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Multiple Solutions of Nonlinear Schrödinger Equation with the Fractional p-Laplacian

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Abstract. We use two variant fountain theorems to prove the existence of infinitely many weak solutions for the following fractional p-Laplace equation

$$(-\Delta)_p^\alpha u + V(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^N,$$

where $N \geq 2$, $p \geq 2$, $\alpha \in (0,1)$, $(-\Delta)_p^\alpha$ is the fractional p-Laplacian and f is either asymptotically linear or subcritical p-superlinear growth. Under appropriate assumptions on V and f, we prove the existence of infinitely many nontrivial high or small energy solutions. Our results generalize and extend some existing results.

1. Introduction

This article is concerned with the fractional p-Laplacian equation

$$(-\Delta)_p^\alpha u + V(x)\,|u|^{p-2}\,u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N, p \geq 2$, $\alpha \in (0,1)$, V is a positive continuous potential and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. The fractional p-Laplacian defined on smooth functions by

$$(-\Delta)_p^\alpha u(x) = 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{N+\alpha p}}\,dy, \quad x \in \mathbb{R}^N,$$

up to some normalization constant depending upon N and α.

When $p = 2$, the equation (1.1) arises in the study of the nonlinear Fractional Shrödinger equation

$$(-\Delta)^\alpha u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

This type of problem arises in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical

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outcome of stochastically stabilization of Lévy processes, see [2, 10, 11, 13, 18–20, 25] and the references therein. The literature on non-local operators and their applications is very interesting and quite large, we refer the interested reader to [1, 3–6, 8, 9, 14, 17, 21–24] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the interested reader to [12].

It is well known that the main difficulty in treating problem (1.2) in $\mathbb{R}^N$ arises from the lack of compactness of the Sobolev embeddings, which prevents from checking directly that the energy functional associated with (1.2) satisfies the C-condition. To overcome the difficulty of the noncompact embedding, Teng [15] and Wei [16] also establish a new compact embedding theorem for the subspace of $H^\alpha(\mathbb{R}^N)$. Furthermore, the authors are able to guarantee the existence and multiplicity of nontrivial weak solutions of (1.2) in $H = \left\{u \in H^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left|(-\Delta)^{\alpha/2}u(x)\right|^2 dx + \int_{\mathbb{R}^N} V(x)u^2\, dx < +\infty\right\}$ provided $\inf V > 0$ and the following condition holds:

(A) For any $M > 0$, there exists $r_0 > 0$ such that

$$\lim_{|y|\to\infty} \operatorname{meas}\left(\left\{x \in \mathbb{R}^N : |x-y| \le r_0, V(x) \le M\right\}\right) = 0,$$

where $\operatorname{meas}(\cdot)$ is the Lebesgue measure on $\mathbb{R}^N$.

Ge [7] established the existence of infinitely many solutions of (1.2) via the variant fountain theorems established in [26]. Inspired by the above facts and aforementioned papers, the main purpose of this paper is to study the existence of infinitely many solutions of (1.1). Before stating our main results, we first make some assumptions on the functions V and f. For the potential V, we make the following assumption:

(B) $V \in C(\mathbb{R}^N)$, $V_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$.

For the nonlinearity f, we divide it into the following two cases. For the asymptotically linear case, we make the following assumptions:

(C1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $f(x,t)t \ge 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and there exist constant $p-1 < r < p$ and positive functions $a \in L^{p/(p-r)}(\mathbb{R}^N)$ such that

$$|f(x,t)| \le a(x)(1 + |t|^{r-1}), \quad \forall\, (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

(C2) $\lim_{t\to 0} f(x,t)/|t|^{p-1} = 0$ uniformly for $x \in \mathbb{R}^N$.

(C3) There exists $\sigma \in [p-1, r)$ such that $\liminf_{t\to\infty} F(x,t)/|t|^\sigma \ge d > 0$ uniformly for $x \in \mathbb{R}^N$, where $F(x,t) = \int_0^t f(x,s)\, ds$.

Now, we are ready to state the first main result of this paper.

Theorem 1.1. *Suppose that* (B) *and* (C1)–(C3) *hold, and that* $f(x,-t) = -f(x,t)$ *for all* $(x,t) \in \mathbb{R}^N \times \mathbb{R}$*. Then problem* (1.1) *possesses infinitely many small energy solutions* $u_k \in E$ *(see* (2.1)*) for every* $k \in \mathbb{N}$*, in the sense that*

$$\frac{1}{p}\left[\int_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{N+\alpha p}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)\,|u_k(x)|^p\,dx\right] - \int_{\mathbb{R}^N} F(x,u_k)\,dx \to 0^-$$

as $k \to \infty$*.*

Here E is a Banach space which is defined in (2.1). For the p-superlinear case, we make the following assumptions:

(D1) $f \in C(\mathbb{R}^N \times \mathbb{R})$, $f(x,t)t \geq 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and there exists a constant $\theta \in (p, p^*_\alpha)$, such that

$$|f(x,t)| \leq b(x)(|t|^{p-1} + |t|^{\theta-1}), \quad \forall\,(x,t) \in \mathbb{R}^N \times \mathbb{R}$$

with a positive function $b \in L^q(\mathbb{R}^N)$ and $q > \frac{pN}{pN+p\alpha\theta-N\theta}$, which implies that $p < \frac{q}{q-1}\theta < p^*_\alpha$, where $p^*_\alpha = \frac{Np}{N-\alpha p}$ (if $N > \alpha p$) or $p^*_\alpha = \infty$ (if $N \leq \alpha p$).

(D2) $\lim_{t\to 0} f(x,t)/|t|^{p-1} = 0$ uniformly for $x \in \mathbb{R}^N$.

(D3) $\lim_{t\to\infty} F(x,t)/|t|^p = \infty$ uniformly for $x \in \mathbb{R}^N$.

(D4) There exist $\mu > p$ such that

$$0 < \mu F(x,t) \leq t f(x,t), \quad \forall\, x \in \mathbb{R}^N,\ t \neq 0.$$

Our second main result reads as follows.

Theorem 1.2. *Suppose that* (B) *and* (D1)–(D4) *hold, and that* $f(x,-t) = -f(x,t)$ *for all* $(x,t) \in \mathbb{R}^N \times \mathbb{R}$*. Then problem* (1.1) *possesses infinitely many high energy solutions* $u^k \in E$ *for all* $k \geq k_0$ *(*$k_0 \in \mathbb{N}$*), in the sense that*

$$\frac{1}{p}\left[\int_{\mathbb{R}^{2N}} \frac{\left|u^k(x)-u^k(y)\right|^p}{|x-y|^{N+\alpha p}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)\left|u^k(x)\right|^p\,dx\right] - \int_{\mathbb{R}^N} F(x,u^k)\,dx \to +\infty$$

as $k \to \infty$*.*

Notation. In this paper we make use of the following notation:

- $\|\cdot\|_p$ the usual norm of the space $L^p(\mathbb{R}^N)$.
- c, C and c_i, C_i denote positive (possibly different) constants.
- We denote the weak convergence in X and its X^* by "$\rightharpoonup$" and the strong convergence by "$\to$".

2. Variational framework

Before stating this section, we define the Cagliardo seminorm by

$$[u]_{\alpha,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\alpha p}} \, dx \, dy \right)^{1/p},$$

where $u \colon \mathbb{R}^N \to \mathbb{R}$ is a measurable function. On one hand, we define fractional Sobolev space by

$$W^{\alpha,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : u \text{ is measurable and } [u]_{\alpha,p} < \infty \right\}$$

endowed with the norm

$$\|u\|_{\alpha,p} = \left([u]_{\alpha,p}^p + \|u\|_p^p \right)^{1/p},$$

where

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u(x)|^p \, dx \right)^{1/p}.$$

On the other hand, we consider the fractional Sobolev space

$$E := \left\{ u \in W^{\alpha,p} : \int_{\mathbb{R}^N} V(x) \, |u|^p \, dx < \infty \right\} \tag{2.1}$$

endowed with the norm

$$\|u\| := \|u\|_E = \left([u]_{\alpha,p}^p + \int_{\mathbb{R}^N} V(x) \, |u|^p \, dx \right)^{1/p}.$$

In order to discuss the problem (1.1), we need to consider the energy functional $\Phi \colon E \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\alpha p}} \, dx \, dy + \frac{1}{p} \int_{\mathbb{R}^N} V(x) \, |u(x)|^p \, dx - \int_{\mathbb{R}^N} F(x,u) \, dx.$$

Under our hypotheses, it follows from Hölder-type inequality and Sobolev embedding theorem that the energy functional Φ is well defined on E. It is well known that $\Phi \in C^1(E,\mathbb{R})$, and its derivative is given by

$$\begin{aligned} \langle \Phi'(u), v \rangle = & \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2} \, (u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+\alpha p}} \, dx \, dy \\ & + \int_{\mathbb{R}^N} V(x) \, |u|^{p-2} \, uv \, dx - \int_{\mathbb{R}^N} f(x,u) v \, dx \end{aligned}$$

for $v \in E$. It is standard to verify that the weak solutions of problem (1.1) correspond to the critical points of the functional Φ. For the readers convenience, we review the main embedding result for this class of fractional Sobolev spaces.

Lemma 2.1. [12] *$W^{\alpha,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $q \in [p, p_\alpha^*]$, and compactly embedded into $L^q_{\rm loc}(\mathbb{R}^N)$ for $q \in [p, p_\alpha^*)$. Assume that* (A) *hold, then E is compactly embedded into $L^q(\mathbb{R}^N)$ for $q \in [p, p_\alpha^*)$.*

In order to assure the existence of infinitely many solutions for the problem (1.1), our main tool will be the two variant fountain theorems (see [26, Theorem 2.2 and Theorem 2.1]), which will be used in our proof.

Let X be a Banach space with the norm $\|\cdot\|$ and $X = \overline{\bigoplus_{i\in\mathbb{N}}^\infty X_i}$ with $\dim X_i < \infty$ for any $i \in \mathbb{N}$. Set

$$Y_k = \bigoplus_{i=0}^{k} X_i, \quad Z_k = \overline{\bigoplus_{i=k}^{\infty} X_i}.$$

Consider the following C^1-functional $\Phi_\lambda \colon X \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1,2],$$

where $A, B \colon X \to \mathbb{R}$ are two functionals.

Lemma 2.2. *Suppose that the functional Φ_λ defined above, and satisfies the following conditions:*

(1) *Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times X$.*

(2) *$B(u) \geq 0$; $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of X.*

(3) *There exist $\rho_k > r_k > 0$ such that*

$$a_k(\lambda) := \inf_{\substack{u\in Z_k \\ \|u\|=\rho_k}} \Phi_\lambda(u) \geq 0 > b_k(\lambda) := \max_{\substack{u\in Y_k \\ \|u\|=r_k}} \Phi_\lambda(u)$$

for all $\lambda \in [1,2]$, $d_k(\lambda) := \inf_{u\in Z_k, \|u\|\leq\rho_k} \Phi_\lambda(u) \to 0$ as $k \to \infty$ uniformly for $\lambda \in [1,2]$. Then there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, \quad \Phi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)] \quad \text{as } n \to \infty.$$

In particular, if $\{u(\lambda_n)\}_{n=1}^\infty$ has a convergent subsequence for every k, then Φ_1 has infinitely many nontrivial critical points $\{u_k\}_{k=1}^\infty \in X \setminus \{0\}$ satisfying $\Phi_1(u_k) \to 0^-$ as $k \to \infty$.

Lemma 2.3. *Suppose that the functional Φ_λ defined above, and satisfies the following conditions:*

(1) *Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times X$.*

(2) *$B(u) \geq 0$; $B(u) \to \infty$ or $A(u) \to \infty$ as $\|u\| \to \infty$ (or $B(u) \leq 0$; $B(u) \to -\infty$ as $\|u\| \to \infty$).*

(3) *There exist $\rho_k > r_k > 0$ such that*

$$b_k(\lambda) := \inf_{\substack{u \in Z_k \\ \|u\| = r_k}} \Phi_\lambda(u) > a_k(\lambda) := \max_{\substack{u \in Y_k \\ \|u\| = \rho_k}} \Phi_\lambda(u) \quad \text{for all } \lambda \in [1,2].$$

Then

$$b_k(\lambda) \leq c_k(\lambda) := \inf_{\gamma \in \Gamma} \max_{u \in B_k} \Phi_\lambda(\gamma(u)), \quad \forall\, \lambda \in [1,2],$$

where $\Gamma_k = \{\gamma \in C(B_k, X) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = \mathrm{id}\}$ *($k \geq 2$) and* $B_k = \{u \in Y_k : \|u\| \leq \rho_k\}$. *Moreover, for almost every $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that*

$$\sup_n \left\| u_n^k(\lambda) \right\| < \infty, \quad \Phi'_\lambda(u_n^k(\lambda)) \to 0 \quad \text{and} \quad \Phi_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \quad \text{as } n \to \infty.$$

3. Proofs of the main results

In this section, for the notation in Lemmas 2.2 and 2.3, the space $X = E$, and related functionals on E are

$$A(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+\alpha p}}\, dx\, dy + \frac{1}{p} \int_{\mathbb{R}^N} V(x)\, |u(x)|^p\, dx, \quad B(u) = \int_{\mathbb{R}^N} F(x,u)\, dx.$$

In order to prove Theorems 1.1 and 1.2, we will consider the following family of functionals

$$\Phi_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+\alpha p}}\, dx\, dy + \frac{1}{p} \int_{\mathbb{R}^N} V(x)\, |u(x)|^p\, dx - \lambda \int_{\mathbb{R}^N} F(x,u)\, dx$$

with $\lambda \in [1,2]$ and $u \in E$. By (B), the energy functional $\Phi_\lambda \colon E \to \mathbb{R}$ is well defined and of class $C^1(E, \mathbb{R})$. Moreover, the derivative of Φ_λ is

$$\begin{aligned} \langle \Phi'_\lambda(u), v \rangle &= \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}\, (u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+\alpha p}}\, dx\, dy \\ &\quad + \int_{\mathbb{R}^N} V(x)\, |u|^{p-2}\, uv\, dx - \lambda \int_{\mathbb{R}^N} f(x,u) v\, dx \end{aligned}$$

for all $u, v \in E$. Since E is a separable and reflexive Banach space, then there exist $\{e_i\}_{i=1}^\infty \subset E$ and $\{e_i^*\}_{i=1}^\infty \subset E^*$ such that

$$E = \overline{\operatorname{span}\{e_i : i = 1, 2, \ldots\}}, \quad E^* = \overline{\operatorname{span}\{e_i^* : i = 1, 2, \ldots\}}$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For convenience, we write $X_i := \mathbb{R} e_i$. Now, we are going to prove our main results.

3.1. Proof of Theorem 1.1

Lemma 3.1. *Suppose that* (B) *and* (C1)–(C3) *are satisfied. Then* $B(u) \geq 0$, *and* $B(u) \to \infty$ *as* $\|u\| \to \infty$ *on any dimensional subspace of* E.

Proof. Evidently, from (C1), we have $B(u) \geq 0$ for all $u \in E$. Let $H \subset E$ be any finite dimensional subspace of E. Next we will show that $B(u) \to \infty$ as $\|u\| \to \infty$ on H. We claim that for any finite dimensional subspace H of E, there exists a constant $\varepsilon_0 > 0$ such that

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : |u(x)| \geq \varepsilon_0 \|u\|\right\} \geq \varepsilon_0, \quad \forall\, u \in H \setminus \{0\}. \tag{3.1}$$

If not, for any $n \in \mathbb{N}$, there exists $u_n \in H \setminus \{0\}$ such that

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : |u_n(x)| \geq \frac{1}{n} \|u_n\|\right\} < \frac{1}{n}, \quad \forall\, n \in \mathbb{N}.$$

Let $w_n = u_n/\|u_n\|$, for all $n \in \mathbb{N}$, then $\|w_n\| = 1$ for all $n \in \mathbb{N}$, and

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : |w_n(x)| \geq \frac{1}{n}\right\} < \frac{1}{n}, \quad \forall\, n \in \mathbb{N}. \tag{3.2}$$

By the boundedness of $\{w_n\}$, passing to a subsequence if necessary, we may assume that $w_n \to w$ with $\|w\| = 1$ in E for some $w \in H$ since H is a finite dimensional space. By Lemma 2.1, we have

$$\int_{\mathbb{R}^N} |w_n(x) - w(x)|^p \, dx \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Since $w \neq 0$, there exists a constant $\delta_0 > 0$ such that

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : |w(x)| \geq \delta_0\right\} \geq \delta_0. \tag{3.4}$$

For any $n \in \mathbb{N}$, we set

$$D_n = \left\{x \in \mathbb{R}^N : |w_n(x)| < \frac{1}{n}\right\}, \quad D_n^c = \left\{x \in \mathbb{R}^N : |w_n(x)| \geq \frac{1}{n}\right\},$$

and $D_0 = \left\{x \in \mathbb{R}^N : |w(x)| \geq \delta_0\right\}$. Thus for $n > (p+1)/\delta_0$, by (3.2) and (3.4), we get

$$\operatorname{meas}(D_n \cap D_0) \geq \operatorname{meas}(D_0) - \operatorname{meas}(D_n^c) > \frac{p\delta_0}{p+1}.$$

Consequently, for $n > (p+1)/\delta_0$, by the inequality $|w - w_n|^p \geq |w|^p - p\,|w|^{p-1}\,|w_n|$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |w_n(x) - w(x)|^p \, dx &\geq \int_{D_n \cap D_0} |w_n(x) - w(x)|^p \, dx \\ &\geq \int_{D_n \cap D_0} \left[|w(x)|^p - p \, |w(x)|^{p-1} \, |w_n(x)| \right] dx \\ &\geq \int_{D_n \cap D_0} |w(x)|^{p-1} \left[|w(x)| - p \, |w_n(x)| \right] dx \\ &\geq \delta_0^{p-1} \left(\delta_0 - \frac{p}{n} \right) \operatorname{meas}(D_n \cap D_0) \\ &> \frac{p}{(p+1)^2} \delta_0^{p+1} > 0.\end{aligned}$$

This is in contradiction with (3.3). Therefore (3.1) holds.

By (C3), there exists $R > 0$ such that

$$F(x, u) \geq d \, |u|^\sigma \quad \text{for all } x \in \mathbb{R}^N \text{ and } |u| \geq R. \tag{3.5}$$

Let $D_u = \left\{ x \in \mathbb{R}^N : |u(x)| \geq \varepsilon_0 \, \|u\| \right\}$ for $u \in H \setminus \{0\}$. By (3.1), we see that for any $u \in H$ with $\|u\| \geq R/\varepsilon_0$, we have $|u(x)| \geq R$ for all $x \in D_u$. Hence, for any $u \in H$ with $\|u\| \geq R/\varepsilon_0$, from (C1) and (3.5), we get

$$\begin{aligned}B(u) = \int_{\mathbb{R}^N} F(x, u(x)) \, dx &\geq \int_{D_u} F(x, u(x)) \, dx \\ &\geq \int_{D_u} d \, |u(x)|^\sigma \, dx \geq d\varepsilon_0^\sigma \, \|u\|^\sigma \operatorname{meas}(D_u) \\ &\geq d\varepsilon_0^{1+\sigma} \, \|u\|^\sigma .\end{aligned}$$

This implies that $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace $H \subset E$. The proof is completed. □

Lemma 3.2. *Suppose that* (B) *and* (C1)–(C3) *are satisfied. Then there exist two sequences* $0 < r_k < \rho_k \to 0$ *as* $k \to \infty$ *such that*

$$a_k(\lambda) = \inf_{\substack{u \in Z_k \\ \|u\| = \rho_k}} \Phi_\lambda(u) \geq 0, \quad b_k(\lambda) = \max_{\substack{u \in Y_k \\ \|u\| = r_k}} \Phi_\lambda(u) < 0$$

and

$$d_k(\lambda) = \inf_{\substack{u \in Z_k \\ \|u\| \leq \rho_k}} \Phi_\lambda(u) \to 0 \quad \text{as } k \to \infty$$

uniformly for $\lambda \in [1, 2]$.

Proof. Let $\alpha_k = \sup_{u \in Z_k, \|u\| = 1} \|u\|_q$ for $q \in [p, p_\alpha^*)$, we see that $\alpha_k \to 0$ as $k \to \infty$. Indeed, suppose that this is not the case, then there exist an ε_0 and $\{u_i\} \subset E$ with $u_i \perp Y_{k_i - 1}$

such that $\|u_i\| = 1$, $\|u_i\|_q \geq \varepsilon_0$, where $k_j \to \infty$ as $j \to \infty$. For any $v \in E$, we may find $w_i \in Y_{k_i - 1}$ such that $w_i \to v$ as $i \to \infty$. Hence,

$$|\langle u_i, v \rangle| = |\langle u_i, w_i - v \rangle| \leq \|w_i - v\| \to 0$$

as $i \to \infty$. Thus, $u_i \rightharpoonup 0$ weakly in E, as a result, by Lemma 2.1, we have $u_i \to 0$ in $L^q(\mathbb{R}^N)$. This is in contradiction with $\|u_i\|_q \geq \varepsilon_0$.

By (C1)–(C3), it is easy to prove that for arbitrary $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$F(x,u) \leq \varepsilon |u|^p + C_\varepsilon a(x) |u|^r, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.6}$$

Therefore, for $u \in Z_k$ and ε small enough, by (3.6), we have

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{p} \|u\|^p - \lambda\varepsilon \|u\|_p^p - \lambda C_\varepsilon \|a\|_{p/(p-r)} \|u\|_p^r \\ &\geq \frac{1}{2p} \|u\|^p - c \|a\|_{p/(p-r)} \alpha_k^r \|u\|^r . \end{aligned}$$

If we choose $\rho_k = (4pc \|a\|_{p/(p-r)} \alpha_k^r)^{1/(p-r)}$, then $\rho_k \to 0^+$ as $k \to \infty$ and by computation, we get

$$a_k(\lambda) = \inf_{\substack{u \in Z_k \\ \|u\| = \rho_k}} \Phi_\lambda(u) \geq \frac{1}{4p} \rho_k^p > 0.$$

In addition, for all $\lambda \in [1,2]$ and $u \in Z_k$ with $\|u\| \leq \rho_k$, we have

$$\Phi_\lambda(u) \geq -c \|a\|_{p/(p-r)} \alpha_k^r \|u\|^r \geq -c \|a\|_{p/(p-r)} \alpha_k^r \rho_k^r \to 0$$

as $k \to \infty$. Therefore,

$$d_k(\lambda) = \inf_{\substack{u \in Z_k \\ \|u\| \leq \rho_k}} \Phi_\lambda(u) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

By (C1)–(C3), we can get

$$F(x,u) \geq d |u|^\sigma - \varepsilon |u|^p - C_\varepsilon a(x) |u|^r, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Hence, if $u \in Y_k$, by the equivalence of any norm in finite dimensional space, Hölder inequality and the above inequality, we get

$$\begin{aligned} \Phi_\lambda(u) &\leq \frac{1}{p} \|u\|^p - \int_{\mathbb{R}^N} F(x,u)\, dx \\ &\leq \frac{1}{p} \|u\|^p - d \int_{\mathbb{R}^N} |u|^\sigma \, dx + \varepsilon \int_{\mathbb{R}^N} |u|^p \, dx + C_\varepsilon \|a\|_{p/(p-r)} \left(\int_{\mathbb{R}^N} |u|^p \, dx \right)^{r/p} \\ &\leq \|u\|^p + c \|a\|_{p/(p-r)} \|u\|^r - c_1 \|u\|^\sigma . \end{aligned}$$

Therefore, we choose $r_k > 0$ small enough satisfying $r_k < \rho_k$ such that

$$b_k(\lambda) = \max_{\substack{u \in Y_k \\ \|u\| = r_k}} \Phi_\lambda(u) < 0 \quad \text{for all } \lambda \in [1,2].$$

The proof is completed. □

Proof of Theorem 1.1. It follows from (3.6) and Lemma 2.1 that Φ_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1,2]$. Evidently, $\Phi_\lambda(u) = \Phi_\lambda(-u)$ for all $(\lambda, u) \in [1,2] \times E$. From Lemma 3.2, we see that all the conditions of Lemma 2.2 have been verified. Consequently, we know from Lemma 2.2 that there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, \quad \Phi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)] \quad \text{as } n \to \infty.$$

For simplicity, we denote $u(\lambda_n)$ by u_n for all $n \in \mathbb{N}$. We claim that the sequence $\{u_n\}_{n=1}^\infty$ is bounded in E. In fact, by (B), (3.6) and the Hölder inequality, we have

$$\begin{aligned}
\frac{1}{p}\|u_n\|^p &= \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\alpha p}}\,dx\,dy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|u(x)|^p\,dx \\
&= \Phi_{\lambda_n}(u_n) + \lambda_n \int_{\mathbb{R}^N} F(x,u_n)\,dx \\
&\le \Phi_\lambda(u_n) + 2\varepsilon\|u_n\|_p^p + 2C_\varepsilon \int_{\mathbb{R}^N} a(x)\,|u_n|^r\,dx \\
&\le M_0 + 2\varepsilon C_1^p\|u_n\|^p + 2C_\varepsilon \int_{\mathbb{R}^N} \frac{a(x)}{V(x)^{r/p}} V(x)^{r/p}\,|u_n|^r\,dx \\
&\le M_0 + 2\varepsilon C_1^p\|u_n\|^p + 2C_\varepsilon \frac{1}{V_0^{r/p}} \int_{\mathbb{R}^N} a(x)V(x)^{r/p}\,|u_n|^r\,dx \\
&\le M_0 + 2\varepsilon C_1^p\|u_n\|^p + 2C_\varepsilon \frac{1}{V_0^{r/p}} \|a\|_{p/(p-r)} \left\|V^{r/p}\,|u_n|^r\right\|_{p/r} \\
&\le M_0 + 2\varepsilon C_1^p\|u_n\|^p + 2C_\varepsilon \frac{1}{V_0^{r/p}} \|a\|_{p/(p-r)} \|u_n\|^r,
\end{aligned} \tag{3.7}$$

where M_0 is some positive constant and C_0 is the embedding constant for $\|u_n\|_p \le C_1\|u_n\|$ (by Lemma 2.1). Since $p-1 < r < p$, (3.7) implies that $\{u_n\}$ is bounded in E. So we can find $M > 0$ such that $\|u_n\| \le M$ for all $n \in \mathbb{N}$.

Our next step is to show that there is a strongly convergent subsequence of $\{u_n\}_{n=1}^\infty$ in E. Indeed, in view of the boundedness of $\{u_n\}_{n=1}^\infty$, passing to a subsequence if necessary, still denoted by $\{u_n\}_{n=1}^\infty$, we may assume that $u_n \rightharpoonup u_0$ weakly in E.

Let $P_n \colon E \to Y_n$ denote the projection operator for all $n \in \mathbb{N}$, then we have

$$0 = \Phi'_{\lambda_n}|_{Y_n}(u_n) = P_n\Phi'_{\lambda_n}(u_n).$$

Thus, $\left\langle P_n \Phi'_{\lambda_n}(u_n), u_n - u_0 \right\rangle = 0$ and from $u_n \rightharpoonup u_0$, we see that $\langle \Phi'_1(u_0), u_n - u_0 \rangle \to 0$ as $n \to \infty$. Therefore, by the inequality $\left| |u|^{p-2} u - |v|^{p-2} v \right| |u - v| \geq c |u - v|^p$ (where c is a constant independent from the variable u and v), we conclude that

$$
\begin{aligned}
&\left\langle P_n \Phi'_{\lambda_n}(u_n) - \Phi'_1(u_0), u_n - u_0 \right\rangle \\
&= \int_{\mathbb{R}^{2N}} \frac{P_n \left\{ |u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) - |u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) \right\}}{|x - y|^{N+\alpha p}} \\
&\qquad \times (u_n(x) - u_n(y) - u_0(x) + u_0(y)) \, dx \, dy \\
&\quad + \int_{\mathbb{R}^N} V(x) P_n \left[|u_n|^{p-2} u_n - |u_0|^{p-2} u \right] (u_n - u_0) \, dx \\
&\quad - \lambda_n \int_{\mathbb{R}^N} f(x, u_n) P_n (u_n - u_0) \, dx + \int_{\mathbb{R}^N} f(x, u_0)(u_n - u_0) \, dx \\
&\geq c_1 \int_{\mathbb{R}^{2N}} \frac{P_n |(u_n(x) - u_n(y)) - (u_0(x) - u_0(y))|^p}{|x - y|^{N+\alpha p}} \, dx \, dy + c_2 \int_{\mathbb{R}^N} V(x) P_n |u_n - u_0|^p \, dx \\
&\quad - \lambda_n \int_{\mathbb{R}^N} f(x, u_n) P_n (u_n - u_0) \, dx + \int_{\mathbb{R}^N} f(x, u_0)(u_n - u_0) \, dx \\
&\geq c_3 \| P_n (u_n - u_0) \|^p - \lambda_n \int_{\mathbb{R}^N} f(x, u_n) P_n (u_n - u_0) \, dx + \int_{\mathbb{R}^N} f(x, u_0)(u_n - u_0) \, dx \\
&= c_3 \| (u_n - u_0) \|^p - \lambda_n \int_{\mathbb{R}^N} f(x, u_n) P_n (u_n - u_0) \, dx + \int_{\mathbb{R}^N} f(x, u_0)(u_n - u_0) \, dx.
\end{aligned} \tag{3.8}
$$

From hypotheses (C1) and (C2), we see that given $\varepsilon > 0$, we can find $C_\varepsilon > 0$ such that

$$
|f(x, t)| \leq \frac{\varepsilon}{M^p} |t|^{p-1} + C_\varepsilon a(x) |t|^{r-1}, \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.9}
$$

From the choice of the function $a \in L^{p/(p-r)}(\mathbb{R}^N)$, we can choose $R_\varepsilon > 0$ such that

$$
\|a\|_{L^{p/(p-r)}(\Omega_\varepsilon^c)} < \frac{\varepsilon}{M^r C_\varepsilon}, \tag{3.10}
$$

where $\Omega_\varepsilon^c = \mathbb{R}^N \setminus \Omega_\varepsilon$ and $\Omega_\varepsilon = \left\{ x \in \mathbb{R}^N : |x| \leq R_\varepsilon \right\}$.

Since the embedding $E \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^N)$ is compact, $u_n \rightharpoonup u_0$ in E implies $u_n \to u_0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$, and hence there exists $n_0 \in N$ such that

$$
\|u_n - u_0\|_{L^p(\Omega_\varepsilon)} < \frac{\varepsilon}{M^{r-1} C_\varepsilon \|a\|_{L^{p/(p-r)}(\mathbb{R}^N)}} \quad \text{for } n \geq n_0. \tag{3.11}
$$

Using (3.9)–(3.11) and the Hölder inequality, we can estimate the last line of (3.8) as follows:

$$
\begin{aligned}
&\left| \lambda_n \int_{\mathbb{R}^N} f(x, u_n) P_n (u_n - u_0) \, dx \right| \\
&\leq 2 \left[\frac{\varepsilon}{M^p} \int_{\mathbb{R}^N} |u_n|^{p-1} |P_n (u_n - u_0)| \, dx + C_\varepsilon \int_{\mathbb{R}^N} a(x) |u_n|^{r-1} |P_n (u_n - u_0)| \, dx \right] \\
&\leq \frac{2\varepsilon}{M^p} \|u_n\|^{p-1}_{L^p(\mathbb{R}^N)} \|P_n (u_n - u_0)\|_{L^p(\mathbb{R}^N)}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2C_\varepsilon \|a\|_{L^{p/(p-r)}(\Omega_\varepsilon)} \|u_n\|^{r-1}_{L^p(\Omega_\varepsilon)} \|P_n(u_n - u_0)\|_{L^p(\Omega_\varepsilon)} \\
&+ 2C_\varepsilon \|a\|_{L^{p/(p-r)}(\Omega_\varepsilon^c)} \|u_n\|^{r-1}_{L^p(\Omega_\varepsilon^c)} \|P_n(u_n - u_0)\|_{L^p(\Omega_\varepsilon^c)} \qquad (3.12)\\
\le\ & \frac{c_4\varepsilon}{M^p} \|u_n\|^{p-1} \|P_n(u_n - u_0)\| \\
&+ c_5 C_\varepsilon \|a\|_{L^{p/(p-r)}(\mathbb{R}^N)} \|u_n\|^{r-1} \frac{\varepsilon}{M^{r-1} C_\varepsilon \|a\|_{L^{p/(p-r)}(\mathbb{R}^N)}} \\
&+ c_6 C_\varepsilon \frac{\varepsilon}{M^r C_\varepsilon} \|u_n\|^{r-1} \|P_n(u_n - u_0)\| \\
\le\ & c_7 \varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^N} f(x, u_0)(u_n - u_0)\, dx \right| \\
\le\ & \frac{\varepsilon}{M^p} \int_{\mathbb{R}^N} |u_0|^{p-1} |u_n - u_0|\, dx + C_\varepsilon \int_{\mathbb{R}^N} a(x) |u_0|^{r-1} |u_n - u_0|\, dx \\
=\ & \frac{\varepsilon}{M^p} \int_{\mathbb{R}^N} |u_0|^{p-1} |(u_n - u_0)|\, dx + C_\varepsilon \int_{\Omega_\varepsilon} a(x) |u_0|^{r-1} |u_n - u_0|\, dx \\
&+ C_\varepsilon \int_{\Omega_\varepsilon^c} a(x) |u_0|^{r-1} |u_n - u_0|\, dx \\
\le\ & \frac{\varepsilon}{M^p} \|u_0\|^{p-1}_{L^p(\mathbb{R}^N)} \|u_n - u_0\|_{L^p(\mathbb{R}^N)} \qquad (3.13)\\
&+ C_\varepsilon \|a\|_{L^{p/(p-r)}(\Omega_\varepsilon)} \|u_0\|^{r-1}_{L^p(\Omega_\varepsilon)} \|u_n - u_0\|_{L^p(\Omega_\varepsilon)} \\
&+ C_\varepsilon \|a\|_{L^{p/(p-r)}(\Omega_\varepsilon^c)} \|u_0\|^{r-1}_{L^p(\Omega_\varepsilon^c)} \|u_n - u_0\|_{L^p(\Omega_\varepsilon^c)} \\
\le\ & \frac{c_8\varepsilon}{M^p} \|u_0\|^{p-1} \|u_n - u_0\| \\
&+ c_9 C_\varepsilon \|a\|_{L^{p/(p-r)}(\mathbb{R}^N)} \|u_0\|^{r-1} \frac{\varepsilon}{M^{r-1} C_\varepsilon \|a\|_{L^{p/(p-r)}(\mathbb{R}^N)}} \\
&+ c_{10} C_\varepsilon \frac{\varepsilon}{M^r C_\varepsilon} \|u_0\|^{r-1} \|u_n - u_0\| \\
\le\ & c_{11}\varepsilon.
\end{aligned}
$$

Since ε is arbitrary, it follows from (3.8), (3.12)–(3.13) that

$$u_n \to u_0 \quad \text{in } E \text{ as } n \to +\infty.$$

Thus, from the last assertion of Lemma 2.2, we know that $\Phi = \Phi_1$ has infinitely many nontrivial critical points. Therefore, problem (1.1) possesses infinitely many nontrivial solutions. The proof of Theorem 1.1 is completed. □

3.2. Proof of Theorem 1.2

In this section, we use Lemma 2.3 to prove Theorem 1.2. Next, we will verify that all the conditions of Lemma 2.3 are fulfilled. In fact, it is obvious that $B(u) \ge 0$ from the definition of the functional B and (D1). Moreover, $A(u) \to \infty$ as $\|u\| \to \infty$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$. From the hypotheses (D1) and (D2), we known that Φ_λ maps bounded set into bounded sets uniformly for $\lambda \in [1,2]$. Thus, conditions (1) and (2) in Lemma 2.3 have been verified. Moreover, we will verify that condition (3) of Lemma 2.3 is fulfilled.

Lemma 3.3. *Suppose that* (D1)–(D4) *are satisfied. Then there exist two sequences* $0 < r_k < \rho_k$ *as such that*

$$b_k(\lambda) = \inf_{\substack{u \in Z_k \\ \|u\| = r_k}} \Phi_\lambda(u) > a_k(\lambda) = \max_{\substack{u \in Y_k \\ \|u\| = \rho_k}} \Phi_\lambda(u), \quad \forall\, \lambda \in [1,2].$$

Proof. By (D1) and (D2), for any $\varepsilon > 0$, there exists a $c_\varepsilon > 0$ such that

$$|f(x,u)| \le \varepsilon \, |u|^{p-1} + c_\varepsilon \, |u|^{\theta - 1} \quad \text{for all } x \in \mathbb{R}^N,\ u \in \mathbb{R}. \tag{3.14}$$

Let $\alpha_k = \sup_{u \in Z_k, \|u\|=1} \|u\|_{L^\theta(\mathbb{R}^N)}$ $(\theta \in [p, p_\alpha^*))$, from Lemma 3.2, we see that $\alpha_k \to 0$ as $k \to \infty$. Therefore, for $u \in Z_k$ and ε small enough, by (3.14), we have

$$\begin{aligned} \Phi_\lambda(u) &\ge \frac{1}{p} \|u\|^p - \frac{\lambda \varepsilon}{p} \|u\|_p^p - \frac{\lambda c_\varepsilon}{\theta} \|u\|_\theta^\theta \\ &\ge \frac{1}{2p} \|u\|^p - c_{12} \|u\|_\theta^\theta \\ &\ge \frac{1}{2p} \|u\|^p - c_{12} \alpha_k^\theta \|u\|^\theta . \end{aligned}$$

If we choose $r_k = (4pc_{12}\alpha_k^\theta)^{1/(p-\theta)}$, then for any $u \in Z_k$ with $\|u\| = r_k$, we get that

$$\Phi_\lambda(u) \ge (4p)^{\theta/(p-\theta)} (c_{12}\alpha_k^\theta)^{p/(p-\theta)} > 0.$$

This inequality implies that

$$b_k(\lambda) = \inf_{\substack{u \in Z_k \\ \|u\| = r_k}} \Phi_\lambda(u) \ge (4p)^{\theta/(p-\theta)} (c_{12}\alpha_k^\theta)^{p/(p-\theta)} > 0, \quad \forall\, \lambda \in [1,2].$$

Note that the proof of (3.1) does not involve the conditions (C1) and (C2), we only use the condition (C3). Therefore, we replace it by the condition (D3), it still hold here. Hence, for any $k \in \mathbb{N}$, there exists a constant $\varepsilon_k > 0$ such that

$$\operatorname{meas}(S_u) \ge \varepsilon_k, \quad \forall\, |u| \in Y_k \setminus \{0\}, \tag{3.15}$$

where $S_u = \{x \in \mathbb{R}^N : |u(x)| \ge \varepsilon_k \|u\|\}$. By (D3), for any $k \in \mathbb{N}$, there exists a constant $R_k > 0$ such that

$$F(x,u) \ge \frac{1}{\varepsilon_k^{p+1}} |u|^p, \quad \forall\, |u| \ge R_k. \tag{3.16}$$

Hence, by (3.15), we see that for any $u \in Y_k$ with $\|u\| \geq R_k/\varepsilon_k$, we have $|u(x)| \geq R_k$, for all $x \in S_u$. Therefore, for any $u \in Y_k$ with $\|u\| \geq R_k/\varepsilon_k$ and $\lambda \in [1,2]$, by (3.15) and (3.16), we have

$$\begin{aligned}\Phi_\lambda(u) &\leq \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} F(x,u)\,dx \\ &\leq \frac{1}{p}\|u\|^p - \int_{S_u} F(x,u)\,dx \\ &\leq \frac{1}{p}\|u\|^p - \int_{S_u} \frac{1}{\varepsilon_k^{p+1}}|u|^p\,dx \\ &\leq \frac{1}{p}\|u\|^p - \varepsilon_k^p\|u\|^p\frac{\operatorname{meas}(S_u)}{\varepsilon_k^{p+1}} \\ &\leq \frac{1}{p}\|u\|^p - \|u\|^p = -\frac{p-1}{p}\|u\|^p.\end{aligned}$$

If we choose $\rho_k > \max\{r_k, R_k/\varepsilon_k\}$, we get that

$$a_k(\lambda) = \max_{\substack{u\in Y_k \\ \|u\|=\rho_k}} \Phi_\lambda(u) \leq -\frac{(p-1)r_k^p}{p} < 0, \quad \forall\, k \in \mathbb{N} \text{ and for all } \lambda \in [1,2].$$

The proof is completed. $\square$

Proof of Theorem 1.2. By Lemma 3.3, the third condition of Lemma 2.3 have been verified. Hence, for almost every $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that

$$\sup_n \left\|u_n^k(\lambda)\right\| < \infty, \quad \Phi'_\lambda(u_n^k(\lambda)) \to 0, \quad \Phi_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \quad \text{as } n \to +\infty. \tag{3.17}$$

By (D1) and (D2), for any $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon|t|^{p-1} + C_\varepsilon b(x)|t|^{\theta-1}, \quad \forall\,(x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.18}$$

Let $\beta_k = \sup_{u\in Z_k,\|u\|=1}\int_{\mathbb{R}^N} b(x)|u|^\theta\,dx$. We claim that $\beta_k \to 0$ as $k \to +\infty$. In fact, it is obvious that $\beta_k \geq \beta_{k+1} \geq 0$, so $\beta_k \to \beta_0 \geq 0$ as $k \to +\infty$. For each $k = 1,2,\ldots$, taking $u_k \in Z_k$, $\|u_k\| = 1$ such that

$$0 \leq \beta_k - \int_{\mathbb{R}^N} b(x)|u_k|^\theta\,dx < \frac{1}{k}. \tag{3.19}$$

As E is reflexive, $\{u_k\}$ has a weakly convergent subsequence, without loss of generality, suppose $u_k \rightharpoonup u$ weakly in E, that is,

$$\langle e_i^*, u\rangle = \lim_{k\to\infty}\langle e_i^*, u_k\rangle = 0, \quad i = 1,2,\ldots,$$

which implies that $u = 0$, and so $u_k \rightharpoonup 0$ weakly in E.

Take $\Omega_k = \{x \in \mathbb{R}^N : |x| < k\}$ and $\Omega_k^c = \mathbb{R}^N \setminus B_k$. From the choice of the function $b \in L^q(\mathbb{R}^N)$, for any given number $\varepsilon > 0$, we may find $k_1 > 0$ big enough such that

$$\|b\|_{L^q(\Omega_{k_1}^c)} < \frac{\varepsilon}{2C_2^\theta}, \tag{3.20}$$

where C_2 is the embedding constant for $\|u\|_{L^{q\theta/(q-1)}(\mathbb{R}^N)} \leq C_2 \|u\|$ (by Lemma 2.1).

Since the embedding $E \hookrightarrow L_{\mathrm{loc}}^{q\theta/(q-1)}(\mathbb{R}^N)$ is compact, $u_k \rightharpoonup 0$ in E implies $u_k \to 0$ in $L_{\mathrm{loc}}^{q\theta/(q-1)}(\mathbb{R}^N)$, and hence there exists $k_2 \in \mathbb{N}$ such that

$$\|u_k\|_{L^{q\theta/(q-1)}(\Omega_{k_1})} < \frac{\varepsilon}{2\|b\|_{L^q(\mathbb{R}^N)}} \quad \text{for } k \geq k_2. \tag{3.21}$$

Using (3.20) and (3.21), we get

$$\begin{aligned}
\int_{\mathbb{R}^N} b(x)\,|u_k|^\theta\,dx &= \int_{\Omega_{k_1}} b(x)\,|u_k|^\theta\,dx + \int_{\Omega_{k_1}^c} b(x)\,|u_k|^\theta\,dx \\
&\leq \|b\|_{L^q(\Omega_{k_1})} \|u_k\|_{L^{q\theta/(q-1)}(\Omega_{k_1})}^\theta + \|b\|_{L^q(\Omega_{k_1}^c)} \|u_k\|_{L^{q\theta/(q-1)}(\Omega_{k_1}^c)}^\theta \\
&\leq \|b\|_{L^q(\mathbb{R}^N)} \|u_k\|_{L^{q\theta/(q-1)}(\Omega_{k_1})}^\theta + \|b\|_{L^q(\Omega_{k_1}^c)} \|u_k\|_{L^{q\theta/(q-1)}(\mathbb{R}^N)}^\theta \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{3.22}$$

Therefore, from (3.19) and (3.22), we conclude that $\beta_k \to 0$ as $k \to \infty$. Thus, for $u \in Z_k$ and ε small enough, by (3.18), we have

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\alpha p}}\,dx\,dy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|u(x)|^p\,dx - \lambda\int_{\mathbb{R}^N} F(x,u)\,dx \\
&\geq \frac{1}{p}\|u\|^p - \frac{\lambda\varepsilon}{p}\|u\|_{L^p(\mathbb{R}^N)}^p - \frac{\lambda C_\varepsilon}{\theta}\int_{\mathbb{R}^N} b(x)\,|u|^\theta\,dx \\
&\geq \frac{1}{p}\|u\|^p - \frac{\lambda\varepsilon}{p}C_1^p\|u\|^p - \frac{\lambda C_\varepsilon}{\theta}\beta_k\|u\|^\theta \\
&\geq \frac{1}{p}\|u\|^p - \varepsilon C_1^p\|u\|^p - C_\varepsilon\beta_k\|u\|^\theta \\
&\geq \frac{1}{2p}\|u\|^p - C_\varepsilon\beta_k\|u\|^\theta.
\end{aligned}$$

If we choose $r_k = (4pC_\varepsilon\beta_k)^{1/(p-\theta)}$, then for any $u \in Z_k$ with $\|u\| = r_k$, we get that

$$\Phi_\lambda(u) = (4p)^{\theta/(p-\theta)}(C_\varepsilon\beta_k)^{p/(p-\theta)} > 0,$$

which implies that

$$b_k(\lambda) \geq (4p)^{\theta/(p-\theta)}(C_\varepsilon\beta_k)^{p/(p-\theta)} > 0, \quad \forall\,\lambda \in [1,2]. \tag{3.23}$$

Then by virtue of (3.23) and Lemma 2.3, we have

$$c_k(\lambda) \geq b_k(\lambda) \geq (4p)^{\theta/(p-\theta)}(C_\varepsilon\beta_k)^{p/(p-\theta)} =: \bar{b}_k \to \infty \quad \text{as } k \to \infty.$$

However, note that

$$c_k(\lambda) = \inf_{\gamma\in\Gamma} \max_{u\in B_k} \Phi_\lambda(\gamma(u)) \le \max_{u\in B_k} \Phi_1(u) =: \bar{c}_k.$$

So we have

$$\bar{b}_k \le c_k(\lambda) \le \bar{c}_k \quad \text{for } k \ge k_0. \tag{3.24}$$

Moreover, using (3.17), we see that if we choose a sequence $\lambda_m \to 1$, then the sequence $\{u_n^k(\lambda_m)\}_{n=1}^{\infty}$ is bounded. Using the similar arguments of the proof of Theorem 1.1, we can prove that the sequence $\{u_n^k(\lambda_m)\}_{n=1}^{\infty}$ has a strong convergent subsequence as $n \to \infty$. Hence, we may assume that $\lim_{n\to\infty} u_n^k(\lambda_m) = u^k(\lambda_m)$ for every $m \in \mathbb{N}$ and $k \ge k_0$. Thus, combining (3.17) and (3.24), we obtain

$$\Phi'_{\lambda_m}(u^k(\lambda_m)) = 0 \quad \text{and} \quad \Phi_{\lambda_m}(u^k(\lambda_m)) \in [\bar{b}_k, \bar{c}_k] \quad \text{for } k \ge k_0. \tag{3.25}$$

Next, we will prove that $\{u^k(\lambda_m)\}_{m=1}^{\infty}$ is bounded in E. If it is unbounded we define $w_m = u^k(\lambda_m)/\|u^k(\lambda_m)\|$. Without loss of generality, suppose that there is $w \in E$ such that

$$\begin{aligned} w_m &\rightharpoonup w && \text{in } E,\\ w_m &\to w && \text{in } L^{\tau}_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } \tau \in (p, p^*_\alpha),\\ w_m(x) &\to w(x) && \text{a.e. } x \in \mathbb{R}^N. \end{aligned}$$

Here, two cases appear.

Case 1. If $w = 0$ in E, using (3.25), we have

$$\begin{aligned} \frac{\mu}{p} - 1 &= \frac{\mu\Phi_{\lambda_m}(u^k(\lambda_m)) - \langle \Phi'_{\lambda_m}(u^k(\lambda_m)), u^k(\lambda_m)\rangle}{\|u^k(\lambda_m)\|^p}\\ &\quad + \lambda_m \int_{\mathbb{R}^N} |w_m(x)|^p \, \frac{\mu F(x, u^k(\lambda_m)) - f(x, u^k(\lambda_m))u^k(\lambda_m)}{|u^k(\lambda_m)|^p}\, dx. \end{aligned}$$

Hence,

$$\lambda_m \int_{\mathbb{R}^N} |w_m(x)|^p \, \frac{\mu F(x, u^k(\lambda_m)) - f(x, u^k(\lambda_m))u^k(\lambda_m)}{|u^k(\lambda_m)|^p}\, dx \to \frac{\mu}{p} - 1 \quad \text{as } m \to \infty. \tag{3.26}$$

But by hypothesis (D4),

$$\limsup_{m\to\infty} \frac{\mu F(x, u^k(\lambda_m)) - f(x, u^k(\lambda_m))u^k(\lambda_m)}{|u^k(\lambda_m)|^p} \, |w_m(x)|^p \le 0. \tag{3.27}$$

Combining with (3.26) and (3.27), we get $\mu/p - 1 \le 0$, i.e., $\mu \le p$, and this is in contradiction with the assumption.

Case 2. If $w \neq 0$ in E, we have

$$\begin{aligned}\frac{1}{p} - \frac{\Phi_{\lambda_m}(u^k(\lambda_m))}{\|u^k(\lambda_m)\|^p} &= \lambda_m \int_{\mathbb{R}^N} \frac{F(x, u^k(\lambda_m))}{\|u^k(\lambda_m)\|^p}\, dx \\ &= \lambda_m \int_{\{w_m(x)\neq 0\}} |w_m(x)|^p \frac{F(x, u^k(\lambda_m))}{|u^k(\lambda_m)|^p}\, dx.\end{aligned}$$

By (3.25), (D3) and Fatou's Lemma, we deduce a contradiction that

$$\frac{1}{p} = \liminf_{m\to\infty} \lambda_m \int_{\{w_m(x)\neq 0\}} |w_m(x)|^p \frac{F(x, u^k(\lambda_m))}{|u^k(\lambda_m)|^p}\, dx \to \infty \quad \text{as } m \to \infty.$$

Hence, $\left\{u^k(\lambda_m)\right\}_{m=1}^{\infty}$ is bounded in E. Thus, as in the proof of Theorem 1.1, $\left\{u^k(\lambda_m)\right\}_{m=1}^{\infty}$ possesses a strong convergent subsequence with the limit $u^k \in E$ for all $k \geq k_0$. Therefore, the limit u^k is a critical point of $\Phi = \Phi_1$ with $\Phi(u^k) \in [\overline{b}_k, \overline{c}_k]$. Since $\overline{b}_k \to +\infty$ as $k \to +\infty$, we get infinitely many nontrivial critical points of Φ. Consequently, problem (1.1) possesses infinitely many nontrivial solutions with high energy.

The proof is completed. □

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