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Multiplicity of Solutions for a Class of Quasilinear Elliptic Systems in Orlicz-Sobolev Spaces

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Abstract. In this paper, we investigate the following nonlinear and non-homogeneous elliptic system

$$\begin{cases} -\operatorname{div}(a_1(|\nabla u|)\nabla u) = \lambda_1 F_u(x,u,v) - \lambda_2 G_u(x,u,v) - \lambda_3 H_u(x,u,v) & \text{in } \Omega, \\ -\operatorname{div}(a_2(|\nabla v|)\nabla v) = \lambda_1 F_v(x,u,v) - \lambda_2 G_v(x,u,v) - \lambda_3 H_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\partial\Omega$, λ_1, λ_2, λ_3 are three parameters, $\phi_i(t) = a_i(|t|)t$ ($i = 1,2$) are two increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, and functions F, G, H are of class $C^1(\Omega \times \mathbb{R}^2, \mathbb{R})$ and satisfy some reasonable growth conditions. By using a three critical points theorem due to B. Ricceri, we obtain that system has at least three solutions. With some additional conditions, by using a four critical points theorem due to G. Anello, we obtain that system has at least four solutions.

1. Introduction and main results

Consider the following nonlinear and non-homogeneous elliptic system in Orlicz-Sobolev spaces:

$$\begin{cases} -\operatorname{div}(a_1(|\nabla u|)\nabla u) = \lambda_1 F_u(x,u,v) - \lambda_2 G_u(x,u,v) - \lambda_3 H_u(x,u,v) & \text{in } \Omega, \\ -\operatorname{div}(a_2(|\nabla v|)\nabla v) = \lambda_1 F_v(x,u,v) - \lambda_2 G_v(x,u,v) - \lambda_3 H_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\partial\Omega$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, $F, G, H\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are three C^1 functions which satisfy some reasonable growth conditions, $a_i\colon]0,+\infty[\to \mathbb{R}$ ($i = 1,2$) are two functions satisfying

(ϕ_1) $\phi_i\colon \mathbb{R} \to \mathbb{R}$ ($i = 1,2$) defined by

$$\phi_i(t) = \begin{cases} a_i(|t|)t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0, \end{cases} \tag{1.2}$$

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are two increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. Therefore, functions $\Phi_i\colon [0,+\infty[\to [0,+\infty[$ $(i=1,2)$ defined by $\Phi_i(t) := \int_0^t \phi_i(s)\,ds$ are strictly convex in $[0,+\infty[$.

Set $a_2 = a_1$, $v = u$, $F(x,u,v) = F(x,v,u)$, $G(x,u,v) = F(x,v,u)$ and $H(x,u,v) = F(x,v,u)$. Then system (1.1) reduces to the following quasilinear elliptic type equation:

$$\begin{cases} -\operatorname{div}(a_1(|\nabla u|)\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

When $a_1(|t|)t = p\,|t|^{p-2}\,t$ $(p > 1)$, equation (1.3) becomes the well-known p-Laplacian equation which has been studied extensively (see [3, 15–17, 23, 24] and references therein). In fact, under the assumption (ϕ_1), equations like (1.3) may be allowed to possess complicated non-homogeneous operator Φ_1 which can be used for modeling many phenomena (see [2, 19]):

(1) (p,q)-Laplacian: $\Phi_1(t) = t^p + t^q$, $q > p > 1$;

(2) nonlinear elasticity: $\Phi_1(t) = (1+t^2)^\gamma - 1$, $\gamma > 1/2$;

(3) plasticity: $\Phi_1(t) = t^\alpha(\log(1+t))^\beta$, $\alpha \geq 1$, $\beta > 0$;

(4) generalized Newtonian fluids: $\Phi_1(t) = \int_0^t s^{1-\alpha}(\sinh^{-1} s)^\beta\,ds$, $0 \leq \alpha \leq 1$, $\beta > 0$.

Based on these interesting facts, equations like (1.3) have aroused keen interest among scholars in recent years. In Clément et al. [14], the authors firstly proved that equation (1.3) has a nontrivial solution by variational method. From then on, variational method has been used widely to study the existence and multiplicity of solutions for this type of nonlinear or non-homogeneous elliptic equations (see [9,13,18,25,26] and references therein).

To study the existence or multiplicity of solutions for equations like (1.3), some appropriate Orlicz-Sobolev spaces might be defined. For this purpose, in most of references, the authors assumed at least one of the following conditions holds:

$(\mathcal{E}_1)$ $m_1 < \min\{N, l_1^*\}$;

$(\mathcal{E}_2)$ $N < l_1$;

$(\mathcal{E}_3)$ $m_1 < l_1^*$;

$(\mathcal{E}_4)$ the function $t \to \Phi_1(\sqrt{t})$ is convex for all $t \in [0,+\infty[$,

where N denotes the dimension of the space $\mathbb{R}^N$ and

$$l_1 := \inf_{t>0} \frac{t\phi_1(t)}{\Phi_1(t)}, \quad m_1 := \sup_{t>0} \frac{t\phi_1(t)}{\Phi_1(t)} \quad \text{and} \quad l_1^* := \begin{cases} \frac{l_1 N}{N-l_1} & \text{if } l_1 < N, \\ +\infty & \text{if } l_1 \geq N, \end{cases}$$

where ϕ_1 and Φ_1 are defined by (1.2). To be precise, $(\mathcal{E}_1)$ is assumed in [13, 18, 25, 26], $(\mathcal{E}_2)$ is assumed in [8, 9], $(\mathcal{E}_3)$ is assumed in [11], and $(\mathcal{E}_4)$ is assumed in [8, 11–13, 26].

To the best of our knowledge, there are few papers to consider the systems like (1.1) except for [22, 31, 32, 34]. In [22], for systems (1.1) with $\lambda_2 = \lambda_3 = 0$ and F has the form

$$F(x,u,v) = A_1(x,u) + b(x)\Gamma_1(u)\Gamma_2(v) + A_2(x,v),$$

Huentutripay-Manásevich translated the existence of solution into a suitable minimizing problem and proved the existence of nontrivial solution under some reasonable restriction. In [32], for systems (1.1) with $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$ and F satisfies the so-called subcritical and super-linear Orlicz-Sobolev growth conditions at infinity, by using the mountain pass theorem, Xia-Wang proved the existence of nontrivial solution. In [34], for system

$$\begin{cases} \operatorname{div}(a_1(|\nabla u|)\nabla u) = a(|x|)f(v) & \text{in } \mathbb{R}^N, \\ \operatorname{div}(a_2(|\nabla v|)\nabla v) = b(|x|)g(u) & \text{in } \mathbb{R}^N, \\ (u,v) \in C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N), \end{cases}$$

by using a monotone iterative method and Arzela-Ascoli theorem, Zhang proved the existence of positive radial solution. In [31], we investigated the following system in Orlicz-Sobolev spaces:

$$\begin{cases} -\operatorname{div}(\phi_1(|\nabla u|)\nabla u) + V_1(x)\phi_1(|u|)u = F_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(\phi_2(|\nabla v|)\nabla v) + V_2(x)\phi_2(|v|)v = F_v(x,u,v) & \text{in } \mathbb{R}^N, \\ (u,v) \in W^{1,\Phi_1}(\mathbb{R}^N) \times W^{1,\Phi_2}(\mathbb{R}^N) & \text{with } N \ge 2, \end{cases}$$

where the functions $V_i(x)$ $(i = 1,2)$ are bounded and positive in $\mathbb{R}^N$, the functions $\phi_i(t)t$ $(i = 1,2)$ satisfy (ϕ_1) and

$(\phi_2)'$ $1 < l_i := \inf_{t>0} \frac{t^2\phi_i(t)}{\Phi_i(t)} \le \sup_{t>0} \frac{t^2\phi_i(t)}{\Phi_i(t)} =: m_i < \min\{N, l_i^*\}$, where $l_i^* := \frac{l_i N}{N-l_i}$.

By using the least action principle, we proved that system possesses at least one nontrivial solution if $F\colon \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function, $F(x,0,0) = 0$ and satisfies

(F$_1$) there exist constants $p_i \in [m_i, l_i^*)$ $(i = 1,2)$, $\max\{1/p_1, 1/p_2\} \le q_1 < q_2 < \cdots < q_k < \min\{l_1/p_1, l_2/p_2\}$, and functions $a_{1j}, a_{2j}, a_{3j}, a_{4j} \in L^{1/(1-q_j)}(\mathbb{R}^N, [0,+\infty))$ $(j = 1,2,\ldots,k)$ such that

$$|F_u(x,u,v)| \le \sum_{j=1}^k a_{1j}(x)\,|u|^{p_1q_j-1} + \sum_{j=1}^k a_{2j}(x)\,|v|^{\frac{p_2(p_1q_j-1)}{p_1}},$$

$$|F_v(x,u,v)| \le \sum_{j=1}^k a_{3j}(x)\,|u|^{\frac{p_1(p_2q_j-1)}{p_2}} + \sum_{j=1}^k a_{4j}(x)\,|v|^{p_2q_j-1}$$

for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$;

(F_2) there exist an open set $\Omega \subset \mathbb{R}^N$ with $|\Omega| > 0$, and constants $\alpha_0 \in [1, l_1)$, $\beta_0 \in [1, l_2)$, $\delta > 0$, $c > 0$ and $\iota, \kappa \in \mathbb{R}$ with $\iota^2 + \kappa^2 \neq 0$ such that

$$F(x, \iota t, \kappa t) \geq c \left(|\iota t|^{\alpha_0} + |\kappa t|^{\beta_0} \right) \quad \text{for all } (x, t) \in \Omega \times [0, \delta].$$

Moreover, suppose that F also satisfies the symmetric condition

$$F(x, -u, -v) = F(x, u, v) \quad \text{for all } (x, u, v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}.$$

Then, by using the genus theory, we proved that system possesses infinitely many solutions.

In recent years, Ricceri, Anello and Bonanno have turned their interests to the multiplicity of critical points for a class of functional on a reflexive real Banach space. With the aid of variational method, they have worked out a series of abstract multiplicity theorems (see [4–7, 28–30]). Then, by using those theorems, some scholars studied the multiplicity of nontrivial solutions for equations like (1.3) even if the nonlinear term f is without symmetry (see [8, 11, 12]). Next, for readers' convenience, we recall the two abstract critical theorems in [30] and [5], which will be used to prove our results.

Theorem 1.1. [30, Theorem 3] *Let X be a reflexive real Banach space; $I\colon X \to \mathbb{R}$ a sequentially weakly lower semicontinuous, coercive, bounded on each bounded subset of X, C^1 functional whose derivative admits a continuous inverse on X^*; $\Psi, \Phi\colon X \to \mathbb{R}$ two C^1 functionals with compact derivative. Suppose also that the functional $\Psi + \lambda\Phi$ is bounded below for all $\lambda > 0$ and that*

$$\liminf_{\|x\| \to +\infty} \frac{\Psi(x)}{I(x)} = -\infty.$$

Then, for each $r > \sup_M \Phi$, where M is the set of all global minima of I, each $\mu > \max\{0, \mu^*(I, \Psi, \Phi, r)\}$, and each compact interval $[a, b] \subset]0, \beta(\mu I + \Psi, \Phi, r)[$, there exists a constant $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $\Gamma\colon X \to \mathbb{R}$ with compact derivative, there exists a constant $\delta > 0$ such that, for each $\nu \in [0, \delta]$, the equation

$$\mu I'(x) + \Psi'(x) + \lambda\Phi'(x) + \nu\Gamma'(x) = 0$$

has at least three solutions in X whose norms are less than ρ, where

$$\beta(\mu I + \Psi, \Phi, r) = \sup_{x \in \Phi^{-1}(]r, +\infty[)} \frac{\mu I(x) + \Psi(x) - \inf_{\Phi^{-1}(]-\infty, r])}(\mu I + \Psi)}{r - \Phi(x)}$$

and

$$\mu^*(I, \Psi, \Phi, r) = \inf\left\{ \frac{\Psi(x) - \gamma + r}{\eta_r - I(x)} : x \in X, \Phi(x) < r, I(x) < \eta_r \right\},$$

where $\gamma = \inf_X(\Psi(x) + \Phi(x))$ and $\eta_r = \inf_{x \in \Phi^{-1}(r)} I(x)$.

Theorem 1.2. [5, Theorem 1] *Let X be a reflexive real Banach space and $I\colon X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous and coercive C^1 functional whose derivative admits a continuous inverse on X^*. Assume also that $\Gamma, \Psi, \Phi\colon X \to \mathbb{R}$ are three C^1 functionals with compact derivative satisfying the following conditions:*

(a) $\liminf_{\|x\|\to\infty} \frac{\Gamma(x)}{I(x)} \geq 0$;

(b) $\limsup_{\|x\|\to\infty} \frac{\Gamma(x)}{I(x)} < +\infty$;

(c) $\liminf_{\|x\|\to\infty} \frac{\Psi(x)}{I(x)} = -\infty$;

(d) $\inf_{x\in X}(\Psi(x) + \lambda\Phi(x)) > -\infty$ *for all* $\lambda > 0$;

(e) *there exists a strict local minimum* $x_0 \in X$ *of* I *such that*

 (e$_1$) $I(x_0) = \Gamma(x_0) = \Psi(x_0) = \Phi(x_0) = 0$;

 (e$_2$) $\liminf_{x\to x_0} \frac{\Gamma(x)}{I(x)} \geq 0$;

 (e$_3$) $\liminf_{x\to x_0} \frac{\Psi(x)}{I(x)} > -\infty$;

 (e$_4$) $\liminf_{x\to x_0} \frac{\Phi(x)}{I(x)} > -\infty$;

(f) *there exists* $y_0 \in X$ *such that* $\Gamma(y_0) < 0$.

Then, for each $\nu \in]0,\infty[$ *with* $\nu > -I(y_0)/\Gamma(y_0)$, *there exists a constant* $\lambda_0 > 0$ *with the following property: for all* $\lambda \in]0,\lambda_0]$, *there exists a constant* $\sigma_\lambda > 0$ *such that, for all* $\sigma \in]0,\sigma_\lambda[$, *there exist four pairwise distinct critical points including* x_0 *of* $I+\nu\Gamma+\lambda\Psi+\sigma\Phi$.

In this paper, we also consider system (1.1) in Orlicz-Sobolev spaces, and by using Theorem 1.1, we obtain that system (1.1) has at least three solutions, and by using Theorem 1.2, we obtain that system (1.1) has at least four solutions which include the trivial solution.

Next, we prepare to present our results. For this purpose, we need to make the following two assumptions:

(ϕ_2) functions $\phi_i, \Phi_i\colon [0,+\infty[\to [0,+\infty[$ $(i=1,2)$ defined by (ϕ_1) satisfy

$$1 < l_i := \inf_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)} \leq \sup_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)} =: m_i < l_i^* := \begin{cases} \frac{l_i N}{N-l_i} & \text{if } l_i < N, \\ +\infty & \text{if } l_i \geq N; \end{cases}$$

(ϕ_3) functions $\phi_i, \Phi_i\colon [0,+\infty[\to [0,+\infty[$ $(i=1,2)$ defined by (ϕ_1) satisfy

$$1 < l_i := \inf_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)} \leq \sup_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)} =: m_i < \min\{N, e_i^*\},$$

where

$$e_i := \liminf_{t\to+\infty} \frac{t\phi_i(t)}{\Phi_i(t)} \quad \text{and} \quad e_i^* := \frac{e_i N}{N-e_i}. \tag{1.4}$$

Remark 1.3. $(\mathcal{E}_1)$ and $(\mathcal{E}_3)$ imply that N can not be large enough when $l_1 \neq m_1$, and $(\mathcal{E}_2)$ implies N is less than l_1. However, our assumption (ϕ_2) implies that N can be arbitrary positive integer even if system (1.1) reduces to the equation case.

Next, we fix two notations. Assume that functions ϕ_i $(i = 1, 2)$ defined by (1.2) satisfy (ϕ_1) and (ϕ_2). We denote by $\mathcal{A}_1$ the class of C^1 functions $A\colon \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ which possess the following properties:

(i) if $N \geq \min\{l_1, l_2\}$, then $A(x, 0, 0) \in L^\infty(\Omega)$ and there exist constants $C_1 > 0$, $a_i \in]m_i, l_i^*[$ such that

$$\begin{cases} |A_y(x,y,z)| \leq C_1 \left(1 + |y|^{a_1-1} + |z|^{\frac{a_2(a_1-1)}{a_1}}\right), \\ |A_z(x,y,z)| \leq C_1 \left(1 + |y|^{\frac{a_1(a_2-1)}{a_2}} + |z|^{a_2-1}\right) \end{cases} \tag{1.5}$$

for all $(x, y, z) \in \Omega \times \mathbb{R} \times \mathbb{R}$;

(ii) if $N < \min\{l_1, l_2\}$, then $A(x, 0, 0) \in L^1(\Omega)$ and for each $K > 0$, the functions

$$x \to \sup_{|(y,z)|\leq K} |A_y(x,y,z)| \quad \text{and} \quad x \to \sup_{|(y,z)|\leq K} |A_z(x,y,z)| \text{ belong to } L^1(\Omega). \tag{1.6}$$

When $A \in \mathcal{A}_1$, by a simple computation, it is easy to obtain that

(i) if $N \geq \min\{l_1, l_2\}$, then there exists $C_2 > 0$ such that

$$|A(x,y,z)| \leq C_2(1 + |y|^{a_1} + |z|^{a_2}) \tag{1.7}$$

for all $(x, y, z) \in \Omega \times \mathbb{R} \times \mathbb{R}$;

(ii) if $N < \min\{l_1, l_2\}$, then for each $K \geq 0$, the function

$$x \to \sup_{|(y,z)|\leq K} |A(x,y,z)| \text{ belongs to } L^1(\Omega). \tag{1.8}$$

Assume that functions ϕ_i $(i = 1, 2)$ defined by (1.2) satisfy (ϕ_1) and (ϕ_3). We denote by $\mathcal{A}_2$ the class of C^1 functions $A\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which possess the following properties: if $A(x, 0, 0) \in L^\infty(\Omega)$ and there exist constants $C_3 > 0$ and $\overline{a}_i \in]m_i, e_i^*[$, $(i = 1, 2)$ such that

$$\begin{cases} |A_y(x,y,z)| \leq C_3 \left(1 + |y|^{\overline{a}_1-1} + |z|^{\frac{\overline{a}_2(\overline{a}_1-1)}{\overline{a}_1}}\right), \\ |A_z(x,y,z)| \leq C_3 \left(1 + |y|^{\frac{\overline{a}_1(\overline{a}_2-1)}{\overline{a}_2}} + |z|^{\overline{a}_2-1}\right) \end{cases} \tag{1.9}$$

for all $(x, y, z) \in \Omega \times \mathbb{R} \times \mathbb{R}$.

Now, it is time to present all assumptions on potential functions F, G and H.

(I_1) functions F and G belong to $\mathcal{A}_1$;

(I_2) functions F, G and H belong to $\mathcal{A}_1$;

(I_3) functions F and G belong to $\mathcal{A}_2$;

(I_4) functions F, G and H belong to $\mathcal{A}_2$;

(II) there exist an open set $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$, $a_3 > m_1$, $a_4 > m_2$ and $\iota, \kappa \in \mathbb{R}$ with $\iota^2 + \kappa^2 = 1$ such that

$$\liminf_{t \to +\infty} \frac{F(x, \iota t, \kappa t)}{|\iota t|^{a_3} + |\kappa t|^{a_4}} > 0 \quad \text{uniformly in } x \in \Omega_0; \tag{1.10}$$

(III) for each $\lambda > 0$, there exists a function $\lambda(x) \in L^1(\Omega)$ such that

$$\lambda G(x, y, z) - F(x, y, z) \geq \lambda(x)$$

for all $(x, y, z) \in \Omega \times \mathbb{R} \times \mathbb{R}$;

(IV) $\liminf_{|(y,z)| \to \infty} \frac{H(x,y,z)}{|y|^{l_1} + |z|^{l_2}} \geq 0$ uniformly in $x \in \Omega$;

(V) $\limsup_{|(y,z)| \to \infty} \frac{H(x,y,z)}{|y|^{l_1} + |z|^{l_2}} < +\infty$ uniformly in $x \in \Omega$;

(VI) $H(x, 0, 0) = 0$ for $x \in \Omega$ and $\int_\Omega F(x, 0, 0)\, dx = \int_\Omega G(x, 0, 0)\, dx = 0$;

(VII) $\liminf_{|(y,z)| \to 0} \frac{H(x,y,z)}{|y|^{m_1} + |z|^{m_2}} \geq 0$ uniformly in $x \in \Omega$;

(VIII) $\limsup_{|(y,z)| \to 0} \frac{F(x,y,z)}{|y|^{m_1} + |z|^{m_2}} < +\infty$ uniformly in $x \in \Omega$;

(IX) $\liminf_{|(y,z)| \to 0} \frac{G(x,y,z)}{|y|^{m_1} + |z|^{m_2}} > -\infty$ uniformly in $x \in \Omega$;

(X) there exist a closed set $\Omega_1 \subset \Omega$ with $|\Omega_1| > 0$, a point $(b_1, b_2) \in \mathbb{R}^2$ and a constant $C_4 > 0$ such that

$$H(x, b_1, b_2) \leq -C_4$$

for all $x \in \Omega_1$.

Define

$$\begin{aligned} I(u, v) &= \int_\Omega \Phi_1(|\nabla u|)\, dx + \int_\Omega \Phi_2(|\nabla v|)\, dx, \quad J_F(u, v) = -\int_\Omega F(x, u, v)\, dx, \\ J_G(u, v) &= \int_\Omega G(x, u, v)\, dx, \quad J_H(u, v) = \int_\Omega H(x, u, v)\, dx, \quad u \in W, \end{aligned} \tag{1.11}$$

where the definition of W is given in Section 3 below. We also fix some notations that will be used in our results. For each $\lambda_1 > 0$ and $r > \inf_W J_G$, we put

$$\widetilde{\beta}(\lambda_1, I, J_F, J_G, r) = \frac{1}{\lambda_1} \sup_{(u,v)\in {J_G}^{-1}(]r,+\infty[)} \frac{I(u,v) + \lambda_1 J_F(u,v) - \inf_{{J_G}^{-1}(]-\infty,r])}(I + \lambda_1 J_F)}{r - J_G(u,v)}$$

and

$$\widetilde{\mu}(I, J_F, J_G, r) = \inf \left\{ \frac{J_F(u,v) - \widetilde{\gamma} + r}{\widetilde{\eta}_r - I(u,v)} : (u,v) \in W, J_G(u,v) < r, I(u,v) < \widetilde{\eta}_r \right\},$$

where $\widetilde{\gamma} = \inf_W (J_F(u,v) + J_G(u,v))$ and $\widetilde{\eta}_r = \inf_{(u,v)\in J_G^{-1}(r)} I(u,v)$.

Theorem 1.4. *Assume that functions ϕ_i, F and G ($i = 1, 2$) satisfy (ϕ_1), (ϕ_2), (I_1), (II) and (III). Then, for each $r > \int_\Omega G(x,0,0)\,dx$, each $\lambda_1 \in]0, \frac{1}{\max\{0,\widetilde{\mu}(I,J_F,J_G,r)\}}[$ and each compact interval $[a,b] \subset]0, \widetilde{\beta}(\lambda_1, I, J_F, J_G, r)[$, there exists a constant $\rho > 0$ with the following property: for every $\lambda_2/\lambda_1 \in [a,b]$ and every function $H \in \mathcal{A}_1$, there exists a constant $\delta > 0$ such that, for each $\lambda_3 \in [0, \delta]$, system (1.1) has at least three weak solutions in W whose norms are less than ρ.*

Theorem 1.5. *Assume that functions ϕ_i, F, G and H ($i = 1, 2$) satisfy (ϕ_1), (ϕ_2), (I_2) and (II)–(X). Then, there exists a point $(u_0, v_0) \in W$ such that $J_H(u_0, v_0) < 0$ and for each $\lambda_3 > -I(u_0, v_0)/J_H(u_0, v_0)$, there exists a constant $\lambda_1^* > 0$ with the following property: for all $\lambda_1 \in]0, \lambda_1^*]$, there exists a constant $\lambda_{2\lambda_1}^* > 0$ such that, for all $\lambda_2 \in]0, \lambda_{2\lambda_1}^*[$, system (1.1) has at least a trivial weak solution and three nontrivial weak solutions in W.*

Theorem 1.6. *Assume that functions ϕ_i, F and G ($i = 1, 2$) satisfy (ϕ_1), (ϕ_3), (I_3), (II) and (III). Then the same conclusion of Theorem 1.4 holds.*

Theorem 1.7. *Assume that functions ϕ_i, F, G and H ($i = 1, 2$) satisfy (ϕ_1), (ϕ_3), (I_4) and (II)–(X). Then the same conclusion of Theorem 1.5 holds.*

2. Preliminaries

In this section, we recall Orlicz and Orlicz-Sobolev spaces and some important properties about them. For more details, we refer the reader to the books [1, 27] and references therein.

First, we recall the notion and some properties of N-function which will be used to define Orlicz space. Let $\phi\colon [0, +\infty[\to [0, +\infty[$ be a right continuous, monotone increasing function satisfying

(1) $\phi(0) = 0$;

(2) $\lim_{t\to+\infty} \phi(t) = +\infty$;

(3) $\phi(t) > 0$ whenever $t > 0$.

Then the function defined by $\Phi(t) = \int_0^t \phi(s)\,ds$, $t \in [0,+\infty[$ is called an N-function. N-function Φ satisfies a global Δ_2-condition if it holds that $\sup_{t>0} \frac{\Phi(2t)}{\Phi(t)} < +\infty$. For N-function Φ, the complement of Φ is defined by

$$\widetilde{\Phi}(t) = \max_{s\geq 0} \{ts - \Phi(s)\} \quad \text{for } t \geq 0.$$

Then, $\widetilde{\Phi}$ is also an N-function and $\widetilde{\widetilde{\Phi}} = \Phi$. Moreover, the following Young's inequality holds:

$$st \leq \Phi(s) + \widetilde{\Phi}(t) \quad \text{for all } s, t \geq 0.$$

Now, we recall the Orlicz space $L^{\Phi}(\Omega)$ correlated with the N-function Φ. When Φ satisfies a global Δ_2-condition, the Orlicz space $L^{\Phi}(\Omega)$ is the vector space of the measurable functions $u\colon \Omega \to \mathbb{R}$ with

$$\int_{\Omega} \Phi(|u|)\,dx < +\infty,$$

where Ω is a domain in $\mathbb{R}^N$. Moreover, $L^{\Phi}(\Omega)$ is a Banach space equipped with the Luxemburg norm

$$\|u\|_{\Phi} := \inf\left\{\lambda > 0 : \int_{\Omega} \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1\right\} \quad \text{for } u \in L^{\Phi}(\Omega).$$

In particular, when $\Phi(t) = |t|^p$ $(1 < p < +\infty)$, the corresponding Orlicz space $L^{\Phi}(\Omega)$ and the Luxemburg norm $\|u\|_{\Phi}$ reduce to the classical Lebesgue space $L^p(\Omega)$ and the norm

$$\|u\|_{L^p(\Omega)} := \left(\int_{\Omega} |u(x)|^p\,dx\right)^{1/p} \quad \text{for } u \in L^p(\Omega),$$

respectively. In this paper, we denote $\|u\|_{L^p(\Omega)}$ by $\|u\|_p$.

Moreover, the Orlicz-Sobolev space defined by

$$W^{1,\Phi}(\Omega) := \left\{u \in L^{\Phi}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{\Phi}(\Omega), i = 1, 2, \ldots, N\right\}$$

is a Banach space equipped with the norm

$$\|u\|_{1,\Phi} := \|u\|_{\Phi} + \|\nabla u\|_{\Phi}.$$

When Ω is bounded, $W_0^{1,\Phi}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\Phi}(\Omega)$ has an equivalent norm

$$\|u\|_{0,\Phi} := \|\nabla u\|_{\Phi},$$

which can be obtained by using the Poincaré inequality in [21] given as

$$\|u\|_{\Phi} \leq 2d\,\|\nabla u\|_{\Phi} \quad \text{for all } u \in W_0^{1,\Phi}(\Omega), \tag{2.1}$$

where $d = \mathrm{diam}(\Omega)$.

Next, we summarize some important properties about N-function, Orlicz and Orlicz-Sobolev spaces.

Lemma 2.1. [1,18] *Assume that Φ is an N-function. Then, the following three conditions are equivalent:*

(1)

$$1 \leq l = \inf_{t>0} \frac{t\phi(t)}{\Phi(t)} \leq \sup_{t>0} \frac{t\phi(t)}{\Phi(t)} = m < +\infty; \tag{2.2}$$

(2) *let* $\zeta_0(t) = \min\left\{t^l, t^m\right\}$, $\zeta_1(t) = \max\left\{t^l, t^m\right\}$ *for* $t \geq 0$. Φ *satisfies*

$$\zeta_0(t)\Phi(\rho) \leq \Phi(\rho t) \leq \zeta_1(t)\Phi(\rho) \quad \textit{for all } \rho, t \geq 0;$$

(3) Φ *satisfies a global Δ_2-condition.*

Lemma 2.2. [18] *Assume that Φ is an N-function and (2.2) holds. Then*

$$\zeta_0(\|u\|_\Phi) \leq \int_\Omega \Phi(|u|)\, dx \leq \zeta_1(\|u\|_\Phi) \quad \textit{for all } u \in L^\Phi(\Omega).$$

Lemma 2.3. [18] *Assume that Φ is an N-function and (2.2) holds with $l > 1$. Let $\widetilde{\Phi}$ be the complement of Φ and $\zeta_2(t) = \min\left\{t^{\widetilde{l}}, t^{\widetilde{m}}\right\}$, $\zeta_3(t) = \max\left\{t^{\widetilde{l}}, t^{\widetilde{m}}\right\}$ for $t \geq 0$, where $\widetilde{l} := l/(l-1)$, $\widetilde{m} := m/(m-1)$. Then*

(1) $\widetilde{m} = \inf_{t>0} \frac{t\widetilde{\Phi}'(t)}{\widetilde{\Phi}(t)} \leq \sup_{t>0} \frac{t\widetilde{\Phi}'(t)}{\widetilde{\Phi}(t)} = \widetilde{l}$;

(2) $\zeta_2(t)\widetilde{\Phi}(\rho) \leq \widetilde{\Phi}(\rho t) \leq \zeta_3(t)\widetilde{\Phi}(\rho)$ *for all* $\rho, t \geq 0$;

(3) $\zeta_2(\|u\|_{\widetilde{\Phi}}) \leq \int_\Omega \widetilde{\Phi}(|u|)\, dx \leq \zeta_3(\|u\|_{\widetilde{\Phi}})$ *for all* $u \in L^{\widetilde{\Phi}}(\Omega)$.

If

$$\int_0^1 \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\, ds < +\infty \quad \text{and} \quad \int_1^{+\infty} \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\, ds = +\infty, \tag{2.3}$$

then the Sobolev conjugate N-function function Φ_* of Φ is given in [1] by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\, ds \quad \text{for } t \geq 0.$$

Lemma 2.4. [18] *Assume that Φ is an N-function and (2.2) holds with $l, m \in]1, N[$. Then (2.3) holds. Let $\zeta_4(t) = \min\left\{t^{l^*}, t^{m^*}\right\}$, $\zeta_5(t) = \max\left\{t^{l^*}, t^{m^*}\right\}$ for $t \geq 0$, where $l^* := lN/(N-l)$, $m^* := mN/(N-m)$. Then*

(1) $l^* = \inf_{t>0} \frac{t\Phi_*'(t)}{\Phi_*(t)} \le \sup_{t>0} \frac{t\Phi_*'(t)}{\Phi_*(t)} = m^*$;

(2) $\zeta_4(t)\Phi_*(\rho) \le \Phi_*(\rho t) \le \zeta_5(t)\Phi_*(\rho)$ *for all* $\rho, t \ge 0$;

(3) $\zeta_4(\|u\|_{\Phi_*}) \le \int_\Omega \Phi_*(|u|)\,dx \le \zeta_5(\|u\|_{\Phi_*})$ *for all* $u \in L^{\Phi_*}(\Omega)$.

Lemma 2.5. [1, 27] *Assume that* Φ *is an* N*-function and* (2.2) *holds with* $l > 1$. *Then the embedding* $W_0^{1,\Phi}(\Omega) \hookrightarrow W_0^{1,l}(\Omega)$ *is continuous, where* $W_0^{1,l}(\Omega)$ *is the classical Sobolev space. So the embedding from* $W_0^{1,\Phi}(\Omega)$ *into* $L^p(\Omega)$ *is continuous for* $1 \le p \le l^*$ *and into* $L^q(\Omega)$ *is compact for* $1 \le q < l^*$, *where*

$$l^* = \begin{cases} \frac{lN}{N-l} & \text{if } l < N, \\ +\infty & \text{if } l \ge N. \end{cases}$$

Therefore, when $1 \le p \le l^*$, *there exists a constant* $C_p > 0$ *such that*

$$\|u\|_p \le C_p \|\nabla u\|_\Phi \quad \text{for all } u \in W_0^{1,\Phi}(\Omega). \tag{2.4}$$

Lemma 2.6. [1, 27] *Assume that* Φ *is an* N*-function and* (2.2) *holds with* $l, m \in]1, N[$. *Then the embedding from* $W_0^{1,\Phi}(\Omega)$ *into* $L^{\Phi_*}(\Omega)$ *is continuous and into* $L^{\Upsilon}(\Omega)$ *is compact for any* N*-function* Υ *increasing essentially more slowly than* Φ_* *near infinity, that is*

$$\lim_{t\to+\infty} \frac{\Upsilon(ct)}{\Phi_*(t)} = 0$$

for any constant $c > 0$.

Remark 2.7. Assume that Φ is an N-function and (2.2) holds with $l > 1$. Then Lemmas 2.1 and 2.3 imply that both Φ and $\widetilde{\Phi}$ satisfy a global Δ_2-condition. Thus the Banach spaces $L^\Phi(\Omega)$, $W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are separable and reflexive (see [1, 27]).

3. Proofs

By (ϕ_1) and (ϕ_2) or (ϕ_1) and (ϕ_3), we define space $W := W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$ with norm

$$\|(u,v)\| := \|u\|_{0,\Phi_1} + \|v\|_{0,\Phi_2} = \|\nabla u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2}.$$

Then W is a separable and reflexive Banach space by Remark 2.7.

On W, define functional J by

$$\begin{aligned} J(u,v) &:= \int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla v|)\,dx - \lambda_1 \int_\Omega F(x,u,v)\,dx \\ &\quad + \lambda_2 \int_\Omega G(x,u,v)\,dx + \lambda_3 \int_\Omega H(x,u,v)\,dx, \quad (u,v) \in W. \end{aligned} \tag{3.1}$$

By (1.11), we have

$$J(u,v) = I(u,v) + \lambda_1 J_F(u,v) + \lambda_2 J_G(u,v) + \lambda_3 J_H(u,v), \quad (u,v) \in W.$$

Moreover, the critical points of J on W are weak solutions of system (1.1). With a similar argument as [20], (ϕ_1) and (ϕ_2) assure that $I\colon W \to \mathbb{R}$ is of class $C^1(W,\mathbb{R})$ and

$$\left\langle I'(u,v), (\widetilde{u},\widetilde{v}) \right\rangle = \int_\Omega a_1(|\nabla u|)\nabla u \cdot \nabla \widetilde{u}\, dx + \int_\Omega a_2(|\nabla v|)\nabla v \cdot \nabla \widetilde{v}\, dx \tag{3.2}$$

for all $(\widetilde{u},\widetilde{v}) \in W$.

We point out that C is used for denoting a positive constant that may be variable in different places.

Lemma 3.1. *Assume that (ϕ_1) and (ϕ_2) hold. Then C^1 functional $I\colon W \to \mathbb{R}$ is sequentially weakly lower semicontinuous, coercive, bounded on each bounded subset of X, and whose derivative I' admits a continuous inverse I'^{-1} on the dual space W^* of W.*

Proof. First, we prove that I is weakly lower semicontinuous. It is sufficient to prove that I is convex and (strongly) continuous by Remark 6 in Chapter 3 of [10]. In fact, it is easy to check that I is strictly convex by (ϕ_1). This, together with the continuity of I, implies that I is weakly lower semicontinuous. So I is sequentially weakly lower semicontinuous (see [10]). Now, we prove that I is coercive. By Lemma 2.2, we have

$$I(u,v) \geq \min\left\{ \|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1} \right\} + \min\left\{ \|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2} \right\} \geq \|\nabla u\|_{\Phi_1}^{l_1} + \|\nabla v\|_{\Phi_2}^{l_2} - 2,$$

which implies that $I(u,v) \to +\infty$ as $\|(u,v)\| = \|\nabla u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2} \to +\infty$. Moreover, by Lemma 2.2, we also have

$$I(u,v) \leq \max\left\{ \|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1} \right\} + \max\left\{ \|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2} \right\} \leq \|\nabla u\|_{\Phi_1}^{m_1} + \|\nabla v\|_{\Phi_2}^{m_2} + 2,$$

which implies that I is bounded on each bounded subset of X. Next, we prove that $I'\colon W \to W^*$ admits an inverse $I'^{-1}\colon W^* \to W$ and I'^{-1} is continuous on W^*. By (3.2), (ϕ_2) and Lemma 2.2, we have

$$\begin{aligned}
\frac{\langle I'(u,v), (u,v) \rangle}{\|(u,v)\|} &= \frac{\int_\Omega a_1(|\nabla u|)\,|\nabla u|^2\, dx + \int_\Omega a_2(|\nabla v|)\,|\nabla v|^2\, dx}{\|\nabla u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2}} \\
&\geq \frac{l_1 \int_\Omega \Phi_1(|\nabla u|)\, dx + l_2 \int_\Omega \Phi_2(|\nabla v|)\, dx}{\|\nabla u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2}} \\
&\geq \frac{l_1 \min\left\{ \|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1} \right\} + l_2 \min\left\{ \|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2} \right\}}{\|\nabla u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2}} \\
&\geq \frac{l_1 \|\nabla u\|_{\Phi_1}^{l_1} + l_2 \|\nabla v\|_{\Phi_2}^{l_2} - l_1 - l_2}{\|\nabla u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2}}
\end{aligned}$$

for all $(u,v)\in W$. Then $\lim_{\|(u,v)\|\to\infty}\frac{\langle I'(u,v),(u,v)\rangle}{\|(u,v)\|}=+\infty$, that is, I' is coercive in W. Furthermore, the continuity of I' implies that I' is hemicontinuous and the strictly convexity of I implies that I' is strictly monotone in W. Thus by Theorem 26.A(d) in [33], we know that the inverse I'^{-1} of I' exists and is bounded in W^*. We now prove that I'^{-1} is continuous by showing that it is sequentially continuous. Let $\{w_n\}\subset W^*$ be any given sequence such that $w_n\to w\in W^*$. Set $(u_n,v_n)=I'^{-1}(w_n)$, $n=1,2,\dots$, and $(u,v)=I'^{-1}(w)$. We claim that $(u_n,v_n)\to(u,v)$ in W. Since I'^{-1} is bounded and $w_n\to w$ in W^*, then $\{(u_n,v_n)\}$ is bounded in W. Without loss of generality, we assume that $(u_n,v_n)\rightharpoonup(u_0,v_0)$ in W, which implies that $u_n\rightharpoonup u_0$ in $W_0^{1,\Phi_1}(\Omega)$ and $v_n\rightharpoonup v_0$ in $W_0^{1,\Phi_2}(\Omega)$, respectively. Since $w_n\to w$ in W^* and $\{(u_n,v_n)\}$ is bounded in W, then

$$\langle w_n-w,(u_n,v_n)-(u_0,v_0)\rangle\to 0\quad\text{as } n\to\infty,$$

which, together with the fact that

$$\langle w,(u_n,v_n)-(u_0,v_0)\rangle\to 0\quad\text{and}\quad\big\langle I'(u_0,v_0),(u_n,v_n)-(u_0,v_0)\big\rangle\to 0\quad\text{as } n\to\infty,$$

implies that

$$\begin{aligned}0&=\lim_{n\to\infty}\langle w_n,(u_n,v_n)-(u_0,v_0)\rangle-\big\langle I'(u_0,v_0),(u_n,v_n)-(u_0,v_0)\big\rangle\\&=\lim_{n\to\infty}\big\langle I'(u_n,v_n)-I'(u_0,v_0),(u_n-u_0,v_n-v_0)\big\rangle\\&=\lim_{n\to\infty}\int_\Omega(a_1(|\nabla u_n|)\nabla u_n-a_1(|\nabla u_0|)\nabla u_0)\cdot(\nabla u_n-\nabla u_0)\,dx\\&\quad+\lim_{n\to\infty}\int_\Omega(a_2(|\nabla v_n|)\nabla v_n-a_2(|\nabla v_0|)\nabla v_0)\cdot(\nabla v_n-\nabla v_0)\,dx.\end{aligned}\tag{3.3}$$

Define operators $\mathcal{T}_i\colon W_0^{1,\Phi_i}(\Omega)\to W_0^{1,\Phi_i}(\Omega)^*$ $(i=1,2)$ by

$$\langle\mathcal{T}_1(u),\widetilde{u}\rangle:=\int_\Omega a_1(|\nabla u|)\nabla u\nabla\widetilde{u}\,dx,\quad u,\widetilde{u}\in W_0^{1,\Phi_1}(\Omega)$$

and

$$\langle\mathcal{T}_2(v),\widetilde{v}\rangle:=\int_\Omega a_2(|\nabla v|)\nabla v\nabla\widetilde{v}\,dx,\quad v,\widetilde{v}\in W_0^{1,\Phi_2}(\Omega).$$

(ϕ_1) implies that $\mathcal{T}_i$ $(i=1,2)$ are strictly monotone in $W_0^{1,\Phi_i}(\Omega)$ $(i=1,2)$, respectively. Then it follows from (3.3) that

$$\lim_{n\to\infty}\int_\Omega(a_1(|\nabla u_n|)\nabla u_n-a_1(|\nabla u_0|)\nabla u_0)\cdot(\nabla u_n-\nabla u_0)\,dx=0$$

and

$$\lim_{n\to\infty}\int_\Omega(a_2(|\nabla v_n|)\nabla v_n-a_2(|\nabla v_0|)\nabla v_0)\cdot(\nabla v_n-\nabla v_0)\,dx=0,$$

which imply that

$$\lim_{n\to\infty}\int_\Omega a_1(|\nabla u_n|)\nabla u_n\cdot(\nabla u_n-\nabla u_0)\,dx=0$$

and

$$\lim_{n\to\infty}\int_\Omega a_2(|\nabla v_n|)\nabla v_n\cdot(\nabla v_n-\nabla v_0)\,dx=0$$

because $u_n \rightharpoonup u_0$ in $W_0^{1,\Phi_1}(\Omega)$ and $v_n \rightharpoonup v_0$ in $W_0^{1,\Phi_2}(\Omega)$, respectively. Now we can conclude that $u_n\to u_0$ in $W_0^{1,\Phi_1}(\Omega)$ and $v_n\to v_0$ in $W_0^{1,\Phi_2}(\Omega)$, respectively, from Lemma 5 in [26]. Thus, $(u_n,v_n)\to(u_0,v_0)$ in W, which implies that $I'(u_n,v_n)\to I'(u_0,v_0)=I'(u,v)$ in W^*. The injectivity of I' implies that $(u_0,v_0)=(u,v)$. Therefore, the claim is valid and I'^{-1} is continuous. □

Lemma 3.2. *Assume that $A\in\mathcal{A}_1$. Then $J_A\colon W\to\mathbb{R}$ defined by*

$$J_A(u,v)=\int_\Omega A(x,u,v)\,dx$$

is a C^1 functional with compact derivative. Moreover,

$$\left\langle J_A'(u,v),(\widetilde{u},\widetilde{v})\right\rangle=\int_\Omega A_y(x,u,v)\widetilde{u}\,dx+\int_\Omega A_z(x,u,v)\widetilde{v}\,dx \tag{3.4}$$

for all $(\widetilde{u},\widetilde{v})\in W$.

Proof. First, suppose $N\geq\min\{l_1,l_2\}$. By (1.7) and Lemma 2.5, we have

$$J_A(u,v)\leq\int_\Omega|A(x,u,v)|\,dx\leq C_2\left(|\Omega|+\|u\|_{a_1}^{a_1}+\|v\|_{a_2}^{a_2}\right)\leq C\left(1+\|\nabla u\|_{\Phi_1}^{a_1}+\|\nabla v\|_{\Phi_2}^{a_2}\right).$$

Thus J_A is well defined in W. We now prove that (3.4) holds. For any given $(u,v),(\widetilde{u},\widetilde{v})\in W$, we have

$$\begin{aligned}\left\langle J_A'(u,v),(\widetilde{u},\widetilde{v})\right\rangle&=\lim_{h\to0}\frac{1}{h}(J_A(u+h\widetilde{u},v+h\widetilde{v})-J_A(u,v))\\&=\lim_{h\to0}\int_\Omega\frac{A(x,u+h\widetilde{u},v+h\widetilde{v})-A(x,u,v+h\widetilde{v})}{h}\,dx\\&\quad+\lim_{h\to0}\int_\Omega\frac{A(x,u,v+h\widetilde{v})-A(x,u,v)}{h}\,dx\\&=\lim_{h\to0}\int_\Omega A_y(x,u+\theta_1(x)h\widetilde{u},v+h\widetilde{v})\widetilde{u}\,dx\\&\quad+\lim_{h\to0}\int_\Omega A_z(x,u,v+\theta_2(x)h\widetilde{v})\widetilde{v}\,dx,\end{aligned}\tag{3.5}$$

where $\theta_1,\theta_2\colon\Omega\to]0,1[$. By the continuity of A_y and A_z, we obtain that

$$A_y(x,u+\theta_1(x)h\widetilde{u},v+h\widetilde{v})\widetilde{u}\to A_y(x,u,v)\widetilde{u} \tag{3.6}$$

and

$$A_z(x,u,v+\theta_2(x)h\widetilde{v})\widetilde{v} \to A_z(x,u,v)\widetilde{v} \tag{3.7}$$

as $h \to 0$ for a.e. $x \in \Omega$. Moreover, for all $h \in]-1,1[$, by (1.5) and the Young's inequality, we have

$$\begin{aligned} &|A_y(x,u+\theta_1(x)h\widetilde{u},v+h\widetilde{v})\widetilde{u}| \\ &\le C_1\left(1+|u+\theta_1(x)h\widetilde{u}|^{a_1-1}+|v+h\widetilde{v}|^{\frac{a_2(a_1-1)}{a_1}}\right)|\widetilde{u}| \\ &\le C\left(1+|u|^{a_1-1}+|\widetilde{u}|^{a_1-1}+|v|^{\frac{a_2(a_1-1)}{a_1}}+|\widetilde{v}|^{\frac{a_2(a_1-1)}{a_1}}\right)|\widetilde{u}| \\ &\le C\left(|\widetilde{u}|+|u|^{a_1}+|\widetilde{u}|^{a_1}+|v|^{a_2}+|\widetilde{v}|^{a_2}\right) =: g_1(x). \end{aligned} \tag{3.8}$$

By Lemma 2.5, we have

$$\int_\Omega g_1(x)\,dx = C\left(\|\widetilde{u}\|_1+\|u\|_{a_1}^{a_1}+\|\widetilde{u}\|_{a_1}^{a_1}+\|v\|_{a_2}^{a_2}+\|\widetilde{v}\|_{a_2}^{a_2}\right) < +\infty. \tag{3.9}$$

Then it follows from (3.6), (3.8), (3.9) and Lebesgue's dominated convergence theorem that

$$\lim_{h\to 0}\int_\Omega A_y(x,u+\theta_1(x)h\widetilde{u},v+h\widetilde{v})\widetilde{u}\,dx = \int_\Omega A_y(x,u,v)\widetilde{u}\,dx. \tag{3.10}$$

Similarly, by (3.7), we can also obtain that

$$\lim_{h\to 0}\int_\Omega A_z(x,u,v+\theta_2(x)h\widetilde{v})\widetilde{v}\,dx = \int_\Omega A_z(x,u,v)\widetilde{v}\,dx. \tag{3.11}$$

Combining (3.10) and (3.11) with (3.5), we can conclude that (3.4) holds. Next, we prove the continuity of J_A'. Let $(u_n,v_n) \to (u_0,v_0)$ in W. For all $(\widetilde{u},\widetilde{v}) \in W$, by (3.4), Hölder's inequality and Lemma 2.5, we have

$$
\begin{aligned}
&\left|\left\langle J_A'(u_n,v_n)-J_A'(u_0,v_0),(\widetilde{u},\widetilde{v})\right\rangle\right| \\
&=\left|\int_\Omega A_y(x,u_n,v_n)\widetilde{u}\,dx+\int_\Omega A_z(x,u_n,v_n)\widetilde{v}\,dx-\int_\Omega A_y(x,u_0,v_0)\widetilde{u}\,dx\right. \\
&\qquad\left.-\int_\Omega A_z(x,u_0,v_0)\widetilde{v}\,dx\right| \\
&\le\int_\Omega |A_y(x,u_n,v_n)-A_y(x,u_0,v_0)|\,|\widetilde{u}|\,dx \\
&\quad+\int_\Omega |A_z(x,u_n,v_n)-A_z(x,u_0,v_0)|\,|\widetilde{v}|\,dx \\
&\le\left(\int_\Omega |A_y(x,u_n,v_n)-A_y(x,u_0,v_0)|^{a_1/(a_1-1)}\,dx\right)^{(a_1-1)/a_1}\|\widetilde{u}\|_{a_1} \\
&\quad+\left(\int_\Omega |A_z(x,u_n,v_n)-A_z(x,u_0,v_0)|^{a_2/(a_2-1)}\,dx\right)^{(a_2-1)/a_2}\|\widetilde{v}\|_{a_2} \\
&\le C\left[\left(\int_\Omega |A_y(x,u_n,v_n)-A_y(x,u_0,v_0)|^{a_1/(a_1-1)}\,dx\right)^{(a_1-1)/a_1}\right. \\
&\qquad\left.+\left(\int_\Omega |A_z(x,u_n,v_n)-A_z(x,u_0,v_0)|^{a_2/(a_2-1)}\,dx\right)^{(a_2-1)/a_2}\right]\|(\widetilde{u},\widetilde{v})\|.
\end{aligned}
\tag{3.12}
$$

We claim that

$$
\int_\Omega |A_y(x,u_n,v_n)-A_y(x,u_0,v_0)|^{a_1/(a_1-1)}\,dx\to 0\quad\text{as } n\to\infty. \tag{3.13}
$$

Otherwise, there exist a constant $\varepsilon_0>0$ and a subsequence of $\{(u_n,v_n)\}$ denoted by $\{(u_{n_i},v_{n_i})\}$ such that

$$
\int_\Omega |A_y(x,u_{n_i},v_{n_i})-A_y(x,u_0,v_0)|^{a_1/(a_1-1)}\,dx\ge\varepsilon_0\quad\text{for all } n_i\in\mathbb{N}. \tag{3.14}
$$

Since $(u_{n_i},v_{n_i})\to(u_0,v_0)$ in W, then $u_{n_i}\to u_0$ in $W_0^{1,\Phi_1}(\Omega)$ and $v_{n_i}\to v_0$ in $W_0^{1,\Phi_2}(\Omega)$, respectively. It follows from Lemma 2.5 that $u_{n_i}\to u_0$ in $L^{a_1}(\Omega)$ and $v_{n_i}\to v_0$ in $L^{a_2}(\Omega)$, respectively. By [10, Theorem 4.9], there exist subsequences of $\{u_{n_i}\}$ and $\{v_{n_i}\}$, still denoted by $\{u_{n_i}\}$ and $\{v_{n_i}\}$, respectively, and functions $h_1\in L^{a_1}(\Omega)$ and $h_2\in L^{a_2}(\Omega)$ such that

$$
u_{n_i}(x)\to u_0(x),\quad v_{n_i}(x)\to v_0(x)\quad\text{a.e. } x\in\Omega \tag{3.15}
$$

and

$$
|u_{n_i}(x)|\le h_1(x),\quad |v_{n_i}(x)|\le h_2(x)\quad\text{for all } n_i\in\mathbb{N},\ \text{a.e. } x\in\Omega.
$$

By (3.15) and the continuity of A_y, we have

$$
|A_y(x,u_{n_i}(x),v_{n_i}(x))-A_y(x,u_0(x),v_0(x))|^{a_1/(a_1-1)}\to 0\quad\text{a.e. } x\in\Omega. \tag{3.16}
$$

By (1.5) and (3.15), for all $n_i \in \mathbb{N}$, a.e. $x \in \Omega$, we have

$$
\begin{aligned}
&|A_y(x, u_{n_i}, v_{n_i}) - A_y(x, u_0, v_0)|^{a_1/(a_1-1)} \\
&\le C \left(|A_y(x, u_{n_i}, v_{n_i})|^{a_1/(a_1-1)} + |A_y(x, u_0, v_0)|^{a_1/(a_1-1)} \right) \\
&\le C \Big[C_1^{a_1/(a_1-1)} \left(1 + |u_{n_i}|^{a_1-1} + |v_{n_i}|^{a_2(a_1-1)/a_1} \right)^{a_1/(a_1-1)} \\
&\quad + C_1^{a_1/(a_1-1)} \left(1 + |u_0|^{a_1-1} + |v_0|^{a_2(a_1-1)/a_1} \right)^{a_1/(a_1-1)} \Big] \\
&\le C \left(1 + |u_{n_i}|^{a_1} + |v_{n_i}|^{a_2} + |u_0|^{a_1} + |v_0|^{a_2} \right) \\
&\le C \left(1 + h_1^{a_1} + h_2^{a_2} + |u_0|^{a_1} + |v_0|^{a_2} \right) =: g_2(x).
\end{aligned} \tag{3.17}
$$

By Lemma 2.5, we have

$$\int_\Omega g_2(x)\, dx < +\infty. \tag{3.18}$$

Then it follows from (3.16)–(3.18) and Lebesgue's dominated convergence theorem that

$$\int_\Omega |A_y(x, u_{n_i}, v_{n_i}) - A_y(x, u_0, v_0)|^{a_1/(a_1-1)}\, dx \to 0 \quad \text{as } n_i \to \infty,$$

which contradicts (3.14). Then (3.13) holds. Similarly, we can also obtain that

$$\int_\Omega |A_z(x, u_n, v_n) - A_z(x, u_0, v_0)|^{a_2/(a_2-1)}\, dx \to 0 \quad \text{as } n \to \infty. \tag{3.19}$$

Combining (3.13) and (3.19) with (3.12), we can conclude that J'_A is continuous. To prove the compactness of J'_A, we take any sequence $\{(u_n, v_n)\} \subset W$ which is bounded. By the reflexivity of W and Lemma 2.5, we obtain that there exists a subsequence $\{(u_{n_i}, v_{n_i})\}$ of $\{(u_n, v_n)\}$ such that $(u_{n_i}, v_{n_i}) \rightharpoonup (u_0, v_0) \in W$, and $u_{n_i} \to u_0$ in $L^{a_1}(\Omega)$ and $v_{n_i} \to v_0$ in $L^{a_2}(\Omega)$, respectively. Then, with the same discussion as above, we can prove that $J'_A(u_{n_i}, v_{n_i}) \to J'_A(u_0, v_0)$ in W^*. So J'_A is compact.

Secondly, suppose $N < \min\{l_1, l_2\}$. Lemma 2.5 implies that the embeddings $W_0^{1,\Phi_i}(\Omega) \hookrightarrow L^\infty(\Omega)$ $(i = 1, 2)$ are continuous. Then for any given $(u, v) \in W$, we have $\|u\|_\infty + \|v\|_\infty < +\infty$, which, together with (1.8), implies that

$$
\begin{aligned}
J_A(u, v) &= \int_\Omega A(x, u, v)\, dx \le \int_\Omega |A(x, u, v)|\, dx \\
&\le \int_\Omega \sup_{|(y,z)| \le \|u\|_\infty + \|v\|_\infty} |A(x, y, z)|\, dx < +\infty.
\end{aligned}
$$

So J_A is well defined in W. Now, we prove that (3.4) holds. It is easy to see that (3.5)–(3.7) are still hold for this case. Moreover, for all $h \in]-1, 1[$, by (1.6) and Lemma 2.5, we have

$$|A_y(x, u + \theta_1(x) h\widetilde{u}, v + h\widetilde{v})\widetilde{u}| \le \|\widetilde{u}\|_\infty \sup_{|(y,z)| \le \|u\|_\infty + \|\widetilde{u}\|_\infty + \|v\|_\infty + \|\widetilde{v}\|_\infty} |A_y(x, y, z)| \in L^1(\Omega) \tag{3.20}$$

and

$$|A_z(x,u,v+\theta_2(x)h\widetilde{v})\widetilde{v}| \le \|\widetilde{v}\|_\infty \sup_{|(y,z)|\le \|u\|_\infty+\|v\|_\infty+\|\widetilde{v}\|_\infty} |A_z(x,y,z)| \in L^1(\Omega). \tag{3.21}$$

Combining (3.5)–(3.7), (3.20) and (3.21) with Lebesgue's dominated convergence theorem, we can conclude that (3.4) holds. Next, we prove the continuity of J'_A. Let $(u_n,v_n) \to (u_0,v_0)$ in W. For all $(\widetilde{u},\widetilde{v}) \in W$, by (3.4) and Lemma 2.5, we have

(3.22)
$$\begin{aligned}
&\left|\left\langle J'_A(u_n,v_n) - J'_A(u_0,v_0), (\widetilde{u},\widetilde{v})\right\rangle\right| \\
&= \bigg|\int_\Omega A_y(x,u_n,v_n)\widetilde{u}\,dx + \int_\Omega A_z(x,u_n,v_n)\widetilde{v}\,dx - \int_\Omega A_y(x,u_0,v_0)\widetilde{u}\,dx \\
&\qquad - \int_\Omega A_z(x,u_0,v_0)\widetilde{v}\,dx\bigg| \\
&\le \|\widetilde{u}\|_\infty \int_\Omega |A_y(x,u_n,v_n) - A_y(x,u_0,v_0)|\,dx + \|\widetilde{v}\|_\infty \int_\Omega |A_z(x,u_n,v_n) - A_z(x,u_0,v_0)|\,dx \\
&\le C\left(\int_\Omega |A_y(x,u_n,v_n) - A_y(x,u_0,v_0)|\,dx + \int_\Omega |A_z(x,u_n,v_n) - A_z(x,u_0,v_0)|\,dx\right) \\
&\qquad \times \|(\widetilde{u},\widetilde{v})\|.
\end{aligned}$$

Moreover, because the embeddings $W_0^{1,\Phi_i}(\Omega) \hookrightarrow L^\infty(\Omega)$ $(i=1,2)$ are continuous, $(u_n,v_n) \to (u_0,v_0)$ in W implies that $u_n \to u_0$ and $v_n \to v_0$ in $L^\infty(\Omega)$. Then

$$u_n(x) \to u_0(x), \quad v_n(x) \to v_0(x) \quad \text{a.e. } x \in \Omega \tag{3.23}$$

and there exists a $K_1 > 0$ such that

$$\|u_n\|_\infty + \|v_n\|_\infty \le K_1 \quad \text{for all } n = 0,1,2,\ldots. \tag{3.24}$$

By (3.23) and the continuity of A_y and A_z, we have

$$|A_y(x,u_n(x),v_n(x)) - A_y(x,u_0(x),v_0(x))| \to 0 \quad \text{a.e. } x \in \Omega \tag{3.25}$$

and

$$|A_z(x,u_n(x),v_n(x)) - A_z(x,u_0(x),v_0(x))| \to 0 \quad \text{a.e. } x \in \Omega. \tag{3.26}$$

By (3.24) and (1.6), we have

$$\begin{aligned}
|A_y(x,u_n,v_n) - A_y(x,u_0,v_0)| &\le |A_y(x,u_n,v_n)| + |A_y(x,u_0,v_0)| \\
&\le 2\sup_{|(y,z)|\le K_1} |A_y(x,y,z)| \in L^1(\Omega)
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}|A_z(x,u_n,v_n)-A_z(x,u_0,v_0)| &\le |A_z(x,u_n,v_n)|+|A_z(x,u_0,v_0)| \\ &\le 2\sup_{|(y,z)|\le K_1}|A_z(x,y,z)|\in L^1(\Omega).\end{aligned} \tag{3.28}$$

Combining (3.25)–(3.28) with (3.22), by Lebesgue's dominated convergence theorem, we can conclude that J'_A is continuous. To prove the compactness of J'_A, we take any sequence $\{(u_n,v_n)\}\subset W$ which is bounded. By the reflexivity of W, there exists a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$ such that $(u_n,v_n)\rightharpoonup(u_0,v_0)\in W$. By Lemma 2.5, we can assume that (3.23) and (3.24) hold. Then, with a similar discussion as above, we can prove that $J'_A(u_n,v_n)\to J'_A(u_0,v_0)$ in W^*. Then J'_A is compact. □

Lemma 3.3. *Assume that* (ϕ_1)*,* (ϕ_2)*,* (I$_1$)*,* (II) *and* (III) *hold. Then functionals* $I,J_F,J_G\colon W\to\mathbb{R}$ *satisfy*

(1) $\liminf_{\|(u,v)\|\to\infty} J_F(u,v)/I(u,v)=-\infty$*;*

(2) *functional* $J_F+\lambda J_G\colon W\to\mathbb{R}$ *is bounded below for all* $\lambda>0$*.*

Proof. (1) By the definitions of I and J_F, it is sufficient to prove

$$\limsup_{\|(u,v)\|\to\infty}\frac{\int_\Omega F(x,u,v)\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx+\int_\Omega\Phi_2(|\nabla v|)\,dx}=+\infty. \tag{3.29}$$

Now, we take $u_0\in C_0^\infty(\Omega_0)\setminus\{0\}$ with $u_0(x)\ge 0$, which, together with the Poincaré inequality (2.1), implies that

$$\|\nabla u_0\|_{\Phi_1}\ne 0,\quad \|\nabla u_0\|_{\Phi_2}\ne 0,\quad \|u_0\|_{a_3}\ne 0\quad\text{and}\quad \|u_0\|_{a_4}\ne 0.$$

Let $(u_1,v_1)=(\iota u_0,\kappa u_0)$. Then $(u_1,v_1)\in W$ satisfying $\|(tu_1,tv_1)\|\to\infty$ as $t\to+\infty$. Moreover, by Lemma 2.2, we have

$$\begin{aligned}&\lim_{t\to+\infty}\left[\int_\Omega\Phi_1(|\iota t\nabla u_0|)\,dx+\int_\Omega\Phi_2(|\kappa t\nabla u_0|)\,dx\right]\\ &\ge\lim_{t\to+\infty}\left(\iota^{l_1}t^{l_1}\|\nabla u_0\|_{\Phi_1}^{l_1}+\kappa^{l_2}t^{l_2}\|\nabla u_0\|_{\Phi_2}^{l_2}-2\right)=+\infty.\end{aligned} \tag{3.30}$$

By (II), there exist $\epsilon>0$ and $t_0>0$ such that

$$F(x,\iota t,\kappa t)\ge\epsilon\left(|\iota t|^{a_3}+|\kappa t|^{a_4}\right)\quad\text{for all } x\in\Omega_0,\ t>t_0. \tag{3.31}$$

First, suppose $N\ge\min\{l_1,l_2\}$. Since F belongs to $\mathcal{A}_1$, then by (3.31) and (1.7), we have

$$F(x,\iota t,\kappa t)\ge\epsilon\left(|\iota t|^{a_3}+|\kappa t|^{a_4}\right)-C_5\quad\text{for all } x\in\Omega_0,\ t\ge 0, \tag{3.32}$$

where $C_5 = C_2(1+|\iota t_0|^{a_1}+|\kappa t_0|^{a_2})+\epsilon(|\iota t_0|^{a_3}+|\kappa t_0|^{a_4})$. Then by (3.32), (3.30), Lemma 2.2 and the fact that $a_3 > m_1$, $a_4 > m_2$ and $u_0 \in C_0^\infty(\Omega_0) \setminus \{0\}$, we have

$$\begin{aligned}
&\lim_{t\to+\infty} \frac{\int_\Omega F(x,tu_1,tv_1)\,dx}{\int_\Omega \Phi_1(|\nabla tu_1|)\,dx + \int_\Omega \Phi_2(|\nabla tv_1|)\,dx} \\
&= \lim_{t\to+\infty} \frac{\int_\Omega F(x,\iota tu_0,\kappa tu_0)\,dx}{\int_\Omega \Phi_1(|\iota t\nabla u_0|)\,dx + \int_\Omega \Phi_2(|\kappa t\nabla u_0|)\,dx} \\
&= \lim_{t\to+\infty} \frac{\int_{\Omega_0} F(x,\iota tu_0,\kappa tu_0)\,dx + \int_{\Omega\setminus\Omega_0} F(x,\iota tu_0,\kappa tu_0)\,dx}{\int_\Omega \Phi_1(|\iota t\nabla u_0|)\,dx + \int_\Omega \Phi_2(|\kappa t\nabla u_0|)\,dx} \\
&\geq \lim_{t\to+\infty} \frac{\int_{\Omega_0} [\epsilon\,(|\iota tu_0|^{a_3} + |\kappa tu_0|^{a_4}) - C_5]\,dx + \int_{\Omega\setminus\Omega_0} F(x,0,0)\,dx}{\int_\Omega \Phi_1(|\iota t\nabla u_0|)\,dx + \int_\Omega \Phi_2(|\kappa t\nabla u_0|)\,dx} \\
&\geq \lim_{t\to+\infty} \frac{\int_{\Omega} [\epsilon\,(|\iota tu_0|^{a_3} + |\kappa tu_0|^{a_4}) - C_5]\,dx - \int_{\Omega} |F(x,0,0)|\,dx}{\int_\Omega \Phi_1(|\iota t\nabla u_0|)\,dx + \int_\Omega \Phi_2(|\kappa t\nabla u_0|)\,dx} \\
&= \lim_{t\to+\infty} \frac{\epsilon\iota^{a_3}t^{a_3}\,\|u_0\|_{a_3}^{a_3} + \epsilon\kappa^{a_4}t^{a_4}\,\|u_0\|_{a_4}^{a_4} - C}{\int_\Omega \Phi_1(|\iota t\nabla u_0|)\,dx + \int_\Omega \Phi_2(|\kappa t\nabla u_0|)\,dx} \\
&\geq \lim_{t\to+\infty} \frac{\epsilon\iota^{a_3}t^{a_3}\,\|u_0\|_{a_3}^{a_3} + \epsilon\kappa^{a_4}t^{a_4}\,\|u_0\|_{a_4}^{a_4}}{\iota^{m_1}t^{m_1}\,\|\nabla u_0\|_{\Phi_1}^{m_1} + \kappa^{m_2}t^{m_2}\,\|\nabla u_0\|_{\Phi_2}^{m_2} + 2} = +\infty,
\end{aligned}$$

which implies that (3.29) holds.

Secondly, suppose $N < \min\{l_1, l_2\}$. By (3.31), we have

$$F(x,\iota t,\kappa t) \geq \epsilon\,(|\iota t|^{a_3} + |\kappa t|^{a_4}) - C_6 - \sup_{|(y,z)|\leq t_0} |F(x,y,z)| \quad \text{for all } x \in \Omega_0,\ t \geq 0, \tag{3.33}$$

where $C_6 = \epsilon\,(|\iota t_0|^{a_3} + |\kappa t_0|^{a_4})$. Note that F belongs to $\mathcal{A}_1$, $a_3 > m_1$, $a_4 > m_2$ and $u_0 \in C_0^\infty(\Omega_0) \setminus \{0\}$. Then by (3.33), (1.8), (3.30) and Lemma 2.2, we have

$$\begin{aligned}
&\lim_{t\to+\infty} \frac{\int_\Omega F(x,tu_1,tv_1)\,dx}{\int_\Omega \Phi_1(|\nabla tu_1|)\,dx + \int_\Omega \Phi_2(|\nabla tv_1|)\,dx} \\
&= \lim_{t\to+\infty} \frac{\int_{\Omega_0} F(x,\iota tu_0,\kappa tu_0)\,dx + \int_{\Omega\setminus\Omega_0} F(x,\iota tu_0,\kappa tu_0)\,dx}{\int_\Omega \Phi_1(|\iota t\nabla u_0|)\,dx + \int_\Omega \Phi_2(|\kappa t\nabla u_0|)\,dx} \\
&\geq \lim_{t\to+\infty} \frac{\int_{\Omega_0} [\epsilon\,(|\iota tu_0|^{a_3} + |\kappa tu_0|^{a_4}) - C_6]\,dx - \int_{\Omega_0} \sup_{|(y,z)|\leq t_0} |F(x,y,z)|\,dx + \int_{\Omega\setminus\Omega_0} F(x,0,0)\,dx}{\int_\Omega \Phi_1(|\iota t\nabla u_0|)\,dx + \int_\Omega \Phi_2(|\kappa t\nabla u_0|)\,dx} \\
&\geq \lim_{t\to+\infty} \frac{\int_{\Omega} [\epsilon\,(|\iota tu_0|^{a_3} + |\kappa tu_0|^{a_4}) - C_6]\,dx - \int_{\Omega} \sup_{|(y,z)|\leq t_0} |F(x,y,z)|\,dx - \int_{\Omega} |F(x,0,0)|\,dx}{\int_\Omega \Phi_1(|\iota t\nabla u_0|)\,dx + \int_\Omega \Phi_2(|\kappa t\nabla u_0|)\,dx} \\
&= \lim_{t\to+\infty} \frac{\epsilon\iota^{a_3}t^{a_3}\,\|u_0\|_{a_3}^{a_3} + \epsilon\kappa^{a_4}t^{a_4}\,\|u_0\|_{a_4}^{a_4} - C}{\int_\Omega \Phi_1(|\iota t\nabla u_0|)\,dx + \int_\Omega \Phi_2(|\kappa t\nabla u_0|)\,dx} \\
&\geq \lim_{t\to+\infty} \frac{\epsilon\iota^{a_3}t^{a_3}\,\|u_0\|_{a_3}^{a_3} + \epsilon\kappa^{a_4}t^{a_4}\,\|u_0\|_{a_4}^{a_4}}{\iota^{m_1}t^{m_1}\,\|\nabla u_0\|_{\Phi_1}^{m_1} + \kappa^{m_2}t^{m_2}\,\|\nabla u_0\|_{\Phi_2}^{m_2} + 2} = +\infty,
\end{aligned}$$

which implies that (3.29) holds.

(2) For any given $\lambda > 0$, by (III), we have

$$\inf_{(u,v)\in W}(J_F+\lambda J_G)=\inf_{(u,v)\in W}\int_\Omega(\lambda G(x,u,v)-F(x,u,v))\,dx\geq\int_\Omega\lambda(x)\,dx$$
$$\geq-\int_\Omega|\lambda(x)|\,dx>-\infty.$$

Then functional $J_F+\lambda J_G$ is bounded below. □

Proof of Theorem 1.4. To apply Theorem 1.1, let $X = W$, I defined by (1.11), $\Psi = J_F$, $\Phi = J_G$, $\Gamma = J_H$, $\mu = 1/\lambda_1$, $\lambda = \lambda_2/\lambda_1$ and $\nu = \lambda_3/\lambda_1$. Then $\beta = \widetilde{\beta}$, $\mu^* = \widetilde{\mu}$, $\gamma = \widetilde{\gamma}$, $\eta_r = \widetilde{\eta_r}$ and J given by (3.1) satisfies $\mu J = \mu I + \Psi + \lambda\Phi + \nu\Gamma$. By ($\mathrm{I}_1$), Lemmas 3.1, 3.2 and 3.3, all conditions of Theorem 1.1 hold. Moreover, it is easy to see that $M = \{(0,0)\}$, and $H \in \mathcal{A}_1$ implies that J_H is C^1 functional with compact derivative. Then Theorem 1.1 shows that for each $r > \int_\Omega G(x,0,0)\,dx$, each $\lambda_1 \in]0, \frac{1}{\max\{0,\widetilde{\mu}(I,J_F,J_G,r)\}}[$ and each compact interval $[a,b] \subset]0, \widetilde{\beta}(\lambda_1, I, J_F, J_G, r)[$, there exists a constant $\rho > 0$ with the following property: for every $\lambda_2/\lambda_1 \in [a,b]$ and every function $H \in \mathcal{A}_1$, there exists a constant $\delta > 0$ such that, for each $\lambda_3 \in [0,\delta]$, $\frac{1}{\lambda_1}J' = \mu I' + \Psi' + \lambda\Phi' + \nu\Gamma' = 0$ has at least three solutions whose norms are less than ρ. □

Proof of Theorem 1.5. To apply Theorem 1.2, we let $X = W$, I defined by (1.11), $\Psi = J_F$, $\Phi = J_G$, $\Gamma = J_H$, and $\nu = \lambda_3$, $\lambda = \lambda_1$, $\sigma = \lambda_2$. Then J given by (3.1) satisfies $J = I + \nu\Gamma + \lambda\Psi + \sigma\Phi$. By definition of W, X is a reflexive real Banach space. Lemma 3.1 implies that I is sequentially weakly lower semicontinuous and coercive C^1 functional whose derivative admits a continuous inverse on X^*. (I_2) together with Lemma 3.2 implies that Γ, Ψ, Φ are three C^1 functionals with compact derivative. Lemma 3.3 implies that conditions (c) and (d) of Theorem 1.2 hold. Next, we prove the remaining conditions of Theorem 1.2 one by one.

(a) By (IV), for any given $\epsilon > 0$, there exists $K_\epsilon > 0$ such that

$$H(x,y,z)\geq-\epsilon\left(|y|^{l_1}+|z|^{l_2}\right)\quad\text{for all } x\in\Omega,\ (y,z)\in\mathbb{R}\times\mathbb{R}\text{ with }|(y,z)|>K_\epsilon. \tag{3.34}$$

When $N \geq \min\{l_1, l_2\}$. Since H belongs to $\mathcal{A}_1$, then by (3.34) and (1.7), we have

$$H(x,y,z)\geq-\epsilon\left(|y|^{l_1}+|z|^{l_2}\right)-C_7\quad\text{for all }(x,y,z)\in\Omega\times\mathbb{R}\times\mathbb{R}, \tag{3.35}$$

where $C_7 = C_2\,(1 + |K_\epsilon|^{a_1} + |K_\epsilon|^{a_2})$. Then by (3.35), Lemmas 2.2 and 2.5, we have

$$\liminf_{\|(u,v)\|\to\infty}\frac{\Gamma(u,v)}{I(u,v)}=\liminf_{\|(u,v)\|\to\infty}\frac{\int_\Omega H(x,u,v)\,dx}{\int_\Omega\Phi_1(|\nabla u|)\,dx+\int_\Omega\Phi_2(|\nabla u|)\,dx}$$
$$\geq\liminf_{\|(u,v)\|\to\infty}\frac{\int_\Omega\left[-\epsilon\left(|u|^{l_1}+|v|^{l_2}\right)-C_7\right]dx}{\int_\Omega\Phi_1(|\nabla u|)\,dx+\int_\Omega\Phi_2(|\nabla v|)\,dx}$$

$$= \liminf_{\|(u,v)\|\to\infty} \frac{-\epsilon\left(\|u\|_{l_1}^{l_1}+\|v\|_{l_2}^{l_2}\right)-C}{\int_\Omega \Phi_1(|\nabla u|)\,dx+\int_\Omega \Phi_2(|\nabla v|)\,dx}$$

$$\geq \liminf_{\|(u,v)\|\to\infty} \frac{-\epsilon\left(\|u\|_{l_1}^{l_1}+\|v\|_{l_2}^{l_2}\right)-C}{\min\left\{\|\nabla u\|_{\Phi_1}^{l_1},\|\nabla u\|_{\Phi_1}^{m_1}\right\}+\min\left\{\|\nabla v\|_{\Phi_2}^{l_2},\|\nabla v\|_{\Phi_2}^{m_2}\right\}}$$

$$\geq \liminf_{\|(u,v)\|\to\infty} \frac{-\epsilon\max\left\{C_{l_1}^{l_1},C_{l_2}^{l_2}\right\}\left(\|\nabla u\|_{\Phi_1}^{l_1}+\|\nabla v\|_{\Phi_2}^{l_2}\right)-C}{\min\left\{\|\nabla u\|_{\Phi_1}^{l_1},\|\nabla u\|_{\Phi_1}^{m_1}\right\}+\min\left\{\|\nabla v\|_{\Phi_2}^{l_2},\|\nabla v\|_{\Phi_2}^{m_2}\right\}}$$

$$= -\epsilon\max\left\{C_{l_1}^{l_1},C_{l_2}^{l_2}\right\}.$$

Since ϵ is arbitrary, then (a) holds.

When $N<\min\{l_1,l_2\}$. By (3.34), we have

$$H(x,y,z)\geq -\epsilon\left(|y|^{l_1}+|z|^{l_2}\right)-\sup_{|(y,z)|\leq K_\epsilon}|H(x,y,z)| \quad \text{for all } (x,u,v)\in\Omega\times\mathbb{R}\times\mathbb{R}. \tag{3.36}$$

Since H belongs to $\mathcal{A}_1$, then by (3.36), (1.8), Lemmas 2.2 and 2.5, we have

$$\liminf_{\|(u,v)\|\to\infty}\frac{\Gamma(u,v)}{I(u,v)} = \liminf_{\|(u,v)\|\to\infty} \frac{\int_\Omega H(x,u,v)\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx+\int_\Omega \Phi_2(|\nabla u|)\,dx}$$

$$\geq \liminf_{\|(u,v)\|\to\infty} \frac{-\epsilon\int_\Omega\left(|u|^{l_1}+|v|^{l_2}\right)dx-\int_\Omega \sup_{|(y,z)|\leq K_\epsilon}|H(x,y,z)|\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx+\int_\Omega \Phi_2(|\nabla v|)\,dx}$$

$$= \liminf_{\|(u,v)\|\to\infty} \frac{-\epsilon\left(\|u\|_{l_1}^{l_1}+\|v\|_{l_2}^{l_2}\right)-C}{\int_\Omega \Phi_1(|\nabla u|)\,dx+\int_\Omega \Phi_2(|\nabla v|)\,dx}$$

$$\geq \liminf_{\|(u,v)\|\to\infty} \frac{-\epsilon\max\left\{C_{l_1}^{l_1},C_{l_2}^{l_2}\right\}\left(\|\nabla u\|_{\Phi_1}^{l_1}+\|\nabla v\|_{\Phi_2}^{l_2}\right)-C}{\min\left\{\|\nabla u\|_{\Phi_1}^{l_1},\|\nabla u\|_{\Phi_1}^{m_1}\right\}+\min\left\{\|\nabla v\|_{\Phi_2}^{l_2},\|\nabla v\|_{\Phi_2}^{m_2}\right\}}$$

$$= -\epsilon\max\left\{C_{l_1}^{l_1},C_{l_2}^{l_2}\right\}.$$

Since ϵ is arbitrary, then (a) holds.

(b) By (V), there exist $\zeta>0$ and $K_\zeta>0$ such that

$$H(x,y,z)\leq \zeta\left(|y|^{l_1}+|z|^{l_2}\right) \quad \text{for all } x\in\Omega,\ (y,z)\in\mathbb{R}\times\mathbb{R} \text{ with } |(y,z)|>K_\zeta. \tag{3.37}$$

When $N\geq\min\{l_1,l_2\}$. Since H belongs to $\mathcal{A}_1$, then by (3.37) and (1.7), we have

$$H(x,y,z)\leq \zeta\left(|y|^{l_1}+|z|^{l_2}\right)+C_8 \quad \text{for all } (x,y,z)\in\Omega\times\mathbb{R}\times\mathbb{R}, \tag{3.38}$$

where $C_8=C_2\left(1+|K_\zeta|^{a_1}+|K_\zeta|^{a_2}\right)$. Then by (3.38), Lemmas 2.2 and 2.5, we have

$$\limsup_{\|(u,v)\|\to\infty}\frac{\Gamma(u,v)}{I(u,v)} = \limsup_{\|(u,v)\|\to\infty} \frac{\int_\Omega H(x,u,v)\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx+\int_\Omega \Phi_2(|\nabla u|)\,dx}$$

$$
\begin{aligned}
&\le \limsup_{\|(u,v)\|\to\infty} \frac{\int_\Omega \left[\zeta\left(|u|^{l_1}+|v|^{l_2}\right)+C_8\right]dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx+\int_\Omega \Phi_2(|\nabla v|)\,dx}\\
&= \limsup_{\|(u,v)\|\to\infty} \frac{\zeta\left(\|u\|_{l_1}^{l_1}+\|v\|_{l_2}^{l_2}\right)+C}{\int_\Omega \Phi_1(|\nabla u|)\,dx+\int_\Omega \Phi_2(|\nabla v|)\,dx}\\
&\le \limsup_{\|(u,v)\|\to\infty} \frac{\zeta\left(\|u\|_{l_1}^{l_1}+\|v\|_{l_2}^{l_2}\right)}{\|\nabla u\|_{\Phi_1}^{l_1}+\|\nabla v\|_{\Phi_2}^{l_2}-2}\\
&\le \limsup_{\|(u,v)\|\to\infty} \frac{\zeta\max\left\{C_{l_1}^{l_1},C_{l_2}^{l_2}\right\}\left(\|\nabla u\|_{\Phi_1}^{l_1}+\|\nabla v\|_{\Phi_2}^{l_2}\right)+C}{\min\left\{\|\nabla u\|_{\Phi_1}^{l_1},\|\nabla u\|_{\Phi_1}^{m_1}\right\}+\min\left\{\|\nabla v\|_{\Phi_2}^{l_2},\|\nabla v\|_{\Phi_2}^{m_2}\right\}}\\
&= \zeta\max\left\{C_{l_1}^{l_1},C_{l_2}^{l_2}\right\}<+\infty.
\end{aligned}
$$

When $N<\min\{l_1,l_2\}$. By (3.37), we have

$$
(3.39)\quad H(x,y,z)\le \zeta\left(|y|^{l_1}+|z|^{l_2}\right)+\sup_{|(y,z)|\le K_\zeta}|H(x,y,z)| \quad \text{for all } (x,y,z)\in\Omega\times\mathbb{R}\times\mathbb{R}.
$$

Since H belongs to $\mathcal{A}_1$, then by (3.39), (1.8), Lemmas 2.2 and 2.5, we have

$$
\begin{aligned}
\limsup_{\|(u,v)\|\to\infty}\frac{\Gamma(u,v)}{I(u,v)} &= \limsup_{\|(u,v)\|\to\infty} \frac{\int_\Omega H(x,u,v)\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx+\int_\Omega \Phi_2(|\nabla u|)\,dx}\\
&\le \limsup_{\|(u,v)\|\to\infty} \frac{\zeta\int_\Omega\left(|u|^{l_1}+|v|^{l_2}\right)dx+\int_\Omega \sup_{|(y,z)|\le K_\zeta}|H(x,y,z)|\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx+\int_\Omega \Phi_2(|\nabla v|)\,dx}\\
&= \limsup_{\|(u,v)\|\to\infty} \frac{\zeta\left(\|u\|_{l_1}^{l_1}+\|v\|_{l_2}^{l_2}\right)+C}{\int_\Omega \Phi_1(|\nabla u|)\,dx+\int_\Omega \Phi_2(|\nabla v|)\,dx}\\
&\le \limsup_{\|(u,v)\|\to\infty} \frac{\zeta\max\left\{C_{l_1}^{l_1},C_{l_2}^{l_2}\right\}\left(\|\nabla u\|_{\Phi_1}^{l_1}+\|\nabla v\|_{\Phi_2}^{l_2}\right)+C}{\min\left\{\|\nabla u\|_{\Phi_1}^{l_1},\|\nabla u\|_{\Phi_1}^{m_1}\right\}+\min\left\{\|\nabla v\|_{\Phi_2}^{l_2},\|\nabla v\|_{\Phi_2}^{m_2}\right\}}\\
&= \zeta\max\left\{C_{l_1}^{l_1},C_{l_2}^{l_2}\right\}<+\infty.
\end{aligned}
$$

(e) By Lemma 2.2, it is easy to see that $(0,0)$ a strict local minimum of I and $I(0,0)=0$.

(e_1) (VI) directly shows that $\Gamma(0,0)=\Psi(0,0)=\Phi(0,0)=0$.

(e_2) By (VII), for any given $\epsilon>0$, there exists $K_\epsilon>0$ such that

$$
(3.40)\quad H(x,y,z)\ge -\epsilon(|y|^{m_1}+|z|^{m_2}) \quad \text{for all } x\in\Omega,\ (y,z)\in\mathbb{R}\times\mathbb{R} \text{ with } |(y,z)|\le K_\epsilon.
$$

When $N\ge\min\{l_1,l_2\}$. Since H belongs to $\mathcal{A}_1$, then by (3.40) and (1.7), for ϵ given above, there exists a constant $C_\epsilon>0$ such that

$$
(3.41)\quad H(x,y,z)\ge -\epsilon(|y|^{m_1}+|z|^{m_2})-C_\epsilon(|y|^{a_1}+|z|^{a_2}) \quad \text{for all } (x,y,z)\in\Omega\times\mathbb{R}\times\mathbb{R}.
$$

Then by (3.41), Lemmas 2.2, 2.5 and the fact that $m_i < a_i$ $(i = 1, 2)$, we have

$$\begin{aligned}
&\liminf_{\|(u,v)\|\to 0} \frac{\Gamma(u,v)}{I(u,v)} \\
&= \liminf_{\|(u,v)\|\to 0} \frac{\int_\Omega H(x,u,v)\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla u|)\,dx} \\
&\geq \liminf_{\|(u,v)\|\to 0} \frac{-\epsilon \int_\Omega (|u|^{m_1} + |v|^{m_2})\,dx - C_\epsilon \int_\Omega (|u|^{a_1} + |v|^{a_2})\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla v|)\,dx} \\
&= \liminf_{\|(u,v)\|\to 0} \frac{-\epsilon(\|u\|_{m_1}^{m_1} + \|v\|_{m_2}^{m_2}) - C_\epsilon(\|u\|_{a_1}^{a_1} + \|v\|_{a_2}^{a_2})}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla v|)\,dx} \\
&\geq \liminf_{\|(u,v)\|\to 0} \frac{-\epsilon(\|u\|_{m_1}^{m_1} + \|v\|_{m_2}^{m_2}) - C_\epsilon(\|u\|_{a_1}^{a_1} + \|v\|_{a_2}^{a_2})}{\min\left\{\|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1}\right\} + \min\left\{\|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2}\right\}} \\
&\geq \liminf_{\|(u,v)\|\to 0} \frac{-\epsilon \max\left\{C_{m_1}^{m_1}, C_{m_2}^{m_2}\right\}(\|\nabla u\|_{\Phi_1}^{m_1} + \|\nabla v\|_{\Phi_2}^{m_2}) - C_\epsilon \max\left\{C_{a_1}^{a_1}, C_{a_2}^{a_2}\right\}(\|\nabla u\|_{\Phi_1}^{a_1} + \|\nabla v\|_{\Phi_2}^{a_2})}{\min\left\{\|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1}\right\} + \min\left\{\|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2}\right\}} \\
&= -\epsilon \max\left\{C_{m_1}^{m_1}, C_{m_2}^{m_2}\right\}.
\end{aligned}$$

Since ϵ is arbitrary, then (e_2) holds.

When $N < \min\{l_1, l_2\}$. It follows from Lemma 2.5 that the embeddings $W_0^{1,\Phi_i}(\Omega) \hookrightarrow L^\infty(\Omega)$ $(i = 1, 2)$ are continuous. Then (2.4) implies that $\|u\|_\infty + \|v\|_\infty \to 0$ as $\|(u,v)\| = \|\nabla u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2} \to 0$, which, together with (3.40), implies that

$$\begin{aligned}
\liminf_{\|(u,v)\|\to 0} \frac{\Gamma(u,v)}{I(u,v)} &= \liminf_{\|(u,v)\|\to 0} \frac{\int_\Omega H(x,u,v)\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla u|)\,dx} \\
&\geq \liminf_{\|(u,v)\|\to 0} \frac{-\epsilon \int_\Omega (|u|^{m_1} + |v|^{m_2})dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla v|)\,dx} \\
&= \liminf_{\|(u,v)\|\to 0} \frac{-\epsilon(\|u\|_{m_1}^{m_1} + \|v\|_{m_2}^{m_2})}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla v|)\,dx} \\
&\geq \liminf_{\|(u,v)\|\to 0} \frac{-\epsilon \max\left\{C_{m_1}^{m_1}, C_{m_2}^{m_2}\right\}(\|\nabla u\|_{\Phi_1}^{m_1} + \|\nabla v\|_{\Phi_2}^{m_2})}{\min\left\{\|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1}\right\} + \min\left\{\|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2}\right\}} \\
&= -\epsilon \max\left\{C_{m_1}^{m_1}, C_{m_2}^{m_2}\right\}.
\end{aligned}$$

Since ϵ is arbitrary, then (e_2) holds.

(e_3) By (VIII), there exist $\xi > 0$ and $K_\xi > 0$ such that

$$F(x,y,z) \leq \xi(|y|^{m_1} + |z|^{m_2}) \quad \text{for all } x \in \Omega,\ (y,z) \in \mathbb{R} \times \mathbb{R} \text{ with } |(y,z)| \leq K_\xi. \tag{3.42}$$

When $N \geq \min\{l_1, l_2\}$. Since F belongs to $\mathcal{A}_1$, then by (3.42) and (1.7), for ξ given above, there exists a constant $C_\xi > 0$ such that

$$F(x,y,z) \leq \xi(|y|^{m_1} + |z|^{m_2}) + C_\xi(|y|^{a_1} + |z|^{a_2}) \quad \text{for all } (x,y,z) \in \Omega \times \mathbb{R} \times \mathbb{R}. \tag{3.43}$$

Then by (3.43), Lemmas 2.2, 2.5 and the fact that $m_i < a_i$ $(i = 1, 2)$, we have

$$\begin{aligned}
&\limsup_{\|(u,v)\|\to 0} -\frac{\Psi(u,v)}{I(u,v)} \\
&= \limsup_{\|(u,v)\|\to 0} \frac{\int_\Omega F(x,u,v)\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla u|)\,dx} \\
&\le \limsup_{\|(u,v)\|\to 0} \frac{\xi \int_\Omega (|u|^{m_1} + |v|^{m_2})\,dx + C_\xi \int_\Omega (|u|^{a_1} + |v|^{a_2})\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla v|)\,dx} \\
&= \limsup_{\|(u,v)\|\to 0} \frac{\xi(\|u\|_{m_1}^{m_1} + \|v\|_{m_2}^{m_2}) + C_\xi(\|u\|_{a_1}^{a_1} + \|v\|_{a_2}^{a_2})}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla v|)\,dx} \\
&\le \limsup_{\|(u,v)\|\to 0} \frac{\xi \max\left\{C_{m_1}^{m_1}, C_{m_2}^{m_2}\right\}(\|\nabla u\|_{\Phi_1}^{m_1} + \|\nabla v\|_{\Phi_2}^{m_2}) + C_\xi \max\left\{C_{a_1}^{a_1}, C_{a_2}^{a_2}\right\}(\|\nabla u\|_{\Phi_1}^{a_1} + \|\nabla v\|_{\Phi_2}^{a_2})}{\min\left\{\|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1}\right\} + \min\left\{\|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2}\right\}} \\
&= \xi \max\left\{C_{m_1}^{m_1}, C_{m_2}^{m_2}\right\} < +\infty,
\end{aligned}$$

which is equivalent to (e_3).

When $N < \min\{l_1, l_2\}$. By the discussion above, we know that $\|u\|_\infty + \|v\|_\infty \to 0$ as $\|(u,v)\| = \|\nabla u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2} \to 0$. Then by (3.42), we have

$$\begin{aligned}
\limsup_{\|(u,v)\|\to 0} -\frac{\Psi(u,v)}{I(u,v)} &= \limsup_{\|(u,v)\|\to 0} \frac{\int_\Omega F(x,u,v)\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla u|)\,dx} \\
&\le \limsup_{\|(u,v)\|\to 0} \frac{\xi \int_\Omega (|u|^{m_1} + |v|^{m_2})\,dx}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla v|)\,dx} \\
&= \limsup_{\|(u,v)\|\to 0} \frac{\xi(\|u\|_{m_1}^{m_1} + \|v\|_{m_2}^{m_2})}{\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla v|)\,dx} \\
&\le \limsup_{\|(u,v)\|\to 0} \frac{\xi \max\left\{C_{m_1}^{m_1}, C_{m_2}^{m_2}\right\}(\|\nabla u\|_{\Phi_1}^{m_1} + \|\nabla v\|_{\Phi_2}^{m_2})}{\min\left\{\|\nabla u\|_{\Phi_1}^{l_1}, \|\nabla u\|_{\Phi_1}^{m_1}\right\} + \min\left\{\|\nabla v\|_{\Phi_2}^{l_2}, \|\nabla v\|_{\Phi_2}^{m_2}\right\}} \\
&= \xi \max\left\{C_{m_1}^{m_1}, C_{m_2}^{m_2}\right\} < +\infty,
\end{aligned}$$

which is equivalent to (e_3).

(e_4) By (IX), a similar argument as (e_3) shows that (e_4) holds.

(f) By (X), we can choose a closed set $\Omega_2 \subset \Omega_1$ with $\partial\Omega_1 \cap \Omega_2 = \emptyset$ and

$$|\Omega_1 \setminus \Omega_2| \le \frac{1}{2} \frac{C_4 |\Omega_2|}{\sup_{x\in\Omega_1, |(u,v)| \le \sqrt{b_1^2 + b_2^2}} |H(x,u,v)|}. \tag{3.44}$$

Take $(u_0, v_0) \in W$ which satisfies that $(u_0(x), v_0(x)) = (0,0)$ in $\Omega \setminus \Omega_1$, $(u_0(x), v_0(x)) =$

(b_1, b_2) in Ω_2 and $\|u_0\|_\infty + \|v_0\|_\infty \le \sqrt{b_1^2 + b_2^2}$. Then by (VI) and (3.44), we have

$$\begin{aligned}
\Gamma(u_0, v_0) &= \int_\Omega H(x, u_0(x), v_0(x))\, dx \\
&= \int_{\Omega \setminus \Omega_1} H(x, u_0(x), v_0(x))\, dx + \int_{\Omega_2} H(x, u_0(x), v_0(x))\, dx \\
&\quad + \int_{\Omega_1 \setminus \Omega_2} H(x, u_0(x), v_0(x))\, dx \\
&= \int_{\Omega \setminus \Omega_1} H(x, 0, 0)\, dx + \int_{\Omega_2} H(x, b_1, b_2)\, dx + \int_{\Omega_1 \setminus \Omega_2} H(x, u_0(x), v_0(x))\, dx \\
&\le -C_4 |\Omega_2| + |\Omega_1 \setminus \Omega_2| \sup_{x \in \Omega_1, |(u,v)| \le \sqrt{b_1^2 + b_2^2}} |H(x, u, v)| \\
&\le -\frac{1}{2} C_4 |\Omega_2| < 0.
\end{aligned}$$

Moreover, it is obvious that $I(u_0, v_0) > 0$.

Thus we verify that all conditions of Theorem 1.2 hold. Then Theorem 1.2 shows that for each $\lambda_3 > \max\{0, -I(u_0, v_0)/J_H(u_0, v_0)\} = -I(u_0, v_0)/J_H(u_0, v_0)$, there exists a constant $\lambda_1^* > 0$ with the following property: for all $\lambda_1 \in]0, \lambda_1^*]$ there exists $\lambda_{2\lambda_1}^* > 0$ such that, for all $\lambda_2 \in]0, \lambda_{2\lambda_1}^*[$, system (1.1) has at least a trivial weak solution and three pairwise distinct nontrivial weak solutions in W. □

Proofs of Theorems 1.6 *and* 1.7. Our results show that the conditions (ϕ_2), (I_1) and (I_2) can be replaced by (ϕ_3), (I_3) and (I_4), respectively. To prove Theorems 1.6 and 1.7, from all arguments in both Theorems 1.4 and 1.5, it is only needed to prove that the embeddings $W_0^{1,\Phi_i}(\Omega) \hookrightarrow L^{\overline{a}_i}(\Omega)$ $(i = 1, 2)$ are compact when (ϕ_2), (I_1) and (I_2) are replaced by (ϕ_3), (I_3) and (I_4), respectively. In fact, by Lemma 2.6, it is sufficient to prove that functions $\Upsilon_i(t) := |t|^{\overline{a}_i}$ $(i = 1, 2)$ increase essentially more slowly than Φ_{i*} $(i = 1, 2)$ near infinity, respectively. Let $a_i^* := \frac{a_i N}{N - a_i} = \overline{a}_i$ $(i = 1, 2)$. It follows from the fact $\overline{a}_i \in]m_i, e_i^*[$ $(i = 1, 2)$ that $a_i < e_i$ $(i = 1, 2)$ and $\overline{a}_i < (\frac{a_i + e_i}{2})^* := \frac{\frac{a_i + e_i}{2} N}{N - \frac{a_i + e_i}{2}}$ $(i = 1, 2)$. Then (1.4) implies that there exists a constant $K > 0$ such that

$$\frac{t\phi_i(t)}{\Phi_i(t)} \ge \frac{1}{2}(a_i + e_i) \quad \text{for all } t \ge K,$$

which implies that

$$\Phi_i(t) \ge C_9 |t|^{\frac{1}{2}(a_i + e_i)} \quad \text{for all } t \ge K$$

for some $C_9 > 0$. So, by Lemma 2.4 and the definition of Φ_{i*} $(i = 1, 2)$, when $t \ge \Phi_i(K)$

we have

$$\begin{aligned}
\Phi_{i*}^{-1}(t) &= \Phi_{i*}^{-1}(\Phi_i(K)) + \int_{\Phi_i(K)}^{t} \frac{\Phi_i^{-1}(s)}{s^{\frac{N+1}{N}}}\, ds \\
&\le \Phi_{i*}^{-1}(\Phi_i(K)) + \left(\frac{1}{C_9}\right)^{\frac{2}{a_i+e_i}} \int_{\Phi_i(K)}^{t} s^{\left(\frac{2}{a_i+e_i} - \frac{N+1}{N}\right)}\, ds \\
&= \Phi_{i*}^{-1}(\Phi_i(K)) + \left(\frac{1}{C_9}\right)^{\frac{2}{a_i+e_i}} \frac{N(a_i+e_i)}{2N-(a_i+e_i)} \left(t^{\frac{2N-(a_i+e_i)}{N(a_i+e_i)}} - \Phi_i(K)^{\frac{2N-(a_i+e_i)}{N(a_i+e_i)}} \right) \\
&\le C_{10} t^{\frac{2N-(a_i+e_i)}{N(a_i+e_i)}}
\end{aligned}$$

for some $C_{10} > 0$, which implies that

$$\Phi_{i*}(t) \ge \left(\frac{1}{C_{10}}\right)^{\frac{N(a_i+e_i)}{2N-(a_i+e_i)}} t^{\frac{N(a_i+e_i)}{2N-(a_i+e_i)}} = \left(\frac{1}{C_{10}}\right)^{(\frac{a_i+e_i}{2})^*} t^{(\frac{a_i+e_i}{2})^*} \quad \text{for all } t \ge \Phi_{i*}^{-1}(\Phi_i(K)).$$

Thus, for any constant $c > 0$, we have

$$\lim_{t\to+\infty} \frac{\Upsilon_i(ct)}{\Phi_{i*}(t)} \le \lim_{t\to+\infty} c^{\overline{a}_i} C_{10}^{(\frac{a_i+e_i}{2})^*} t^{\left[\overline{a}_i - (\frac{a_i+e_i}{2})^*\right]} = 0,$$

which implies that $\Upsilon_i(t) := |t|^{\overline{a}_i}$ $(i = 1,2)$ increase essentially more slowly than Φ_{i*} $(i = 1,2)$ near infinity, respectively. Hence the embeddings $W_0^{1,\Phi_i}(\Omega) \hookrightarrow L^{\overline{a}_i}(\Omega)$ $(i = 1,2)$ are compact. □

4. Remarks

Remark 4.1. (i) Assume that (ϕ_1) and (ϕ_2) hold and $N > \max\{m_1, m_2\}$. Then by the definitions of l_i, e_i, m_i $(i = 1,2)$, it is easy to see that $l_i \le e_i$ $(i = 1,2)$ and thus $l_i^* \le e_i^*$ $(i = 1,2)$, which, together with (ϕ_2), implies that (ϕ_3) holds. Moreover, (1.5) and (1.9) directly imply that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ if $l_i \le e_i$ $(i = 1,2)$, and $\mathcal{A}_1 = \mathcal{A}_2$ if and only if $l_i = e_i$ $(i = 1,2)$. Hence, Theorems 1.4 and 1.5 are corollaries of Theorems 1.6 and 1.7, respectively, if $N > \max\{m_1, m_2\}$ which shows that N can be large enough. There exist examples satisfying (ϕ_3) but not satisfying (ϕ_2). For example, let

$$\phi_i(t) = \begin{cases} a_i(|t|)t = p\,|t|^{p_i-2}\, t + q\,|t|^{q_i-2}\, t & \text{for } t \ne 0, \\ 0 & \text{for } t = 0, \end{cases}$$

where $1 < p_i < q_i < +\infty$ $(i = 1,2)$. On one hand, by a simple computation, we get

$$\Phi_i(t) = |t|^{p_i} + |t|^{q_i}, \quad t \in \mathbb{R},\ i = 1,2$$

and

$$l_i = p_i < e_i = m_i = q_i, \quad i = 1,2.$$

Then (ϕ_3) holds for all $N > \max\{m_1, m_2\}$. On the other hand, it is easy to see that $\lim_{N\to\infty} l_i^* = l_i$. Hence, we can choose N large enough such that $l_i^* \le m_i$ which contradicts (ϕ_2).

(ii) If $N < \max\{m_1, m_2\}$, then it is obvious that (ϕ_3) does not hold and so (ϕ_2) is not different from (ϕ_3).

Remark 4.2. In (II), let $y = \iota t$ (or $z = \kappa t$) if $\iota \neq 0$ (or $\kappa \neq 0$). Then (1.10) is equivalent to

$$\liminf_{y\to\operatorname{sgn}(\iota)\infty} \frac{F(x, y, \frac{\kappa}{\iota}y)}{|y|^{a_3} + \left|\frac{\kappa}{\iota}y\right|^{a_4}} > 0 \left(\text{or } \liminf_{z\to\operatorname{sgn}(\kappa)\infty} \frac{F(x, \frac{\iota}{\kappa}z, z)}{\left|\frac{\iota}{\kappa}z\right|^{a_3} + |z|^{a_4}} > 0\right), \text{uniformly in } x \in \Omega_0,$$

which clearly implies that $F(x, \cdot, \cdot)$ is only needed to satisfy the so-called super-linear Orlicz-Sobolev growth condition at infinity on a certain half-line which passes through origin in y-z plane for all $x \in \Omega_0$.

Remark 4.3. We present an example to verify our results. Let $N = 5$, Ω is a bounded domain in $\mathbb{R}^5$ with smooth boundary $\partial\Omega$. Assume that

$$a_1(t) = 2 + 3t, \quad a_2(t) = 3t\log(1+t) + \frac{t^2}{1+t} \quad \text{for } t > 0,$$

$F(x, y, z) = y^3 + z^6 + yz^3$, $G(x, y, z) = |y|^{\frac{19}{6}} + |z|^7$ and $H(x, y, z) = y\sin^3 y + z\sin^3 z$ for $(x, y, z) \in \Omega \times \mathbb{R}^2$. Then

$$\phi_1(t) = a_1(|t|)t = (2 + 3|t|)t, \ \phi_2(t) = a_2(|t|)t = \left(3|t|\log(1+|t|) + \frac{t^2}{1+|t|}\right)t \quad \text{for } t \in \mathbb{R}$$

and

$$\Phi_1(t) = t^2 + t^3, \quad \Phi_2(t) = t^3\log(1+t) \quad \text{for } t \ge 0.$$

By some simple computations, it is easy to obtain that (ϕ_1) holds and

$$l_1 = 2, \ m_1 = 3, \ e_1 = 3, \ l_2 = 3, \ m_2 = 4, \ e_2 = 3,$$
$$l_1^* = \frac{10}{3}, \ m_1^* = \frac{15}{2}, \ l_2^* = \frac{15}{2} \quad \text{and} \quad m_2^* = 20,$$

which shows that (ϕ_2) holds. Since $N > \max\{m_1, m_2\}$, then Remark 4.1 implies that (ϕ_3) holds and $\mathcal{A}_1 \subset \mathcal{A}_2$. Next, we show that $F, G, H \in \mathcal{A}_1$. Choose $a_1 = 19/6$ and $a_2 = 7$ in (1.5). Then

$$|F_y(x, y, z)| = \left|3y^2 + z^3\right| \le 3y^2 + |z|^3, \ |F_z(x, y, z)| = \left|6z^5 + 3yz^2\right| \le \frac{3}{2}y^2 + \frac{3}{2}z^4 + 6|z|^6,$$
$$|G_y(x, y, z)| = \frac{19}{6}|y|^{\frac{13}{6}}, \quad |G_z(x, y, z)| = 7|z|^6,$$
$$|H_y(x, y, z)| = \left|\sin^3 y + 3y\sin^2 y\cos y\right| \le 1 + 3|y|,$$
$$|H_z(x, y, z)| = \left|\sin^3 z + 3z\sin^2 z\cos z\right| \le 1 + 3|z|,$$

which imply that (1.5) holds. So, $F, G, H \in \mathcal{A}_1$. Choose $a_3 = 19/6$, $a_4 = 6$, $\iota = 0$ and $\kappa = 1$. Then

$$\liminf_{t\to+\infty} \frac{F(x,\iota t,\kappa t)}{|\iota t|^{a_3} + |\kappa t|^{a_4}} = \liminf_{t\to+\infty} \frac{t^6}{t^6} = 1 > 0,$$

which shows that (II) holds. For each $\lambda > 0$, we have

$$\begin{aligned} \lambda G(x,y,z) - F(x,y,z) &= \lambda |y|^{\frac{19}{6}} + \lambda |z|^7 - y^3 - z^6 - yz^3 \\ &\geq \lambda |y|^{\frac{19}{6}} + \lambda |z|^7 - |y|^3 - z^6 - \frac{1}{2}y^2 - \frac{1}{2}z^6, \end{aligned}$$

which shows that function $\lambda G(x,y,z) - F(x,y,z)$ is coercive. Then there exists a constant $C_\lambda < 0$ such that

$$\lambda G(x,y,z) - F(x,y,z) \geq C_\lambda =: \lambda(x) \in L^1(\Omega).$$

So (III) holds. Moreover,

$$\lim_{|(y,z)|\to\infty} \frac{H(x,y,z)}{|y|^{l_1} + |z|^{l_2}} = \lim_{|(y,z)|\to\infty} \frac{y\sin^3 y + z\sin^3 z}{|y|^2 + |z|^3} = 0,$$

which shows that (IV) and (V) hold. Obviously, (VI) holds and

$$\liminf_{|(y,z)|\to 0} \frac{H(x,y,z)}{|y|^{m_1} + |z|^{m_2}} = \liminf_{|(y,z)|\to 0} \frac{y\sin^3 y + z\sin^3 z}{|y|^3 + |z|^4} = 0,$$

$$\limsup_{|(y,z)|\to 0} \frac{F(x,y,z)}{|y|^{m_1} + |z|^{m_2}} = \limsup_{|(y,z)|\to 0} \frac{y^3 + z^6 + yz^3}{|y|^3 + |z|^4} \leq \limsup_{|(y,z)|\to 0} \frac{\frac{4}{3}|y|^3 + \frac{2}{3}|z|^{\frac{9}{2}} + z^6}{|y|^3 + |z|^4} = \frac{4}{3}$$

and

$$\liminf_{|(y,z)|\to 0} \frac{G(x,y,z)}{|y|^{m_1} + |z|^{m_2}} = \liminf_{|(y,z)|\to 0} \frac{|y|^{\frac{19}{6}} + |z|^7}{|y|^3 + |z|^4} = 0$$

show (VII), (VIII) and (IX), respectively. Finally, choose $(b_1, b_2) = (\frac{3}{2}\pi, \frac{3}{2}\pi) \in \mathbb{R}^2$. Then $H(x, b_1, b_2) = -3\pi < 0$, which implies that (X) holds.

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