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Growth of Solutions of Higher Order Complex Linear Differential Equation

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Abstract. Some new conditions on coefficient functions $A_i(z)$, which will guarantee all nontrivial solutions of $f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_0(z)f = 0$ are of infinite order, are found in this paper. The first condition involves two classes of extremal functions for some inequalities about finite asymptotic values and deficient values. The second condition assumes that a coefficient itself is a nontrivial solution of another differential equation $w'' + P(z)w = 0$, where $P(z)$ is a polynomial.

1. Introduction and main results

In this paper, all the functions considered are meromorphic function on the complex plane $\mathbb{C}$. Let f be a meromorphic function, its order of growth $\rho(f)$ and lower order of growth $\mu(f)$ (defined in terms of the Nevanlinna characteristic functions $T(r,f)$) are given by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}$$

and

$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r},$$

respectively. If f is entire, then the Nevanlinna characteristic $T(r,f)$ can be replaced with $\log M(r,f)$, where $M(r,f) = \max_{|z|=r} |f(z)|$. We assume that the reader is familiar with the fundamental results and standard notations in Nevanlinna theory, see [12, 16, 28] for more details.

Here we consider the differential equation

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

where $A_0(z) \not\equiv 0, A_1(z), \ldots, A_{n-1}(z)$ are entire functions, and $n \geq 2$ is an integer. It is well known that every meromorphic solution of (1.1) is entire and from now on by a solution of (1.1), we mean an entire solution and nontrivial solutions the nonconstant entire

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solutions. For the case of polynomial coefficients, a classical result due to Wittich [24]: If $A_0(z), A_1(z), \ldots, A_{n-1}(z)$ are entire functions, then all solutions of (1.1) are of finite order if and only if all coefficients $A_0(z), A_1(z), \ldots, A_{n-1}(z)$ are polynomials. Another result due to Frei [6] for (1.1) is: if $A_0(z), \ldots, A_j(z)$ are transcendental coefficients while $A_{j+1}(z), \ldots, A_{n-1}(z)$ are polynomials, then there can exist at most j linearly independent finite order solutions of (1.1). Thus it can be deduced that "most" of the solutions of (1.1) with at least one $A_i(z)$ transcendental have infinite order. On the other hand, there exist equations of the form (1.1) that possess one or more nontrivial solutions of finite order. For example: (a) $f(z) = -z$ solves $f'' - ze^z f' + e^z f = 0$, (b) $f(z) = c_1 \sin z + c_2 \cos z$ solves $f''' + e^z f'' + f' + e^z f = 0$, where c_1, c_2 are arbitrary constants, and (c) $f(z) = e^z + 1$ solves $f''' + 2e^{-z} f'' - e^z f' + (e^z - 2)f = 0$.

A natural question to ask is: What conditions on $A_0(z), A_1(z), \ldots, A_{n-1}(z)$ will guarantee that every solution of (1.1) is of infinite order? We mention that [10, 13, 21] contain results on this question for equation (1.1) when $n = 2$. Some recent papers that investigate the growth properties of solution of (1.1) include [3, 11, 18, 20]. In this paper, we obtain some results on this question by using two distinct approaches. To state our results, we will need some definitions. For $\theta \in \mathbb{R}$, let $\Delta(\theta) = \left\{re^{i\theta} : r \geq 0\right\}$. For $\alpha < \beta$ and $r, r_1, r_2 \in (0, \infty)$, set

$$\begin{aligned} S(\alpha, \beta) &= \{z : \alpha < \arg z < \beta\}, \\ S(\alpha, \beta; r) &= \{z : |z| < r, \alpha < \arg z < \beta\}, \\ S(\alpha, \beta; r_1, r_2) &= \{z : r_1 < |z| < r_2, \alpha < \arg z < \beta\}. \end{aligned}$$

Let $\overline{F}$ denote the closure of $F \subset \mathbb{C}$. We first recall the following definition due to Yang [27].

Definition 1.1. Let f be a meromorphic function of finite lower order $\mu(f) > 0$ in $\mathbb{C}$. A ray $\Delta(\theta)$ is called a Borel direction of order $\geq \mu(f)$ of f, if for each $\varepsilon > 0$,

$$\limsup_{r\to\infty} \frac{\log^+ n(S(\theta - \varepsilon, \theta + \varepsilon; r), 1/(f - a))}{\log r} \geq \mu(f) \tag{1.2}$$

for all $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, with at most two exceptions, where $n(S(\theta - \varepsilon, \theta + \varepsilon; r), 1/(f - a))$ denotes the number of zeros, counting the multiplicities, of $f(z) - a$ in $S(\theta - \varepsilon, \theta + \varepsilon; r)$.

The definition of Borel direction of order $\rho(f)$ of f can be found in [31, p. 78], it is defined similarly with the only exception that "$\geq \mu(f)$" in (1.2) is to be replaced with "$= \rho(f)$". From [27], the following result can be derived immediately, see also [25, Theorem A].

Theorem 1.2. *Let f be an entire function of finite lower order $\mu > 0$. Let q denote the number of Borel directions of order $\geq \mu$ of f, and let p denote the number of finite deficient values of f. Then $p \leq q/2$.*

An entire function f is called an extremal function for Yang's inequality if f satisfies the assumptions of Theorem 1.2 with $p = q/2$.

For extremal functions for Yang's inequality, the Borel directions of order $\rho(f)$ and Borel directions of order $\geq \mu(f)$ are one and the same by Lemma 4.1 below. For brevity, we call a Borel direction of order $\rho(f)$ simply as a Borel direction, unless otherwise specified. A simple example of an extremal function for Yang's inequality is $f(z) = e^z$. Then $z = 0$ is the one finite deficient value, $\Delta(\theta) = \arg z = \pm\pi/2$ are the two Borel directions. A slightly more complicated example is stated in Example 2.1 below.

The idea of this paper comes from [26]. The first result in this paper is obtained by using the properties of extremal functions for Yang's inequality.

Theorem 1.3. *Let $A_0(z), A_1(z), \ldots, A_{n-1}(z)$ be entire functions. Suppose that there exists an integer $l \in \{1, 2, \ldots, n-1\}$ such that $A_l(z)$ is an extremal function for Yang's inequality. Suppose that $A_0(z)$ is an entire function such that $\mu(A_0) \neq \rho(A_l)$ and $\mu(A_0) > \rho(A_i)$ for all $i \neq l$, $1 \leq i \leq n-1$. Then every nontrivial solution of* (1.1) *is of infinite order.*

An analogue of Theorem 1.3 in which the assumptions $\mu(A_0) \neq \rho(A_l)$ and $\mu(A_0) > \rho(A_i)$ are replaced with $\rho(A_0) \neq \rho(A_l)$ and $\rho(A_0) > \rho(A_i)$ respectively, can be found in [18, Theorem 5]. The proof of Theorem 1.3 deviates from that of [18, Theorem 5] in the sense that we require a modification of the Phragmén-Lindelöf principle, see Lemma 3.3 below.

Theorem 1.5 below relating with a conjecture due to Denjoy [4] in 1907 and a second order differential equation. To this end, we begin with recalling the conjecture.

Conjecture 1.4 (Denjoy's Conjecture)**.** *Let f be an entire function of finite order ρ. If f has k distinct finite asymptotic values, then $k \leq 2\rho$.*

It is verified by Ahlfors [1] in 1930. An entire function f is called an extremal function for Denjoy's conjecture if it is of finite order ρ and has $k = 2\rho$ distinct finite asymptotic values. This kind of functions are investigated by Ahlfors [1], Drasin [5], Kennedy [15] and Zhang [30]. An example of an extremal function for Denjoy's conjecture is stated in Example 2.2 below. Coefficient function $A_i(z)$ itself is a nontrivial solution of second order differential equation

$$w'' + P(z)w = 0, \tag{1.3}$$

where $P(z)$ is a polynomial. This assumption yields stability on the behavior of $A_i(z)$ via Hille's result, which is stated in Lemma 3.6 below. We get the following result by combining the extremal functions for Denjoy's conjecture with Lemma 3.6.

Theorem 1.5. *Let $A_0(z), A_1(z), \ldots, A_{n-1}(z)$ be entire functions. Suppose that there exists an integer $l \in \{1, 2, \ldots, n-1\}$ such that $A_l(z)$ is a nontrivial solution of* (1.3)*, where $P(z) = a_m z^m + \cdots + a_0$, $a_m \neq 0$. Suppose that $A_0(z)$ is an extremal function for Denjoy's conjecture such that $\rho(A_0) \neq \rho(A_l)$ and $\rho(A_0) > \rho(A_i)$ for all $i \neq l$, $1 \leq i \leq n-1$. Then every transcendental solution of* (1.1) *is of infinite order.*

We proceed to consider conditions on the coefficient $A_0(z)$ by restricting its growth. However, we mention that the $\cos \pi\rho$ theorem is not working if the order of growth (or lower order) is greater than $1/2$, we need new idea to cope with the case of the order of growth (or lower order) greater than $1/2$. Here we use a modification of the Phragmén-Lindelöf principle which is stated Lemma 3.3 to prove the following result.

Theorem 1.6. *Let $A_0(z), A_1(z), \ldots, A_{n-1}(z)$ be entire functions. Suppose that there exists an integer $l \in \{1, 2, \ldots, n-1\}$ such that $A_l(z)$ is a nontrivial solution of* (1.3)*, where $P(z) = a_m z^m + \cdots + a_0$, $a_m \neq 0$. Suppose that $A_0(z)$ is an entire function with $\mu(A_0) < 1/2 + 1/[2(m+1)]$ such that $\rho(A_0) \neq \rho(A_l)$ and $\mu(A_0) > \rho(A_i)$ for all $i \neq l$, $1 \leq i \leq n-1$. Then every transcendental solution of* (1.1) *is of infinite order.*

We are not sure whether the transcendental solution is necessary in the statements of Theorems 1.5 and 1.6, it is just needed in proving Theorems 1.5 and 1.6 when we apply Laine-Yang's result [17, Theorem 2.1]. For the case $n = 2$, an analogue of Theorem 1.7 below in which the assumptions $\mu(A_0) < \pi/\nu$ and $|A_1(z)| = O(e^{|z|^\eta})$ are replaced with $\rho(A_0) < \pi/\nu$ and $|A_1(z)| = O(|z|^\eta)$ respectively, is proved in [10, Theorem 7]. The proof of Theorem 1.7 is different from that of [10, Theorem 7] in the sense that we require Lemma 3.4 in Section 3.

Theorem 1.7. *Let $A_0(z), A_1(z), \ldots, A_{n-1}(z)$ be entire functions. Let $\{\phi_k\}$ and $\{\theta_k\}$ be finite collections of real numbers that satisfying $\phi_1 < \theta_1 < \phi_2 < \theta_2 < \cdots < \phi_m < \theta_m < \phi_{m+1}$ where $\phi_{m+1} = \phi_1 + 2\pi$, and set $\nu = \max_{1 \leq k \leq m} \{\phi_{k+1} - \theta_k\}$. Suppose that there exists an integer $l \in \{1, 2, \ldots, n-1\}$ such that for some constant $\eta \in [0, \rho(A_0))$, $A_l(z)$ satisfies*

$$|A_l(z)| = O\left(e^{|z|^\eta}\right) \tag{1.4}$$

as $z \to \infty$ in $\phi_k \leq \arg z \leq \theta_k$ for $k = 1, 2, \ldots, m$. Suppose that $A_0(z)$ satisfies $\mu(A_0) < \pi/\nu$ and $\rho(A_0) > \rho(A_i)$ for all $i \neq l$, $1 \leq i \leq n-1$. Then every nontrivial solution of (1.1) *is of infinite order.*

2. Examples

We will give examples which illustrate the existence of extremal functions for Yang's inequality and extremal functions for Denjoy's conjecture. The following example can be found in [28, pp. 210–211].

Example 2.1. For an integer $n \geq 2$, let

$$f(z) = \int_0^z e^{-t^n}\, dt.$$

Then $\rho(f) = n$, and f has $p = n$ finite deficient values

$$a_l = e^{i\frac{2\pi l}{n}} \int_0^\infty e^{-t^n}\, dt, \quad l = 1, 2, \ldots, n,$$

and $q = 2n$ Borel directions $\Delta(\theta_k) = \arg z = \frac{2k-1}{2n}\pi$, $k = 1, 2, \ldots, 2n$. Therefore, f is an extremal function for Yang's inequality.

The following example shows the existence of extremal functions for Denjoy's conjecture, which can be found in [31, p. 210].

Example 2.2. Let

$$f(z) = \int_0^z \frac{\sin t^q}{t^q}\, dt,$$

where $q > 0$ is an integer. Then $\rho(f) = q$, and f has $2q$ distinct finite asymptotic values

$$a_l = e^{l\pi i/q} \int_0^\infty \frac{\sin r^q}{r^q}\, dr$$

with its corresponding $2q$ asymptotic paths being

$$\arg z = \frac{l\pi}{q},$$

where $l = 1, 2, \ldots, 2q$.

3. Auxiliary results

The following lemma of Gundersen [9] on an estimation of logarithmic derivatives plays an important role in proving our results.

Lemma 3.1. *Let f be a transcendental meromorphic function of finite order $\rho(f)$. Let $\varepsilon > 0$ be given real constant, and let k and j be integers such that $k > j \geq 0$. Then there exists a set $E \subset [0, 2\pi)$ of linear measure zero, such that if $\varphi \in [0, 2\pi) - E$, there is a constant $r_0 = r_0(\varphi) > 1$ such that for all z satisfying $\arg z = \varphi$ and $|z| \geq r_0$,*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}.$$

The Lebesgue linear measure of a set $E \subset [0,\infty)$ is $\mathrm{m}(E) = \int_E dt$, and the logarithmic measure of a set $F \subset [1,\infty)$ is $\mathrm{m_l}(F) = \int_F \frac{dt}{t}$. The upper and lower logarithmic densities of F are given, respectively, by

$$\overline{\log\mathrm{dens}}(F) = \limsup_{r\to\infty} \frac{\mathrm{m_l}(F\cap[1,r])}{\log r}, \quad \underline{\log\mathrm{dens}}(F) = \liminf_{r\to\infty} \frac{\mathrm{m_l}(F\cap[1,r])}{\log r}.$$

The following result is due to Barry [2].

Lemma 3.2. *Let f be an entire function of lower order $\mu(f) \in [0,1)$, and denote $m(r) = \inf_{|z|=r} \log|f(z)|$ and $M(r) = \sup_{|z|=r} \log|f(z)|$. Then, for every $\alpha \in (\mu(f),1)$,*

$$\overline{\log\mathrm{dens}}(\{r \in [1,\infty) : m(r) > M(r)\cos\pi\alpha\}) \ge 1 - \frac{\mu(f)}{\alpha}.$$

The following result is a modified Phragmén-Lindelöf principle, which can be found in [26, Lemma 3.2].

Lemma 3.3. *Let f be an entire function of lower order $\mu(f) \in [1/2,\infty)$. Then there exists a sector domain $S(\alpha,\beta)$ with $\beta-\alpha \ge \pi/\mu(f)$, such that*

$$\limsup_{r\to\infty} \frac{\log\log\left|f(re^{i\theta})\right|}{\log r} \ge \mu(f)$$

for all the rays $\arg z = \theta \in (\alpha,\beta)$, where $0 \le \alpha < \beta \le 2\pi$.

The following lemma comes from [7, p. 177].

Lemma 3.4. *Let f be an analytic function in D and continuous in $\overline{D}$, where $D = S(\alpha,\beta)\cap \{z : |z| > r_0\}$, and α, β, r_0 are constants such that $0 < \beta-\alpha \le 2\pi$ and $r_0 > 0$. Suppose that there exists a constant $M > 0$ such that $|f(z)| \le M$ for $z \in \partial D$. If*

$$\liminf_{r\to\infty} \frac{\log\log M(r,D,f)}{\log r} < \frac{\pi}{\beta-\alpha},$$

where $M(r,D,f) = \max_{|z|=r,z\in D}|f(z)|$, then $|f(z)| \le M$ for all $z \in D$.

Remark 3.5. Let f be an analytic function in D and continuous in $\overline{D}$. Suppose that there exists a constant $M > 0$ such that $|f(z)| \le M$ for $z \in \partial D$. If $\mu(f) < \pi/(\beta-\alpha)$, then the conclusion of Lemma 3.4 holds.

In order to state the following lemma, we need some concepts. Let f be an entire function of order $\rho(f) \in (0,\infty)$. For simplicity, set $\rho = \rho(f)$ and $S = S(\alpha,\beta)$. We say that f blows up exponentially in $\overline{S}$ if for any $\theta \in (\alpha,\beta)$

$$\lim_{r\to\infty} \frac{\log\log\left|f(re^{i\theta})\right|}{\log r} = \rho.$$

We also say that f decays to zero exponentially in $\overline{S}$ if for any $\theta \in (\alpha, \beta)$

$$\lim_{r\to\infty} \frac{\log\log\left|f(re^{i\theta})\right|^{-1}}{\log r} = \rho.$$

The following lemma, originally due to Hille [14, Chapter 7.4], see also [8,22], plays an important role in proving Theorems 1.5 and 1.6.

Lemma 3.6. *Let w be a nontrivial solution of (1.3), where $P(z) = a_m z^m + \cdots + a_0$, $a_m \neq 0$. Set $\theta_j = \frac{2j\pi - \arg(a_m)}{m+2}$ and $S_j = S(\theta_j, \theta_{j+1})$, where $j = 0, 1, \dots, m+1$ and $\theta_{m+2} = \theta_0 + 2\pi$. Then w has the following properties.*

(1) *In each sector S_j, w either blows up or decays to zero exponentially.*

(2) *If, for some j, w decays to zero in S_j, then it must blow up in S_{j-1} and S_{j+1}. However, it is possible for w to blow up in many adjacent sectors.*

(3) *If w decays to zero in S_j, then w has at most finitely many zeros in any closed subsector within $S_{j-1} \cup \overline{S_j} \cup S_{j+1}$.*

(4) *If w blows up in S_{j-1} and S_j, then for each $\varepsilon > 0$, w has infinitely many zeros in each sector $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon)$, and furthermore, as $r \to \infty$,*

$$n(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon; r), w = 0) = (1 + o(1)) \frac{2\sqrt{|a_m|}}{\pi(m+2)} r^{(m+2)/2},$$

where $n(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon; r), w = 0)$ is the number of zeros of w in the region $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon; r)$.

4. Proof of Theorem 1.3

We begin by recalling some basic properties of extremal functions for Yang's inequality. To this end, if $A_l(z)$ is an extremal function for Yang's inequality, then the rays $\arg z = \theta_k$ are the q distinct Borel directions of $A_l(z)$, where $k = 1, 2, \dots, q$ and $0 \le \theta_1 < \theta_2 < \cdots < \theta_q < \theta_{q+1} = \theta_1 + 2\pi$.

Lemma 4.1. [25, Theorem 4] *Suppose that A_l is an extremal function for Yang's inequality. Then $\mu(A_l) = \rho(A_l)$. Moreover, for every finite deficient value a_i, $i = 1, 2, \dots, p$, there exists a corresponding sector domain $S(\theta_{k_i}, \theta_{k_i+1})$ such that for every $\varepsilon > 0$ the inequality*

$$\log \frac{1}{|A_l(z) - a_i|} > C(\theta_{k_i}, \theta_{k_i+1}, \varepsilon, \delta(a_i, A_l)) T(|z|, A_l)$$

holds for $z \in S(\theta_{k_i} + \varepsilon, \theta_{k_i+1} - \varepsilon; r, +\infty)$, where $C(\theta_{k_i}, \theta_{k_i+1}, \varepsilon, \delta(a_i, A_l))$ is a positive constant depending only on θ_{k_i}, θ_{k_i+1}, ε and $\delta(a_i, A_l)$.

Lemma 4.2. [19] *Let A_l be an extremal function for Yang's inequality. Suppose that there exists a ray* $\arg z = \theta$ *with* $\theta_j < \theta < \theta_{j+1}$, $1 \le j \le q$, *such that*

$$\limsup_{r\to\infty} \frac{\log\log\left|A_l(re^{i\theta})\right|}{\log r} = \rho(A_l).$$

Then $\theta_{j+1} - \theta_j = \pi/\rho(A_l)$.

Proof of Theorem 1.3. Since the case $\mu(A_0) > \rho(A_l)$, $A_0(z)$ is a dominant coefficient in equation (1.1), by using similar arguments of proving [10, Theorem 2], the assertion is trivial, so we may assume $\mu(A_0) < \rho(A_l)$. Suppose on the contrary to the assertion that there is a nontrivial solution f of (1.1) with $\rho(f) < \infty$. We aim for a contradiction. We consider two cases: $\mu(A_0) \ge 1/2$ and $0 < \mu(A_0) < 1/2$.

(1) Suppose that $\mu(A_0) \ge 1/2$. By Lemma 3.3, there exists a sector domain $S(\alpha,\beta)$ with $\beta - \alpha \ge \pi/\mu(A_0)$, $0 \le \alpha < \beta \le 2\pi$, such that

$$\limsup_{r\to\infty} \frac{\log\log\left|A_0(re^{i\theta})\right|}{\log r} \ge \mu(A_0) \tag{4.1}$$

for all the rays $\arg z = \theta \in (\alpha,\beta)$.

Let $\varepsilon \in (0, \mu(A_0)/2)$ be given constant. Since $\rho(A_i) < \mu(A_0)$ for all $i \ne l$ and $1 \le i \le n-1$, then there exists an $R_1 > 1$ such that

$$|A_i(z)| < \exp(r^{\mu(A_0)-2\varepsilon}) \tag{4.2}$$

for all $|z| = r > R_1$.

Applying the assumption on $A_l(z)$, suppose that a_i, $i = 1,2,\ldots,p$, are all the finite deficient values of $A_l(z)$. Thus we have $2p$ sectors $S_j = S(\theta_j,\theta_{j+1})$, $j = 1,2,\ldots,2p$, such that $A_l(z)$ has the following properties. In each sector S_j, either there exists an a_i such that

$$\log\frac{1}{|A_l(z) - a_i|} > C(\theta_j,\theta_{j+1},\varepsilon,\delta(a_i,A_l))T(|z|,A_l) \tag{4.3}$$

for $z \in S(\theta_j + \varepsilon, \theta_{j+1} - \varepsilon; r, +\infty)$, where $C(\theta_j,\theta_{j+1},\varepsilon,\delta(a_i,A_l))$ is a positive constant depending only on θ_j, θ_{j+1}, ε and $\delta(a_i,A_l)$, or there exists a ray $\arg z = \theta \in (\theta_j,\theta_{j+1})$ such that

$$\limsup_{r\to\infty} \frac{\log\log\left|A_l(re^{i\theta})\right|}{\log r} = \rho(A_l). \tag{4.4}$$

For the sake of simplicity, in the sequel we use C to represent $C(\theta_j,\theta_{j+1},\varepsilon,\delta(a_i,A_l))$. Note that if there exists an a_i such that (4.3) holds in S_j, then there exist two rays $\arg z = \theta$ (θ') such that (4.4) holds in S_{j-1} and S_{j+1}, respectively. If there exists a ray

$\arg z = \theta \in (\theta_j, \theta_{j+1})$ such that (4.4) holds, then there are a_i ($a_{i'}$) such that (4.3) holds in S_{j-1} and S_{j+1}, respectively.

Without loss of generality, we assume that there is a ray $\arg z = \theta$ in S_1 such that (4.4) holds. Therefore, there exists a ray in each sector $S_3, S_5, \ldots, S_{2p-1}$, such that (4.4) holds. By using Lemma 4.2, we know that all the sectors have the same magnitude $\pi/\rho(A_l)$. Note that $\rho(A_l) > \mu(A_0)$. It is not hard to see that there exists a subsector $S(\alpha', \beta')$, where $\alpha < \alpha' < \beta' < \beta$, and finite deficient value a_{j_0} such that

$$\log \frac{1}{|A_l(re^{i\theta}) - a_{j_0}|} > CT(r, A_l) \tag{4.5}$$

for all $\theta \in [\alpha', \beta']$. By using Lemma 3.1, there exists $\theta_0 \in [\alpha', \beta']$ and $R_2 > 1$ such that

$$\left| \frac{f^{(k)}(re^{i\theta_0})}{f(re^{i\theta_0})} \right| \le r^{n\rho(f)}, \quad k = 1, 2, \ldots, n, \tag{4.6}$$

for all $r > R_2$. Note that (4.1) and (4.2) hold for $\theta = \theta_0$. Thus there exists a sequence r_j in the set $F = (\max\{R_1, R_2\}, \infty)$ with $r_j \to \infty$ as $j \to \infty$, such that (4.2) and (4.5) hold for $|z| = r = r_j$, and

$$\left| A_0(r_j e^{i\theta_0}) \right| \ge \exp(r_j^{\mu(A_0)-\varepsilon}). \tag{4.7}$$

It follows from (4.2), (4.5)–(4.7) and (1.1) that

$$\begin{aligned}
\exp(r_j^{\mu(A_0)-\varepsilon}) &\le \left| A_0(r_j e^{i\theta_0}) \right| \\
&\le \left| \frac{f^{(n)}(r_j e^{i\theta_0})}{f(r_j e^{i\theta_0})} \right| + \cdots + \left| \frac{f^{(l)}(r_j e^{i\theta_0})}{f(r_j e^{i\theta_0})} \right| \left(\left| A_l(r_j e^{i\theta_0}) - a_{j_0} \right| + |a_{j_0}| \right) \\
&\quad + \cdots + \left| \frac{f'(r_j e^{i\theta_0})}{f(r_j e^{i\theta_0})} \right| \left| A_1(r_j e^{i\theta_0}) \right| \\
&\le r_j^{n\rho(f)} \left(1 + |a_{j_0}| + \exp(-CT(r_j, A_l)) + (n-2)\exp(r_j^{\mu(A_0)-2\varepsilon}) \right)
\end{aligned}$$

for all sufficiently large j. Obviously, this is a contradiction.

(2) Suppose that $0 < \mu(A_0) < 1/2$. By Lemma 3.2, for any $\alpha \in (\mu(A_0), 1)$, there exists a set $E_1 \subset [1, \infty)$ with $\overline{\log\operatorname{dens}}(E_1) \ge 1 - \mu(A_0)/\alpha$, such that

$$|A_0(z)| > \exp(r^{\mu(A_0)-\varepsilon}) \tag{4.8}$$

for all $|z| = r \in E_1 \cap (R_3, \infty)$, where $R_3 > 1$ is a constant.

By using Lemma 3.1, there exists $\theta_0 \in [0, 2\pi) - E_2$, where $E_2 \subset [0, 2\pi)$ and $\mathrm{m}(E_2) = 0$, such that (4.6) holds. Note that (4.2), (4.5) and (4.8) hold for $\theta = \theta_0$. Thus there exists a sequence r_j in $E_1 \cap (R_3, \infty)$ with $r_j \to \infty$ as $j \to \infty$, such that (4.2), (4.5), (4.6) and (4.8) hold for $|z| = r = r_j$. By the similar reasoning of the case of $\mu(A_0) \ge 1/2$, we get a contradiction. Hence every nontrivial solution of (1.1) is of infinite order, and this completes the proof. □

5. Proof of Theorem 1.5

We begin by recalling some properties of extremal functions for Denjoy's conjecture.

Lemma 5.1. [31, Theorem 4.11] *Let f be an extremal function for Denjoy's conjecture. Then, for any $\theta \in (0, 2\pi)$, either $\Delta(\theta)$ is a Borel direction of f, or there exists a constant $\sigma \in (0, \pi/4)$, such that*

$$\lim_{\substack{|z|\to\infty \\ z\in(S(\theta-\sigma,\theta+\sigma)-E)}} \frac{\log\log|f(z)|}{\log|z|} = \rho(f),$$

where E denotes a subset of $S(\theta-\sigma, \theta+\sigma)$, and satisfies

$$\lim_{r\to\infty} \mathrm{m}(S(\theta-\sigma, \theta+\sigma; r, \infty) \cap E) = 0.$$

For proving Theorem 1.5, we also need the following auxiliary result.

Lemma 5.2. *Let f be an entire function of order $\rho \in (0, \infty)$, and let $S(\phi_1, \phi_2)$ be a sector with $\phi_2 - \phi_1 < \pi/\rho$. If there exists a Borel direction of f in $S(\phi_1, \phi_2)$, then for at least one of the two rays $L_j : \arg z = \phi_j$, $j = 1, 2$, say L_2, we have*

$$\limsup_{r\to\infty} \frac{\log\log\left|f(re^{i\phi_2})\right|}{\log r} = \rho.$$

Lemma 5.2 is Lemma 1 in [29], which can be proved by using a result in [23, pp. 119–120].

Proof of Theorem 1.5. We may assume $\rho(A_0) < \rho(A_l)$ due to the proof of Theorem 1.3. Suppose on the contrary to the assertion that there is a transcendental solution f of (1.1) with $\rho(f) < \infty$. We aim for a contradiction. We consider two cases appearing in Lemma 3.6.

(1) Suppose that $A_l(z)$ blows up exponentially in each sector $S_j = S(\theta_j, \theta_{j+1})$, where $\theta_j = \frac{2j\pi-\arg(a_m)}{m+2}$, $j = 0, 1, \ldots, m+1$ and $\theta_{m+2} = \theta_0 + 2\pi$. That is, for any $\theta \in (\theta_j, \theta_{j+1})$, we have

$$\lim_{r\to\infty} \frac{\log\log\left|A_l(re^{i\theta})\right|}{\log r} = \rho(A_l) = \frac{m+2}{2}.$$

Then for any given constant $\varepsilon \in (0, \pi/(8\rho(A_l)))$ and $\eta \in (0, (\rho(A_l) - \rho(A_0))/4)$, we have

$$|A_l(z)| \geq \exp\left\{(1+\delta)\alpha\,|z|^{(m+2)/2-\eta}\right\}$$

and

$$|A_t(z)| \leq \exp(|z|^{\rho(A_0)+\eta}) \leq \exp(|z|^{\rho(A_l)-2\eta}) \leq \exp\left\{\delta\alpha\,|z|^{(m+2)/2-\eta}\right\},$$

as $z \to \infty$ in $\overline{S_j}(\varepsilon/2) = \{z : \theta_j + \varepsilon/2 \le \arg z \le \theta_{j+1} - \varepsilon/2\}$, $j = 0, 1, \ldots, m+1$, $t = 0, \ldots, l-1, l+1, \ldots, n-1$, where δ is a positive constant satisfying $\delta n < 1$, and $\alpha > 0$ is also constant.

For each S_j, $j = 0, 1, \ldots, m+1$, applying [17, Theorem 2.1], there exists $s \in \{1, 2, \ldots, l-1\}$ and $b_s \neq 0$ such that

$$\left| f^{(s)}(z) - b_s \right| \le \exp\left\{ -(1 - n\delta)\alpha \left| z \right|^{(m+2)/2 - \eta} \right\},$$

as $z \to \infty$ in $\overline{S_j}(\varepsilon)$. For each integer $p \ge s+1$,

$$\left| f^{(p)}(z) \right| \le \exp\left\{ -(1 - n\delta)\alpha \left| z \right|^{(m+2)/2 - \eta} \right\},$$

as $z \to \infty$ in $\overline{S_j}(3\varepsilon/2)$. Hence $\left| f^{(l)}(z) \right|$ must be bounded in the whole complex plane by the Phragmén-Lindelöf principle. By Liouville theorem, f has to be a polynomial. This is a contradiction.

(2) There exists at least one sector of the $m+2$ sectors, such that $A_l(z)$ decays to zero exponentially, say $S_{j_0} = \{z : \theta_{j_0} < \arg z < \theta_{j_0+1}\}$, $0 \le j_0 \le m+1$. This shows that for any $\theta \in (\theta_{j_0}, \theta_{j_0+1})$,

$$\lim_{r\to\infty} \frac{\log\log \frac{1}{\left|A_l(re^{i\theta})\right|}}{\log r} = \frac{m+2}{2}. \tag{5.1}$$

Since $\rho(A_i) < \rho(A_0)$ for all $i \neq l$ and $1 \le i \le n-1$, then there exists an $R_1 > 1$ such that

$$|A_i(z)| < \exp(r^{\rho(A_0) - 2\varepsilon}) \tag{5.2}$$

for all $|z| = r > R_1$. Applying Lemma 3.1, there exists a set $E_1 \subset [0, 2\pi)$ with $\mathrm{m}(E_1) = 0$ such that if $\theta_0 \in [0, 2\pi) - E_1$, then there is a constant $R_2 = R_2(\theta_0) > 1$ such that for all z satisfying $\arg z = \theta_0$ and $|z| \ge R_2$, the inequality (4.6) holds. Next we consider the two cases appearing in Lemma 5.1.

(i) Suppose that the ray $\arg z = \theta$ is a Borel direction of $A_0(z)$, where $\theta \in (\theta_{j_0}, \theta_{j_0+1})$. Choose $\phi_1 \in (\theta_{j_0}, \theta) - E_1$ and $\phi_2 \in (\theta, \theta_{j_0+1}) - E_1$. Then $\phi_2 - \phi_1 < \pi/\rho(A_l) < \pi/\rho(A_0)$. By Lemma 5.2, at least one of two rays $L_1 : \arg z = \phi_1$ and $L_2 : \arg z = \phi_2$, say L_1, satisfies

$$\limsup_{r\to\infty} \frac{\log\log \left|A_0(re^{i\phi_1})\right|}{\log r} = \rho(A_0).$$

Note that (5.1) holds for $\theta = \phi_1$. Thus there exists a sequence of points $z_j = r_j e^{i\phi_1}$ with $r_j \to \infty$ as $j \to \infty$, such that

$$\lim_{j\to\infty} \frac{\log\log \left|A_0(r_j e^{i\phi_1})\right|}{\log r_j} = \rho(A_0), \tag{5.3}$$

and (4.6), (5.1) and (5.2) hold for $z = z_j = r_j e^{i\phi_1}$. Combining (4.6), (5.1)–(5.3) and (1.1), we arrive at a contradiction as in the proof of Theorem 1.3.

(ii) Suppose that the ray $\arg z = \theta$ is not a Borel direction of $A_0(z)$, where $\theta \in (\theta_{j_0}, \theta_{j_0+1})$. By Lemma 5.1, there exists a constant $\sigma \in (0, \zeta)$, where $\zeta = \min\{(\theta - \theta_{j_0})/2, (\theta_{j_0+1} - \theta)/2, \pi/4\}$, such that

$$\lim_{\substack{|z|\to\infty \\ z\in(S(\theta-\sigma,\theta+\sigma)-E_2)}} \frac{\log\log|A_0(z)|}{\log|z|} = \rho(A_0),$$

where E_2 denotes a subset of $S(\theta - \sigma, \theta + \sigma)$, and satisfies

$$\lim_{r\to\infty} \mathrm{m}(S(\theta - \sigma, \theta + \sigma; r, \infty) \cap E_2) = 0.$$

Let $\Delta = \{z : \arg z = \psi, \psi \in E_1\}$. We can easily see that there exists a sequence of points z_j with $z_j \to \infty$ as $j \to \infty$, $\{z_j\} \subset (S(\theta - \sigma, \theta + \sigma) - E_2) \cap (S_{j_0} - \Delta)$, such that

$$\lim_{j\to\infty} \frac{\log\log|A_0(z_j)|}{\log|z_j|} = \rho(A_0), \tag{5.4}$$

$$\lim_{j\to\infty} \frac{\log\log\frac{1}{|A_l(z_j)|}}{\log|z_j|} = \frac{m+2}{2} \tag{5.5}$$

and

$$\left|\frac{f^{(k)}(z_j)}{f(z_j)}\right| \le |z_j|^{n\rho(f)}, \quad k = 1, 2, \ldots, n, \tag{5.6}$$

$$|A_i(z_j)| < \exp(|z_j|^{\rho(A_0)-2\varepsilon}). \tag{5.7}$$

Combining (5.4)–(5.7) and (1.1), we arrive at a contradiction as in the proof of Theorem 1.3. This completes the proof. □

6. Proof of Theorem 1.6

We may assume $\rho(A_0) < \rho(A_l)$ due to the proof of Theorem 1.3. Suppose on the contrary to the assertion that there is a transcendental solution f of (1.1) with $\rho(f) < \infty$. We aim for a contradiction. We consider two cases appearing in Lemma 3.6.

(1) Suppose that $A_l(z)$ blows up exponentially in each sector $S_j = S(\theta_j, \theta_{j+1})$, where $\theta_j = \frac{2j\pi - \arg(a_m)}{m+2}$, $j = 0, 1, \ldots, m+1$ and $\theta_{m+2} = \theta_0 + 2\pi$. We get a contradiction by the similar reasoning in the proving Theorem 1.5.

(2) There exists at least one sector of the $m + 2$ sectors, such that $A_l(z)$ decays to zero exponentially, say $S_{j_0} = \{z : \theta_{j_0} < \arg z < \theta_{j_0+1}\}$, $0 \le j_0 \le m + 1$. That is, for any $\theta \in (\theta_{j_0}, \theta_{j_0+1})$, (5.1) holds.

Let $\varepsilon \in (0, \mu(A_0)/2)$ be given constant. Since $\rho(A_i) < \mu(A_0)$ for all $i \neq l$ and $1 \leq i \leq n-1$, then there exists an $R_1 > 1$ such that (4.2) holds for all $|z| = r > R_1$.

We proceed to divide into two situations. If $1/2 \leq \mu(A_0) < 1/2 + 1/[2(m+1)]$, by Lemma 3.3, there exists a sector $S(\alpha, \beta)$ with $\beta - \alpha \geq \pi/\mu(A_0) > \pi/[\frac{1}{2} + \frac{1}{2(m+1)}] = 2\pi - 2\pi/(m+2)$, such that

$$\limsup_{r\to\infty} \frac{\log\log\left|A_0(re^{i\theta})\right|}{\log r} \geq \mu(A_0) \tag{6.1}$$

for all $\theta \in (\alpha, \beta)$. Thus, there exists a subsector $S(\alpha', \beta') \subset S_{j_0} \cap S(\alpha, \beta)$, where $\alpha < \alpha' < \beta' < \beta$, such that (5.1) and (6.1) hold for any $\theta \in (\alpha', \beta')$.

By Lemma 3.1, there exists a set $E_1 \subset [0, 2\pi)$ of linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E_1$, then there is a constant $R_2 = R_2(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_2$, we have (4.6). Thus there exists a sequence of points $z_j = r_j e^{i\psi_0}$ with $r_j \to \infty$ as $j \to \infty$, where $\psi_0 \in (\alpha', \beta') - E_1$ and $r_j \in (\max\{R_1, R_2\}, \infty)$, such that

$$\left|A_0(r_j e^{i\psi_0})\right| > r_j^{\mu(A_0)-\varepsilon}, \tag{6.2}$$

and (4.2), (4.6) and (5.1) hold for $z = z_j = r_j e^{i\psi_0}$. Combining (4.2), (4.6), (5.1), (6.2) and (1.1), we get a contradiction.

If $0 < \mu(A_0) < 1/2$, by Lemma 3.2, for every $\alpha \in (\mu(A_0), 1)$, there exists a set $E_2 \subset [1, \infty)$ with $\overline{\log\operatorname{dens}}(E_2) \geq 1 - \mu(A_0)/\alpha$, such that

$$|A_0(z)| \geq |z|^{\mu(A_0)-\varepsilon} \tag{6.3}$$

for all $|z| = r \in E_2 - [0, R_3]$, where $R_3 > 1$ is a constant. Thus, there exists a sequence of points $z_j = r_j e^{i\theta_0}$ with $r_j \to \infty$ as $j \to \infty$, such that (4.2), (4.6), (5.1) and (6.3) hold for $z = z_j = r_j e^{i\theta_0}$, where $\theta_0 \in (\theta_{j_0}, \theta_{j_0+1}) - E_1$, $r_j \in E_2 - [0, \max\{R_1, R_2, R_3\}]$. Combining (4.2), (4.6), (5.1), (6.3) and (1.1), we get a contradiction. The proof is completed.

7. Proof of Theorem 1.7

Suppose on the contrary to the assertion that there is a nontrivial solution f of (1.1) with $\rho(f) < \infty$. We aim for a contradiction. Let η be a constant satisfying

$$\max_{\substack{1\leq i\leq n-1\\ i\neq l}} \{\rho(A_i)\} < \eta < \rho(A_0).$$

Then there exists a $R > 1$ such that

$$|A_i(z)| < \exp(r^\eta), \quad i \neq l,\ 1 \leq i \leq n-1,$$

for all $|z| = r > R$.

Applying Lemma 3.1, there exists a set $E \subset [0, 2\pi)$ with $\mathrm{m}(E) = 0$ such that

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq O(|z|^{n\rho(f)}), \quad j = 1, 2, \ldots, n,$$

as $z \to \infty$ along $\arg z = \theta_0 \in [\phi_k, \theta_k] - E$, $k = 1, 2, \ldots, m$. It follows from (1.1) and (1.4) that

$$|A_0(z)| \leq \left|\frac{f^{(n)}(z)}{f(z)}\right| + \cdots + \left|\frac{f'(z)}{f(z)}\right| |A_1(z)| \leq O(\exp(|z|^{\eta})) \tag{7.1}$$

as $z \to \infty$ along $\arg z = \theta_0$.

Let $\varepsilon > 0$ be a small constant that satisfies $\mu(A_0) < \pi/(\nu + 2\varepsilon)$. Applying Phragmén-Lindelöf principle on (7.1),

$$|A_0(z)| = O(\exp(|z|^{\eta})) \tag{7.2}$$

as $z \to \infty$ in $\phi_k + \varepsilon \leq \arg z \leq \theta_k - \varepsilon$, $k = 1, 2, \ldots, m$. Since $\mu(A_0) < \pi/(\nu + 2\varepsilon)$, this implies that $\phi_{k+1} - \theta_k + 2\varepsilon < \pi/\mu(A_0)$ for all k, $1 \leq k \leq m$. Hence by using Lemma 3.4, (7.2) holds as $z \to \infty$ in $\theta_k - \varepsilon \leq \arg z \leq \phi_{k+1} + \varepsilon$, $k = 1, 2, \ldots, m$. Applying Phragmén-Lindelöf principle again, (7.2) holds in the whole complex plane. This means that $\rho(A_0) \leq \eta$. This is a contradiction with the choice of η, and the proof is complete.

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