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# On Abelian Canonical $n$-folds of General Type

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Abstract. Let $X$ be a Gorenstein minimal projective $n$-fold with at worst locally factorial terminal singularities, and suppose that the canonical map of $X$ is generically finite onto its image. When $n < 4$, the canonical degree is universally bounded. While the possibility of obtaining a universal bound on the canonical degree of $X$ for $n \geq 4$ may be inaccessible, we give a uniform upper bound for the degrees of certain abelian covers. In particular, we show that if the canonical divisor $K_X$ defines an abelian cover over $\mathbb{P}^n$, i.e., when $X$ is an *abelian canonical $n$-fold*, then the canonical degree of $X$ is universally upper bounded by a constant which only depends on $n$ for $X$ nonsingular. We also construct two examples of nonsingular minimal projective 4-folds of general type with canonical degrees 81 and 128.

## 1. Introduction

The study of the canonical maps of $n$-dimensional projective varieties of general type is one of the central problems in algebraic geometry. For nonsingular algebraic surfaces, Beauville [1] proved that the degree of the canonical map is less than or equal to 36 and that equality holds if and only if $X$ is a ball quotient surface with $K_X^2 = 36$, $p_g = 3$, $q = 0$, and $|K_X|$ is base point free. For $n = 3$, Chen posed an open problem in [5] as follows. Let $X$ be a minimal projective 3-fold with at worst locally factorial terminal singularities, and suppose that the canonical map is generically finite onto its image. Is the generic degree of the canonical map universally bounded from above? Later, Hacon [10] gave some examples of 3-folds of general type with terminal singularities such that the canonical degrees of these 3-folds can be arbitrarily large, but he also showed that the answer of Chen's question is "yes" if one adds the Gorenstein condition. More precisely, he showed

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that if $X$ is a Gorenstein minimal projective 3-fold with at worst locally factorial terminal singularities, then the canonical degree of $X$ is at most 576. Recently, the first and the second authors improved Hacon's upper bound to 360, and showed that equality holds if and only if $p_g(X) = 4$, $q(X) = 2$, $\chi(\omega_X) = 5$, $K_X^3 = 360$ and $|K_X|$ is base point free. For $n < 4$, the Miyaoka-Yau inequality plays a vital role in the proof. For $n \geq 4$ however, the Miyaoka-Yau inequality is not effective enough to give a universal bound to control $K_X^n$.

Another open question in this direction is to determine the positive degree of the canonical map. Progress in this direction for surfaces appears in [1, 3, 7, 12–17] and for 3-folds in [2, 8]. Explicit examples are often constructed by taking abelian covers. Since the canonical degree may not be universally bounded, it is natural to ask the following question: Can one construct nonsingular minimal projective $n$-folds whose canonical divisors define abelian covers over $\mathbb{P}^n$ with arbitrarily large degrees?

**Definition 1.1.** Let $X$ be a minimal projective $n$-fold of general type with at worst locally factorial terminal singularities. If $|K_X|$ defines abelian cover over $\mathbb{P}^n$, then we call $X$ an *abelian canonical $n$-fold*.

In [7], the first and the second authors obtained a complete classification of abelian canonical surfaces. The universal upper bound for such surfaces is 16. In [8], they showed that for Gorenstein minimal projective 3-folds of general type with at worst locally factorial terminal singularities, the upper bound of the canonical degrees of such 3-folds is 32. Although we do not know if the canonical degree is universally bounded or not for higher dimensional projective varieties, there is evidence suggesting that the canonical degrees of nonsingular abelian canonical $n$-folds may be universally bounded due to the fact that there are very strong restrictions on the defining data of abelian covers. In this paper, we show that the degrees of certain abelian covers are uniformly bounded. In particular, we give a negative answer to the above question. More precisely, we show that the canonical degrees of nonsingular abelian canonical $n$-folds are universally bounded. We also construct two examples of nonsingular minimal projective 4-folds of general type with canonical degrees 81 and 128 in the last section.

## 2. Basics on abelain covers

The theory of covering is a very important tool in algebraic geometry. The cyclic covers of algebraic surfaces were first studied by Comessatti in [6]. Later, F. Catanese [4] studied smooth abelian covers in the case $(\mathbb{Z}_2)^{\oplus 2}$. While R. Pardini analyzed the general case in [11], F. Catanese [4] pointed out that it is difficult to give the defining data of abelian covers by Pardini's method. Recently, the second author studied abelian covers of algebraic varieties from another point of view by calculating the normalization bases of the covering spaces which made the constructions more explicit [9]. In this section, we shall recall some basic definitions and results of abelian covers which will facilitate our subsequent discussion. As our goal is to determine the defining equations for covering spaces by explicit calculation, we use the method appearing in [9].

Let $X$ and $Y$ be projective algebraic varieties such that $X$ is normal, $Y$ is nonsingular and $\varphi\colon X \to Y$ is an abelian cover associated with the abelian group $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, where $n_1 \mid n_2 \mid \cdots \mid n_k$ (i.e., the function field $\mathbb{C}(X)$ of $X$ is an abelian extension of the rational function field $\mathbb{C}(Y)$ with Galois group $G$).

**Definition 2.1.** Let $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$. The data of an abelian cover over $Y$ with group $G$ consists of $k$ effective divisors $D_1, \ldots, D_k$, and $k$ linear equivalence relations

$$D_1 \sim n_1 L_1, \ \ldots, \ D_k \sim n_k L_k.$$

Let $\mathscr{L}_i = \mathscr{O}_Y(L_i)$ and $f_i$ be the defining equation of $D_i$, i.e., $D_i = \operatorname{div}(f_i)$, where $f_i \in H^0(Y, \mathscr{L}_i^{n_i})$. Denote by $V(\mathscr{L}_i) = \operatorname{Spec} S(\mathscr{L}_i)$ the line bundle corresponding to $\mathscr{L}_i$, where $S(\mathscr{L}_i)$ is the sheaf of the symmetric $\mathscr{O}_Y$ algebra, and let $z_i$ be the fiber coordinate of $V(\mathscr{L}_i)$. Then the abelian cover can be realized by the normalization of the variety $V$ defined by the system of equations

$$z_1^{n_1} = f_1, \ \ldots, \ z_k^{n_k} = f_k. \tag{2.1}$$

The above paragraph is summarized schematically via the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\text{normalization}"] \arrow[drr, "\varphi"'] & V \arrow[r, hook] \arrow[dr, "f"] & \bigoplus_{i=1}^{k} V(\mathscr{L}_i) \arrow[d, "p"] \\
& & Y
\end{tikzcd}$$

We often make the abuse of saying $X$ is defined by equations (2.1), although it should be clear from the context that $X$ is in fact the normalization of the solution $V$ of these equations.

We now list some useful results which will be crucial later on.

**Theorem 2.2.** [9] *Denote by $[Z]$ the integral part of a $\mathbb{Q}$-divisor $Z$, and $-L_g = -\sum_{i=1}^k g_i L_i + \left[\sum_{i=1}^k \frac{g_i}{n_i} D_i\right]$, where $g = (g_1, \ldots, g_k) \in G$. Then*

$$\varphi_* \mathcal{O}_X = \bigoplus_{g \in G} \mathcal{O}_Y(-L_g). \tag{2.2}$$

We remark that equation (2.2) implies the decomposition of $\varphi_* \mathcal{O}_X$ is completely determined by the data of an abelian cover.

**Corollary 2.3.** [9] *If $X$ is nonsingular, $D$ is a divisor on $Y$, then*

$$h^i(X, \varphi^*\mathcal{O}_Y(D)) = \sum_{g\in G} h^i(Y, \mathcal{O}_Y(D - L_g)).$$

The following result will be used to calculate the ramification index, since the branching locus is uniformly ramified for an abelian cover.

**Theorem 2.4.** [9] *Let $P$ be an irreducible and reduced hypersurface in $Y$, let $\overline{P} = \pi^{-1}(P)$ be the reduced preimage of $P$ in $X$, and let $a_i$ be the multiplicity of $P$ in $D_i = \operatorname{div}(f_i)$. Then*

$$\pi^* P = \frac{|G|}{d_P}\overline{P},$$

*where*

$$d_P = \gcd\left(|G|, |G|\frac{a_1}{n_1}, \dots, |G|\frac{a_k}{n_k}\right)$$

*is the number of points in the preimage of a generic point on $P$.*

## 3. Abelian canonical $n$-folds

Let $X$ and $Y$ be projective algebraic varieties, with $X$ normal and $Y$ nonsingular, and let $\varphi\colon X \to Y$ be an abelian cover associated to the abelian group $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, where $n_1 \mid n_2 \mid \cdots \mid n_k$. Using the notations after Definition 2.1, we have that $X$ is the normalization of the $n$-fold defined by

$$z_1^{n_1} = f_1, \ \dots, \ z_k^{n_k} = f_k. \tag{3.1}$$

We say that an abelian cover $\varphi\colon X \to Y$ is *totally ramified* if the inertia subgroups of the divisorial components of the branch locus of $\varphi$ generate $G$, or, equivalently, if $\varphi$ does not factorize through a cover $X' \to Y$ that is étale over $Y$. Note that if $\operatorname{Cl}(Y)$ has no torsion, then every connected abelian cover of $Y$ is totally ramified.

**Lemma 3.1.** *With notations as above, suppose that the cover is totally ramified and $D_1, D_2, \dots, D_m$ are the irreducible components of the branch loci of the abelian cover $\varphi$ with ramification indices $r_1, r_2, \dots, r_m$ respectively. Then $n_k \le \prod_{i=1}^m r_i$.*

*Proof.* Suppose $n_t = p_1^{e_{t_1}} \cdots p_s^{e_{t_s}}$, $1 \le t \le k$, is the prime decomposition of $n_t$. Without loss of generality, we only need to show that there exists $D_j$ such that its ramification index $r_j$ is divisible by $p_s^{e_{k_s}}$. Otherwise, for any irreducible component of the branch locus, we would have for the ramification index

$$\frac{|G|}{d} = p_1^{e_1} \cdots p_s^{e_s},$$

such that $e_s < e_{k_s}$, where

$$d = \gcd\left(|G|, |G|\frac{a_1}{n_1}, \dots, |G|\frac{a_k}{n_k}\right),$$

and $a_i$ is the multiplicity of the irreducible component in $\mathrm{div}(f_i)$ by Theorem 2.4. Thus

$$d = p_1^{\sum_{j=1}^{k} e_{j_1} - e_1} \cdots p_s^{\sum_{j=1}^{k} e_{j_s} - e_s},$$

which yields

$$d \,\Big|\, p_1^{\sum_{j=1}^{k-1} e_{j_1}} \cdots p_s^{\sum_{j=1}^{k-1} e_{j_s}} a_k.$$

Hence $p_s^{e_{k_s}-e_s} \mid a_k$, which implies equations (3.1) split and so $X$ is not irreducible, a contradiction. □

**Theorem 3.2.** *Let $Y$ be a nonsingular projective $n$-fold such that the divisor class group* $\mathrm{Cl}(Y)$ *torsion free, and fix both a divisor $L$ and an ample divisor $A$ on $Y$. Consider the following set of abelian covers*

$$\mathscr{C}_L := \{\varphi\colon X \to Y \mid X \text{ is nonsingular and } K_X = \varphi^* L\}.$$

*Then there exists a constant $C_{L,A}$ such that for any $\varphi \in \mathscr{C}_L$,* $\deg\varphi \le C_{L,A}$.

*Proof.* Since $\mathrm{Cl}(Y)$ is torsion free, all $D_i$'s in the defining data of each abelian cover over $Y$ as in Definition 2.1 are nonzero. Suppose that $D_1, D_2, \dots, D_m$ are the irreducible components of the branch locus of an abelian cover $\varphi \in \mathscr{C}_L$ with the ramification indices $r_1, r_2, \dots, r_m$, where $r_i \ge 2$, $i = 1, 2, \dots, m$. Fix an ample divisor $A$ on $Y$ and write $d_i = D_i A^{n-1}$. By the Hurwitz formula,

$$K_X = \varphi^*\left(K_Y + \sum_{i=1}^{m} \frac{r_i - 1}{r_i} D_i\right).$$

On the other hand, $K_X = \varphi^*(L)$, thus

$$(L - K_Y)A^{n-1} = \sum_{i=1}^{m} \frac{r_i - 1}{r_i} d_i,$$

i.e.,

$$\sum_{i=1}^{m} \frac{1}{r_i} d_i = m - (L - K_Y)A^{n-1}. \tag{3.2}$$

We then have

$$(L - K_Y)A^{n-1} < m \le \sum_{i=1}^{m} d_i \le 2 \cdot (L - K_Y)A^{n-1}.$$

Without loss of generality suppose $r_1 \leq r_2 \leq \cdots \leq r_m$. Then from (3.2) we have

$$\frac{1}{r_1}\sum_{i=1}^{m} d_i \geq m - (L-K_Y)A^{n-1},$$

thus

$$r_1 \leq \sum_{i=1}^{m} d_i \leq 2\cdot(L-K_Y)A^{n-1} =: C_1.$$

For fixed $m$, $r_1$ and $d_1$,

$$\frac{1}{r_2}\sum_{i=2}^{m} d_i \geq \sum_{i=2}^{m}\frac{1}{r_i}d_i = m - (L-K_Y)A^{n-1} - \frac{d_1}{r_1},$$

which yields

$$r_2 \leq \frac{2\cdot(L-K_Y)A^{n-1}}{m-(L-K_Y)A^{n-1}-d_1/r_1}.$$

As such, there exists a constant $C_2$ with $r_2 \leq C_2$. By induction, there exists a constant $C_m$ such that $r_m \leq C_m$. By Lemma 3.1, $n_k \leq \prod_{i=1}^{m} r_i \leq \prod_{i=1}^{m} C_i$, and since $n_k$, $m$ and $\sum_{i=1}^{m} d_i$ are finite, the number of the equations (3.1) is also finite, say $w$. So for any $\varphi \in \mathscr{C}_L$, $\deg\varphi$ is bounded by the constant

$$C_{L,A} := \left(\prod_{i=1}^{m} C_i\right)^{w}. \qquad \square$$

Now let $X$ be a minimal projective $n$-fold of general type with at worst locally factorial terminal singularities. As we have defined in Section 1, if $|K_X|$ defines an abelian cover over $\mathbb{P}^n$ then we call $X$ an abelian canonical $n$-fold.

**Corollary 3.3.** *If $X$ is an abelian canonical $n$-fold, then there exists a constant $C(n)$ depending only on $n$ such that the canonical degree of $X$ is universally bounded by $C(n)$.*

*Proof.* Take $L = H$ in the theorem above, where $H$ corresponds to a hyperplane in $\mathbb{P}^n$. $\square$

**Proposition 3.4.** *Let $X$ be a nonsingular abelian canonical $n$-fold and $\varphi$ be the finite abelian cover of degree $d$ over $\mathbb{P}^n$ defined by $|K_X|$, then $c_1((\varphi_*\mathcal{O}_X)^\vee) = \frac{n+2}{2}\cdot d$ and*

$$\begin{aligned}\varphi_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} &\oplus \mathcal{O}_{\mathbb{P}^n}(-n-2) \oplus (\mathcal{O}_{\mathbb{P}^n}(-2)\oplus\mathcal{O}_{\mathbb{P}^n}(-n))^{\oplus k_2}\\ &\oplus (\mathcal{O}_{\mathbb{P}^n}(-3)\oplus\mathcal{O}_{\mathbb{P}^n}(-n+1))^{\oplus k_3}\oplus\cdots\\ &\oplus(\mathcal{O}_{\mathbb{P}^n}(-t)\oplus\mathcal{O}_{\mathbb{P}^n}(-n-2+t))^{\oplus k_t}\oplus\cdots,\end{aligned}$$

*where $\mathcal{O}_{\mathbb{P}^n}(-t)$ appears the same number of times as $\mathcal{O}_{\mathbb{P}^n}(-n-2+t)$ in the direct sum.*

*Proof.* If $\varphi$ is a finite abelian cover, $\varphi_*\mathcal{O}_X$ is a direct sum of line bundles by Theorem 2.2, i.e.,

$$\varphi_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^n} \oplus \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^n}(-l_i).$$

Now assume $0 < l_1 \le l_2 \le \cdots \le l_{d-1}$. By relative duality,

$$\varphi_*\omega_X \cong (\varphi_*\mathcal{O}_X)^\vee \otimes \omega_{\mathbb{P}^n} \cong (\varphi_*\mathcal{O}_X)^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1),$$

and since $\omega_X = \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$, by the projection formula we have

$$\varphi_*\omega_X \cong \varphi_*\varphi^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong \mathcal{O}_{\mathbb{P}^n}(1) \otimes \varphi_*\mathcal{O}_X,$$

so that

$$(\varphi_*\mathcal{O}_X)^\vee \cong \varphi_*\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^n}(n+2),$$

from which it follows $\mathcal{O}_{\mathbb{P}^n}(-t)$ appears the same number of times as $\mathcal{O}_{\mathbb{P}^n}(-n-2+t)$ in the direct sum. Therefore $2c_1((\varphi_*\mathcal{O}_X)^\vee) = (n+2)d$, i.e., $c_1((\varphi_*\mathcal{O}_X)^\vee) = \frac{n+2}{2} \cdot d$, so we only need to show that $t \ge 2$. For this, it follows by the definition of an abelian canonical $n$-fold that

$$p_g(X) = h^0(K_X) = h^0(\varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))) = n+1,$$

thus by the projection formula we have

$$h^0(\mathcal{O}_{\mathbb{P}^n}(1)) + \sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^n}(1-l_i)) = n+1.$$

We then have

$$h^0(\mathcal{O}_{\mathbb{P}^n}(1-l_i)) = 0, \quad 1 \le i \le d-1,$$

thus $l_i \ge 2$. □

Now suppose $\varphi\colon X \to \mathbb{P}^n$ is an abelian cover associated to an abelian group $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, such that $n_1 \mid n_2 \mid \cdots \mid n_k$. Then $X$ is the normalization of the $n$-fold defined by

$$z_1^{n_1} = f_1 = \prod_\alpha p_\alpha^{\alpha_1}, \; \ldots, \; z_k^{n_k} = f_k = \prod_\alpha p_\alpha^{\alpha_k},$$

where the $p_\alpha$'s are coprime and $\alpha = (\alpha_1, \ldots, \alpha_k) \in G$, $\alpha_1, \ldots, \alpha_k < n_k$. Denote by $x_\alpha$ the degree of $p_\alpha$, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in G$ with $1 \le i \le k$, and denote by $l_g$ the degree of $L_g$. The $x_\alpha$ and $l_g$ are then integers. We then have

$$n_i l_{e_i} = \sum_\alpha \alpha_i x_\alpha, \quad i = 1, \ldots, k, \tag{3.3}$$

$$l_g = \sum_{i=1}^k g_i l_{e_i} - \sum_\alpha \left[\sum_{i=1}^k \frac{g_i \alpha_i}{n_i}\right] x_\alpha. \tag{3.4}$$

## 4. Abelian canonical 4-folds

Since the canonical degrees of nonsingular abelian canonical 4-folds have universal bounds, it is interesting to find some examples of such 4-folds such that the canonical degree is as close to the bound as possible.

By Proposition 3.4 and equations (3.3) and (3.4), we construct two examples of non-singular abelian canonical 4-folds (cf. [8, Theorem 3.6]).

**Example 4.1.** Let $X$ be the canonical abelian 4-fold which is the normalization of the variety whose defining equations are given by

$$\begin{aligned} &z_1^2 = h_1h_8h_9h_{10}, && z_2^2 = h_2h_8h_9h_{11}, && z_3^2 = h_3h_8h_9h_{12}, && z_4^2 = h_4h_8h_{10}h_{11}, \\ &z_5^2 = h_5h_8h_{10}h_{12}, && z_6^2 = h_6h_9h_{10}h_{11}, && z_7^2 = h_7h_9h_{10}h_{12}, \end{aligned}$$

where the hyperplanes $H_i$'s defined by $h_i$'s are normal crossing in $\mathbb{P}^4$.

First, we will show that $X$ is nonsingular. Obviously, the possible singularities of $X$ lie in the preimages of the intersections of the branch locus. Without loss of generality, we assume that $P$ is the intersection point of $H_8$, $H_9$, $H_{10}$ and $H_{11}$. The cover is locally defined by the following equations at $P$:

$$z_1^2 = xyu,\ z_2^2 = xyt,\ z_3^2 = xy,\ z_4^2 = xut,\ z_5^2 = xu,\ z_6^2 = yut,\ z_7^2 = yu.$$

After normalization, the cover $X$ is locally defined by

$$\bar{z}_1^2 = x, \quad \bar{z}_2^2 = t, \quad \bar{z}_3^2 = u, \quad \bar{z}_4^2 = y,$$

thus $X$ is smooth at the preimages of $P$.

Similarly, it is easy to show that $X$ is nonsingular on the preimage of the sets $H_i \cap H_j \cap H_k \cap H_l$, $H_i \cap H_j \cap H_k$ and $H_i \cap H_j$ for all $i$, $j$, $k$, $l$. So $X$ is nonsingular.

Next, we want to calculate $l_g$ by (3.4). Note that in this example $x_i = \deg(h_i) = 1$ for $1 \le i \le 12$. It is easy to check by (3.3) that $l_{e_i} = 2$ for all $i$. For other $l_g$'s, we take $g = (1,1,0,0,0,0,0)$ and $g' = (1,1,1,1,1,1,1)$ for example:

$$\begin{aligned} l_g &= l_{e_1} + l_{e_2} - \left[\frac{1+0}{2}\right] x_1 - \left[\frac{0+1}{2}\right] x_2 - \sum_{i=3}^{7} \left[\frac{0+0}{2}\right] x_i \\ &\quad - \left[\frac{1+1}{2}\right] x_8 - \left[\frac{1+1}{2}\right] x_9 - \left[\frac{1+0}{2}\right] x_{10} - \left[\frac{0+1}{2}\right] x_{11} - \left[\frac{0+0}{2}\right] x_{12} \\ &= 4 - x_8 - x_9 \\ &= 2, \end{aligned}$$

$$\begin{aligned}
l_{g'} &= \sum_{i=1}^{7} l_{e_i} - \sum_{i=1}^{7} \left[\frac{1}{2}\right] x_i - \left[\frac{1+1+1+1+1+0+0}{2}\right] x_8 \\
&\quad - \left[\frac{1+1+1+0+0+1+1}{2}\right] x_9 - \left[\frac{1+0+0+1+1+1+1}{2}\right] x_{10} \\
&\quad - \left[\frac{0+1+0+1+0+1+0}{2}\right] x_{11} - \left[\frac{0+0+1+0+1+0+1}{2}\right] x_{12} \\
&= 14 - \left[\frac{5}{2}\right] x_8 - \left[\frac{5}{2}\right] x_9 - \left[\frac{5}{2}\right] x_{10} - \left[\frac{3}{2}\right] x_{11} - \left[\frac{3}{2}\right] x_{12} \\
&= 6.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
& l_{(0,0,0,0,0,0,1)} = l_{(0,0,0,0,0,1,0)} = l_{(0,0,0,0,0,1,1)} = l_{(0,0,0,0,1,0,0)} = l_{(0,0,0,0,1,0,1)} \\
&= l_{(0,0,0,1,0,0,0)} = l_{(0,0,0,1,0,1,0)} = l_{(0,0,0,1,1,0,0)} = l_{(0,0,0,1,1,1,1)} = l_{(0,0,1,0,0,0,0)} \\
&= l_{(0,0,1,0,0,0,1)} = l_{(0,0,1,0,1,0,0)} = l_{(0,0,0,0,1,0,1)} = l_{(0,0,0,0,1,1,0)} = l_{(0,0,1,1,0,0,1)} \\
&= l_{(0,0,1,1,0,1,0)} = l_{(0,1,0,0,0,0,0)} = l_{(0,1,0,0,0,1,0)} = l_{(0,1,0,0,1,0,1)} = l_{(0,1,0,0,1,1,0)} \\
&= l_{(0,1,0,1,0,0,0)} = l_{(0,1,0,0,0,0,1)} = l_{(0,1,0,1,0,1,0)} = l_{(0,1,1,0,0,0,0)} = l_{(0,1,1,0,0,1,1)} \\
&= l_{(0,1,1,1,1,0,0)} = l_{(1,0,0,0,0,0,0)} = l_{(1,0,0,0,0,0,1)} = l_{(1,0,0,0,0,1,0)} = l_{(1,0,0,0,1,0,0)} \\
&= l_{(1,0,0,0,1,0,1)} = l_{(1,0,0,1,0,0,0)} = l_{(1,0,0,1,0,1,0)} = l_{(1,0,1,0,0,0,0)} = l_{(1,0,1,0,0,0,1)} \\
&= l_{(1,0,1,0,1,0,0)} = l_{(1,1,0,0,0,0,0)} = l_{(1,1,0,0,0,1,0)} = l_{(1,1,0,1,0,0,0)} = 2,
\end{aligned}$$

$$\begin{aligned}
& l_{(0,0,0,0,1,1,0)} = l_{(0,0,0,0,1,1,1)} = l_{(0,0,0,1,0,0,1)} = l_{(0,0,0,1,0,1,1)} = l_{(0,0,0,1,1,0,1)} \\
&= l_{(0,0,0,1,1,1,0)} = l_{(0,0,1,0,0,1,0)} = l_{(0,0,1,0,0,1,1)} = l_{(0,0,1,1,0,0,0)} = l_{(0,0,1,1,0,1,1)} \\
&= l_{(0,0,1,1,1,0,0)} = l_{(0,0,1,1,1,1,0)} = l_{(0,1,0,0,0,0,1)} = l_{(0,1,0,0,0,1,1)} = l_{(0,1,0,0,1,0,0)} \\
&= l_{(0,1,0,0,1,1,1)} = l_{(0,1,0,1,1,0,0)} = l_{(0,1,0,1,1,0,1)} = l_{(0,1,1,0,0,0,1)} = l_{(0,1,1,0,0,1,0)} \\
&= l_{(0,1,1,0,1,0,0)} = l_{(0,1,1,0,1,1,0)} = l_{(0,1,1,1,0,0,0)} = l_{(0,1,1,1,0,0,1)} = l_{(1,0,0,0,1,1,0)} \\
&= l_{(1,0,0,0,1,1,1)} = l_{(1,0,0,1,0,0,1)} = l_{(1,0,0,1,0,1,1)} = l_{(1,0,0,1,1,0,1)} = l_{(1,0,0,1,1,1,0)} \\
&= l_{(1,0,1,0,0,1,0)} = l_{(1,0,1,0,0,1,1)} = l_{(1,0,1,1,0,0,0)} = l_{(1,0,1,1,0,1,1)} = l_{(1,0,1,1,1,0,0)} \\
&= l_{(1,0,1,1,1,1,0)} = l_{(1,1,0,0,0,0,1)} = l_{(1,1,0,0,0,1,1)} = l_{(1,1,0,0,1,0,0)} = l_{(1,1,0,0,1,1,1)} \\
&= l_{(1,1,0,1,1,0,0)} = l_{(1,1,0,1,1,0,1)} = l_{(1,1,1,0,0,0,1)} = l_{(1,1,1,0,0,1,0)} = l_{(1,1,1,0,1,0,0)} \\
&= l_{(1,1,1,0,1,1,0)} = l_{(1,1,1,1,0,0,0)} = l_{(1,1,1,1,0,0,1)} = 3,
\end{aligned}$$

$$\begin{aligned}
& l_{(0,0,1,0,1,1,1)} = l_{(0,0,1,1,1,0,1)} = l_{(0,0,1,1,1,1,1)} = l_{(0,1,0,1,0,1,1)} = l_{(0,1,0,1,1,1,0)} \\
&= l_{(0,1,0,1,1,1,1)} = l_{(0,1,1,0,1,0,1)} = l_{(0,1,1,0,1,1,1)} = l_{(0,1,1,1,0,1,0)} = l_{(0,1,1,1,0,1,1)} \\
&= l_{(0,1,1,1,1,0,1)} = l_{(0,1,1,1,1,1,0)} = l_{(0,1,1,1,1,1,1)} = l_{(1,0,0,0,0,1,1)} = l_{(1,0,0,1,1,0,0)}
\end{aligned}$$

$$\begin{aligned}
&= l_{(1,0,0,1,1,1,1)} = l_{(1,0,1,0,1,0,1)} = l_{(1,0,1,0,1,1,0)} = l_{(1,0,1,0,1,1,1)} = l_{(1,0,1,1,0,0,1)} \\
&= l_{(1,0,1,1,0,1,0)} = l_{(1,0,1,1,1,0,1)} = l_{(1,0,1,1,1,1,1)} = l_{(1,1,0,0,1,0,1)} = l_{(1,1,0,0,1,1,0)} \\
&= l_{(1,1,0,1,0,0,1)} = l_{(1,1,0,1,0,1,0)} = l_{(1,1,0,1,0,1,1)} = l_{(1,1,0,1,1,1,0)} = l_{(1,1,0,1,1,1,1)} \\
&= l_{(1,1,1,0,0,0,0)} = l_{(1,1,1,0,0,1,1)} = l_{(1,1,1,0,1,1,0)} = l_{(1,1,1,0,1,1,1)} = l_{(1,1,1,1,0,1,0)} \\
&= l_{(1,1,1,1,0,1,1)} = l_{(1,1,1,1,1,0,0)} = l_{(1,1,1,1,1,0,1)} = l_{(1,1,1,1,1,1,0)} = 4,
\end{aligned}$$

$$l_{(0,0,0,0,0,0,0)} = 0 \quad \text{and} \quad l_{(1,1,1,1,1,1,1)} = 6.$$

Therefore, by Theorem 2.2,

$$\varphi_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(-6) \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 39} \oplus \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 48} \oplus \mathcal{O}_{\mathbb{P}^4}(-4)^{\oplus 39}.$$

Thus, by Corollary 2.3,

$$p_g(X) = 5, \quad q = h^{2,0} = h^{3,0} = 0, \quad \chi(\omega_X) = 6$$

and by the Hurwitz formula, $K_X = \varphi^*\mathcal{O}_{\mathbb{P}^4}(1)$. So $X$ is minimal and the canonical degree of $X$ is $K_X^4 = 128$.

Similarly, we have the following example.

**Example 4.2.** Let $X$ be the canonical abelian 4-fold which is the normalization of the variety whose defining equations are given by

$$z_1^3 = h_1h_5^2h_6^2h_7, \quad z_2^3 = h_2h_6^2h_7h_8^2, \quad z_3^3 = h_3h_7h_8^2h_9^2, \quad z_4^3 = h_4h_5^2h_7h_9^2,$$

where the hyperplanes $H_i$'s defined by $h_i$'s intersect with normal crossings in $\mathbb{P}^4$.

The arguments of smoothness of $X$ and the calculation of $l_g$'s are as similar as the above example. Finally, we have

$$\begin{aligned}
&l_{(0,0,0,1)} = l_{(0,0,0,2)} = l_{(0,0,1,0)} = l_{(0,0,1,2)} = l_{(0,0,2,0)} = l_{(0,0,2,1)} = l_{(0,1,0,0)} \\
&= l_{(0,1,2,0)} = l_{(0,2,0,0)} = l_{(0,2,1,0)} = l_{(1,0,0,0)} = l_{(1,0,0,2)} = l_{(1,2,0,0)} = l_{(1,2,1,2)} \\
&= l_{(2,0,0,0)} = l_{(2,0,0,1)} = l_{(2,1,0,0)} = l_{(2,1,2,1)} = 2,
\end{aligned}$$

$$\begin{aligned}
&l_{(0,0,1,1)} = l_{(0,0,2,2)} = l_{(0,1,0,2)} = l_{(0,1,1,0)} = l_{(0,1,1,1)} = l_{(0,1,1,2)} = l_{(0,1,2,1)} \\
&= l_{(0,2,0,1)} = l_{(0,2,0,2)} = l_{(0,2,1,1)} = l_{(0,2,1,2)} = l_{(0,2,2,0)} = l_{(1,0,0,1)} = l_{(1,0,1,1)} \\
&= l_{(1,0,1,2)} = l_{(1,0,2,0)} = l_{(1,0,2,1)} = l_{(1,1,0,0)} = l_{(1,1,0,1)} = l_{(1,1,0,2)} = l_{(1,1,1,0)} \\
&= l_{(1,1,1,1)} = l_{(1,1,1,2)} = l_{(1,1,2,0)} = l_{(1,1,2,1)} = l_{(1,1,2,2)} = l_{(1,2,0,1)} = l_{(1,2,0,2)} \\
&= l_{(1,2,1,0)} = l_{(1,2,1,1)} = l_{(1,2,2,1)} = l_{(2,0,0,2)} = l_{(2,0,1,0)} = l_{(2,0,1,1)} = l_{(2,0,2,0)} \\
&= l_{(2,0,2,1)} = l_{(2,1,0,1)} = l_{(2,1,1,0)} = l_{(2,1,1,1)} = l_{(2,1,1,2)} = l_{(2,1,2,0)} = l_{(2,2,0,0)} \\
&= l_{(2,2,1,1)} = 3,
\end{aligned}$$

$$\begin{aligned} & l_{(0,1,0,1)} = l_{(0,1,2,2)} = l_{(0,2,2,1)} = l_{(0,2,2,2)} = l_{(1,0,1,0)} = l_{(1,0,2,2)} = l_{(1,2,2,0)} \\ & = l_{(1,2,2,2)} = l_{(2,0,1,2)} = l_{(2,0,2,2)} = l_{(2,1,0,2)} = l_{(2,1,2,2)} = l_{(2,2,0,1)} = l_{(2,2,0,2)} \\ & = l_{(2,2,1,0)} = l_{(2,2,1,2)} = l_{(2,2,2,0)} = l_{(2,2,2,1)} = 4, \end{aligned}$$

$$l_{(0,0,0,0)} = 0 \quad \text{and} \quad l_{(1,1,1,1)} = 6.$$

So

$$\varphi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(-6) \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 18} \oplus \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 43} \oplus \mathcal{O}_{\mathbb{P}^4}(-4)^{\oplus 18}.$$

Thus, by Corollary 2.3, $p_g(X) = 5$, $q = h^{2,0} = h^{3,0} = 0$, $\chi(\omega_X) = 6$ and by the Hurwitz formula, $K_X = \varphi^* \mathcal{O}_{\mathbb{P}^4}(1)$. So $X$ is minimal and the canonical degree of $X$ is $K_X^4 = 81$.

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