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Reducibility of the Hilbert Scheme of Smooth Curves and Families of Double Covers

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Abstract. Let $\mathcal{I}_{d,g,r}$ be the union of irreducible components of the Hilbert scheme whose general points represent smooth irreducible complex curves of degree d and genus g in $\mathbb{P}^r$. Severi claimed in [15] that $\mathcal{I}_{d,g,r}$ is irreducible if $d \geq g + r$. His statement turned out to be correct for $r = 3$ and 4, while for $r \geq 6$, counterexamples have been found by using families of m-sheeted covers of rational curves with $m \geq 3$. In this work we show the existence of an additional component of $\mathcal{I}_{d,g,r}$ whose general elements are double covers of curves of positive genus. In addition, we find upper bounds for the dimension of the possible components of $\mathcal{I}_{d,g,r}$.

1. Introduction

We denote by $\mathcal{I}_{d,g,r}$ the union of irreducible components of the Hilbert scheme whose general points correspond to smooth irreducible non-degenerate complex curves of degree d and genus g embedded in $\mathbb{P}^r$. It can be decomposed into a union of irreducible components, see [14, 11-13, 11-14], as

$$\mathcal{I}_{d,g,r} = R_1 \cup \cdots \cup R_k \cup S_1 \cup \cdots \cup S_l$$

where

(1) the components R_i, $i = 1, \ldots, k$, are called *regular* and they are characterized by being generically smooth and having the expected dimension $\lambda_{d,g,r} := (r+1)d-(r-3)(g-1)$;

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(2) the components S_j, $j = 1, \dots, l$, are called *superabundant*, and they are characterized by being non-reduced or having dimension greater than $\lambda_{d,g,r}$.

When the Brill-Noether number $\rho(d,g,r) := g - (r+1)(g-d+r)$ is positive, it is known that $\mathcal{I}_{d,g,r}$ has the unique component dominating the moduli space $\mathcal{M}_g$ of smooth curves of genus g, see [6, p. 70]. It is usually referred to as *distinguished* component. It is indeed regular. Severi claimed in [15] that $\mathcal{I}_{d,g,r}$ is irreducible if $d \geq g + r$. Ein proved the conjecture for $r = 3$ and 4, see [4] and [5], while Mezzetti and Sacchiero [12], and Keem [8] gave examples where the conjecture doesn't hold for $r \geq 6$. We remark that all of these counterexamples have used curves which are m-sheeted coverings of $\mathbb{P}^1$ with $m \geq 3$. Ein also showed in [5] that $\mathcal{I}_{d,g,r}$ is irreducible for $d > \frac{2r-2}{r+2}g + \frac{r+8}{r+2}$ and $r \geq 5$. Subsequently, Kim extended in [10] the irreducibility range to $d > \eta_3 := \frac{2r-4}{r+1}g + \frac{r+13}{r+1}$ for $r \geq 8$. In her work she also found that for $\eta_4 := \frac{2r-6}{r+1}g + \frac{2r+26}{r+1} < d \leq \eta_3$ and $r \geq 15$, the scheme $\mathcal{I}_{d,g,r}$ is irreducible if and only if $2g-2-d$ is divisible by 3 and the additional components must parameterize triple coverings.

Our approach has been to look at the morphism Φ determined by the residual of the line bundle embedding a curve into $\mathbb{P}^r$. Namely, a general point of $\mathcal{I}_{d,g,r}$ corresponds to an embedding $C \hookrightarrow \mathbb{P}^r$ of an abstract smooth curve C in $\mathbb{P}^r$. Then we use the morphism Φ determined by the linear series $|\omega_C \otimes \mathcal{O}_C(-1)|$ on C, after possibly removing its base locus. This allows us to characterize the families of curves giving rise to a component of $\mathcal{I}_{d,g,r}$ in terms of degree of Φ, the geometric genus $g(\Phi(C))$ and $h^1(\Phi(C), \mathcal{O}_{\Phi(C)}(1))$. In Proposition 3.2 we establish numerical constraints on the degree d implied by the existence of an additional component of $\mathcal{I}_{d,g,r}$. We focus on the interval

$$\eta_5 := \frac{2r-8}{r+1}g + \frac{3r+43}{r+1} < d \leq \frac{2r-4}{r+1}g + \frac{r+13}{r+1} =: \eta_3,$$

where we can see a transitional nature of the geometry of $\mathcal{I}_{d,g,r}$ while tracing the decrease of d from η_3. Namely, the results in [10] suggest that for $\eta_4 < d \leq \eta_3$ different components over families of triple covers of curves of genus γ exist for different values of γ. By Proposition 3.2, for $\eta_5 < d \leq \eta_4$, components whose general elements are quadruple covers of curves of genus γ might possibly exist. Theorems 4.3 and 4.4 show that for $\eta_5 < d < \left(2 - \frac{8}{r}\right) g + 2 + \frac{8}{r} < \eta_4$, there exist irreducible components parameterizing curves which are double covers of curves of positive genus. Finally, Example 4.5 shows that there is a non-distinguished component of the minimal dimension, while by [12] there exist generically smooth superabundant components parameterizing trigonal curves.

As mentioned already, the main results of this work are contained in Theorems 4.3 and 4.4, where we prove the existence of a non-distinguished component of $\mathcal{I}_{d,g,r}$. In the next section we recall some preliminary facts and statements, and outline the approach that we follow. The main result in the third section is Proposition 3.2 in which we find necessary numerical conditions for the existence of other components of $\mathcal{I}_{d,g,r}$.

2. Basic notions and preliminary results

We recall some basic notions which will be used in our study of $\mathcal{I}_{d,g,r}$. Consider a family of genus $g \geq 2$ curves

$$p\colon \mathcal{C} \to S$$

parameterized by a scheme S which is a finite, possibly ramified covering of an open subset of the moduli space $\mathcal{M}_g$ of genus g curves. Assume that the family admits a section. Then for each integer d there exist a relative Picard S-scheme $\operatorname{Pic}^d(p)$ and a universal relative line bundle $\mathcal{L}_d(p)$ over $\mathcal{C} \times_S \operatorname{Pic}^d(p)$, which restricts on the fibers $p^{-1}(s)$ to the usual Poincaré line bundle on $\mathcal{C}_s \times \operatorname{Pic}^d(p)_s$ for each $s \in S$. For this family, there exist S-schemes $\mathcal{W}^r_d(p)$ and $\mathcal{G}^r_d(p)$, whose closed points appear as

$$\begin{aligned}\mathcal{W}^r_d(p) &= \Big\{(s, L) \mid s \in S,\, L \in \operatorname{Pic}^d(\mathcal{C}_s) \text{ such that } h^0(\mathcal{C}_s, L) \geq r+1\Big\},\\ \mathcal{G}^r_d(p) &= \Big\{(s, g^r_d) \mid s \in S,\, g^r_d = (L, V) \text{ with } L \in \operatorname{Pic}^d(\mathcal{C}_s),\, V \subset H^0(C_s, L)\\ &\qquad\qquad \text{such that } h^0(\mathcal{C}_s, L) \geq r+1 \text{ and } \dim V = r+1\Big\}.\end{aligned}$$

For the precise definitions and other properties of these varieties consult [3, Chapter XXI]. We remark that for every smooth irreducible curve C of genus g, there exists an open neighborhood $D \subset \mathcal{M}_g$ of the isomorphism class $[C] \in \mathcal{M}_g$ of C over which there exists a family $p\colon \mathcal{C} \to S$ admitting a section. More precisely, the variety S is a finite ramified covering $\nu\colon S \to D$, such that for every $s \in S$ we have $\mathcal{W}^r_d(p)_s \cong W^r_d(\nu(s))$ and $\mathcal{G}^r_d(p)_s \cong G^r_d(\nu(s))$, where W^r_d and G^r_d are the varieties of line bundles and linear series, correspondingly, defined for a fixed curve, see [2, Chapter IV] for details.

Since in what follows we are pursuing upper bounds for the dimensions of the schemes $\mathcal{W}^r_d(p)$ and $\mathcal{G}^r_d(p)$ defined for all families $p\colon \mathcal{C} \to S$ like above, i.e., admitting a section and S being a finite ramified covering of an open subset of $\mathcal{M}_g$, we will omit the explicit reference to the family $p\colon \mathcal{C} \to S$. We will denote them just by $\mathcal{W}^r_d$ and $\mathcal{G}^r_d$ implying that the results hold for any such families p of curves of genus g. When we work with families of curves of different genus, say γ, we will use notations like $\mathcal{W}^r_{d,\gamma}$ or $\mathcal{G}^r_{d,\gamma}$ to specify the genus. We will also sometimes defy rigor and call $\mathcal{W}^r_d$ and $\mathcal{G}^r_d$ "varieties", although they are not, in order to stress their similarities with W^r_d and G^r_d defined for a fixed curve.

The geometry of $\mathcal{W}^r_d$ and $\mathcal{G}^r_d$ is closely related to the properties of $\mathcal{I}_{d,g,r}$. An open set of a component of $\mathcal{I}_{d,g,r}$ is a $\mathrm{PGL}_{r+1}(\mathbb{C})$-bundle over an open subset of a component $\mathcal{G} \subset \mathcal{G}^r_d$ whose general elements correspond to pairs (C, g^r_d) such that g^r_d is very ample on C. An irreducible component $\mathcal{G} \subset \mathcal{G}^r_d$ is generically a Grassmannian fiber bundle with fiber $\operatorname{Grass}(r+1, \alpha+1)$ over an irreducible component $\mathcal{W} \subset \mathcal{W}^\alpha_d$ for some $\alpha \geq r$. When

$\rho(d,g,r) > 0$, it is shown by Brill-Noether theory that $\mathcal{I}_{d,g,r}$ has a unique component $\mathcal{I}_0$ dominating $\mathcal{M}_g$, see [13] where a more general statement is proved. The component $\mathcal{I}_0$ is referred to as the *distinguished* component and

$$\dim \mathcal{I}_0 = \lambda_{d,g,r} := (r+1)d - (r-3)(g-1),$$

where $\lambda_{d,g,r}$ is called *expected* dimension of $\mathcal{I}_{d,g,r}$. It is the minimal dimension that a component of $\mathcal{I}_{d,g,r}$ could have. We remark also that $\mathcal{I}_0$ is a regular component.

We recall that a linear series g^r_d defined on an algebraic curve X is called *birationally very ample* if the morphism defined by its base point free part is birational mapping. As we will use this notion a number of times in what follows, we remark that it is weaker than very ampleness.

The following proposition will be used in the proofs of Lemma 3.1 and Proposition 3.2.

Proposition 2.1. [9] *Let $\mathcal{G} \subset \mathcal{G}^r_d$ be an irreducible closed subvariety of $\mathcal{G}^r_d$, $r \geq 2$, whose general element $(C, g^r_d) \in \mathcal{G}$ is such that g^r_d is complete, special and birationally very ample on C. Then*

$$\dim \mathcal{G} \leq 3d + g - 4r - 1.$$

The next proposition will be used in the proof of Proposition 3.2 to produce one of our upper bound estimates.

Proposition 2.2. [10] *Suppose that a smooth curve of genus g has a birationally very ample linear series g^r_d with $d \leq 2g/(k-1) - k + 2$ for an integer $k \geq 2$. Then $r \leq (d+k-1)/k$.*

We use the following two well-known genus bounds, which we recall for a convenience of the reader. The first one is known as *Castelnuovo genus bound.*

Proposition 2.3. [6, p. 87] or [2, p. 116] *Let $C \hookrightarrow \mathbb{P}^r$ be an integral curve properly contained in $\mathbb{P}^r$, $r \geq 3$. Then for its arithmetic genus $p_a(C)$ we have*

$$p_a(C) \leq \pi(d,r),$$

where $\pi(d,r) = \binom{m}{2}(r-1) + m\varepsilon$, with $m := \left[\frac{d-1}{r-1}\right]$ and $\varepsilon = d - 1 - m(r-1)$. Further, equality is possible only if C lies on a surface of minimal degree in $\mathbb{P}^r$.

The next statement is often referred to as *Castelnuovo-Severi inequality.*

Proposition 2.4. [1, p. 21] *Let C be a smooth integral curve of genus g, and D_j be a smooth integral curve of genus g_j for $j = 1, 2$. Suppose that $\psi_j \colon C \to D_j$, $j = 1, 2$, are morphisms that do not admit a proper factorization, i.e., there do not exist a smooth curve Γ of genus $\gamma < g$ and morphisms $\varphi_0 \colon C \to \Gamma$, $\varphi_j \colon \Gamma \to D_j$, $j = 1, 2$, such that $\psi_j = \varphi_j \circ \varphi_0$ for $j = 1, 2$. Then*

$$g \leq (\deg \psi_1 - 1)(\deg \psi_2 - 1) + g_1 \deg \psi_1 + g_2 \deg \psi_2.$$

We work over $\mathbb{C}$ and if not explicitly said otherwise, we understand by *curve* a smooth integral projective algebraic curve. We denote by $L^\vee$ the dual line bundle for a given line bundle L defined on an algebraic variety X. As usual, ω_X will stand for the canonical line bundle on X. We denote $|L|$ the complete linear series $\mathbb{P}\left(H^0(X,L)\right)$. For definitions and properties of the objects not explicitly introduced in the paper we refer to [2] and [3].

3. Upper bounds

Every irreducible component $\mathcal{I} \subset \mathcal{I}_{d,g,r}$ is generically a fiber bundle over an irreducible component $\mathcal{W} \subset \mathcal{W}^\alpha_d$ for some $\alpha \geq r$, where the general fiber has dimension $\dim \operatorname{Grass}(r+1, \alpha+1) + \dim \operatorname{Aut}(\mathbb{P}^r)$. Thus, an upper bound for the dimension of $\mathcal{W}^\alpha_d$ gives an upper bound for the dimension of $\mathcal{I} \subset \mathcal{I}_{d,g,r}$. We will use the following claim which provides an upper bound for the dimension of a variety of line bundles defined for a family of curves.

Lemma 3.1. *Let $\mathcal{W}$ be an irreducible closed subvariety of $\mathcal{W}^l_{e,g}$ with $l \geq 1$ such that its general element (C,L) defines an m-fold covering morphism $\varphi := \varphi_L \colon C \to \Gamma := \varphi_L(C) \subset \mathbb{P}^l$ with $g(\Gamma) = \gamma$ for some positive integers γ and $m \geq 2$. Then*

$$\dim \mathcal{W} \leq \begin{cases} 2g-3-(2m-1)\gamma + \frac{3e}{m} - 4l + 2m & \text{if } h^1(\Gamma, \mathcal{O}_\Gamma(1)) > 0, \\ 2g-5+2m-2(m-2)\gamma & \text{if } h^1(\Gamma, \mathcal{O}_\Gamma(1)) = 0. \end{cases}$$

Proof. Consider the incidence variety:

$$T = \Big\{ \Big((C,L),(\widetilde{\Gamma},M)\Big) \in \mathcal{W} \times \mathcal{W}^l_{e/m,\gamma} \;\big|\; (C,L) \in \mathcal{W}, \Gamma = \varphi_L(C), M = \iota^*\mathcal{O}_\Gamma(1),$$
$$\text{where } \iota\colon \widetilde{\Gamma} \to \Gamma \text{ is the normalization of } \Gamma \Big\}.$$

Let p_1 and p_2 be the projections from T to the first factor and the second one respectively.

$$\begin{tikzcd} & T \subset \mathcal{W} \times \mathcal{W}^l_{e/m,\gamma} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{W} & & \mathcal{W}^l_{e/m,\gamma} \end{tikzcd}$$

From this we get $\dim \mathcal{W} \leq \dim p_2(T) + \dim p_2^{-1}(\cdot)$, where $\dim p_2^{-1}(\cdot)$ is the dimension of a fiber of p_2 over a point $(\widetilde{\Gamma}, M) \in p_2(T)$. In addition, $\dim p_2^{-1}(\widetilde{\Gamma}, M)$ is at most the dimension of the family of m-fold covers of $\widetilde{\Gamma}$ since $\Big((C,L),(\widetilde{\Gamma},M)\Big)$ is an element of T only if there is an m-fold covering morphism $\varphi\colon C \to \widetilde{\Gamma}$ and $L = \varphi^* M$. Thus we have

$$\dim p_2^{-1}(\widetilde{\Gamma}, M) \leq 2g-2-2m(\gamma-1).$$

If $h^1(\Gamma, \mathcal{O}_\Gamma(1)) > 0$, then we apply Proposition 2.1 to get

$$\dim \mathcal{W} \leq 2g-2-2m(\gamma-1) + 3\frac{e}{m} + \gamma - 4l - 1 = 2g-3-(2m-1)\gamma + \frac{3e}{m} - 4l + 2m.$$

If $h^1(\Gamma, \mathcal{O}_\Gamma(1)) = 0$, we use $\dim \mathcal{W}^l_{e/m,\gamma} = 3\gamma - 3 + \gamma$ to get

$$\dim \mathcal{W} \leq 2g - 2 - 2m(\gamma - 1) + 4\gamma - 3 = 2g - 5 + 2m - 2(m-2)\gamma. \qquad \square$$

The functions $\eta_m(\gamma)$ and $\xi_m(\gamma)$ defined as

$$\eta_m(\gamma) := \left(2 - \frac{2m}{r+1}\right) g - \frac{mr + 2m^2 - 3m}{r+1}\gamma + \frac{(m-2)r + 2m^2 - m - 2}{r+1},$$
$$\xi_m(\gamma) := \left(2 - \frac{4m}{r+3}\right) g - \frac{4m^2 - 2m}{r+3}\gamma + \frac{(2m-2)r + 4m^2 + 2m - 6}{r+3},$$

will play a role in identifying the ranges for d in which different types of components of $\mathcal{I}_{d,g,r}$ occur. For convenience, we denote

$$\eta_m := \eta_m(0) = \left(2 - \frac{2m}{r+1}\right) g + \frac{(m-2)r + 2m^2 - m - 2}{r+1},$$
$$\xi_m := \xi_m(0) = \left(2 - \frac{4m}{r+3}\right) g + \frac{(2m-2)r + 4m^2 + 2m - 6}{r+3}.$$

Notice that for fixed positive numbers g and r, both $\eta_m(\gamma)$ and $\xi_m(\gamma)$ are decreasing functions of γ. The following proposition shows that $\eta_m(\gamma)$ and $\xi_m(\gamma)$ give criteria to tell some moduli properties for the possible components of $\mathcal{I}_{d,g,r}$.

Proposition 3.2. *Assume that $\mathcal{I}$ is an irreducible component of $\mathcal{I}_{d,g,r}$ with $r \geq 6$ such that $h^1(C, \mathcal{O}_C(1)) = l + 1 \geq 2$ for a general $C \in \mathcal{I}$. Let*

b : *the degree of the base locus of $|\omega_C \otimes \mathcal{O}_C(-1)|$,*

m : *the degree of $\Phi \colon C \to \mathbb{P}^l$ defined by the moving part of $|\omega_C \otimes \mathcal{O}_C(-1)|$,*

γ : *the geometric genus of $\Gamma := \Phi(C)$.*

Then

(1) *If $m = 1$ and $d \geq \left(2 - \frac{2}{k-1}\right) g + k - 4$ for some integer $k \geq 3$, then*

$$d \leq \left(2 - \frac{3k}{r - 3 + 3k}\right) g + \left(\frac{2rk - r + 3}{r - 3 + 3k} - 2\right) - \left(1 - \frac{k}{r - 3 + 3k}\right) b.$$

(2.1) *If $m \geq 2$ and $h^1(\Gamma, \mathcal{O}_\Gamma(1)) > 0$, then*

(i) *$d \leq \xi_m(\gamma) - \left(1 - \frac{2m}{r+3}\right) b$, and*

(ii) *$\dim \mathcal{I} \leq \lambda_{d,g,r} - \frac{r+3}{2m}(d - \xi_m(\gamma)) - \left(\frac{r+3}{2m} - 1\right) b$.*

with equality being possible only if $l = \frac{2g-2-d-b}{2m}$.

(2.2) *If $m \geq 2$ and $h^1(\Gamma, \mathcal{O}_\Gamma(1)) = 0$, then*

(i) $m \geq 3$,

(ii) $d \leq \eta_m(\gamma) - \left(1 - \frac{m}{r+1}\right) b$, *and*

(iii) $\dim \mathcal{I} \leq \lambda_{d,g,r} - \frac{r+1}{m}(d - \eta_m(\gamma)) - \left(\frac{r+1}{m} - 1\right) b$,

with equality being possible only if $l = \frac{2g-2-d-b}{m} - \gamma$.

Proof. The irreducible component $\mathcal{I}$ of $\mathcal{I}_{d,g,r}$ is generically an $\mathrm{Aut}(\mathbb{P}^r)$-bundle over an irreducible component $\mathcal{G} \subset \mathcal{G}^r_{d,g}$, which of its turn is fibered over an irreducible closed subset $\mathcal{W} \subset \mathcal{W}^\alpha_{d,g}$ with generic fiber being the Grassmannian $\mathrm{Grass}(r+1, \alpha+1)$, where $\alpha = h^0(C, \mathcal{O}_C(1)) - 1$ for a general $C \in \mathcal{I}$. Therefore,

$$\lambda_{d,g,r} \leq \dim \mathcal{I} \leq \dim \mathcal{W} + \dim \mathrm{Grass}(r+1, \alpha+1) + \dim \mathrm{Aut}(\mathbb{P}^r).$$

Thus proving the above statements focuses on estimating $\dim \mathcal{W}$. Let $\widetilde{\mathcal{W}} \subset \mathcal{W}^l_{2g-2-d,g}$, $l := h^1(C, \mathcal{O}_C(1)) - 1 = g - d + \alpha - 1 \geq 1$, be the residual of $\mathcal{W}$ component in the relative Jacobian of the universal curve. The fact that $l \geq 1$ is explained in the proof of [10, Proposition 3.6]. Let b be the degree of the base locus of $|\omega_C \otimes \mathcal{O}_C(-1)|$ for a general $(C, \omega_C \otimes \mathcal{O}_C(-1)) \in \widetilde{\mathcal{W}}$. Removing the base locus of $|\omega_C \otimes \mathcal{O}_C(-1)|$ gives a rational mapping

$$\begin{aligned} \mathcal{W}^l_{2g-2-d,g} \times \mathcal{C}_b \supset \widetilde{\mathcal{W}} \times \mathcal{C}_b &\longrightarrow \mathcal{W}^{l-b}_{2g-2-d-b,g} \\ ((C,L),(C,p_1),\ldots,(C,p_b)) &\longmapsto (C, L(-p_1 - \cdots - p_b)), \end{aligned}$$

where $\mathcal{C}_b$ denotes the b-fold symmetric product $\mathcal{C} \times_S \cdots \times_S \mathcal{C}$ for a family $p \colon \mathcal{C} \to S$. Hence there is a component $\mathcal{U} \subset \mathcal{W}^l_{2g-2-d-b,g}$ such that its general element represents a base point free linear series and

$$\dim \widetilde{\mathcal{W}} \leq \dim \mathcal{U} + b.$$

Now for a general element $(C, L) \in \mathcal{U}$ the linear series $|L|$ on C defines a morphism of degree m (this follows from assumptions in the proposition, the irreducibility of $\mathcal{U}$ and the upper-semicontinuity of the degree of a morphism). In case (1), Proposition 2.1 gives

$$\dim \mathcal{U} \leq 3(2g - 2 - d - b) + g - 4l - 1,$$

therefore

$$\dim \mathcal{I} \leq 3(2g-2-d-b) + g - 4l - 1 + b + (r+1)(\alpha - r) + (r+1)^2 - 1.$$

The condition $d \geq (2 - \frac{2}{k-1})g + k - 4$ is equivalent to $2g - 2 - d \leq \frac{2g}{k-1} - k + 2$. Applying Proposition 2.2 to a birationally very ample $g^l_{2g-2-d-b}$, we deduce that $l \leq (2g - 2 - d -$

$b+k-1)/k$. Since $\alpha = l+d-g+1$ it follows that

$$\begin{aligned}
\dim \mathcal{I} &\leq 3(2g-2-d-b)+g-4l-1+b+(r+1)(l+d-g+2)-1 \\
&= (r+1)d-(r+1)(g-1)+rl+r+l+3(2g-2-d-b)+g-4l-1+b \\
&= (r+1)d-(r-3)(g-1)-3(g-1)+3(2g-2-d-b)+(r-3)l+b+r \\
&\leq \lambda_{d,g,r}-3(g-1)+3(2g-2-d-b)+(r-3)\frac{2g-2-d-b+k-1}{k}+b+r \\
&= \lambda_{d,g,r}-\frac{r-3+3k}{k}d+\frac{2(r-3+3k)-3k}{k}g-\frac{r-3+2k}{k}b \\
&\quad +\frac{-2(r-3+3k)+2rk-r+3}{k}.
\end{aligned}$$

In order for $\mathcal{I}$ to be a component of $\mathcal{I}_{d,g,r}$ we must have

$$-(r-3+3k)d+(2(r-3+3k)-3k)g-(r-3+2k)b-2(r-3+3k)+2rk-r+3 \geq 0,$$

which gives precisely the upper bound for d in part (1).

In case (2.1) an upper bound for $\dim \mathcal{I}$ is obtained from Lemma 3.1, the Riemann-Roch theorem and Clifford inequality $l \leq (2g-2-d-b)/(2m)$ applied to the linear series on the curve $\Gamma = \Phi(C)$, namely

$$\begin{aligned}
\dim \mathcal{I} &\leq \left[\frac{3(2g-2-d-b)}{m}-4l+2g-2-(2m-1)(\gamma-1)+b\right] \\
&\quad +(r+1)(\alpha-r)+(r+1)^2-1 \\
&= \left[\frac{3(2g-2-d-b)}{m}-4l+2g-2-(2m-1)(\gamma-1)+b\right] \\
&\quad +(r+1)d-(r-3)(g-1)-4(g-1)+(r+1)l+r \\
&= \lambda_{d,g,r}+\frac{3(2g-2-d-b)}{m}+(r-3)l-2g+2+r-(2m-1)\gamma+2m-1+b \\
&\leq \lambda_{d,g,r}+\frac{(r+3)(2g-2-d-b)}{2m}-2g-(2m-1)\gamma+(2m+1+r)+b \\
&= \lambda_{d,g,r}-\frac{r+3}{2m}d+\left(\frac{r+3}{m}-2\right)g-(2m-1)\gamma \\
&\quad +\left(2m+1+r-\frac{r+3}{m}\right)-\left(\frac{r+3}{2m}-1\right)b.
\end{aligned}$$

In case (2.2), the very ampleness of $|\mathcal{O}_C(1)|$ yields $m \geq 3$. Further, an upper bound for $\dim \mathcal{I}$ is obtained as follows. Since the line bundles $\omega_C \otimes \mathcal{O}_C(-1)$ are pull-backs of non-special line bundles on a base curve of genus γ, we have $\dim \mathcal{U} \leq 2g-5-(2m-4)\gamma+2m$

by Lemma 3.1. Hence

$$
\begin{aligned}
\dim \mathcal{I} &\le [2g-5-(2m-4)\gamma+2m+b]+(r+1)(\alpha-r)+(r+1)^2-1\\
&= [2g-5-(2m-4)\gamma+2m+b]+(r+1)d-(r-3)(g-1)\\
&\quad -4(g-1)+(r+1)l+r\\
&= \lambda_{d,g,r}-2g-1-(2m-4)\gamma+(r+1)\left(\frac{2g-2-d-b}{m}-\gamma\right)+2m+r+b\\
&= \lambda_{d,g,r}-\frac{r+1}{m}d+\left(\frac{2r+2}{m}-2\right)g-(r+2m-3)\gamma\\
&\quad +\frac{2m^2+rm-m-2r-2}{m}-\left(\frac{r+1}{m}-1\right)b.
\end{aligned}
$$

Since $\dim \mathcal{I} \ge \lambda_{d,g,r}$, we conclude immediately the upper bounds for d in both part (2.1) and part (2.2) of the proposition. Therefore the proof is completed. $\square$

4. Irreducible components using families of double coverings

Consider the subset $\mathcal{M}_g(\gamma,m) \subset \mathcal{M}_g$, $g \ge 3$, of points corresponding to curves admitting a rational map of degree $m \ge 2$ to a curve of genus γ. In [11] it was shown that $\mathcal{M}_g(\gamma,m)$ is constructible of dimension $2g-2+(2m-3)(1-\gamma)$. Let $\Sigma_g(\gamma,m) \subset \mathcal{M}_g(\gamma,m)$ be an irreducible component of maximal dimension dominating $\mathcal{M}_\gamma$, i.e., $\dim \Sigma_g(\gamma,m) = 2g-2+(2m-3)(1-\gamma)$. In this section, we show the existence of an irreducible component of $\mathcal{I}_{d,g,r}$, which is different from the distinguished component and projects onto $\Sigma_g(\gamma,2)$. In particular, the curves that it parametrizes are embedded in $\mathbb{P}^r$ by the residual to the pull-back of the canonical line bundle of the base curve. For this purpose we prove the following two lemmas.

The first two statements in this section are about the very ampleness of specific line bundles arising when dealing with double coverings of curves. In the first lemma we prove that the line bundle associated to the ramification divisor on the covering curve is very ample if its genus is sufficiently bigger than the genus of the base.

Lemma 4.1. *Let $\varphi\colon X \to Y$ be a double cover of a smooth curve Y of genus $g_Y \ge 2$ by a smooth curve X of genus g_X such that $g_X > 6g_Y - 2$. Then*

(a) $h^0(X,\varphi^*\omega_Y) = h^0(Y,\omega_Y) = g_Y$,

(b) *the line bundle $\omega_X \otimes (\varphi^*\omega_Y)^\vee$ is very ample and determines a very ample linear series $g^{g_X-3g_Y+2}_{2g_X+2-4g_Y} = |\omega_X \otimes (\varphi^*\omega_Y)^\vee|$ on X.*

Proof. In its essence, the proof consists of repeated applications of the Castelnuovo-Severi inequality and the Castelnuovo genus bound.

Regarding (a): Obviously, $h^0(X, \varphi^*\omega_Y) \geq h^0(Y, \omega_Y) = g_Y$. Suppose that $h^0(X, \varphi^*\omega_Y) > g_Y$. Then there exist independent sections $s_1, s_2 \in H^0(X, \varphi^*\omega_Y)$, which are not pull-backs of sections in $H^0(X, \omega_Y)$. Let ψ be the morphism defined by the base point free part of the pencil spanned by s_1 and s_2. By the choices of s_1 and s_2, the morphism ψ can not factor through φ. This implies that there is no proper factorization involving the morphisms $\psi\colon X \to \psi(X) \cong \mathbb{P}^1$ and $\varphi\colon X \to Y$ since $\deg\varphi = 2$. Therefore by the Castelnuovo-Severi inequality and $\deg\psi \leq 4g_Y - 4$, we have

$$g \leq (\deg\psi_s - 1)(2-1) + 2g_Y \leq 4g_Y - 5 + 2g_Y = 6g_Y - 5,$$

which is a contradiction with $g \geq 6g_Y - 4$.

Regarding (b): Let $L := \omega_X \otimes (\varphi^*\omega_Y)^\vee$, which is the line bundle on X defined by the ramification divisor of the covering. First we check that for an arbitrary point $p \in X$ we have $h^0(X, L(-p)) = h^0(X, L) - 1$. Notice that by Riemann-Roch theorem this is equivalent to $h^0(X, \varphi^*\omega_Y(p)) = g_Y$. Suppose that to the contrary, there exists $p \in X$ such that $h^0(X, \varphi^*\omega_Y(p)) = g_Y + 1$. Then the linear series $\mathcal{D} := |\varphi^*\omega_Y(p)|$ is base point free and so defines a morphism

$$\Psi_{\mathcal{D}}\colon X \to \Gamma \subset \mathbb{P}^{g_Y},$$

where $\Gamma = \Psi_{\mathcal{D}}(X)$. Let $\delta := \deg\mathcal{D} = 4g_Y - 3$. If $\Psi_{\mathcal{D}}$ is a birational morphism, we find by Castelnuovo's genus bound that

$$g_X \leq p_a(\Gamma) \leq \binom{4}{2}(g_Y - 1) = 6g_Y - 6,$$

which is indeed a contradiction. Hence $n := \deg\Psi_{\mathcal{D}} \geq 2$ and there exists a linear series $g^{g_Y}_{\delta/n}$ on the desingularization $\widetilde{\Gamma}$ of Γ induced by $\mathcal{D}$. Clifford's theorem implies that $g^{g_Y}_{\delta/n}$ must be nonspecial, and also that $n = 2$ or $n = 3$. By the Riemann-Roch theorem we get

$$g(\widetilde{\Gamma}) \leq \frac{\delta}{n} - g_Y \leq \frac{4g_Y - 3}{n} - g_Y.$$

Since the linear series $|\varphi^*\omega_Y(p)|$ is base point free and $\deg\varphi^*\omega_Y(p) = 4g_Y - 3$ is odd, the morphism $\Psi_{\mathcal{D}}$ can not factor through the degree 2 morphism φ. This implies that $\Psi_{\mathcal{D}}$ and φ do not admit a proper factorization. Thus we deduce by Castelnuovo-Severi inequality

$$g_X \leq 2g_Y + n\left(\frac{4-n}{n}g_Y - \frac{3}{n}\right) + n - 1 = (6-n)g_Y + n - 4 \leq 4g_Y - 2,$$

which is impossible since $g_Y \geq 2$ and $g_X > 6g_Y - 2$. Therefore we have $h^0(X, \varphi^*\omega_Y(p)) = g_Y$, equivalently, $h^0(X, L(-p)) = h^0(X, L) - 1$ for any $p \in X$.

Next we show that for arbitrary point $p, q \in X$ we have $h^0(X, L(-p-q)) = h^0(X, L) - 2$. Notice that this is equivalent to show that $h^0(X, \varphi^*\omega_Y(p+q)) = h^0(X, \varphi^*\omega_Y) = g_Y$. Assume that this is not the case. Then there must exist points $p, q \in X$ such that

$h^0(X, \varphi^*\omega_Y(p+q)) = g_Y + 1$. Then the equality $h^0(X, \varphi^*\omega_Y(p)) = g_Y$ implies that $\mathcal{D} := |\varphi^*\omega_Y(p+q)|$ is base point free and defines a morphism $\Psi_{\mathcal{D}}\colon X \to \Psi_{\mathcal{D}}(X) =: \Gamma$. If $\Psi_{\mathcal{D}}$ factors through φ, then there is a line bundle M of degree $2g_Y - 1$ on Y such that $\varphi^*\omega_Y(p+q) \simeq \varphi^*M$ and $h^0(X, \varphi^*\omega_Y(p+q)) = h^0(Y, M)$. This cannot occur since $h^0(X, \varphi^*\omega_Y(p+q)) = g_Y + 1$ and $h^0(Y, M) = g_Y$. Therefore $\Psi_{\mathcal{D}}$ does not factor through φ. Just as in the previous paragraph, if $n := \deg \Psi_{\mathcal{D}} = 1$, Castelnuovo's genus bound would yield

$$g_X \leq p_a(\Gamma) \leq \binom{4}{2}(g_Y - 1) + 4,$$

which is impossible due to $g_X > 6g_Y - 2$. Therefore the morphism $\Psi_{\mathcal{D}}$ is of degree $n \geq 2$ and thus there exists a linear series $g^{g_Y}_{(4g_Y-2)/n}$ on the desingularization $\widetilde{\Gamma}$ of Γ. By Clifford's theorem we must have that $g^{g_Y}_{(4g_Y-2)/n}$ is nonspecial, and also $n = 2$ or $n = 3$. Since $\Psi_{\mathcal{D}}$ and φ do not admit a proper factorization, we conclude by Castelnuovo-Severi inequality that

$$g_X \leq 2g_Y + n\left(\frac{4-n}{n}g_Y - \frac{2}{n}\right) + n - 1 = (6-n)g_Y + n - 3 < 4g_Y,$$

which can not occur for $g_X > 6g_Y - 2$ with $g_Y \geq 2$. This completes the proof. □

The next lemma gives one more very ample line bundle that is related with the ramification divisor for the double coverings.

Lemma 4.2. *Let $\varphi\colon X \to Y$ be a double cover of a smooth curve Y of genus $g_Y \geq 2$ by a smooth curve X of genus g_X such that $g_X > 6g_Y - 1$. Let $p_0 \in X$ be arbitrary point. Then*

(a) $h^0(X, \varphi^*\omega_Y(p_0)) = h^0(Y, \omega_Y) = g_Y$,

(b) *the line bundle $\omega_X \otimes (\varphi^*\omega_Y)^\vee(-p_0)$ is very ample and determines a very ample linear series $g^{g_X-3g_Y+1}_{2g_X+1-4g_Y}$ $(= |\omega_X \otimes (\varphi^*\omega_Y)^\vee(-p_0)|)$ on X.*

Proof. The statement of (a) was in fact established in the proof of Lemma 4.1(b).

Regarding (b): Denote $L := \omega_X \otimes (\varphi^*\omega_Y)^\vee$ as before. It suffices to show that for arbitrary point $p_1, p_2 \in X$ we have $h^0(X, L(-p_0-p_1-p_2)) = h^0(X, L(-p_0)) - 2 = h^0(X, L) - 3$. Notice that this is equivalent to show that $h^0(X, \varphi^*\omega_Y(p_0+p_1+p_2)) = h^0(X, \varphi^*\omega_Y) = g_Y$. Remark that by the proof of Lemma 4.1(b) we have $h^0(X, \varphi^*\omega_Y(p_0+p_1)) = h^0(X, \varphi^*\omega_Y) = g_Y$ for arbitrary $p_0, p_1 \in X$ and that the base points of $|\varphi^*\omega_Y(p_0+p_1)|$ are precisely p_0 and p_1. Assume now that $h^0(X, \varphi^*\omega_Y(p_0+p_1+p_2)) = g_Y + 1$. Then the complete linear series $\mathcal{D} := |\varphi^*\omega_Y(p_0+p_1+p_2)|$ must be base point free, and since $\deg \mathcal{D} = 4g_Y - 1$ the morphism $\psi_{\mathcal{D}}$ that it determines can't factor through φ. Then by Castelnuovo-Severi inequality

$$g_X \leq (4g_Y - 2)(2-1) + 2g_Y = 6g_Y - 2,$$

which is impossible in view of the condition $g_X \geq 6g_Y - 1$ in the lemma. □

Using the lemmas above, we construct in the next two theorems a non-distinguished component of $\mathcal{I}_{d,g,r}$ over $\Sigma_g(\gamma,2)$ for an odd γ and d within a certain range. We remark that the reducibility of $\mathcal{I}_{d,g,r}$ for d in this range has already been confirmed in [10].

Theorem 4.3. *Assume that g, d and $r \geq 21$ are positive integers such that $\rho(d,g,r) \geq 0$, $2g-2-d \equiv 0 \pmod 4$ and*

$$\eta_5 < d \leq \min\left\{\left(2-\frac{8}{r}\right)g + \left(2+\frac{8}{r}\right), 2g-28\right\}.$$

If $\gamma := (2g+2-d)/4$ is odd, then $\mathcal{I}_{d,g,r}$ possesses in addition to its distinguished component an irreducible component $\mathcal{D}_{d,g,r} \subset \mathcal{I}_{d,g,r}$ whose general elements $C \hookrightarrow \mathbb{P}^r$ are such that

(i) $[C] \in \Sigma_g(\gamma,2)$ *is a double cover* $\Phi\colon C \to \Gamma$ *of a general* $[\Gamma] \in \mathcal{M}_\gamma$,

(ii) *the embedding* $C \hookrightarrow \mathbb{P}^r$ *is given by a general series* $g^r_d \subset \left|\omega_C \otimes \Phi^*(\omega_\Gamma)^{-1}\right|$.

Furthermore,

$$\begin{aligned}\dim \mathcal{D}_{d,g,r} &= \lambda_{d,g,r} + \frac{r}{4}\left[\left(2-\frac{8}{r}\right)g + 2 + \frac{8}{r} - d\right]\\ &= \lambda_{d,g,r} + \frac{r+3}{4}(\xi_2(\gamma) - d),\end{aligned}$$

i.e., $\dim \mathcal{D}_{d,g,r}$ *attains the upper bound in Proposition* 3.2(2.1) *with* $b = 0$.

Proof. Remark first that the condition $\rho(d,g,r) \geq 0$ guarantees the existence of the distinguished component of $\mathcal{I}_{d,g,r}$. Since $d = 2g+2-4\gamma > \eta_5 \geq \frac{4}{3}g + 3$, we have $g \geq 6\gamma - 1$. Therefore for any double cover $\Phi\colon C \to \Gamma$ of a smooth genus γ curve, the line bundle $\omega_C \otimes \Phi^*(\omega_\Gamma)^{-1}$ is very ample and $h^0(C, \omega_C \otimes \Phi^*(\omega_\Gamma)^{-1}) = g - 3\gamma + 3$ according to Lemma 4.1. Let $\mathcal{D}_{d,g,r}$ be the closure of the irreducible family of curves $C \hookrightarrow \mathbb{P}^r$, constructed over the irreducible subset $\Sigma_g(\gamma,2)$ and embedded by a general linear series $g^r_d \subset \left|\omega_C \otimes \Phi^*(\omega_\Gamma)^{-1}\right|$. Then

$$\begin{aligned}\dim \mathcal{D}_{d,g,r} &= \dim \Sigma_g(\gamma,2) + \dim \operatorname{Grass}(r+1, g-3\gamma+3) + \dim \operatorname{Aut}(\mathbb{P}^r)\\ &= 2g-1-\gamma + (r+1)(g-3\gamma+3) - 1\\ &= (r+3)g - (3r+4)\gamma + 3r + 1.\end{aligned}$$

Using $d = 2g+2-4\gamma$, we obtain

$$\begin{aligned}\dim \mathcal{D}_{d,g,r} - \lambda_{d,g,r} &= (r+3)g - (3r+4)\gamma + 3r + 1 - (r+1)d + (r-3)(g-1)\\ &= -2g + r\gamma + 2\\ &= \frac{r}{4}\left[\left(2-\frac{8}{r}\right)g + 2 + \frac{8}{r} - d\right],\end{aligned}$$

which is equivalent to

$$(*)\qquad \dim \mathcal{D}_{d,g,r} = \lambda_{d,g,r} + \frac{r+3}{4}(\xi_2(\gamma) - d).$$

It remains to show that $\mathcal{D}_{d,g,r}$ is not properly contained in other components of $\mathcal{I}_{d,g,r}$. For this suppose the opposite, i.e., there is an irreducible component $\mathcal{E} \subset \mathcal{I}_{d,g,r}$ such that $\mathcal{D}_{d,g,r} \subsetneq \mathcal{E}$. Let ϕ be the morphism defined by the moving part g^s_e of $|\omega_E \otimes \mathcal{O}_E(-1)|$ for a general element E of $\mathcal{E}$. Remark that we have $s \geq 1$ since in the Severi's range the line bundles embedding the curves of the additional components of Hilbert scheme have speciality at least 2, see [10] or [7, Theorem C]. Let

$$n := \deg \phi.$$

Notice that $n \geq 2$, because if $n = 1$ then by Proposition 3.2(1), in which we could consider $k = 5$ as $d \geq \frac{3}{2}g + 1$, it would follow that $d \leq (2 - \frac{15}{r+12})g + (\frac{9r+3}{r+12} - 2)$. This is impossible since $d > \eta_5 = (2 - \frac{10}{r+1})g + \frac{3r+43}{r+1}$ and the condition $r \geq 21$. Thus $\phi\colon E \to \phi(E) =: T \subset \mathbb{P}^s$ is a multiple covering of an integral curve T. By Proposition 3.2, if $h^1(T, \mathcal{O}_T(1)) > 0$ then $n = 2$ since $\xi_3 < \eta_5$, and if $h^1(T, \mathcal{O}_T(1)) = 0$ then $n = 3$ or $n = 4$ because $\eta_5 < d$. We will show that none of these cases is possible.

Suppose first that $n = 2$. Then for the general element $E \hookrightarrow \mathbb{P}^r$ of $\mathcal{E}$ we get that E is a double cover of a curve of fixed genus, say τ. By the equality $(*)$ and Proposition 3.2(2.1) it follows that

$$\dim \mathcal{D}_{d,g,r} < \dim \mathcal{E} \leq \dim \mathcal{D}_{d,g,r} + \frac{r+3}{4}(\xi_2(\tau) - \xi_2(\gamma)).$$

Since $\xi_m(t)$ is a decreasing function in t, it follows that $\tau < \gamma$. Consider the natural projection map $\mu\colon \mathcal{E} \to \overline{\mathcal{M}_g}$. Then a general element of $\mathcal{E}$ projects to a general element of $\mu(\mathcal{E})$ and also $\mu(\mathcal{D}_{d,g,r}) \subset \mu(\mathcal{E})$. Therefore for a general $[E] \in \mu(\mathcal{E})$ the inequality $\tau < \gamma$ gives

$$\operatorname{gon}(E) \leq 2\left[\frac{\tau+3}{2}\right] < 2\left[\frac{\gamma+3}{2}\right] = \gamma + 3$$

since γ is odd. Here, the gonality $\operatorname{gon}(C)$ of a smooth curve C is defined by

$$\operatorname{gon}(C) := \min\left\{n \mid \text{there is a surjective morphism to } \mathbb{P}^1 \text{ of degree } n\right\}.$$

Using the Castelnuovo-Severi inequality it is easy to see that for a general $[C] \in \mu(\mathcal{D}_{d,g,r}) \equiv \Sigma_g(\gamma, 2)$ we have $\operatorname{gon}(C) = \gamma + 3$, and thus $\operatorname{gon}(E) < \operatorname{gon}(C)$. But this contradicts to the lower semi-continuity property of the gonality of a curve.

It remains to consider the case when $h^1(T, \mathcal{O}_T(1)) = 0$ and $3 \leq n \leq 4$. If a general element $E \hookrightarrow \mathbb{P}^r$ of $\mathcal{E}$ is a n-sheeted cover of a curve of genus τ, we obtain from $\mu(\mathcal{D}_{d,g,r}) \subset \mu(\mathcal{E})$ that

$$2g - 1 - \gamma = \dim \mu(\mathcal{D}_{d,g,r}) \leq \dim \mu(\mathcal{E}) \leq 2g - 2 - (2n-3)(\tau - 1).$$

On the other hand, the lower semi-continuity of gonality gives that $\gamma + 3 \leq n(\tau + 3)/2$. Combining these two inequalities, we obtain $(2n-3)(\tau-1)+1 \leq \gamma \leq n(\tau+3)/2 - 3$, which implies

$$\tau \leq \frac{7}{3},$$

hence $\tau \leq 2$. As $3 \leq n \leq 4$, this would imply $\gamma \leq 2(\tau+3) - 3 \leq 7$, which is impossible due to the assumption $d = 2g + 2 - 4\gamma \leq 2g - 27$. This completes the proof. $\square$

The next theorem is obtained in a similar way.

Theorem 4.4. *Assume that g, d and $r \geq 21$ are positive integers such that $\rho(d,g,r) \geq 0$, $2g - 2 - d \equiv 1 \pmod 4$ and*

$$\eta_5 < d \leq \min\left\{ \left(2 - \frac{8}{r}\right) g + \left(1 + \frac{12}{r}\right), 2g - 28 \right\}.$$

If $\gamma := (2g+1-d)/4$ is odd, then $\mathcal{I}_{d,g,r}$ possesses in addition to its distinguished component an irreducible component $\mathcal{D}'_{d,g,r} \subset \mathcal{I}_{d,g,r}$ whose general elements $C \hookrightarrow \mathbb{P}^r$ are such that

(i) *$[C] \in \Sigma_g(\gamma, 2)$ is double cover $\Phi\colon C \to \Gamma$ of a general $[\Gamma] \in \mathcal{M}_\gamma$,*

(ii) *the embedding $C \hookrightarrow \mathbb{P}^r$ is given by a general linear series $g^r_d \subset \left|\omega_C \otimes \Phi^*(\omega_\Gamma)^{-1}(-p)\right|$, where $p \in C$ is an arbitrary point.*

Further,

$$\begin{aligned} \dim \mathcal{D}'_{d,g,r} &= \lambda_{d,g,r} + \frac{r}{4}\left[\left(2 - \frac{8}{r}\right) g + 1 + \frac{12}{r} - d\right] \\ &= \lambda_{d,g,r} + \frac{r+3}{4}(\xi_2(\gamma) - d) - \left(\frac{r+3}{4} - 1\right), \end{aligned}$$

i.e., $\dim \mathcal{D}'_{d,g,r}$ attains the upper bound in Proposition 3.2(2.1) with $b = 1$.

Proof. The proof goes exactly the same way as the proof of Theorem 4.3. The inequality $\eta_5 < d = 2g + 1 - 4\gamma$ together with $r \geq 21$ implies $g > 6\gamma + 2$. This allows us to define $\mathcal{D}'_{d,g,r}$ as the closure of the irreducible family of curves $C \hookrightarrow \mathbb{P}^r$, constructed over the irreducible subset $\Sigma_g(\gamma, 2)$ of double covers $\Phi\colon C \to \Gamma$ and embedded by a general $g^r_d \subset \left|\omega_C \otimes \Phi^*(\omega_\Gamma)^{-1}(-p)\right|$ where $p \in C$. Note that $\left|\omega_C \otimes \Phi^*(\omega_\Gamma)^{-1}(-p)\right|$ is very ample due to Lemma 4.2. Since $p \in C$ can be arbitrary, we get

$$\begin{aligned} \dim \mathcal{D}'_{d,g,r} &= \dim \Sigma_g(\gamma, 2) + 1 + \dim \operatorname{Grass}(r+1, g - 3\gamma + 2) + \dim \operatorname{Aut}(\mathbb{P}^r) \\ &= (r+3)g - (3r+4)\gamma + 2r + 1. \end{aligned}$$

Using $d = 2g + 1 - 4\gamma$, we obtain

$$\begin{aligned}\dim \mathcal{D}'_{d,g,r} - \lambda_{d,g,r} &= (r+3)g - (3r+4)\gamma + 2r + 1 - (r+1)d + (r-3)(g-1) \\ &= -2g + r\gamma + 3 \\ &= \frac{r}{4}\left[\left(2 - \frac{8}{r}\right) g + 1 + \frac{12}{r} - d\right] \\ &= \frac{r+3}{4}(\xi_2(\gamma) - d) - \left(\frac{r+3}{4} - 1\right).\end{aligned}$$

The remaining part is the same as the proof of Theorem 4.3. $\square$

Theorems 4.3 and 4.4 suggest the construction of the component of $\mathcal{I}_{d,g,r}$ having exactly the expected dimension and the construction of superabundant components. We exhibit them in the next examples.

Example 4.5. Let $\gamma \geq 9$ be an odd integer.

(1) If g is an integer such that $2(g-1)$ is divisible by γ and $2(g-1)/\gamma \geq 21$, then $\mathcal{I}_{2g+2-4\gamma,g,2(g-1)/\gamma}$ possesses a non-distinguished component $\mathcal{D}_{2g+2-4\gamma,g,2(g-1)/\gamma}$ as in Theorem 4.3, of the expected dimension $\lambda_{2g+2-4\gamma,g,2(g-1)/\gamma}$.

(2) If g is an integer such that $2g-1$ is divisible by γ and $(2g-1)/\gamma \geq 21$, then $\mathcal{I}_{2g+1-4\gamma,g,(2g-1)/\gamma}$ possesses a non-distinguished component $\mathcal{D}'_{2g+1-4\gamma,g,(2g-1)/\gamma}$ as in Theorem 4.4 of the expected dimension $\lambda_{2g+1-4\gamma,g,(2g-1)/\gamma}$.

Proof. The proof amounts just to checking numerical conditions. We check briefly those related to (1). Notice that for $d = 2g + 2 - 4\gamma$ and $r = 2(g-1)/\gamma$, we have $d > g + r$, from where $\rho(2g+2-4\gamma, g, 2(g-1)/\gamma) > 0$ follows immediately. Hence $\mathcal{I}_{2g+2-4\gamma,g,2(g-1)/\gamma}$ possesses a distinguished component. Further, the right part of the inequality for d in Theorem 4.3 follows directly by $d = 2g + 2 - 4\gamma$, $r = 2(g-1)/\gamma$ and $\gamma \geq 9$, while the left inequality $\eta_5 < d = 2g + 2 - 4\gamma$ can be shown since the condition $\gamma = 2(g-1)/r$ implies $\eta_5 = \frac{2r-8}{r+1} g + \frac{3r+43}{r+1} = 2g + 2 - \frac{5r}{r+1}\gamma + \frac{r+31}{r+1}$. Finally, a direct substitution gives $\dim \mathcal{D}_{2g+2-4\gamma,g,(2g-2)/\gamma} - \lambda_{2g+2-4\gamma,g,(2g-2)/\gamma} = 0$. The check of claim (2) is similar. $\square$

Example 4.6. If we take $g = 10\gamma \geq 130$, $d = 16\gamma + 2$ and $r = 21$, it is easy to see that the numerical conditions of Theorem 4.3 are satisfied and for the family of double covers as in the theorem $\Phi\colon C \to \Gamma$, the line bundles $L = \omega_C \otimes (\Phi^*\omega_\Gamma)^{-1}$ are very ample of degree $d = 16\gamma + 2$, and $h^0(C, L) = 7\gamma + 3$. Thus we obtain an irreducible component $\mathcal{D}_{16\gamma+2,10\gamma,21}$ of $\mathcal{I}_{16\gamma+2,10\gamma,21}$ such that

$$\begin{aligned}\dim \mathcal{D}_{16\gamma+2,10\gamma,21} - \lambda_{16\gamma+2,10\gamma,21} &= \frac{r+3}{4}(\xi_2(\gamma) - d) \\ &= 6\left(\frac{50}{3}\gamma - \frac{\gamma}{2} + \frac{7}{3} - 16\gamma - 2\right) \\ &= \gamma + 2.\end{aligned}$$

At the end we remark that it seems interesting, to us at least, to answer the following two questions as they are related to understanding the geometry of $\mathcal{I}_{d,g,r}$ while tracing the decrease of d from $\frac{2r-4}{r+1}g + \frac{r+13}{r+1}$, at least in the Severi range $d \geq g + r$.

Question 4.7. (1) Are the conclusions of Theorems 4.3 and 4.4 valid without the assumption γ odd?

(2) Are the components $\mathcal{D}_{2g+2-4\gamma,g,2(g-1)/\gamma}$ and $\mathcal{D}'_{2g+1-4\gamma,g,(2g-1)/\gamma}$ obtained in Example 4.5 reduced or non-reduced, in other words, should they be classified as being *regular* or *superabundant*?

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