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τ-rigid Modules over Auslander Algebras

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Abstract. We give a characterization of τ-rigid modules over Auslander algebras in terms of projective dimension of modules. Moreover, we show that for an Auslander algebra Λ admitting finite number of non-isomorphic basic tilting Λ-modules and tilting Λ^{op}-modules, if all indecomposable τ-rigid Λ-modules of projective dimension 2 are of grade 2, then Λ is τ-tilting finite.

1. Introduction

Recently Adachi, Iyama and Reiten [4] introduced τ-tilting theory to generalize the classical tilting theory in terms of mutations. τ-tilting theory is close to the silting theory introduced by [5] and the cluster tilting theory in the sense of [10, 18, 21].

Note that τ-tilting theory depends on τ-rigid modules. So it is very interesting to find all τ-rigid modules for a given algebra. There are some works on this topic (see [1–3, 6, 16, 17, 20, 22, 24–26] and so on). In particular, Iyama and Zhang [19] classified all the support τ-tilting modules and indecomposable τ-rigid modules for the Auslander algebra Γ of $K[x]/(x^n)$. They showed that the number of non-isomorphic basic support τ-tilting Γ-modules is exactly $(n+1)!$. For an arbitrary Auslander algebra Λ, little is known on τ-rigid Λ-modules. So a natural question is:

Question 1.1. How to judge τ-rigid modules over an arbitrary Auslander algebra?

Our first goal in this paper is to give a partial answer to this question. Throughout this paper all algebras are finite-dimensional algebras over a field K and all modules are finitely generated right modules.

For an algebra Λ, denote by $(-)^*$ the functor $\mathrm{Hom}_\Lambda(-, \Lambda)$. For a Λ-module M, denote by $\mathrm{pd}_\Lambda M$ (resp. $\mathrm{id}_\Lambda M$) the projective dimension (resp. injective dimension) of M. Denote by $\mathrm{grade}\, M$ the grade of M. Then we have the following theorem.

Theorem 1.2. (Theorems 3.3 and 3.10, Corollary 3.7) *Let Λ be an Auslander algebra and M a Λ-module. Then we have the following:*

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(1) *Every simple module S is τ-rigid.*

(2) *If* $\mathrm{pd}_\Lambda M = 1$, *then M is (τ-)rigid if and only if* $\mathrm{Ext}^2_\Lambda(N, M) = 0$, *where* $N = M^{**}/M$.

(3) *If* $\operatorname{grade} M = 2$, *then M is τ-rigid if and only if* $\operatorname{Tr} M$ *is τ-rigid with* $\mathrm{pd}_\Lambda \operatorname{Tr} M = 1$.

(4) *If Λ admits a unique simple module S with* $\mathrm{pd}_\Lambda S = 2$, *then*

   (a) *every indecomposable module M with* $\mathrm{pd}_\Lambda M = 1$ *is (τ-)rigid.*

   (b) *all indecomposable τ-rigid Λ-modules N with* $\mathrm{pd}_\Lambda N = 2$ *are of grade* 2.

On the other hand, Demonet, Iyama and Jasso gave a general description of algebras with finite number of support τ-tilting modules in [11] where they call the algebras τ-tilting finite algebras. It is clear that an algebra Λ is τ-tilting finite if and only if so is its opposite algebra Λ^{op}. We should remark that an algebra is τ-tilting finite implies that there are finite number of non-isomorphic basic tilting Λ-modules and tilting Λ^{op}-modules. It is natural to consider the following question.

Question 1.3. When is an algebra admitting finite number of basic tilting Λ-modules and tilting Λ^{op}-modules τ-tilting finite?

It is obvious that algebras of finite representation type are both tilting-finite and τ-tilting finite. However, we need a non-trivial case. Our second goal of this paper is to give a more general answer to this question whenever Λ is an Auslander algebra. We prove the following theorem in which the algebra is not necessary to be an Auslander algebra.

Theorem 1.4. (Theorem 3.8) *Let Λ be an algebra of global dimension* 2 *admitting finite number of basic tilting Λ-modules and tilting Λ^{op}-modules. If all indecomposable τ-rigid modules with projective dimension* 2 *are of grade* 2, *then Λ is τ-tilting finite.*

The paper is organized as follows: In Section 2, we recall some preliminaries. In Section 3, we prove the main results and give some examples to show the main results.

Throughout this paper, all algebras Λ are basic connected finite dimensional algebras over an algebraic closed field K and all Λ-modules are finitely generated right modules. Denote by $\operatorname{mod}\Lambda$ the category of finitely generated right Λ-modules. For $M \in \operatorname{mod}\Lambda$, denote by $\operatorname{add} M$ the subcategory of direct summands of finite direct sum of M. We use $\operatorname{Tr} M$ to denote the Auslander transpose of M. Denote by τ the AR-translation and denote by $|M|$ the number of non-isomorphic indecomposable direct summands of M.

2. Preliminaries

In this section we recall some basic preliminaries for later use. For an algebra Λ, denote by $\operatorname{gl.dim}\Lambda$ the global dimension of Λ. We begin with the definition of Auslander algebras.

Definition 2.1. An algebra R is called an *Auslander algebra* if $\operatorname{gl.dim} R \leq 2$ and $I_i(R)$ is projective for $i = 0, 1$, where $I_i(R)$ is the $(i+1)$-th term in a minimal injective resolution of R.

Let R be a representation-finite algebra and A an additive generator of $\operatorname{mod} R$. Auslander proved that there is a one to one correspondence between representation-finite algebras and Auslander algebras via $R \mapsto \operatorname{End}_R(A)$. In this case, we call $\operatorname{End}_R(A)$ the *Auslander algebra of* R. Furthermore, for $X \in \operatorname{mod} R$ we denote by $P_X = \operatorname{Hom}_R(A, X)$ and $S_X = P_X/\operatorname{rad} P_X$. The following statement [9] is essential in the proof of the main result.

Proposition 2.2. *Let X be an indecomposable R-module. Then*

(1) $\operatorname{pd}_\Lambda S_X \leq 1$ *if and only if X is projective, and* $0 \to P_{\operatorname{rad} X} \to P_X \to S_X \to 0$ *is a minimal projective resolution of* S_X.

(2) $\operatorname{pd}_\Lambda S_X = 2$ *if and only if X is not projective, and the almost split sequence* $0 \to \tau X \to E \to X \to 0$ *gives a minimal projective resolution* $0 \to P_{\tau X} \to P_E \to P_X \to S_X \to 0$ *of* S_X.

For a positive integer k, an algebra Λ is called Auslander's k-Gorenstein if $\operatorname{pd}_\Lambda I_j(\Lambda) \leq j$ for $0 \leq j \leq k-1$. For a Λ-module M and a positive integer i, we call $\operatorname{grade} M \geq i$ if $\operatorname{Ext}^j_\Lambda(M, \Lambda) = 0$ for $0 \leq j \leq i-1$. We need the following result.

Lemma 2.3. *Let Λ be an Auslander algebra and $T \in \operatorname{mod}\Lambda$. For $j = 1, 2$,*

(1) *the subcategory* $\{M \mid \operatorname{grade} M \geq j\}$ *is closed under submodules and factor modules.*

(2) *every simple Λ-module S is either of grade 0 or of grade 2.*

(3) $\operatorname{grade}\operatorname{Ext}^j_\Lambda(T, \Lambda) \geq 2$.

(4) *the projective dimension of any composition factor of* $\operatorname{Ext}^2_\Lambda(T, \Lambda)$ *is* 2.

Proof. (1) is a straight result of [15, Proposition 2.4].

(2) follows from the fact $\operatorname{Ext}^i_\Lambda(S, \Lambda) \simeq \operatorname{Hom}_\Lambda(S, I_i(\Lambda))$ and Λ is an Auslander algebra.

(3) By the definition of Auslander algebra, Λ is Auslander's 2-Gorenstein. Then by [12] Λ is Auslander's k-Gorenstein if and only if for each submodule X of $\operatorname{Ext}^i_\Lambda(T, \Lambda)$ with T in $\operatorname{mod}\Lambda$ and $i \leq k$, we have $\operatorname{grade} X \geq i$. Then we have $\operatorname{grade}\operatorname{Ext}^j_\Lambda(T, \Lambda) \geq j$ for $j = 1, 2$.

By (1) every composition factor S of $\operatorname{Ext}^1_\Lambda(T,\Lambda)$ has grade at least 1, and hence 2 by (2). Then by an induction on the length of $\operatorname{Ext}^1_\Lambda(T,\Lambda)$, we get $\operatorname{grade}\operatorname{Ext}^1_\Lambda(T,\Lambda)\geq 2$.

(4) is a direct result of (1) and (3). $\square$

In the following we recall some basic properties of τ-rigid modules. We start with the following definition [4].

Definition 2.4. We call $M\in \operatorname{mod}\Lambda$ *τ-rigid* if $\operatorname{Hom}_\Lambda(M,\tau M)=0$. In addition, M is called *τ-tilting* if M is τ-rigid and $|M|=|\Lambda|$. Moreover, M is called *support τ-tilting* if there exists an idempotent e of Λ such that M is a τ-tilting $\Lambda/(e)$-module.

It is clear that any τ-rigid Λ-module M is rigid, that is, $\operatorname{Ext}^1_\Lambda(M,M)=0$. In general the converse is not true. But if $\operatorname{pd}_\Lambda M=1$, then M is τ-rigid if and only if M is rigid. Recall that a Λ-module T is called a *(classical) tilting module* if T satisfies (1) $\operatorname{pd}_\Lambda T\leq 1$, (2) $\operatorname{Ext}^1_\Lambda(T,T)=0$ and (3) $|T|=|\Lambda|$. It is showed in [4] that a tilting Λ-module is exactly a faithful support τ-tilting Λ-module.

To judge τ-rigid modules of projective dimension 2 over Auslander algebras, we also need the following lemma in [4].

Lemma 2.5. *Let Λ be an algebra and M a Λ-module without projective direct summands. Then M is τ-rigid in $\operatorname{mod}\Lambda$ if and only if $\operatorname{Tr}M$ is τ-rigid in $\operatorname{mod}\Lambda^{\mathrm{op}}$.*

Recall that a morphism $f\colon M\to N$ is called *right minimal* (resp. *left minimal*) if $fg=f$ (resp. $gf=f$) implies that g is an isomorphism, where g is a homomorphism of the form $M\to M$ (resp. $N\to N$). The following properties of right minimal (resp. left minimal) morphisms in [13] are useful for the proof of the main results.

Lemma 2.6. *Let $0\to A\xrightarrow{g} B\xrightarrow{f} C\to 0$ be a non-split exact sequence in $\operatorname{mod}\Lambda$ with B projective-injective. Then the following are equivalent:*

(1) *A is indecomposable and g is left minimal.*

(2) *C is indecomposable and f is right minimal.*

3. Main results

In this section we give the main results of this paper and some examples to show the main results. Throughout this section, $\Lambda=\operatorname{End}_R A$ is the Auslander algebra of a representation-finite algebra R with an additive generator A.

It is showed by Igusa [14] that S is rigid for any simple module S over an algebra Γ of finite global dimension. However, we give a new direct proof for the rigidness of simple modules whenever Γ is an Auslander algebra.

Proposition 3.1. *Let Λ be an Auslander algebra and S a simple Λ-module. Then* $\mathrm{Ext}^1_\Lambda(S,S)=0$.

Proof. For a simple Λ-module S, we show the assertion by using the projective dimension of S.

If $\mathrm{pd}_\Lambda S=0$, there is nothing to show.

If $\mathrm{pd}_\Lambda S=1$, then we can get a minimal projective resolution $0\to P_1(S)\to P_0(S)\to S\to 0$. Then the length of $P_1(S)$ is smaller than that of $P_0(S)$, and hence $\mathrm{Hom}_\Lambda(P_1(S),S)=0$. So one gets $\mathrm{Ext}^1_\Lambda(S,S)\simeq\mathrm{Hom}_\Lambda(P_1(S),S)=0$.

If $\mathrm{pd}_\Lambda S=2$, then by Proposition 2.2, there is an AR-sequence $0\to\tau X\to E\to X\to 0$ in $\mathrm{mod}\,R$ such that $0\to\mathrm{Hom}_R(A,\tau X)\to\mathrm{Hom}_R(A,E)\to\mathrm{Hom}_R(A,X)\to S\to 0$ is a minimal projective resolution of S. On the contrary, suppose that $\mathrm{Ext}^1_\Lambda(S,S)\neq 0$, then we get that $\mathrm{Hom}_\Lambda(P_1(S),S)\simeq\mathrm{Ext}^1_\Lambda(S,S)\neq 0$. So $P_0(S)=\mathrm{Hom_R(A,X)}$ is a direct summand of $P_1(S)=\mathrm{Hom}_\Lambda(A,E)$. Note that the functor $\mathrm{Hom}_\Lambda(A,-)$ induces an equivalence from $\mathrm{add}\,A$ to $\mathrm{add}\,\Lambda$, then X is a direct summand of E. Since $E\to X$ is right almost split, then we get an irreducible morphism $f\colon X\to X$ by [7, IV, Theorem 1.10(b)], a contradiction. □

Denote by $(-)^*$ the functor $\mathrm{Hom}_\Lambda(-,\Lambda)$, then we have the following lemma [19] with a different shorter proof.

Lemma 3.2. *Let Λ be an Auslander algebra, and let M be a Λ-module with* $\mathrm{pd}_\Lambda M\leq 1$. *Then the canonical map $M\overset{\varphi_M}{\to}M^{**}$ is injective, and the projective dimension of any composition factor of M^{**}/M is* 2.

Proof. By [8], we get an exact sequence

$$0\to\mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\,M,\Lambda)\to M\to M^{**}\to\mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\,M,\Lambda)\to 0.$$

To show the former assertion, it suffices to show that $\mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\,M,\Lambda)=0$. In the following we show $\mathrm{grade}\,\mathrm{Tr}\,M=2$. Since $\mathrm{pd}_\Lambda M\leq 1$, then one gets $\mathrm{Tr}\,M\simeq\mathrm{Ext}^1_\Lambda(M,\Lambda)$ and hence by Lemma 2.3(3), $\mathrm{grade}\,\mathrm{Tr}\,M\geq 2$ holds, and hence $\mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\,M,\Lambda)=0$. We get the desired injection. Then by Lemma 2.3(2), the later assertion holds. □

Now we are in a position to state the following main result on the (τ-)rigidness of modules with projective dimension 1.

Theorem 3.3. *Let Λ be an Auslander algebra and M a Λ-module with* $\mathrm{pd}_\Lambda M=1$. *Then* $\mathrm{Ext}^1_\Lambda(M,M)=0$ *if and only if* $\mathrm{Ext}^2_\Lambda(N,M)=0$ *holds for $N=M^{**}/M$.*

Proof. We show the assertion step by step.

(1) For any $M \in \operatorname{mod}\Lambda$, M^* is projective. Here we only need the condition $\operatorname{gl.dim}\Lambda = 2$.

Let $P_1(M) \to P_0(M) \to M \to 0$ be a projective resolution of M. Applying the functor $(-)^*$, we get an exact sequence $0 \to M^* \to P_0(M)^* \to P_1(M)^*$. Since $\operatorname{gl.dim}\Lambda \leq 2$, one gets that M^* is a projective Λ^{op}-module. Thus M^{**} is a projective Λ-module.

(2) $\operatorname{Ext}^1_\Lambda(M,M) \simeq \operatorname{Ext}^2_\Lambda(M^{**}/M,M)$ holds.

By Lemma 3.2, we get the exact sequence $0 \to M \to M^{**} \to \operatorname{Ext}^2_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M,\Lambda)\,(= M^{**}/M) \to 0$. Applying the functor $\operatorname{Hom}_\Lambda(-,M)$ to the exact sequence, we get the desired isomorphism since M^{**} is projective by (1). □

Immediately, we have the following corollary.

Corollary 3.4. *Let Λ be an Auslander algebra and M a Λ-module with $\operatorname{pd}_\Lambda M = 1$.*

(1) *If $\operatorname{id}_\Lambda M = 1$, then $\operatorname{Ext}^1_\Lambda(M,M) = 0$ holds.*

(2) *If $\operatorname{Ext}^2_\Lambda(S',M) = 0$ holds for any composition factor S' of M^{**}/M, then $\operatorname{Ext}^1_\Lambda(M,M) = 0$ holds.*

Proof. (1) follows from Theorem 3.3 directly. By induction on the length of M^{**}/M, one can get the assertion (2). □

Remark 3.5. We should remark that the converse of Corollary 3.4 are not true in general (see Example 3.11(5)).

Denote by $\mathrm{i}\tau\text{-rig}\,\Lambda$ the set of isomorphism classes of indecomposable τ-rigid Λ-modules. Similarly, one can define $\mathrm{i}\tau\text{-rig}\,\Lambda^{\mathrm{op}}$. Denote by $\mathcal{G}$ the subset of $\mathrm{i}\tau\text{-rig}\,\Lambda$ consisting of isomorphism classes of τ-rigid modules of grade 2 and denote by $\mathcal{S}$ the subset of $\mathrm{i}\tau\text{-rig}\,\Lambda^{\mathrm{op}}$ consisting of isomorphism classes of non-projective τ-rigid submodules of $\operatorname{add}\Lambda^{\mathrm{op}}$. To judge τ-rigid modules of projective dimension 2 over Auslander algebras, we need the following proposition.

Proposition 3.6. *Let Λ be an algebra of global dimension 2. There is a bijection between $\mathcal{G}$ and $\mathcal{S}$ via $\operatorname{Tr}\colon M \mapsto \operatorname{Tr} M$.*

Proof. By Lemma 2.5 M is τ-rigid if and only if $\operatorname{Tr} M$ is τ-rigid. Now it suffices to show that (a) $M \in \mathcal{G}$ implies that $\operatorname{Tr} M \in \mathcal{S}$ and (b) $M \in \mathcal{S}$ implies that $\operatorname{Tr} M \in \mathcal{G}$.

(a) Since $M \in \mathcal{G}$, take the following minimal projective resolution of M: $\cdots \to P_1(M) \to P_0(M) \to M \to 0$. Applying the functor $(-)^*$, we get an exact sequence

$$0 = M^* \to P_0(M)^* \to P_1(M)^* \to \operatorname{Tr} M \to 0, \tag{3.1}$$

which is a minimal projective resolution of $\operatorname{Tr} M$. Then $\operatorname{pd}_\Lambda \operatorname{Tr} M = 1$. On the other hand, since $\operatorname{grade} M = 2$, one gets the following sequences

$$0 = M^* \to P_0(M)^* \to \Omega^1 M^* \to \operatorname{Ext}^1_\Lambda(M, \Lambda) = 0 \tag{3.2}$$

and

$$0 \to \Omega^1 M^* \to P_1(M)^* \to P_2(M)^*. \tag{3.3}$$

Comparing exact sequences (3.1) with (3.2) and (3.3), one gets that $\operatorname{Tr} M$ is a submodule of $P_2(M)^*$.

(b) Since $M \in \mathcal{S}$ is non-projective and $\operatorname{gl.dim} \Lambda = 2$, then $\operatorname{pd}_\Lambda M = 1$. Take a minimal projective resolution of M: $0 \to P_1(M) \to P_0(M) \to M \to 0$. Applying $(-)^*$, we get the following exact sequence $0 \to M^* \to P_0(M)^* \to P_1(M)^* \to \operatorname{Tr} M \to 0$. Note that Tr is a duality and $\operatorname{pd}_\Lambda M = 1$, one gets that $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda) = 0$. Since M can be embedded into a projective module, then M is torsionless, that is, $M \to M^{**}$ is injective. By [8] there is an exact sequence $0 \to \operatorname{Ext}^1_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda) \to M \to M^{**} \to \operatorname{Ext}^2_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda) \to 0$ which implies that $\operatorname{Ext}^1_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda) = 0$. Then $\operatorname{grade} \operatorname{Tr} M = 2$. □

As a corollary, we get the following

Corollary 3.7. *Let Λ be an Auslander algebra and $M \in \operatorname{mod} \Lambda$. If M is of grade 2, then M is τ-rigid if and only if $\operatorname{Tr} M$ is τ-rigid with $\operatorname{pd}_\Lambda \operatorname{Tr} M = 1$ in $\operatorname{mod} \Lambda^{\mathrm{op}}$.*

Proof. By Proposition 3.6, it is enough to show that $\operatorname{pd}_\Lambda M = 1$ if and only if M can be embedded into a projective module. Since $\operatorname{gl.dim} \Lambda = 2$, one gets that M can be embedded into a projective module implies that $\operatorname{pd}_\Lambda M = 1$. The converse follows from Lemma 3.2. □

Recall that from [11] that an algebra Λ is called τ-tilting finite if there are finite number of non-isomorphic indecomposable τ-rigid modules in $\operatorname{mod} \Lambda$. It is clear that a τ-tilting finite algebra admits finite number of tilting Λ-modules and tilting Λ^{op}-modules. To find a way from two-sided tilting finite to τ-tilting finite, we have the following

Theorem 3.8. *Let Λ be an algebra of global dimension 2 admitting finite number of basic tilting Λ-modules and tilting Λ^{op}-modules. If all indecomposable τ-rigid modules M with $\operatorname{pd}_\Lambda M = 2$ are of grade 2, then Λ is τ-tilting finite.*

Proof. By the assumption, there are finite number of tilting modules which implies that there are finite number of indecomposable τ-rigid Λ-modules and Λ^{op}-modules of projective dimension less than or equal to 1. Then by Proposition 3.6, the number of indecomposable

τ-rigid Λ-module of grade 2 is equal to the number of indecomposable non-projective τ-rigid submodules N of Λ^{op}. Since $\operatorname{gl.dim} \Lambda = 2$, we get that $\operatorname{pd}_\Lambda N = 1$, and hence the number of this class of modules is finite. Note that all indecomposable τ-rigid Λ-modules with projective dimension 2 are of grade 2, then the number of indecomposable τ-rigid modules with projective dimension 2 is finite by Proposition 3.6. □

Immediately, we have the following corollary which confirms the τ-tilting finiteness of the Auslander algebra of $K[x]/(x^n)$ showed in [19].

Corollary 3.9. *Let Λ be an Auslander algebra admitting finite number of basic tilting Λ-modules and tilting Λ^{op}-modules. If all indecomposable τ-rigid modules M with $\operatorname{pd}_\Lambda M = 2$ are of grade 2, then Λ is τ-tilting finite.*

For a module M, denote by $\operatorname{rad} M$ and $\operatorname{soc} M$ the radical and the socle of M, respectively. Now we give the following classification of Auslander algebras admitting a unique simple module of projective dimension 2 which gives a support to Theorem 3.3 and Corollary 3.9.

Theorem 3.10. *Let Λ be an Auslander algebra. If Λ admits a unique simple Λ-module S with $\operatorname{pd}_\Lambda S = 2$, then*

(1) *Λ is either the Auslander algebra of the path algebra $R = KQ$ with $Q\colon 1 \to 2$ or the Auslander algebra of the Nakayama local algebra R of radical square zero.*

(2) *every indecomposable Λ-module M with $\operatorname{pd}_\Lambda M \leq 1$ is rigid, and hence τ-rigid.*

(3) *all indecomposable τ-rigid Λ-modules N with $\operatorname{pd}_\Lambda N = 2$ are of grade 2.*

Proof. Since (2) and (3) follow from (1) easily, we only show (1). By Proposition 2.2, there is a unique non-projective indecomposable R-module X such that the AR-sequence $0 \to \tau X \to E \to X \to 0$ in $\operatorname{mod} R$ induces a minimal projective resolution of S: $0 \to \operatorname{Hom}_R(A, \tau X) \to \operatorname{Hom}_R(A, E) \to \operatorname{Hom}_R(A, X) \to S \to 0$. Then all indecomposable modules are projective except X. We claim that X should be simple. Otherwise, there would be a simple factor module Y of X such that $Y \not\simeq X$. By the proof above Y would be projective and hence $X \simeq Y$ is projective, a contradiction. Now we divide the proof in two parts.

(a) If X is not injective, then all indecomposable injective R-modules are projective, and hence R is self-injective. So we get that R is local with a unique simple module X. Otherwise, there would be a simple projective-injective R-module. One gets a contradiction since R is basic and connected. Taking a minimal projective resolution of X, we get the following exact sequence $0 \to \Omega^1 X \to P_0(X)\,(= R) \to X \to 0$. By Lemma 2.6, $\Omega^1 X$ is indecomposable non-projective, and hence $\Omega^1 X \simeq X$. Then $\operatorname{rad}^2 R = 0$ holds. By [9, IV, Proposition 2.16], R is a Nakayama algebra.

(b) If X is injective, then $X \not\simeq \operatorname{soc} P$ for any indecomposable projective R-module. Hence the injective envelope $I^0(R)$ is projective, that is, R is Auslander's 1-Gorenstein [12]. Then $P_0(X)$ is projective-injective since X is injective. Taking a part of minimal projective resolution of X: $0 \to \Omega^1 X \to P_0(X) \to X \to 0$, one gets that $\Omega^1 X$ is indecomposable and projective by Lemma 2.6. Then we conclude that R is a hereditary algebra.

In the following we show R is a Nakayama algebra. One can show that $P_0(X)$ is the unique projective-injective module in $\operatorname{mod} R$ since R is a basic connected hereditary algebra. Then every indecomposable projective R-module is contained in $P_0(X)$ and admits a unique composition series. By [12], R^{op} is also Auslander's 1-Gorenstein. Similarly, every indecomposable projective R^{op}-module admits a unique composition series. So R is a Nakayama algebra. By [7, V, Theorem 3.2] and the fact all indecomposable R-modules are projective except one, we get that $R = KQ$ with $Q\colon 1 \to 2$. $\square$

At the end of this paper we give another two examples to show our main results.

Example 3.11. Let Λ be the Auslander algebra of $K[x]/(x^n)$. Then we have the following:

(1) Λ is given by

$$1 \underset{b_2}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_3}{\overset{a_2}{\rightleftarrows}} 3 \underset{b_4}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-2}}{\rightleftarrows}} n-1 \underset{b_n}{\overset{a_{n-1}}{\rightleftarrows}} n$$

with relations $a_1 b_2 = 0$ and $a_i b_{i+1} = b_i a_{i-1}$ for any $2 \leq i \leq n-1$. Λ is of infinite representation type if $n \geq 5$.

(2) All indecomposable module M with $\operatorname{pd}_\Lambda M = 1 = \operatorname{id}_\Lambda M$ are direct summands of tilting modules, and hence τ-rigid.

(3) All indecomposable τ-rigid modules of projective dimension 2 are of grade 2 (see [19] for details).

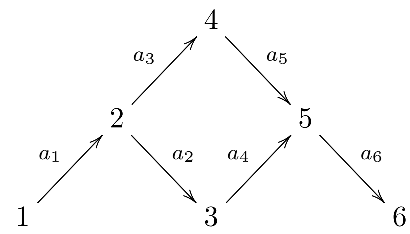
(4) The number of tilting Λ-modules (resp. Λ^{op}-modules) is $n!$ [19,23]. By Theorem 3.8, Λ is τ-tilting finite.

(5) If $n = 4$, then the indecomposable module $M = \begin{smallmatrix} 2 & & 4 \\ & 3 & \\ & & 4 \end{smallmatrix}$ is (τ-)rigid with $\operatorname{pd}_\Lambda M = 1$ and $\operatorname{id}_\Lambda M = 2$ and $M^{**} = \begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \\ & & & 4 \end{smallmatrix}$. But $\operatorname{Ext}^2_\Lambda(S(2), M) \neq 0$.

We should remark that there does exist an Auslander algebra Λ such that an indecomposable τ-rigid Λ-module with projective dimension 2 does not necessarily have grade 2.

Example 3.12. Let Λ be the Auslander algebra of KQ with $Q\colon 1 \overset{a_1}{\to} 2 \overset{a_2}{\to} 3$. Then

(1) Λ is given by the following quiver Q':



with relations $a_2a_1 = 0$, $a_5a_3 = a_4a_2$ and $a_6a_4 = 0$.

(2) All indecomposable modules are τ-rigid.

(3) The indecomposable module $M = {}_3{}^2{}_4$ is of projective dimension 2, but it is not of grade 2 since $\mathrm{pd}_\Lambda S(4) = 1$.

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