,\infty)$ with $\lambda_m > \lambda_{m+1}$, and denote $k_m^\infty := \sqrt{\lambda_m} \in i\mathbb{R}^+$ for $m = \overline{1,N}$, which are all poles of $I(k)$ on $i\mathbb{R}^+$. Let us recall that the spectral functions $\rho_0(\lambda)$ and $\rho(\lambda)$ of the operators $L(0,\infty)$ and $L(q,\infty)$, respectively, satisfy

$$\frac{d\rho_0(\lambda)}{d\lambda} = \begin{cases} \dfrac{\sqrt{\lambda}}{\pi} & \lambda \geq 0, \\ 0 & \lambda < 0 \end{cases} \quad \text{and} \quad \frac{d\rho(\lambda)}{d\lambda} = \begin{cases} \pi^{-1} \operatorname{Im} I(\sqrt{\lambda}) & \lambda \geq 0, \\ \displaystyle\sum_{m=1}^N c_m \delta(\lambda + \lambda_m) & \lambda < 0, \end{cases} \tag{3.1}$$

where $\delta(\cdot)$ is the delta function, and c_m $(m = \overline{1,N})$ are the norming constants, which can be calculated by the formula

$$c_m = -2k_m^\infty \cdot \operatorname{Res}_{k=k_m^\infty} I(k). \tag{3.2}$$

Denote

$$L(x,y)=\int_{-\infty}^{+\infty}\frac{\sin\sqrt{\lambda}x\sin\sqrt{\lambda}y}{\lambda}\,d[\rho(\lambda)-\rho_0(\lambda)]. \tag{3.3}$$

Solving the Gel'fand-Levitan equation

$$K(x,y)+\int_0^x K(x,s)L(s,y)\,ds+L(x,y)=0,\quad 0\le y\le x, \tag{3.4}$$

one gets $K(x,y)$. Then the potential $q(x)$ can be obtained by the formula

$$q(x)=2\frac{dK(x,x)}{dx}. \tag{3.5}$$

Now we shall recover the I-function from the given data $\{\lambda_j(q,h_n)\}_{n\ge1}$ by using the so-called *method of power series analytic continuation*. The following lemma is helpful for us to find k_m^∞, $m=\overline{1,N}$.

Lemma 3.1. *If $R>0$ is the convergence radius of the power series*

$$g(k):=\sum_{j=0}^{\infty}c_j(k-z_0)^j,$$

then the function $g(k)$ is analytic in $\{k:|k-z_0|<R\}$, and has at least one singularity on the circle $\{k:|k-z_0|=R\}$.

For convenience we fix $j=1$, and denote $k_n:=\sqrt{\lambda_1(q,h_n)}\subset i\mathbb{R}^+$ for $n\ge1$. Taking (1.1) into account, together with the definition of I-function, we know that

$$I(k_n)=h_n,\quad n=1,2,\dots. \tag{3.6}$$

Since both the sets $\{k_n\}_{n\ge1}$ and $\{h_n\}_{n\ge1}$ are bounded and infinite, one can choose the accumulation points from these two sets, respectively, denoted by k_* and h_*, such that $I(k_*)=h_*$. From the proof of Theorem 2.2, $\{|k_n|\}_{n\ge n_0}$ is an increasing sequence, where n_0 is some sufficiently large number, which implies $k_*\ne0$. Also, from the monotonicity of the sequences $\{|k_n|\}_{n\ge n_0}$ and $\{h_n\}_{n\ge1}$, we have

$$\lim_{n\to\infty}k_n=k_*,\quad \lim_{n\to\infty}h_n=h_*. \tag{3.7}$$

Set $a_0:=I(k_*)=h_*$ and

$$a_j:=\lim_{n\to\infty}\frac{h_n-\sum_{i=0}^{j-1}a_i(k_n-k_*)^i}{(k_n-k_*)^j},\quad j\ge1. \tag{3.8}$$

From (3.6)–(3.8) we see that a_j $(j\ge0)$ are actually Taylor coefficients of the function $I(k)$. Denote

$$R_*:=\frac{1}{\overline{\lim}_{j\to\infty}|a_j|^{1/j}}. \tag{3.9}$$

Then we have

$$I(k) = \sum_{j=0}^{\infty} a_j (k - k_*)^j, \quad k \in \Omega_* := \{k : |k - k_*| < R_*\}. \tag{3.10}$$

Next we shall use Lemma 3.1 and (3.10) to find k_m^∞, $m = \overline{1, N}$.

(1) If $R_* \geq |k_*|$, then one may choose a point $z_1 \in i\mathbb{R}^+$ such that $R_* + |k_*| > |z_1| > |k_*|$ (see Figure 3.1).

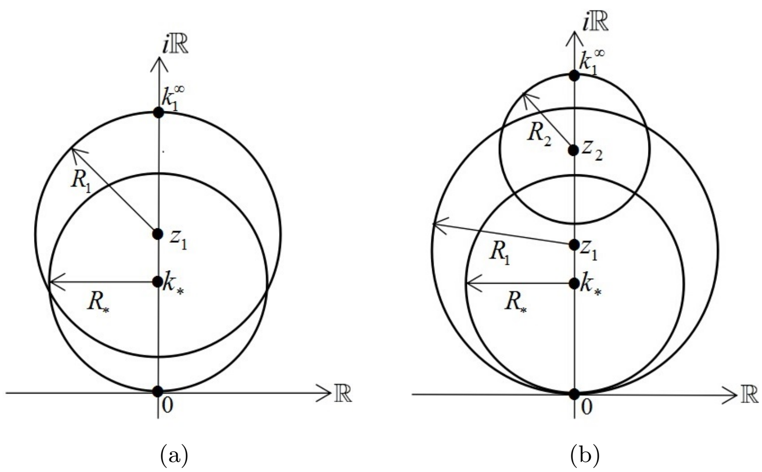

Figure 3.1

Use the formula (3.9) to calculate the convergence radius R_1 of the power series

$$g_1(k) := \sum_{j=0}^{\infty} a_{1,j} (k - z_1)^j, \quad a_{1,j} = \frac{I^{(j)}(z_1)}{j!}.$$

If $R_1 < |z_1|$ then by virtue of Lemma 3.1 we know that $R_1 = |k_1^\infty - z_1|$. Thus k_1^∞ is obtained (see Figure 3.1(a)). If $R_1 \geq |z_1|$, then one may again choose another point $z_2 \in i\mathbb{R}^+$ such that $|z_1| + R_1 > |z_2| > |z_1|$ (see Figure 3.1(b)). By the formula (3.9) we try to calculate the convergence radius R_2 of the power series

$$g_2(k) := \sum_{j=0}^{\infty} a_{2,j} (k - z_2)^j, \quad a_{2,j} = \frac{g_1^{(j)}(z_2)}{j!}.$$

Repeating the same argument for finitely many times one can obtain k_1^∞.

(2) If $R_* < |k_*|$, then there are three possibilities:

(i) $R_* = |k_* - k_1^\infty|$ with $|k_1^\infty| > |k_*|$;

(ii) $R_* = |k_* - k_1^\infty|$ with $|k_1^\infty| < |k_*|$;

(iii) $R_* = |k_* - k_2^\infty|$ (see Figure 3.2).

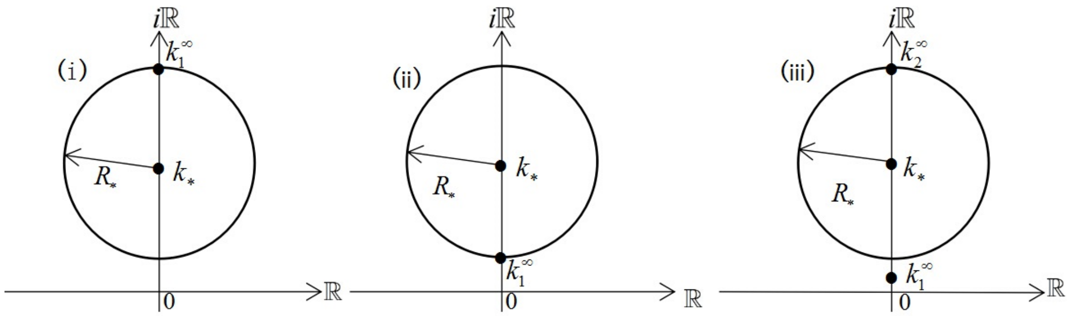

Figure 3.2

Letting $k \to i(|k_*| - R_*)$ in (3.10), if $I(k) \to \infty$ then the possibility (ii) is true, and thus k_1^∞ is obtained; otherwise, either the possibility (i) or (iii) is true. If either (i) or (iii) is true, then one may choose a point $z_3 \in i\mathbb{R}^+$ such that $|k_*| - R_* < |z_3| < |k_*|$ (see Figure 3.3). Calculate the convergence radius R_3 of the power series

$$g_3(k) := \sum_{j=0}^{\infty} a_{3,j}(k - z_3)^j, \quad a_{3,j} = \frac{I^{(j)}(z_3)}{j!}.$$

If $R_3 \geq |z_3|$ then the possibility (i) is true, and thus k_1^∞ is obtained. If $R_3 < |z_3|$ and $R_3 < R_* + |k_* - z_3|$ then the possibility (iii) is true and $R_3 = |z_3 - k_1^\infty|$ (see Figure 3.3(b)), and thus both k_1^∞ and k_2^∞ are obtained. If $R_3 < |z_3|$ and $R_3 = R_* + |k_* - z_3|$, then one may choose another point $z_4 \in i\mathbb{R}^+$ such that $|z_3| - R_3 < |z_4| < |z_3|$ (see, for example, Figure 3.3(c)).

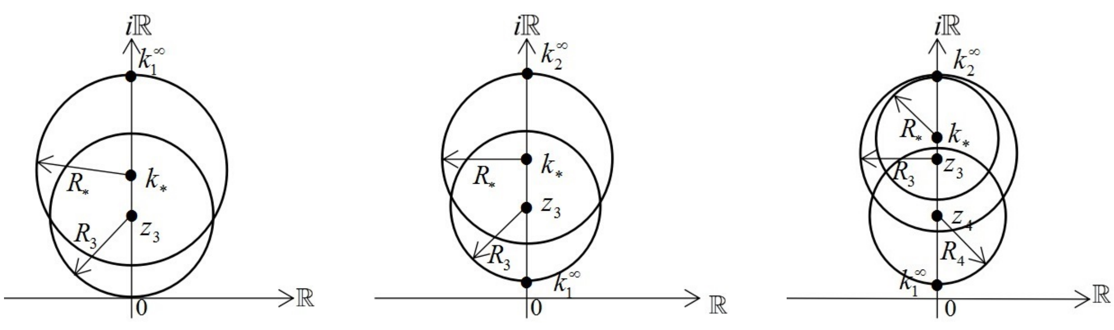

Figure 3.3

Again, calculate the convergence radius R_4 of the power series

$$g_4(k) := \sum_{j=0}^{\infty} a_{4,j}(k - z_4)^j, \quad a_{4,j} = \frac{g_3^{(j)}(z_4)}{j!}.$$

Repeating the same argument above for finitely many times, one can distinguish (i) and (iii), and then gets k_1^∞.

After that k_1^∞ has been obtained, denote $I_1(k) := (k - k_1^\infty)I(k)$. Then $I_1(k)$ is analytic at k_1^∞, but not analytic at k_2^∞. Using the method of power series analytic continuation for $I_1(k)$ as above, one can get k_2^∞. Repeating preceding arguments subsequently for $k_2^\infty, \ldots, k_N^\infty$, one obtains the set $\{k_m^\infty\}_{m=1}^N$.

Note that since there is only a finite (or zero) number of eigenvalues of the operator $L(q, \infty)$, one can always find them, in theory, by repeating the above arguments for N_1 times, where $0 < N_1 \le \infty$ and $N_1 = \infty$ means that there is no eigenvalue of $L(q, \infty)$. However, the case $N_1 = \infty$ can also be avoided if one chooses the minimum eigenvalues of the operators $L(q, h_n)$ as the input data. Indeed, by Lemma 2.1, we see that the minimum eigenvalue $\lambda_{N_2}(q, H_2)$ of the operator $L(q, H_2)$ is always less than the minimum eigenvalue of $L(q, \infty)$ if $H_2 < \infty$. Thus, all eigenvalues of $L(q, \infty)$ lie inside the interval $(0, \lambda_{N_2}(q, H_2))$. In this case the minimum eigenvalue of $L(q, \infty)$ should be found first. In addition, if the number N of the eigenvalues of $L(q, \infty)$ is known a priori then N_1 is finite, because the final ending step is that we find N eigenvalues.

Denote

$$A(k) := kI(k) \prod_{m=1}^{N} (k - k_m^\infty), \quad k \in \Omega_*. \tag{3.11}$$

Then $A(k)$ is analytic in $\mathbb{C}^+$ and continuous on $\overline{\mathbb{C}}^+ := \mathbb{C}^+ \cup \mathbb{R}$. Using the method of power series analytic continuation for $A(k)$ with $k \in \Omega_*$, one can get $A(k)$ for $k \in \mathbb{C}^+$. For arbitrary $x \in \mathbb{R}$, letting $k \to x$ one obtains $A(k)$ for $k \in \mathbb{R}$. Therefore, the recovery of I-function is complete:

$$I(k) = A(k)k^{-1} \prod_{m=1}^{N} (k - k_m^\infty)^{-1}, \quad k \in \overline{\mathbb{C}}^+. \tag{3.12}$$

Now we reformulate the algorithm for reconstructing the potential.

Algorithm 3.2. *Let the data $\{\lambda_j(q, h_n)\}_{n \ge 1}$ be given.*

1. *Define a map $I(k)$ from the set $\left\{k_n : k_n = \sqrt{\lambda_j(q, h_n)}\right\}_{n \ge 1}$ to the set $\{h_n\}_{n \ge 1}$ as* (3.6)*.*

2. *Choose the accumulation points k_* and h_*, respectively, from the two sets $\{k_n\}_{n \ge 1}$ and $\{h_n\}_{n \ge 1}$, and the corresponding subsequences satisfying* (3.7)*.*

3. *Construct the power series* (3.10)*, where* a_j $(j \geq 0)$ *are given by* (3.8) *and* R_* *is calculated by* (3.9)*.*

4. *By Lemma* 3.1 *and* (3.10) *find all the eigenvalues* λ_m $(m = \overline{1, N})$ *of the operator* $L(q, \infty)$*.*

5. *Apply the method of power series analytic continuation, together with* $A(k)$ *defined by* (3.11)*, to find* $I(k)$ *for* $k \in \overline{\mathbb{C}}^+$ *by* (3.12)*.*

6. *Find the norming constants* c_m $(m = \overline{1, N})$ *by the formula* (3.2)*.*

7. *Using the data* $\{I(k), c_m, \lambda_m\}_{k \in \mathbb{R}, m=\overline{1,N}}$ *obtained in the above steps, one can construct the Gel'fand-Levitan equation by* (3.1)*,* (3.3) *and* (3.4)*, and solve it to obtain* $K(x, y)$*, and then* $q(x)$ *is recovered by the formula* (3.5)*.*

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