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Stability of Traveling Wavefronts for a Delayed Lattice System with Nonlocal Interaction

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Abstract. In this paper, we mainly investigate exponential stability of traveling wavefronts for delayed $2D$ lattice differential equation with nonlocal interaction. For all non-critical traveling wavefronts with the wave speed $c > c_*(\theta)$, where $c_*(\theta) > 0$ is the critical wave speed and θ is the direction of propagation, we prove that these traveling waves are asymptotically stable, when the initial perturbation around the traveling waves decay exponentially at far fields, but can be allowed arbitrarily large in other locations. Our approach adopted in this paper is the weighted energy method and the squeezing technique with the help of Gronwall's inequality. Furthermore, from stability result, we prove the uniqueness (up to shift) of the traveling wavefront. Our results can apply to the discrete diffusive Mackey-Glass model and the dicrete diffusive Nicholson's blowflies model on $2D$ lattices.

1. Introduction

In this paper, we study the exponential stability of traveling waves for the following time-delayed population model with the nonlocal interaction on $2D$ spatial lattices

$$\begin{aligned}\frac{dw_{i,j}(t)}{dt} &= D[w_{i+1,j}(t) + w_{i-1,j}(t) + w_{i,j+1}(t) + w_{i,j-1}(t) - 4w_{i,j}(t)] - dw_{i,j}(t) \\ &\quad + \varepsilon \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q) b(w_{i+l,j+q}(t-r)),\end{aligned} \tag{1.1}$$

with the initial condition

$$w_{i,j}(s) = w_{i,j}^0(s), \quad s \in [-r, 0],\ i, j \in \mathbb{Z}. \tag{1.2}$$

This model describes the population distribution of a single species with age-structure in a two-dimensional patchy environment [2, 20], where $w_{i,j}(t) > 0$ is the population density of the species at time t and spatial lattice $(i, j) \in \mathbb{Z} \times \mathbb{Z}$; and $D > 0$ and $d > 0$ are

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diffusion and death rates of the population, respectively; $\varepsilon > 0$ represents the impact of the death rate for the immature; $r > 0$ is the matured age and is called the delay-time mathematically; and $\beta(l)$ and $\gamma(q)$ are non-negative and symmetrical kernels (nonlocal interactions) for directions $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$, respectively.

Throughout the paper, we assume that:

(H1) *Non-negative and symmetrical kernels*: $\beta(l) = \beta(-l) \geq 0$ and $\gamma(l) = \gamma(-l) \geq 0$ for any $l \in \mathbb{Z}$; and $\sum_{l=-\infty}^{\infty} \beta(l) = \sum_{q=-\infty}^{\infty} \gamma(q) = 1$; and there exists $\lambda^\sharp > 0$ such that $\chi(\lambda) =: \sum_{l=0}^{\infty} \sum_{q=0}^{\infty} \beta(l)\gamma(q)e^{\lambda(l+q)}$ is convergent when $\lambda \in [0, \lambda^\sharp)$, but $\lim_{\lambda \to \lambda^\sharp} \chi(\lambda) = +\infty$, where $\lambda^\sharp$ may be $+\infty$;

(H2) *Two constant-equilibria* $w_\pm$: there exist $w_- = 0$ and $w_+ = K > 0$ such that $b(0) = 0$, $\varepsilon b(K) = dK$, $b \in C^2[0, K]$ and for all $w \in (0, K)$, $\varepsilon b(w) > dw$;

(H3) *Mono-stable type*: $\varepsilon b'(0) > d$ (unstable node: $w_- = 0$) and $d > \varepsilon b'(K)$ (stable node: $w_+ = K$);

(H4) *Sub-linearity*: $b'(w) \geq 0$ and $b''(w) \leq 0$ for $w \in [0, K]$.

When the immature population is non-mobile, then (1.1) can be reduced to the delayed local lattice differential equation [1, 3, 7, 26]

$$\begin{aligned}\frac{dw_{i,j}(t)}{dt} &= D[w_{i+1,j}(t) + w_{i-1,j}(t) + w_{i,j+1}(t) + w_{i,j-1}(t) - 4w_{i,j}(t)] \\ &\quad - dw_{i,j}(t) + \varepsilon b(w_{i,j}(t-r)).\end{aligned} \tag{1.3}$$

The main purpose of the present paper is to investigate the stability of traveling wavefronts for the $2D$ time-delayed differential-lattice equation (1.1). A traveling wave solution of (1.1) connecting $w_- = 0$ and $w_+ = K$ is a special solution with the form of

$$\begin{cases} w_{i,j}(t) = \phi(i\cos\theta + j\sin\theta + ct) =: \phi(\xi), \\ \xi := i\cos\theta + j\sin\theta + ct, \end{cases} \quad \text{for } i, j \in \mathbb{Z}, \text{ and } \theta \in [0, \pi/2],$$

i.e., ϕ satisfies the following wave profile equation

$$\begin{aligned} c\frac{d\phi(\xi)}{d\xi} &= D[\phi(\xi + \cos\theta) + \phi(\xi - \cos\theta) + \phi(\xi + \sin\theta) + \phi(\xi - \sin\theta) - 4\phi(\xi)] \\ &\quad - d\phi(\xi) + \varepsilon \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b(\phi(\xi + l\cos\theta + q\sin\theta - cr)), \\ \phi(\pm\infty) &= w_\pm. \end{aligned} \tag{1.4}$$

The constant θ represents the direction of the wave. We call c the wave speed and ϕ the wave profile. Moreover, we say ϕ is a traveling wavefront if $\phi\colon \mathbb{R} \to \mathbb{R}$ is monotone.

The existence of traveling wavefronts for the $2D$ lattice model (1.1) with monotone birth rate function $b(w)$ has been obtained by Cheng et al. [2] and Weng et al. [20] by using the methods developed in [12, 19] for $1D$ lattice differential equations. Recently, when the birth function $b(u)$ is non-monotone, the existence of the wavefronts to (1.1) was further investigated by Yu et al. [27], and the uniqueness of the wavefronts (including the critical wave) was showed by Yu and Mei in [24], recently, where the birth function is non-monotone but the nonlocal interactions $\beta(l)$ and $\gamma(q)$ are compactly supported. Furthermore, the stability of wavefronts for (1.1) (including the local case (1.3)) was studied in [3, 23, 26] by using the weighted energy method (see also [9–11, 13, 14, 16–18, 21, 22, 25]), where, for the local case (1.3), Cheng et al. [3] obtained the stability for the wavefronts with large wave speed $c \gg 1$ and restricted the initial perturbation to be small enough, and Yu and Yuan [26] further showed the stability with large speed $c \gg 1$, but allowed the initial perturbation to be arbitrarily large in certain weighted space due to the comparison principle, while, for the nonlocal case (1.1), Wu and Liu [23] similarly proved stability in the case of the large wave speed with a large initial perturbation. However, the most interesting but also challenging case is to study the stability of the wavefronts with the speed c arbitrarily close to the critical wave speed $c_*(\theta) > 0$, particularly, the case of $c = c_*(\theta) > 0$, because the spreading speed is just the minimum wave speed $c_*(\theta) > 0$. In this paper, for the nonlocal lattice equation (1.1), we shall prove that all non-critical wavefronts with $c > c_*(\theta)$ are globally stable. The exponential convergence rate will be also derived. These essentially improve the existing wave stability results [3, 23, 26]. The approach adopted is still the weighted energy method and the squeezing technique with the help of Gronwall's inequality, but the chosen weight function is optimal and depends on the eigenvalue of the characteristic equation for the wave profiles. As a corollary of the stability result, we also obtain the uniqueness of traveling waves without assuming $\beta(l)$ and $\gamma(q)$ to be compactly supported. This also improves the existing uniqueness obtained in [24] where the compact support needs for the nonlocal interactions $\beta(l)$ and $\gamma(q)$. Regarding the case $c = c_*(\theta)$, unfortunately it still remains open, because the l^2-weighted energy method doesn't work out for this critical case.

The rest of this paper is organized as follows. In Section 2, we state the property of the characteristic equation and the existence result for traveling wavefronts. Based on the property of the characteristic equation, we then introduce a non-piecewise function e^{kx} for some carefully selected positive number k, which is the eigenvalue for the characteristic equation of the wave profiles. Lastly, we list the stability and uniqueness on the traveling wave front. In Section 3, by using the weighted energy method and the comparison principle with the help of Gronwall's inequality, we prove the stability for all traveling wavefronts $c > c_*(\theta)$. There will be no restriction on the delay time r, the wave speed c

and the initial perturbation. This essentially improves and develops the previous stability results in [3, 23, 26]. Section 4 is devoted to proving uniqueness of traveling wavefronts.

Notations. Throughout this paper, l_v^2 denotes the weighted l^2-space with weight $0 < v(\xi) \in C(\mathbb{R})$ and a fixed $\theta \in [0, \pi/2]$, that is,

$$l_v^2 := \left\{ \zeta = \{\zeta_{i,j}\}_{i,j \in Z}, \zeta_{i,j} \in \mathbb{R} \,\middle|\, \sum_{i,j} v(i\cos\theta + j\sin\theta)\zeta_{i,j}^2 < \infty \right\}$$

and its norm is defined by

$$\|\zeta\|_{l_v^2} = \left(\sum_{i,j} v(i\cos\theta + j\sin\theta)\zeta_{i,j}^2 \right)^{1/2} \quad \text{and} \quad \zeta \in l_v^2.$$

In particular, when $v \equiv 1$, we denote l_v^2 by l^2.

For given positive number $T > 0$, we also define a uniform continuous space by

$$\begin{aligned} &C_{\text{unif}}[-r, T] \\ &:= \left\{ \zeta_{i,j}(t) \in C[-r, T], i, j \in Z \,\middle|\, \lim_{i,j\to\infty} \zeta_{i,j}(t) \text{ exists uniformaly in } t \in [-r, T] \right\}. \end{aligned}$$

2. Preliminaries and main theorems

Let $\phi(\xi) = \phi(i\cos\theta + j\sin\theta + ct)$ be the wavefronts satisfying (1.4), and let us linearize (1.4) around $\phi = 0$ to have

$$\begin{aligned} c\frac{d\phi(\xi)}{d\xi} &= D[\phi(\xi + \cos\theta) + \phi(\xi - \cos\theta) + \phi(\xi + \sin\theta) + \phi(\xi - \sin\theta) - 4\phi(\xi)] \\ &\quad - d\phi(\xi) + \varepsilon \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(0)\phi(\xi + l\cos\theta + q\sin\theta - cr). \end{aligned}$$

Consider $\phi(\xi) = e^{\lambda\xi}$ for $\xi \to -\infty$ as the eigenfunction to the above equation, then we obtain the following characteristic equation for (1.4):

$$\begin{aligned} \Delta(c, \lambda) &:= -c\lambda + D\left(e^{\lambda\cos\theta} + e^{-\lambda\cos\theta} + e^{\lambda\sin\theta} + e^{-\lambda\sin\theta} - 4\right) \\ &\qquad + \varepsilon b'(0) \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)e^{\lambda(l\cos\theta + q\sin\theta - cr)} - d \\ &= 0. \end{aligned}$$

Now we recall some properties of $\Delta(c, \lambda)$, which was investigated in [3, 20, 24].

Lemma 2.1. *Assume that* (H1) *and* $\varepsilon b'(0) > d$ *hold. Then, there exist a unique pair of* $c_*(\theta) > 0$ *and* $\lambda_*(\theta) > 0$ *for any fixed* $\theta \in [0, \pi/2]$ *such that the following assertions hold.*

(i) $\Delta(c_*(\theta),\lambda_*(\theta))=0$, $\frac{\partial\Delta(c,\lambda)}{\partial\lambda}\big|_{c=c_*(\theta),\lambda=\lambda_*(\theta)}=0$;

(ii) *For any* $c\in(0,c_*(\theta))$ *and* $\lambda>0$, $\Delta(c,\lambda)>0$;

(iii) *For any* $c>c_*(\theta)$, $\Delta(c,\lambda)=0$ *has two positive roots* $0<\lambda_1<\lambda_2$. *Moreover,* $\Delta(c,\lambda)<0$ *for any* $\lambda\in(\lambda_1,\lambda_2)$.

The result on the existence of traveling wavefronts of (1.1) comes from Theorem 5.4 of [2] and Theorem 3.1 of [20].

Proposition 2.2. *Assume that* (H1)–(H4) *hold. Then, for every* $\theta\in[0,\pi/2]$, *there exists* $c_*(\theta)>0$ *such that for any* $c\ge c_*(\theta)$, (1.1) *admits a traveling wavefront* $\phi(i\cos\theta+j\sin\theta+ct)$ *satisfying the boundary conditions*

$$\lim_{\xi\to-\infty}\phi(\xi)=0 \quad \textit{and} \quad \lim_{\xi\to+\infty}\phi(\xi)=K,$$

and for any $c\in(0,c_*(\theta))$, *there are no non-trivial wavefront* $\phi(i\cos\theta+j\sin\theta+ct)$ *satisfying* $\phi(\xi)\in[0,K]$. *Moreover, for* $c>c_*(\theta)$, ϕ *satisfies*

$$\lim_{\xi\to-\infty}\phi(\xi)e^{-\lambda_1\xi}=1 \quad \textit{and} \quad \lim_{\xi\to-\infty}\phi'(\xi)e^{-\lambda_1\xi}=\lambda_1,$$

where $c_*(\theta)$ *and* λ_1 *are given in Lemma* 2.1.

Define a weight function $v(\xi)$ by

$$v(\xi)=e^{-2\lambda\xi},\quad \lambda\in(\lambda_1,\lambda_2). \tag{2.1}$$

Now we state our main result as follows.

Theorem 2.3 (Stability)**.** *Assume that* (H1)–(H4) *hold. For all traveling wavefronts* $\phi(i\cos\theta+j\sin\theta+ct)$ *with the wave speed* $c>c_*(\theta)$, *if the initial data satisfies*

$$0\le w_{i,j}^0(s)\le K \quad \textit{for } s\in[-r,0],\quad i,j\in\mathbb{Z},$$

and the initial perturbation $w_{i,j}^0(s)-\phi(i\cos\theta+j\sin\theta+cs)$ *is in* $C([-r,0],l_v^2\cap l^\infty)\cap C_{\mathrm{unif}}[-r,0]$, *where* $v=v(i\cos\theta+j\sin\theta+ct)$ *is the weight function given in* (2.1), *then the solution* $\{w_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *of* (1.1) *and* (1.2) *satisfies*

$$0\le w_{i,j}(t)\le K \quad \textit{for } t\in[0,+\infty),\quad i,j\in\mathbb{Z},$$

and

$$\{w_{i,j}(t)-\phi(i\cos\theta+j\sin\theta+ct)\}_{i,j\in\mathbb{Z}}\in C([0,\infty),l_v^2\cap l^\infty)\cap C_{\mathrm{unif}}[0,\infty).$$

In particular, the solution $\{w_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *converges to the traveling wave front* $\phi(i\cos\theta+j\sin\theta+ct)$ *exponentially in time* t, *that is,*

$$\sup_{i,j\in\mathbb{Z}}|w_{i,j}(t)-\phi(i\cos\theta+j\sin\theta+ct)|\le Ce^{-\mu t},\quad t\ge0$$

for some positive constants C *and* μ.

As a corollary of the stability results, we easily prove the following uniqueness.

Corollary 2.4 (Uniqueness)**.** *Assume that* (H1)–(H4) *hold. For any given* $\theta \in [0, \pi/2]$, *let* $\phi_1(i\cos\theta + j\sin\theta + ct)$ *and* $\phi_2(i\cos\theta + j\sin\theta + ct)$ *be two different traveling wavefronts of* (1.1) *with the same speed* $c > c_*(\theta)$. *Then* ϕ_1 *is a translation of* ϕ_2*; more precisely, there exists* $\xi_0 \in \mathbb{R}$ *such that* $\phi_1(\xi) = \phi_2(\xi + \xi_0)$, $\xi \in \mathbb{R}$.

Remark 2.5. (i) Motivated by the weight function in [4,8], we can also choose the same weight function (2.1) to overcome this difficulty caused by the nonlocal terms, which is different from the weight functions in [10,13–15]. Thus, we can obtain the stability of traveling wavefront for (1.1) with the wave speed $c > c_*(\theta)$. This improves the stability results for the case with the larger wave speed in [3, 23, 26].

(ii) The condition $d > \varepsilon b'(K) + 2D(e-1)$ (i.e., (H3) in [3]) and the condition $d > D(e-1) + \frac{1}{2}\varepsilon b'(K)(1 + L_2)$ (i.e., inequality (2.3) in [23]) can be weaken to the present condition (H3): $d > \varepsilon b'(K)$.

(iii) As a corollary of the stability result, the uniqueness of traveling waves is obtained without assuming $\beta(l)$ and $\gamma(q)$ to be compactly supported.

(iv) When $\beta(0) = \gamma(0) = 1$, $\beta(l) = \gamma(l) = 0$ for $l \in \mathbb{Z}/\{0\}$, then (1.1) is reduced to the local delayed lattice differential equation (1.3). Thus Theorem 2.3 and Corollary 2.4 still hold for (1.3).

3. Stability

Throughout this section, we assume that (H1)–(H4) hold. First of all, we recall the existence and the comparison principle presented in [3, 23] for the Cauchy problem (1.1) and (1.2), then prove our stability result by using the weighted energy method and the squeezing technique with the help of Gronwall's inequality.

Lemma 3.1 (Existence)**.** *For any initial function* $w^0(s) = \{w^0_{i,j}(s)\}_{i,j\in\mathbb{Z}} \in C([-\tau, 0], l^\infty)$, (1.1) *with the initial condition* (1.2) *has a unique solution* $w(t) = \{w_{i,j}(t)\}_{i,j\in\mathbb{Z}} \in C([-\tau, 0], l^\infty)$. *Furthermore, if* $\{w^0_{i,j}(s) - \phi(i\cos\theta + j\sin\theta + cs)\}_{i,j\in\mathbb{Z}} \in C([-\tau, 0], l^2)$, *then* $\{w_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)\}_{i,j\in\mathbb{Z}} \in C([0, \infty), l^2)$. *Similarly, if* $\{w^0_{i,j}(s) - \phi(i\cos\theta + j\sin\theta + cs)\}_{i,j\in\mathbb{Z}} \in C_{\mathrm{unif}}[-\tau, 0]$, *then* $\{w_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)\}_{i,j\in\mathbb{Z}} \in C_{\mathrm{unif}}[0, \infty)$.

Lemma 3.2 (Comparison principle)**.** *Let* $\{\overline{w}_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *and* $\{\underline{w}_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *be the solutions of* (1.1) *and* (1.2) *with the initial data* $\{\overline{w}^0_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *and* $\{\underline{w}^0_{i,j}(t)\}_{i,j\in\mathbb{Z}}$, *respectively. If*

$$0 \le \underline{w}^0_{i,j}(s) \le \overline{w}^0_{i,j}(s) \le K \quad \text{for } s \in [-r, 0], \quad i, j \in \mathbb{Z}$$

then

$$0 \le \underline{w}_{i,j}(t) \le \overline{w}_{i,j}(t) \le K \quad \text{for } t \in [0, +\infty), \quad i, j \in \mathbb{Z}.$$

In what follows, we shall prove the stability theorem by using the comparison principle together with the weighed energy method.

We assume that the initial data $w^0_{i,j}(s)$ satisfy $0 \le w^0_{i,j}(s) \le K$ for $s \in [-r,0]$, $i,j \in \mathbb{Z}$, and $w^0_{i,j}(s) - \phi(i\cos\theta + j\sin\theta + cs) \in C([-r,0], l^2_v \cap l^\infty) \cap C_{\text{unif}}[-r,0]$. Take

$$\begin{cases} \overline{W}^0_{i,j}(s) = \max\left\{ w^0_{i,j}(s), \phi(i\cos\theta + j\sin\theta + cs) \right\}, \\ \underline{W}^0_{i,j}(s) = \min\left\{ w^0_{i,j}(s), \phi(i\cos\theta + j\sin\theta + cs) \right\} \end{cases} \quad \text{for } s \in [-r,0], \quad i,j \in \mathbb{Z}.$$

According to Lemmas 3.1 and 3.2, it is easily seen that

$$0 \le \underline{W}_{i,j}(t) \le w_{i,j}(t),\ \phi(i\cos\theta + j\sin\theta + ct) \le \overline{W}_{i,j}(t) \le K \quad \text{for } t \in [0,+\infty), \quad i,j \in \mathbb{Z} \tag{3.1}$$

and

$$\begin{aligned} &\overline{W}_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct),\ \underline{W}_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct) \\ &\in C([-r,0], l^2_v \cap l^\infty) \cap C_{\text{unif}}[-r,0], \end{aligned}$$

where $\overline{W}^0_{i,j}(t)$ and $\underline{W}^0_{i,j}(t)$ are the corresponding solutions of (1.1) and (1.2) with the initial data $\overline{W}^0_{i,j}(s)$ and $\underline{W}^0_{i,j}(s)$, respectively.

3.1. A prior estimates

For the sake of convenience, we always let $\xi = \xi(t,i,j) := i\cos\theta + j\sin\theta + ct$. Take

$$u_{i,j}(t) = \overline{W}_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct), \quad t \in [0,+\infty),\ i,j \in \mathbb{Z}$$

and

$$u^0_{i,j}(s) = \overline{W}^0_{i,j}(s) - \phi(i\cos\theta + j\sin\theta + cs), \quad s \in [-r,0],\ i,j \in \mathbb{Z}.$$

Therefore, it follows from (3.1) that

$$u_{i,j}(t) \ge 0 \quad \text{and} \quad u^0_{i,j}(s) \ge 0.$$

From (1.1) and (1.2), $u_{i,j}(t)$ satisfies

$$\begin{aligned} \frac{du_{i,j}(t)}{dt} &= D[u_{i+1,j}(t) + u_{i-1,j}(t) + u_{i,j+1}(t) + u_{i,j-1}(t) - 4u_{i,j}(t)] - du_{i,j}(t) \\ &\quad + \varepsilon \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q) b'(\phi(\widetilde{\xi}(t,i,j)) u_{i+l,j+q}(t-r) + Q_{i,j}(t-r), \quad t > 0 \\ u^0_{i,j}(s) &= \overline{W}^0_{i,j}(s) - \phi(\xi(s,i,j)), \quad i,j \in \mathbb{Z},\ s \in [-r,0], \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} Q_{i,j}(t-r) = \varepsilon \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)\big[b(u_{i+l,j+q}(t-r) + \phi(\widetilde{\xi}(t,i,j)) - b(\phi(\widetilde{\xi}(t,i,j))) \\ - b'(\phi(\widetilde{\xi}(t,i,j)))u_{i+l,j+q}(t-r)\big], \\ \widetilde{\xi}(t,i,j) = \xi(t,i,j) + l\cos\theta + q\sin\theta - cr \end{aligned}$$

and

$$u_{i\pm1,j}(t) = \overline{W}_{i\pm1,j}(t) - \phi(\xi(t,i,j) \pm \cos\theta), \quad u_{i,j\pm1}(t) = \overline{W}_{i,j\pm1}(t) - \phi(\xi(t,i,j) \pm \sin\theta).$$

By Taylor's formula and assumption (H2), we have $Q_{i,j}(t-r) \le 0$.

Now we are going to establish *a priori* estimate for the solution $u_{i,j}(t)$. Let

$$u_{i,j}(t) \in C([0,T]; l_v^2 \cap l^\infty) \cap C_{\text{unif}}[0,T] =: X(0,T)$$

for any given $T > 0$, where $X(0,T)$ is called the solution space. We are going to prove the exponential decay of the solution in serval lemmas. In order to obtain a weighted energy estimate, we need the following key inequality.

Let

$$\begin{aligned} B_{\mu,v}(\xi(t,i,j)) := A_{\mu,v}(\xi(t,i,j)) - 2\mu \\ - (e^{2\mu r} - 1)\varepsilon \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q) b'(\phi(\widetilde{\xi}(t,i,j))) \frac{v((t+r,i-l,j-q))}{v(\xi(t,i,j))} \frac{1}{\kappa_5}, \end{aligned}$$

where

$$\begin{aligned} &A_{\mu,v}(\xi(t,i,j)) \\ &:= -c\frac{v'_\xi(\xi(t,i,j))}{v(\xi(t,i,j))} - D\bigg\{ \left[\kappa_1 + \frac{1}{\kappa_1}\frac{v(\xi(t,i-1,j))}{v(\xi(t,i,j))}\right] + \left[\kappa_2 + \frac{1}{\kappa_2}\frac{v(\xi(t,i+1,j))}{v(\xi(t,i,j))}\right] \\ &\quad + \left[\kappa_3 + \frac{1}{\kappa_3}\frac{v(\xi(t,i,j-1))}{v(\xi(t,i,j))}\right] + \left[\kappa_4 + \frac{1}{\kappa_4}\frac{v(\xi(t,i,j+1))}{v(\xi(t,i,j))}\right] - 8 \bigg\} + 2d \\ &\quad - \varepsilon \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q) b'(\phi(\widetilde{\xi}(t,i,j))) \left[\kappa_5 + \frac{v((t+r,i-l,j-q))}{v(\xi(t,i,j))}\frac{1}{\kappa_5}\right] \end{aligned}$$

and

$$\kappa_1 = e^{\lambda\cos\theta}, \quad \kappa_2 = e^{-\lambda\cos\theta}, \quad \kappa_3 = e^{\lambda\sin\theta}, \quad \kappa_4 = e^{-\lambda\sin\theta}, \quad \kappa_5 = e^{\lambda(l\cos\theta + j\sin\theta - cr)}.$$

Lemma 3.3. *For any $c > c_*(\theta)$ and $\lambda \in (\lambda_1, \lambda_2)$, $A_{\mu,v}(\xi(t,i,j)) \ge C_2 > 0$ for some positive constant C_2, which is independent on t, i, j, μ.*

Proof. Notice that $0 \le b'(w) \le b'(0)$ for any $w \in [0, K]$ and

$$\frac{v'_\xi(\xi(t,i,j))}{v(\xi(t,i,j))} = -2\lambda, \qquad \frac{v(\xi(t,i-1,j))}{v(\xi(t,i,j))} = e^{2\lambda\cos\theta},$$
$$\frac{v(\xi(t,i+1,j))}{v(\xi(t,i,j))} = e^{-2\lambda\cos\theta}, \qquad \frac{v(\xi(t,i,j-1))}{v(\xi(t,i,j))} = e^{2\lambda\sin\theta},$$
$$\frac{v(\xi(t,i,j+1))}{v(\xi(t,i,j))} = e^{-2\lambda\sin\theta}, \qquad \frac{v(\xi(t+r,i-l,j-q))}{v(\xi(t,i,j))} = e^{2\lambda(l\cos\theta+q\sin\theta-cr)}.$$

Thus, we have

$$\begin{aligned} A_{\mu,v}(\xi(t,i,j)) \ge 2c\lambda - D\bigg\{ & \left[\kappa_1 + \frac{1}{\kappa_1}e^{2\lambda\cos\theta}\right] + \left[\kappa_2 + \frac{1}{\kappa_2}e^{-2\lambda\cos\theta}\right] \\ & + \left[\kappa_3 + \frac{1}{\kappa_3}e^{-2\lambda\cos\theta}\right] + \left[\kappa_4 + \frac{1}{\kappa_4}e^{-2\lambda\sin\theta}\right] - 8\bigg\} + 2d \\ & - \varepsilon \sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(0)\left[\kappa_5 + e^{2\lambda(l\cos\theta+j\sin\theta-cr)}\frac{1}{\kappa_5}\right] \\ = {} & 2c\lambda - 2D\left(e^{\lambda\cos\theta} + e^{-\lambda\cos\theta} + e^{\lambda\sin\theta} + e^{-\lambda\sin\theta} - 4\right) + 2d \\ & - 2\varepsilon \sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(0)e^{\lambda(l\cos\theta+j\sin\theta-cr)} \\ = {} & -2\Delta(c,\lambda) =: C_2 > 0 \quad \text{for } \lambda \in (\lambda_1, \lambda_2) \end{aligned}$$

according to Lemma 2.1. This completes the proof. □

Lemma 3.4 (Key inequality)**.** *For any $c > c_*(\theta)$ and $\lambda \in (\lambda_1, \lambda_2)$, there exists a positive number μ_1 such that $B_{\mu,v}(\xi(t,i,j)) > 0$ for $0 < \mu < \mu_1$, where μ_1 is the unique root of the following equation*

$$C_2 - 2\mu - (e^{2\mu r} - 1)\varepsilon \sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(0)e^{\lambda(l\cos\theta+j\sin\theta-cr)} = 0.$$

Proof. Let

$$F(\mu) = C_2 - 2\mu - (e^{2\mu r} - 1)\varepsilon \sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(0)e^{\lambda(l\cos\theta+j\sin\theta-cr)}$$

for $\mu \in [0, +\infty)$. Since $F(0) = C_2 > 0$ by Lemma 3.3 and $F(+\infty) = -\infty < 0$, there exists a positive number μ_1 such that $F(\mu_1) = 0$ according to Intermediate Value Theorem. In addition, this conclusion holds since $F'(\mu) < 0$. This completes the proof. □

Lemma 3.5. *For any $c > c_*(\theta)$, it holds that*

$$e^{2\mu t}\|u(t)\|_{l_v^2}^2 \le \left\|u^0(0)\right\|_{l_v^2}^2 + C_1 \int_{-r}^{0} \left\|u^0(s)\right\|_{l_v^2}^2 ds, \quad t \ge 0,\ 0 < \mu < \mu_1. \tag{3.3}$$

Proof. Let $v(\xi)$ be the weight function defined in (2.1). Multiplying (3.2) by $e^{2\mu t}u_{i,j}(t)$ $v(\xi(t,i,j))$, where $\mu > 0$ will be given in Lemma 3.4, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[e^{2\mu t}u_{i,j}^2(t)v(\xi(t,i,j))\right] \\ &- De^{2\mu t}u_{i,j}(t)v(\xi(t,i,j))[u_{i+1,j}(t)+u_{i-1,j}(t)+u_{i,j+1}(t)+u_{i,j-1}(t)-4u_{i,j}(t)] \\ &+ e^{2\mu t}u_{i,j}^2(t)v(\xi(t,i,j))\left[-c\frac{v'_\xi(\xi(t,i,j))}{2v(\xi(t,i,j))}+d-\mu\right] \\ &- \varepsilon\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(t,i,j)))e^{2\mu t}v(\xi(t,i,j))u_{i,j}(t)u_{i+l,j+q}(t-r) \\ = {}& Q_{i,j}(t-r)e^{2\mu t}u_{i,j}(t)v(\xi(t,i,j)) \le 0. \end{aligned}$$

By the Cauchy-Schwarz inequality $|xy| \le \frac{\kappa_i}{2}x^2 + \frac{1}{2\kappa_i}y^2$ for any $\kappa_i > 0$, $i = 1,2,3,4,5$, we obtain

$$u_{i,j}(t)u_{i\pm 1,j\pm 1}(t) \le \frac{\kappa_i u_{i,j}^2(t)}{2} + \frac{u_{i\pm 1,j\pm 1}^2(t)}{2\kappa_i}.$$

Summing about all $i,j \in \mathbb{Z}$ and integrating over $[0,t]$, this yields

(3.4)

$$\begin{aligned} &e^{2\mu t}\left\|u(t)\right\|_{l_v^2}^2 - \left\|u^0(0)\right\|_{l_v^2}^2 + \int_0^t\sum_{i,j}e^{2\mu s}u_{i,j}^2(s)v(\xi(s,i,j))\left\{-c\frac{v'_\xi(\xi(s,i,j))}{v(\xi(s,i,j))}+2d-2\mu\right\}ds \\ \le{}& \int_0^t\sum_{i,j}De^{2\mu s}\bigg\{\left[\kappa_1u_{i,j}^2(s)+\frac{1}{\kappa_1}u_{i+1,j}^2(s)\right]+\left[\kappa_2u_{i,j}^2(s)+\frac{1}{\kappa_2}u_{i-1,j}^2(s)\right] \\ &+\left[\kappa_3u_{i,j}^2(s)+\frac{1}{\kappa_3}u_{i,j+1}^2(s)\right]+\left[\kappa_4u_{i,j}^2(s)+\frac{1}{\kappa_4}u_{i,j-1}^2(s)\right]-8u_{i,j}^2(s)\bigg\}v(\xi(s,i,j))\,ds \\ &+2\varepsilon\int_0^t\sum_{i,j}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(t,i,j)))e^{2\mu s}v(\xi(s,i,j))u_{i,j}(s)u_{i+l,j+q}(s-r)\,ds \\ \le{}& \int_0^t\sum_{i,j}De^{2\mu s}u_{i,j}^2(s)v(\xi(s,i,j))\bigg\{\left[\kappa_1+\frac{1}{\kappa_1}\frac{v(\xi(s,i-1,j))}{v(\xi(t,i,j))}\right]+\left[\kappa_2+\frac{1}{\kappa_2}\frac{v(\xi(s,i+1,j))}{v(\xi(t,i,j))}\right] \\ &+\left[\kappa_3+\frac{1}{\kappa_3}\frac{v(\xi(s,i,j-1))}{v(\xi(t,i,j))}\right]+\left[\kappa_4+\frac{1}{\kappa_4}\frac{v(\xi(s,i,j+1))}{v(\xi(t,i,j))}\right]-8\bigg\}ds \\ &+2\varepsilon\int_0^t\sum_{i,j}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(t,i,j)))e^{2\mu s}v(\xi(s,i,j))u_{i,j}(s)u_{i+l,j+q}(s-r)\,ds. \end{aligned}$$

Now we begin to estimate last term

$$2\varepsilon\int_0^t\sum_{i,j}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(s,i,j)))e^{2\mu s}v(\xi(s,i,j))u_{i,j}(s)u_{i+l,j+q}(s-r)\,ds$$

in (3.4).

In view of (H1), there exists some positive number $C_1 > 0$ such that

$$
\begin{aligned}
&\varepsilon \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(0)e^{2\mu r} \frac{v(\xi(s+r,i-l,j-q))}{v(\xi(s,i,j))} \frac{1}{\kappa_5} \\
&= \varepsilon b'(0)e^{2\mu r} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)e^{\lambda(l\cos\theta + q\sin\theta - cr)} \\
&\le \varepsilon b'(0)e^{2\mu r} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)e^{\lambda(|l|+|q|)} \\
&\le C_1 \quad \text{for all } i,j \in \mathbb{Z}.
\end{aligned}
$$

It follows from (H4) that $0 \le b'(w) \le b'(0)$ for any $w \in [0,K]$. Thus, we obtain

(3.5)

$$
\begin{aligned}
&2\varepsilon \int_0^t \sum_{i,j} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(s,i,j)))e^{2\mu s}v(\xi(s,i,j))u_{i,j}(s)u_{i+l,j+q}(s-r)\,ds \\
&\le \varepsilon \int_0^t \sum_{i,j} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(s,i,j))e^{2\mu s}v(\xi(s,i,j))\kappa_5 u_{i,j}^2(s)\,ds \\
&\quad + \varepsilon \int_0^t \sum_{i,j} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(s,i,j))e^{2\mu s}v(\xi(s,i,j))\frac{1}{\kappa_5}u_{i+l,j+q}^2(s-r)\,ds \\
&= \varepsilon \int_0^t \sum_{i,j} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(s,i,j))e^{2\mu s}v(\xi(s,i,j))\kappa_5 u_{i,j}^2(s)\,ds \\
&\quad + \varepsilon \int_{-r}^{t-r} \sum_{i,j} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(s,i,j)))e^{2\mu s}v(\xi(s+r,i-l,j-q))\frac{1}{\kappa_5}u_{i,j}^2(s)e^{2\mu r}\,ds \\
&= \varepsilon \int_0^t \sum_{i,j} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(s,i,j))e^{2\mu s}v(\xi(s,i,j))\kappa_5 u_{i,j}^2(s)\,ds \\
&\quad + \varepsilon \int_0^{t-r} \sum_{i,j} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(s,i,j))e^{2\mu s}v(\xi(s+r,i-l,j-q))\frac{1}{\kappa_5}u_{i,j}^2(s)e^{2\mu r}\,ds \\
&\quad + \varepsilon \int_{-r}^{0} \sum_{i,j} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(s,i,j))e^{2\mu s}\frac{v(\xi(s+r,i-l,j-q))}{v(\xi(s,i,j))}v(\xi(s,i,j)) \\
&\qquad \times \frac{1}{\kappa_5}u_{i,j}^2(s)e^{2\mu r}\,ds \\
&\le \varepsilon \int_0^t \sum_{i,j} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(s,i,j))e^{2\mu s}v(\xi(s,i,j))\kappa_5 u_{i,j}^2(s)\,ds \\
&\quad + \varepsilon \int_0^t \sum_{i,j} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b'(\phi(\widetilde{\xi}(s,i,j))e^{2\mu s}v(\xi(s+r,i-l,j-q)\frac{1}{\kappa_5}u_{i,j}^2(s)e^{2\mu r}\,ds \\
&\quad + C_1 \int_{-r}^{0} \left\|u^0(s)\right\|_{l_v^2}^2\,ds.
\end{aligned}
$$

Thus, it follows from (3.4) and (3.5) that

$$
\begin{aligned}
& e^{2\mu t}\|u(t)\|_{l_v^2}^2+\int_0^t \sum_{i,j} e^{2\mu s} u_{i,j}^2(s) v(\xi(s,i,j)) B_{\mu,v}(\xi(s,i,j))\, ds \\
& \leq \left\|u^0(0)\right\|_{l_v^2}^2+C_1 \int_{-r}^0\left\|u^0(s)\right\|_{l_v^2}^2 ds .
\end{aligned} \tag{3.6}
$$

According to Lemma 3.4 and dropping the positive term

$$
\int_0^t \sum_{i,j} e^{2\mu s} u_{i,j}^2(s) v(\xi(s,i,j)) B_{\mu,\nu}(\xi(s,i,j))\, ds
$$

in (3.6), we obtain the this basic energy estimate. This completes the proof. □

Lemma 3.6. *For any $c > c_*(\theta)$, there exists a large number $N \gg 1$ (independent of t and ξ) such that*

$$
\sup_{i,j\geq N} |u_{i,j}(t)| \leq C_3 e^{-\mu_2 t}, \quad t \geq 0
$$

for some positive constants C_3 and $0 < \mu_2 \leq d - \varepsilon b'(K)$.

Proof. According to $Q_{i,j}(t-r) \leq 0$, (3.2) can be reduced to

$$
\begin{aligned}
\frac{du_{i,j}(t)}{dt} \leq\; & D[u_{i+1,j}(t)+u_{i-1,j}(t)+u_{i,j+1}(t)+u_{i,j-1}(t)-4u_{i,j}(t)]-du_{i,j}(t) \\
& +\varepsilon \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q) b'(\phi(\widetilde{\xi}(t,i,j)) u_{i+l,j+q}(t-r).
\end{aligned} \tag{3.7}
$$

Since $u_{i,j}(t) \in C_{\text{unif}}([0,T]$, namely, $\lim_{i,j\to\infty} u_{i,j}(t) := u_\infty(t)$ exists uniformly in $t \in [0,T]$. Now, let us take limits as $i,j \to \infty$, then from (3.7) we have

$$
\frac{du_\infty(t)}{dt} \leq -du_\infty(t) + \varepsilon b'(K) u_\infty(t-r). \tag{3.8}
$$

Since $b'(K) > 0$, integrating (3.8) over $[0,t]$, we can obtain

$$
\int_0^t u_\infty(s-r)\, ds = \int_{-r}^{t-r} u_\infty(s)\, ds \leq \int_0^t u_\infty(s)\, ds + \int_{-r}^0 u_\infty^0(s)\, ds
$$

and

$$
u_\infty(t) \leq [\varepsilon b'(K) - d] \int_0^t u_\infty(s)\, ds + \varepsilon b'(K) \int_{-r}^0 u_\infty^0(s)\, ds + u_\infty^0(0). \tag{3.9}
$$

By the Gronwall's inequality, (3.9) yields

$$
\lim_{i,j\to\infty} |u_{i,j}(t)| := |u_\infty(t)| \leq C_4 e^{-\mu_2 t},
$$

where $0 < \mu_2 \leq d - \varepsilon b'(K)$. Thus, we immediately obtain that, there exists a large number $N \gg 1$ which is independent of t and ξ such that

$$
\sup_{i,j\geq N} |u_{i,j}(t)| \leq C_3 e^{-\mu_2 t}, \quad t \geq 0.
$$

This completes the proof. □

Lemma 3.7. *For any $c > c_*(\theta)$, it holds that*

$$\sup_{i,j\in(-\infty,N]} |u_{i,j}(t)| \le C_4 e^{-\mu t}\left(\left\|u^0(0)\right\|_{l_v^2}^2 + \int_{-r}^0 \left\|u^0(s)\right\|_{l_v^2}^2 ds\right)^{1/2}, \quad t \ge 0,\ 0 < \mu < \mu_1$$

for all $t \ge 0$ and some positive constant C_4.

Proof. Since the weight function satisfies

$$v(\xi) = e^{-2\lambda\xi} \ge e^{-2\lambda N} > 0 \quad \text{for } \xi \in (-\infty, N],$$

by the standard Sobolev's embedding inequality $l^2 \hookrightarrow l^\infty$, then it follows from (3.3) in Lemma 3.5 that

$$\begin{aligned}\sup_{i,j\in(-\infty,N]} |u_{i,j}(t)| &\le \|u(t)\|_{l^2(-\infty,N]} \le e^{\lambda N}\|u(t)\|_{l_v^2(-\infty,N]} \le e^{\lambda N}\|u(t)\|_{l_v^2(R)} \\ &\le C_4 e^{-\mu t}\left(\left\|u^0(0)\right\|_{l_v^2}^2 + \int_{-r}^0 \left\|u^0(s)\right\|_{l_v^2}^2 ds\right)^{1/2}\end{aligned}$$

for $t \ge 0$, $0 < \mu < \mu_1$ and some positive constant C_4. The proof is complete. □

According to the Lemmas 3.6 and 3.7, one has immediately the conclusion.

Lemma 3.8. *For any $c > c_*(\theta)$, it holds that*

$$\sup_{i,j} \left|\overline{W}_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)\right| = \sup_{i,j} |u_{i,j}(t)| \le C_5 e^{-\mu t}$$

for some positive constant C_5, where $\mu \in (0, \min\{\mu_1, \mu_2\})$.

3.2. Proof of Theorem 2.3

Let

$$v_{i,j}(t) = \underline{W}_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)$$

and

$$v_{i,j}^0(s) = \underline{W}_{i,j}^0(s) - \phi(i\cos\theta + j\sin\theta + cs).$$

We can similarly prove that $\underline{W}_{i,j}(t)$ converges to $\phi(i\cos\theta + j\sin\theta + ct)$, i.e., the following lemma holds.

Lemma 3.9. *For any $c > c_*(\theta)$, it holds that*

$$\sup_{i,j} \left|\underline{W}_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)\right| = \sup_{i,j} |u_{i,j}(t)| \le C_6 e^{-\mu t}$$

for all $t \ge 0$ and some positive constant C_6, where $\mu \in (0, \min\{\mu_1, \mu_2\})$.

Thus, we can easily obtain the following lemma.

Lemma 3.10. *For any $c > c_*(\theta)$, it holds that*

$$\sup_{i,j} |W_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)| \le Ce^{-\mu t}$$

for all $t \ge 0$ and some positive constant C, where $\mu \in (0, \min\{\mu_1, \mu_2\})$.

Proof of Theorem 2.3. Since the initial data satisfy $\underline{W}_{i,j}(s) \le W_{i,j}(s) \le \overline{W}_{i,j}(s)$, $s \in [-r, 0]$, it follows from Lemma 3.2 that the corresponding solutions of (1.1) and (1.2) satisfy

$$\underline{W}_{i,j}(t) \le W_{i,j}(t) \le \overline{W}_{i,j}(t) \quad \text{for all } t \ge 0, \quad i, j \in \mathbb{Z}.$$

According to Lemmas 3.5–3.8, the squeeze method yields

$$\sup_{i,j} |W_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)| \le Ce^{-\mu t}$$

for all $t \ge 0$ and some positive constant C. This completes the proof. □

4. Uniqueness of traveling waves

This section is devoted to the proof of Corollary 2.4, i.e., the uniqueness of the traveling wavefronts can be obtained.

Proof of Corollary 2.4. According to Theorem 2.3 of [24], any two traveling wavefronts possess the same exponential decay at $-\infty$. Therefore, there exist two positive constants A and B such that

$$\phi_1(\xi) \sim Ae^{-\lambda_1|\xi|}, \quad \phi_2(\xi) \sim Be^{-\lambda_1|\xi|} \quad \text{as } \xi \to -\infty$$

where $\lambda_1 = \lambda_1(c) > 0$ is defined in Lemma 2.1 and $\sim$ is the sign of equivalence. Let us shift $\phi_2(i\cos\theta + j\sin\theta + ct)$ to $\phi_2(i\cos\theta + j\sin\theta + ct + m\cos\theta + n\sin\theta)$ with the constant shift $\xi_0 := m\cos\theta + n\sin\theta$. By taking $\xi \to -\infty$, obviously $\xi + \xi_0 < 0$. Choosing $\xi_0 = \frac{1}{\lambda_1}\ln\frac{A}{B}$, we have

$$\phi_2(\xi + \xi_0) \sim Be^{-\lambda_1|\xi+\xi_0|} = Be^{\lambda_1(\xi+\xi_0)} = Be^{\lambda_1\xi_0}e^{-\lambda_1|\xi|} = Ae^{-\lambda_1|\xi|} \quad \text{as } \xi \to -\infty$$

and

$$|\phi_2(\xi + \xi_0) - \phi_1(\xi)| = O(1)e^{-\alpha|\xi|} \in l_v^2, \quad \text{for } \alpha < \lambda_1, \text{ as } \xi \to -\infty.$$

And then, we take the initial data for (1.1) by

$$w_{i,j}^0(s) = \phi_2(i\cos\theta + j\sin\theta + cs + \xi_0), \quad s \in [-r, 0],\ i, j \in \mathbb{Z}.$$

Obviously, the corresponding solution to (1.1) is

$$w_{i,j}(t) = \phi_2(i\cos\theta + j\sin\theta + ct + \xi_0).$$

Applying the stability theorem, i.e., Theorem 2.3, we obtain

$$\lim_{t\to\infty} \sup_{i,j\in\mathbb{Z}} |\phi_2(i\cos\theta + j\sin\theta + ct + \xi_0) - \phi_1(i\cos\theta + j\sin\theta + ct)| = 0,$$

i.e., $\phi_2(i\cos\theta + j\sin\theta + ct + \xi_0) = \phi_1(i\cos\theta + j\sin\theta + ct)$ for all $i, j \in \mathbb{Z}$, as $t \gg 1$. In view of the arbitrary of i, j we have $\xi = i\cos\theta + j\sin\theta + ct \in \mathbb{R}$ and $\phi_2(\xi + \xi_0) = \phi_1(\xi)$. This completes the proof. □

5. Applications

In this section, we apply our results to some biological models.

Example 5.1. We first investigate the discrete diffusive Nicholson's blowflies model on $2D$ lattices

$$(5.1)\quad \begin{aligned} \frac{dw_{i,j}(t)}{dt} &= D[w_{i+1,j}(t) + w_{i-1,j}(t) + w_{i,j+1}(t) + w_{i,j-1}(t) - 4w_{i,j}(t)] - dw_{i,j}(t) \\ &\quad + \varepsilon p \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q) w_{i+l,j+q}(t-r) e^{-aw_{i+l,j+q}(t-r)}, \end{aligned}$$

which is the discrete-version in [5,6,13,14]. Let $b(u) = pue^{-au}$ for $1 < \varepsilon p/d \le e$, where a, p, d are positive constants, then $w_- = 0$ and $w_+ = K := \frac{1}{a}\ln\frac{\varepsilon p}{d}$ are the two equilibria of (5.1). It is easily checked that (H2)–(H4) hold. According to Theorem 2.3 and Corollary 2.4, we can obtain the following results.

Theorem 5.2. *Assume that* (H1) *and* $1 < \varepsilon p/d \le e$ *hold. For all traveling wavefronts* $\phi(i\cos\theta + j\sin\theta + ct)$ *with the wave speed* $c > c_*(\theta)$, *if the initial data satisfies*

$$0 \le w_{i,j}^0(s) \le K \quad \text{for } s \in [-r, 0], \quad i, j \in \mathbb{Z},$$

and the initial perturbation $w_{i,j}^0(s) - \phi(i\cos\theta + j\sin\theta + cs)$ *is in* $C([-r,0], l_v^2 \cap l^\infty) \cap C_{\text{unif}}[-r,0]$, *where* $v = v(i\cos\theta + j\sin\theta + ct)$ *is the weight function given in* (2.1), *then the solution* $\{w_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *converges to the traveling wave front* $\phi(i\cos\theta + j\sin\theta + ct)$ *exponentially in time* t, *that is,*

$$\sup_{i,j\in\mathbb{Z}} |w_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)| \le Ce^{-\mu t}, \quad t \ge 0$$

for some positive constants C *and* μ.

Corollary 5.3. *Assume that* (H1) *and* $1 < \varepsilon p/d \le e$ *hold. For any given* $\theta \in [0, \pi/2]$*, let* $\phi_1(i\cos\theta + j\sin\theta + ct)$ *and* $\phi_2(i\cos\theta + j\sin\theta + ct)$ *be two different traveling wavefronts of* (5.1) *with the same speed* $c > c_*(\theta)$*. Then there exists* $\xi_0 \in \mathbb{R}$ *such that* $\phi_1(\xi) = \phi_2(\xi + \xi_0)$*,* $\xi \in \mathbb{R}$*.*

Example 5.4. Now we consider the discrete diffusive Mackey-Glass model model on $2D$ lattices

$$\begin{aligned}\frac{dw_{i,j}(t)}{dt} &= D[w_{i+1,j}(t) + w_{i-1,j}(t) + w_{i,j+1}(t) + w_{i,j-1}(t) - 4w_{i,j}(t)] - dw_{i,j}(t) \\ &\quad + \varepsilon p \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q) \frac{w_{i+l,j+q}(t-r)}{1 + aw_{i+l,j+q}(t-r)},\end{aligned} \tag{5.2}$$

where D, d, p, a are positive constants. (5.2) can be regarded as a discrete version in [5,6,13,14]. If $\varepsilon p/d > 1$, then (5.2) has two equilibria $w_- = 0$ and $w_+ = K := (\varepsilon p - d)/(ad)$. Moreover, $b(w) = pw/(1 + aw)$ satisfies (H2)–(H4). Thus, we easily obtain the following results.

Theorem 5.5. *Assume that* (H1) *and* $\varepsilon p/d > 1$ *hold. For all traveling wavefronts* $\phi(i\cos\theta + j\sin\theta + ct)$ *with the wave speed* $c > c_*(\theta)$*, if the initial data satisfies*

$$0 \le w_{i,j}^0(s) \le K \quad \text{for } s \in [-r, 0], \quad i, j \in \mathbb{Z},$$

and the initial perturbation $w_{i,j}^0(s) - \phi(i\cos\theta + j\sin\theta + cs)$ *is in* $C([-r, 0], l_v^2 \cap l^\infty) \cap C_{\text{unif}}[-r, 0]$*, where* $v = v(i\cos\theta + j\sin\theta + ct)$ *is the weight function given in* (2.1)*, then the solution* $\{w_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *converges to the traveling wave front* $\phi(i\cos\theta + j\sin\theta + ct)$ *exponentially in time* t*, that is,*

$$\sup_{i,j\in\mathbb{Z}} |w_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)| \le Ce^{-\mu t}, \quad t \ge 0$$

for some positive constants C *and* μ*.*

Corollary 5.6. *Assume that* (H1) *and* $\varepsilon p/d > 1$ *hold. For any given* $\theta \in [0, \pi/2]$*, let* $\phi_1(i\cos\theta + j\sin\theta + ct)$ *and* $\phi_2(i\cos\theta + j\sin\theta + ct)$ *be two different traveling wavefronts of* (5.1) *with the same speed* $c > c_*(\theta)$*. Then there exists* $\xi_0 \in \mathbb{R}$ *such that* $\phi_1(\xi) = \phi_2(\xi + \xi_0)$*,* $\xi \in \mathbb{R}$*.*

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