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A Note on co-Higgs Bundles

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Abstract. We show that for any ample line bundle on a smooth complex projective variety with nonnegative Kodaira dimension, the semistability of co-Higgs bundles of implies the semistability of bundles. Then we investigate the criterion for surface X to have $H^0(T_X) = H^0(S^2T_X) = 0$, which implies that any co-Higgs structure of rank two is nilpotent.

1. Introduction

A co-Higgs bundle on a smooth complex projective variety X is a pair $(\mathcal{E}, \Phi)$, where $\mathcal{E}$ is a vector bundle on X and Φ, the co-Higgs field, is a morphism $\mathcal{E} \to \mathcal{E} \otimes T_X$, satisfying the integrability condition $\Phi \wedge \Phi = 0$. It is introduced and developed in [6, 7] as a generalized vector bundle over X, considered as a generalized complex manifold.

There have been recent interests on the classification of stable co-Higgs bundles on lower dimensional varieties. In [13,14], Rayan describe the moduli spaces of stable co-Higgs bundles of rank two both on the projective line and on the projective plane, and show several non-existence results of them over varieties with nonnegative Kodaira dimension $\kappa(X) \geq 0$. The main philosophy is that the existence of stable co-Higgs bundles determine the position of X toward negative direction in the Kodaira spectrum. Indeed it is shown in [4] that if $\dim(X) = 2$, the existence of semistable co-Higgs bundle $(\mathcal{E}, \Phi)$ of rank two with Φ nilpotent, implies that X is either uniruled, a torus, or a properly elliptic surface, up to finite étale cover. Colmenares also describes the moduli space of semistable co-Higgs bundles of rank two on Hirzebruch surfaces in [2, 3].

Our goal in this article is twofold. We first investigate the relationship between semistability of $(\mathcal{E}, \Phi)$ and semistability of $\mathcal{E}$.

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Theorem 1.1. *With respect to any ample line bundle on a smooth projective variety with nonnegative Kodaira dimension, if a co-Higgs bundle* $(\mathcal{E}, \Phi)$ *is semistable, then* $\mathcal{E}$ *is also semistable.*

In fact, when we disregard the condition on Kodaira dimension, we also get a similar statement on stabilities under the assumption that the strict order of instability is low (see Proposition 2.8). The results give a way to study moduli of semistable co-Higgs bundles, with a base on the study of moduli of semistable bundles. Indeed, Proposition 2.11 asserts that any semistable co-Higgs bundle of rank two over a surface of general type has a trivial co-Higgs field, and so the semistability of co-Higgs bundles of rank two is equivalent to the semistability of bundles.

Then we pay our attention to the case when X is a surface and the rank of co-Higgs bundles is two. Under vanishing of $H^0(T_X)$, the existence of unstable co-Higgs bundle of rank two implies that the co-Higgs field is nilpotent (see Lemma 2.5), while any non-trivial global tangent vector field suggests an example of strictly semistable co-Higgs bundle of arbitrary rank with injective co-Higgs field (see Remark 2.6). It motivates to seek for additional conditions to assure that a semistable co-Higgs bundle has a nilpotent co-Higgs field. By the argument in [13, Theorem 7.1, pp. 148–149], the vanishing condition $H^0(T_X) = H^0(S^2T_X) = 0$ implies that co-Higgs fields are nilpotent. Indeed, any surface can achieve this vanishing after a finite number of blow-ups.

Theorem 1.2. *For a surface* X*, there exists a surface* X' *and a birational morphism* $u\colon X' \to X$ *with the following property. If* $v\colon X'' \to X'$ *is any birational morphism, then every rank two co-Higgs field on* X'' *is nilpotent.*

Since, due to the Enriques-Kodaira classification, we have a list of minimal model Y of X together with information on $H^0(T_Y)$ and $H^0(S^2T_Y)$, we are able to find several classes of surfaces with the prescribed vanishing: blow-ups of the projective plane, Hirzebruch surfaces, abelian surfaces and a type of properly elliptic surfaces.

Let us summarize here the structure of this article. In Section 2 we introduce the definition of co-Higgs bundles and a notion of semistability with respect to a fixed ample line bundle. Then we discuss the relationship between semistability of co-Higgs bundles and semistability of bundles, together with nilpotent co-Higgs fields. Main ingredients are the generic nefness of the cotangent bundle of non-uniruled X and certain extensions that nilpotent co-Higgs fields induce. In Section 3, we study the extensions above via holomorphic foliation in [15] to characterize the base variety X in Kodaira spectrum, when it admits a semistable co-Higgs bundle of rank two whose bundle factor is not semistable with non-zero co-Higgs field. In Section 4, we mainly work on the criterion for vanishing $H^0(T_X)$ and $H^0(S^2T_X)$ and it is observed that the vanishing can be achieved by blow-ups sufficiently many times.

2. Preliminaries

Throughout the article our base field is the field $\mathbb{C}$ of complex numbers. We will always assume that X is a smooth projective variety with a fixed very ample line bundle $\mathcal{O}_X(1)$ and the tangent bundle T_X. For a coherent sheaf $\mathcal{E}$ on a projective scheme X, we denote $\mathcal{E} \otimes \mathcal{O}_X(t)$ by $\mathcal{E}(t)$ for $t \in \mathbb{Z}$. The dimension of cohomology group $H^i(X, \mathcal{E})$ is denoted by $h^i(X, \mathcal{E})$ and we will skip X in the notation, if there is no confusion.

Lemma 2.1. $\Omega^1_X(2)$ *is globally generated.*

Proof. Consider X as a subvariety of a projective space $\mathbb{P}^r$ such that the embedding is given by a complete linear system $|\mathcal{O}_X(1)|$. Then we have an exact sequence

$$0 \to N^\vee_{X|\mathbb{P}^r}(2) \to \Omega^1_{\mathbb{P}^r}(2)_{|X} \to \Omega^1_X(2) \to 0,$$

where $N_{X|\mathbb{P}^r}$ is the normal bundle of X in $\mathbb{P}^r$. Note that $\Omega^1_{\mathbb{P}^r}(2)$ is 0-regular by the Bott formula. Thus it is globally generated and so is $\Omega^1_X(2)$. □

In particular, we have $h^0(T_X(-2)) = 0$. Now we give the definition of co-Higgs bundle, which is the main object of this article.

Definition 2.2. A *co-Higgs* bundle on X is a pair $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is a vector bundle on X and $\Phi \in H^0(\mathrm{End}(\mathcal{E}) \otimes T_X)$ for which $\Phi \wedge \Phi = 0$ as an element of $H^0(\mathrm{End}(\mathcal{E}) \otimes \wedge^2 T_X)$. Here Φ is called the *co-Higgs field* of $(\mathcal{E}, \Phi)$ and the condition $\Phi \wedge \Phi = 0$ is called the *integrability.*

We may constrain ourselves to the co-Higgs bundles $(\mathcal{E}, \Phi)$ with trace zero, i.e., Φ is contained in $H^0(\mathrm{End}_0(\mathcal{E}) \otimes T_X)$, because each co-Higgs field Φ can be decomposed as (ϕ_1, ϕ_2), where $\phi_1 \in H^0(\mathrm{End}_0(\mathcal{E}) \otimes T_X)$ and $\phi_2 \in H^0(T_X)$ due to the splitness of the trace map sequence: $0 \to \mathrm{End}_0(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \stackrel{\mathrm{tr}}{\to} \mathcal{O}_X \to 0$. In particular, if $H^0(T_X) = 0$, then every co-Higgs field has trace-zero.

Definition 2.3. A co-Higgs bundle $(\mathcal{E}, \Phi)$ is *semistable* (resp. *stable*) if

$$\frac{\deg \mathcal{F}}{\mathrm{rank}\, \mathcal{F}} \leq \text{(resp. } <\text{)}\ \frac{\deg \mathcal{E}}{\mathrm{rank}\, \mathcal{E}}$$

for every coherent subsheaf $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ with $\Phi(\mathcal{F}) \subset \mathcal{F} \otimes T_X$.

The following observation shows why to consider (non-)existence results for co-Higgs bundles $(\mathcal{E}, \Phi)$ with $\Phi \neq 0$ one usually assume that $(\mathcal{E}, \Phi)$ is semistable.

Remark 2.4. Take a positive integer k such that $T_X(k)$ is globally generated and choose a non-zero section $\sigma \in H^0(T_X(k))$. For an integer $r \geq 2$, set $\mathcal{E} := \mathcal{O}_X \oplus \mathcal{O}_X(k)^{\oplus(r-1)}$. If we define a map $\Phi \colon \mathcal{E} \to \mathcal{E} \otimes T_X$ to send the factor $\mathcal{O}_X$ to one of the factor, $T_X(k)$, of $\mathcal{E} \otimes T_X$ using σ and send $\mathcal{O}_X(k)^{\oplus(r-1)}$ onto 0. By construction we have $\Phi \circ \Phi = 0$ and so $\Phi \wedge \Phi = 0$.

Lemma 2.5. *Assume that $H^0(T_X) = 0$. If a co-Higgs bundle $(\mathcal{E}, \Phi)$ of rank two on X is not stable, then Φ is nilpotent.*

Proof. We may assume $\Phi \neq 0$. Since $(\mathcal{E}, \Phi)$ is not stable, there is a line subbundle $\mathcal{L} \subset \mathcal{E}$ such that $\Phi(\mathcal{L}) \subset \mathcal{L} \otimes T_X$ and $\deg \mathcal{L} \geq \deg \mathcal{R}/2$, where $\mathcal{R} := \det(\mathcal{E})$. The saturation $\mathcal{L}'$ of $\mathcal{L}$ in $\mathcal{E}$ satisfies $\Phi(\mathcal{L}') \subset \mathcal{L}' \otimes T_X$, because $\mathcal{L}' \otimes T_X$ is the saturation of $\mathcal{L} \otimes T_X$ in $\mathcal{E} \otimes T_X$. Thus we may assume that $\mathcal{L}$ is saturated in $\mathcal{E}$ with an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{I}_Z \otimes \mathcal{R} \otimes \mathcal{L}^\vee \to 0. \tag{2.1}$$

Due to vanishing $H^0(T_X) = 0$, the inclusion $\Phi(\mathcal{L}) \subset \mathcal{L} \otimes T_X$ implies $\Phi(\mathcal{L}) = 0$. So Φ induces a map $u \colon \mathcal{I}_Z \otimes \mathcal{R} \otimes \mathcal{L}^\vee \to \mathcal{E} \otimes T_X$. Composing u with the map $\mathcal{E} \otimes T_X \to \mathcal{I}_Z \otimes \mathcal{R} \otimes T_X$ induced by (2.1) we get a map $v \colon \mathcal{I}_Z \otimes \mathcal{R} \otimes \mathcal{L}^\vee \to \mathcal{I}_Z \otimes \mathcal{R} \otimes \mathcal{L}^\vee \otimes T_X$. Again from $H^0(T_X) = 0$ we get $v = 0$, i.e., $\mathrm{Im}(\Phi) \subset \mathcal{L} \otimes T_X$ and so $\Phi^2 = 0$. □

Remark 2.6. Assume $H^0(T_X) \neq 0$ with a fixed non-zero section $\sigma \in H^0(T_X)$. Fix an open subset $U \subset X$, where T_X is trivial and let $\{\partial_1, \partial_2\}$ be a basis of $T_U \cong \mathcal{O}_U^{\oplus 2}$. If $\Phi \colon \mathcal{L} \to \mathcal{L} \otimes T_X$ is the map induced by σ for a line bundle $\mathcal{L}$, then we may write $\Phi_{|U} = f_1(z)\partial_1 + f_2(z)\partial_2$ with $f_i(z) \in H^0(\mathcal{O}_U)$. We have $\Phi \wedge \Phi = (f_1 f_2 - f_2 f_1)\partial_1 \wedge \partial_2 = 0$ (see [2, Remark 2.29] and [3]) and so $(\mathcal{L}, \Phi)$ is a co-Higgs bundle. For a fixed integer $r \geq 1$, let $(\mathcal{E}, \Phi)$ denote the co-Higgs bundle which is the direct sum of r copies of $(\mathcal{L}^{\oplus r}, \sigma)$. Then Φ is injective and so it is not nilpotent. Notice that $(\mathcal{E}, \Phi)$ is strictly semistable for any polarization on X.

Remark 2.7. Let $\mathcal{E}$ be a vector bundle of rank two on X, which is not semistable with respect to $\mathcal{O}_X(1)$ with the following destabilizing sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{I}_Z \otimes \mathcal{A} \to 0. \tag{2.2}$$

Assume the existence of a co-Higgs field Φ on X such that $(\mathcal{E}, \Phi)$ is semistable. Clearly we get $\Phi \neq 0$. Since $h^0(T_X(-2)) = 0$ by Lemma 2.1, we have $h^0(\mathcal{I}_Z \otimes \mathcal{A} \otimes (\mathcal{L}^\vee)^{\otimes 2}(2)) > 0$. Later in Proposition 3.3 we see that $(\mathcal{E}, \Phi)$ is associated to a foliation by rational curves.

For a vector bundle $\mathcal{E}$ of rank two on X, let us choose $\mathcal{R} \in \mathrm{Pic}(X)$ so that $\mathcal{F} := \mathcal{E} \otimes \mathcal{R}$ has a non-zero section with a 2-codimensional zero locus Z and an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{I}_Z \otimes \mathcal{L} \to 0 \tag{2.3}$$

with $\mathcal{L} := \det(\mathcal{F}) = \det(\mathcal{E}) \otimes \mathcal{R}^{\otimes 2} \in \mathrm{Pic}(X)$. Note that there is an obvious bijection between the co-Higgs fields (and the co-Higgs integrable structures) of $\mathcal{E}$ and $\mathcal{F}$. The *strict order of instability* of $\mathcal{E}$ is the maximal integer k such that $h^0(\mathcal{L}(-k)) > 0$ for all possible sequences (2.3) and $\mathcal{R}$, denoted by $\mathrm{ord}^{\mathrm{in}}(\mathcal{E})$. In particular, the strict order of instability is invariant under the twist by line bundles.

Proposition 2.8. *Let $\mathcal{E}$ be an unstable bundle of rank two on X with* $\mathrm{ord}^{\mathrm{in}}(\mathcal{E}) \leq -3$. *Then $(\mathcal{E}, \Phi)$ is not stable for any co-Higgs field Φ.*

Proof. Take a maximal destabilizing line subbundle $\mathcal{R}$ of $\mathcal{E}$ and consider (2.3). It is enough to show that $\Phi(\mathcal{R}^\vee) \subset \mathcal{R}^\vee \otimes T_X$ for every non-zero co-Higgs field $\Phi\colon \mathcal{E} \to \mathcal{E} \otimes T_X$. Then $\mathcal{R}^\vee$ would destabilize the co-Higgs bundle $(\mathcal{E}, \Phi)$. Note that $\mathrm{ord}^{\mathrm{in}}(\mathcal{E}) \leq -3$ implies that $H^0(\mathcal{L}(2)) = 0$. With notations above, let $\Phi'\colon \mathcal{F} \to \mathcal{F} \otimes T_X$ be the non-zero map induced by Φ. Then we need to prove that $\Phi'(\mathcal{O}_X) \subset \mathcal{O}_X \otimes T_X$, i.e., that the induced map $\mathcal{O}_X \to \mathcal{I}_Z \otimes \mathcal{L} \otimes T_X$ is a zero map. Since $H^0(\mathcal{L}(2)) = 0$, so we have $H^0(\mathcal{I}_Z \otimes \mathcal{L}(2)) = 0$. Since $\Omega^1_X(2)$ is globally generated by Lemma 2.1, we have $H^0(\mathcal{I}_Z \otimes \mathcal{L} \otimes T_X) = 0$. □

Proposition 2.8 is sharp, as shown in [13, Remarks 3.1, 5.1 and Proposition 5.1] and [14, Proposition 5.1]. When $\kappa(X)$ is non-negative, Proposition 2.8 may be improved in the following way, applying Definition 2.3 with $\mathcal{H}$ instead of $\mathcal{O}_X(1)$.

Theorem 2.9. *Let X be a smooth projective variety with $\kappa(X) \geq 0$. For any ample line bundle $\mathcal{H}$ on X, if $\mathcal{E}$ is a torsion-free sheaf which is not $\mathcal{H}$-semistable, then no co-Higgs sheaf $(\mathcal{E}, \Phi)$ is $\mathcal{H}$-semistable.*

Proof. Set $n := \dim(X)$ and $r := \mathrm{rank}(\mathcal{E})$, and let $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$ be the Harder-Narasimhan filtration of $\mathcal{E}$ (see [8]). Since $\mathcal{E}$ is not semistable, we have $r \geq 2$ and $k \geq 2$. We may also assume $n \geq 2$, since the case $n = 1$ is known in [13, 14]. If $\Phi(\mathcal{E}_1) \subset \mathcal{E}_1 \otimes T_X$, then $(\mathcal{E}, \Phi)$ is not semistable and so we may also assume that $\Phi(\mathcal{E}_1) \nsubseteq \mathcal{E}_1 \otimes T_X$. Therefore $\Phi_{|\mathcal{E}_1}$ induces a non-zero map $u\colon \mathcal{E}_1 \to (\mathcal{E}/\mathcal{E}_1) \otimes T_X$. Set $\mathcal{A} := \mathrm{Im}(u)$ and then it is torsion-free. Let $0 = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_s = \mathcal{A}$ be the Harder-Narasimhan filtration of $\mathcal{A}$. Two of the properties of the Harder-Narasimhan filtration say that

- $\mathcal{E}_i/\mathcal{E}_1$ is torsion-free for $i = 2, \ldots, k$ and
- $0 \subset \mathcal{E}_2/\mathcal{E}_1 \subset \cdots \subset \mathcal{E}/\mathcal{E}_1$ is the Harder-Narasimhan filtration of $\mathcal{E}/\mathcal{E}_1$.

Since $\mathcal{A}$ is a quotient of the semistable sheaf $\mathcal{E}_1$, the normalized Hilbert polynomial of $\mathcal{A}_1$ and each subquotients $\mathcal{A}_{i+1}/\mathcal{A}_i$ with $0 \leq i < s$, is at least the one of $\mathcal{E}_1$ and so bigger than the ones of $\mathcal{E}_i/\mathcal{E}_{i-1}$ for $i = 2, \ldots, k$.

Fix an integer $m \gg 0$ so that $\mathcal{H}^{\otimes m}$ is very ample and take a general complete intersection $C \subset X$ of $n-1$ elements of $|\mathcal{H}^{\otimes m}|$. By [11,15] or [12, Theorem 4.1], $\Omega^1_{X|C}$ is nef. Since $\mathcal{E}_i$, $\mathcal{E}_i/\mathcal{E}_{i-1}$, $\mathcal{A}_j$ and $\mathcal{A}_j/\mathcal{A}_{j-1}$ are all torsion-free for $i = 1, \ldots, k$ and $j = 1, \ldots, s$, they are locally free outside a finite union of two-codimensional subvarieties of X. Thus for a general C all these sheaves are locally free in a neighborhood of C. Since u is non-zero, we get a non-zero map $v\colon \mathcal{A}_{|C} \to (\mathcal{E}/\mathcal{E}_1) \otimes T_{X|C}$ and so a non-zero map $w\colon \Omega^1_{X|C} \to \mathcal{A}^\vee_{|C} \otimes \mathcal{E}/\mathcal{E}_{1|C}$.

By [10] the restrictions $\mathcal{A}_{1|C}$, $\mathcal{E}_i/\mathcal{E}_{i-1|C}$, and $\mathcal{A}_j/\mathcal{A}_{j-1|C}$ for $i = 2, \ldots, k$ and $j = 2, \ldots, s$, are all semistable, i.e., the restriction to C of the Harder-Narasimhan filtrations of $\mathcal{E}$ and $\mathcal{A}$ are the Harder-Narasimhan filtrations of the vector bundles $\mathcal{E}_{|C}$ and $\mathcal{A}_{|C}$, respectively. Note that

- the slope of the tensor product of two vector bundles on C is the sum of their slopes and
- the tensor product of two semistable bundles on C is semistable (see [9, Corollary 6.4.14]).

Thus all the Harder-Narasimhan subquotients of $\mathcal{A}^{\vee}_{|C} \otimes (\mathcal{E}/\mathcal{E}_1)_{|C}$ are semistable vector bundles of negative degree. Let $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_h$ be the Harder-Narasishan filtration of $\mathcal{A}^{\vee}_{|C} \otimes (\mathcal{E}/\mathcal{E}_1)_{|C}$ and let l be the minimal integer such that $\mathrm{Im}(w) \subseteq \mathcal{F}_l$. Since $\Omega^1_{X|C}$ is nef, any quotient of $\Omega^1_{X|C}$ has non-negative degree. If $l = 1$, every non-zero subsheaf of $\mathcal{F}_1$ has negative degree, since $\mathcal{F}_1$ is semistable with $\deg(\mathcal{F}_1) < 0$, a contradiction. If $l > 1$, then we get a contradiction taking the composition of w with the surjection $\mathcal{F}_l \to \mathcal{F}_l/\mathcal{F}_{l-1}$. □

As shown in [14], many unstable vector bundles such as decomposable ones may give stable co-Higgs bundles. The next observation and Lemma 2.1 shows in particular that we cannot increase too much the stability. In case of stable bundles this phenomenon may be measured by the following observation.

Remark 2.10. In (2.3) assume $h^0(\mathcal{I}_Z \otimes \mathcal{L} \otimes T_X) = 0$, e.g., $h^0(\mathcal{I}_Z \otimes \mathcal{L}(2)) = 0$, and then every co-Higgs field on $\mathcal{F}$ preserves $\mathcal{O}_X$. Thus for fixed $\mathcal{L}$ and "large" Z we have several examples in which any co-Higgs field cannot be more stable than the order of stability represented by the non-zero map $\mathcal{O}_X \to \mathcal{F}$ in (2.3).

Note that for any line bundle $\mathcal{A} \in \mathrm{Pic}(X)$, a co-Higgs bundle $(\mathcal{E}, \Phi)$ is semistable or stable if and only if $(\mathcal{E} \otimes \mathcal{A}, \Phi_{\mathcal{A}})$ has the same property, where $\Phi_{\mathcal{A}}$ is induced by Φ by tensoring with $\mathcal{A}$. So in case of rank two, we may reduce many problems to the case in which $\det(\mathcal{E})$ is in a prescribed class of $\mathrm{Pic}(X)/(2\,\mathrm{Pic}(X))$. For many surfaces we have $2\,\mathrm{Pic}(X) \subsetneq \mathrm{Pic}(X)$ and so we cannot reduce all problems to the case in which $\det(\mathcal{E}) \cong \mathcal{O}_X$, e.g., if X is a surface of general type, which is not minimal. This explains why we extend [13, Theorem 7.1] in the following way; we also exclude the strictly semistable case.

Proposition 2.11. *Let X be a surface of general type. Then there is no semistable co-Higgs bundle $(\mathcal{E}, \Phi)$ of rank two with $\Phi \neq 0$.*

Proof. Since X is of general type, we have $H^0(T_X) = 0$ and so Φ has trace zero. The integrability $\Phi \wedge \Phi = 0$ gives a map $\Phi \circ \Phi \colon \mathcal{E} \to \mathcal{E} \otimes S^2 T_X$. As in the proof of [13, Theorem 7.1, pp. 148–149], we get $\mathcal{L} := \ker(\Phi) \neq 0$. Since Φ is not trivial, $\mathcal{L}$ is a line bundle and $\mathcal{E}$ fits into an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{I}_Z \otimes \mathcal{A} \to 0 \tag{2.4}$$

with Z a zero-dimensional subscheme and $\mathcal{A}$ a line bundle. The proof of loc. cit. also gives that $\Phi(\mathcal{E}) \subset \mathcal{L} \otimes T_X$ and so $\Phi \circ \Phi = 0$. Since $\Phi(\mathcal{L}) = 0 \subset \mathcal{L} \otimes T_X$ and $(\mathcal{E}, \Phi)$ is semistable, we have $\deg \mathcal{L} \leq \deg \mathcal{A}$. Note that Φ induces a non-zero map $u\colon \mathcal{O}_X \to \mathcal{B} \otimes T_X$ with $\mathcal{B} := \mathcal{L} \otimes \mathcal{A}^\vee$. In particular, for a fixed integer $m \gg 0$ and a general $C \in |\mathcal{O}_X(m)|$, we have $u_{|C} \neq 0$. On the other hand, by [12, Theorem 4.11] and [11] the vector bundle $(\Omega^1_X)_{|C}$ is ample, and so $h^0(C, (\mathcal{B} \otimes T_X)_{|C}) = 0$, a contradiction. □

3. Rank two co-Higgs bundles on surfaces and foliations

From now on we always assume that X is a smooth projective surface.

Remark 3.1. In general, for a co-Higgs bundle $(\mathcal{E}, \Phi)$ with a non-injective $\Phi \neq 0$, the sheaf $\mathcal{L} := \ker(\Phi)$ is saturated of rank one in $\mathcal{E}$ and so $\mathcal{E}$ fits in an exact sequence (2.4) with $\det(\mathcal{E}) \cong \mathcal{L} \otimes \mathcal{A}$ and Z a zero-dimensional subscheme. Note that Φ is nilpotent if and only if $\Phi(\mathcal{E}) \subset \mathcal{L} \otimes T_X$. Assume that Φ is nilpotent. From $\Phi(\mathcal{L}) = 0$, Φ induces a map $\phi\colon \mathcal{I}_Z \otimes \mathcal{A} \to \mathcal{L} \otimes T_X$ with $\mathrm{Im}(\Phi) = \mathrm{Im}(\phi)$. If $\deg \mathcal{L} > \deg \mathcal{A}$ (resp. $\deg \mathcal{L} \geq \deg \mathcal{A}$), then both $\mathcal{E}$ and $(\mathcal{E}, \mathcal{L})$ are not semistable (resp. not stable). Now assume $\deg \mathcal{L} \leq \deg \mathcal{A}$ (resp. $\deg \mathcal{L} < \deg \mathcal{A}$) and that $(\mathcal{E}, \Phi)$ is not stable (resp. semistable). Then there is a saturated line bundle $\mathcal{L}' \subset \mathcal{E}$ such that $\Phi(\mathcal{L}') \subset \mathcal{L}' \otimes T_X$ and $\deg \mathcal{L}' > \deg \mathcal{L}$. Since $\Phi(\mathcal{E}) \subset \mathcal{L} \otimes T_X$, we get $\mathcal{L}' \otimes T_X \subseteq \mathcal{L} \otimes T_X$, contradicting the inequality $\deg \mathcal{L}' > \deg \mathcal{L}$. Hence in case of $\deg \mathcal{L} \leq \deg \mathcal{A}$, we are in the set-up of [4].

Remark 3.2. Assume $H^0(T_X) \neq 0$ and take a non-zero section $\sigma \in H^0(T_X)$. For $\mathcal{E} := \mathcal{O}_X \oplus \mathcal{O}_X$ with a fixed basis $\{e_1, e_2\}$ of $H^0(\mathcal{E})$, define $\Phi\colon \mathcal{E} \to \mathcal{E} \otimes T_X$ to be induced by $\left(\begin{smallmatrix} 0 & \sigma \\ 0 & 0 \end{smallmatrix}\right)$. Then Φ is a non-trivial nilpotent field whose associated foliation is the saturated foliation associated to σ.

Let us assume for the moment that X is a smooth and non-rational projective surface X with negative Kodaira dimension $\kappa(X) = -\infty$. Let $u\colon X \to Y$ be the minimal model of X and then u is a finite sequence of blow-ups of points. Denoting the Albanese variety of X by C, we get that C is a smooth curve of positive genus $q = h^1(\mathcal{O}_X)$ and there is a vector bundle $\mathcal{F}$ of rank two on C such that $Y = \mathbb{P}(\mathcal{F})$. Here we may assume that $\mathcal{F}$ is initialized, i.e., $h^0(\mathcal{F}) > 0$ and $h^0(\mathcal{F} \otimes \mathcal{L}^\vee) = 0$ for all $\mathcal{L} \in \mathrm{Pic}^d(C)$ with $d \geq 1$. If $\pi\colon Y \to C$ denotes the projection, then $f = \pi \circ u\colon X \to C$ is the Albanese map of X. Letting T_f be the relative tangent sheaf of f, we have an exact sequence

$$0 \to T_f \to T_X \to \mathcal{I}_W \otimes (\omega_X \otimes T_f)^\vee \to 0$$

with W a zero-dimensional subscheme of X supported by the critical locus of f.

Proposition 3.3. *Let $(\mathcal{E}, \Phi)$ be a semistable co-Higgs bundle of rank two on a surface X with $\mathcal{E}$ not semistable and $\Phi \neq 0$. Then we have $\kappa(X) = -\infty$ and Φ induces a meromorphic foliation on X whose general leaf is a smooth rational curve. Moreover if we assume that X is not rational, then there exists a nonnegative divisor D and a line bundle $\mathcal{A}$ such that $\mathcal{E}$ fits into an exact sequence*

$$0 \to T_f(-D) \otimes \mathcal{A} \to \mathcal{E} \to \mathcal{I}_Z \otimes \mathcal{A} \to 0 \tag{3.1}$$

with $\deg T_f(-D) > 0$, where $f \colon X \to C$ is the Albanese mapping.

Proof. Set $\mathcal{R} := \det(\mathcal{E})$. Since $(\mathcal{E}, \Phi)$ is semistable and $\mathcal{E}$ is not, then there exists a line subbundle $\mathcal{L} \subset \mathcal{E}$ such that $\deg \mathcal{L} > \deg \mathcal{R}/2$ with an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{I}_Z \otimes \mathcal{R} \otimes \mathcal{L}^\vee \to 0 \tag{3.2}$$

with Z a zero-dimensional scheme. The assumption on $(\mathcal{E}, \Phi)$ gives $\Phi(\mathcal{L}) \nsubseteq \mathcal{L} \otimes T_X$. Composing $\Phi_{|\mathcal{L}}$ with the map $\mathcal{E} \otimes T_X \to \mathcal{I}_Z \otimes \mathcal{R} \otimes \mathcal{L}^\vee \otimes T_X$ induced by (3.2), we get a non-zero map $v \colon \mathcal{L} \to \mathcal{I}_Z \otimes \mathcal{R} \otimes \mathcal{L}^\vee \otimes T_X$ and so an injective map $j \colon \mathcal{L}^{\otimes 2} \otimes \mathcal{R}^\vee \to T_X$. If we let $\mathcal{N} \subset T_X$ be the saturation of $j(\mathcal{L}^{\otimes 2} \otimes \mathcal{R}^\vee)$, then we have $\mathcal{N} \cong \mathcal{L}^{\otimes 2} \otimes \mathcal{R}^\vee(D)$ with either $D = \emptyset$ or D an effective divisor. In particular, we get $\deg \mathcal{N} \geq \deg \mathcal{L}^{\otimes 2} \otimes \mathcal{R}^\vee > 0$. By a theorem of Miyaoka and Shepherd-Barron (see [15]), $\mathcal{N}$ is the tangent sheaf of a meromorphic foliation by curves with a rational curve as a general leaf. Since q is positive, this foliation is induced by the Albanese map $f \colon X \to C$, i.e., $\mathcal{N} \cong T_f$, and we may take $\mathcal{A} := \mathcal{R} \otimes \mathcal{L}^\vee$. □

4. Surfaces with $H^0(T_X) = H^0(S^2 T_X) = 0$

Recall that if there is no non-trivial global tangent vector field, then the existence of unstable co-Higgs bundle of rank two implies that the co-Higgs field is nilpotent by Lemma 2.5. On the other hand, it is observed in Lemma 2.6 that any non-trivial global tangent vector field suggests an example of strictly semistable co-Higgs bundle of arbitrary rank with injective co-Higgs field.

Lemma 4.1. [4, 13] *Let X be a smooth surface such that $H^0(T_X) = H^0(S^2 T_X) = 0$. If $(\mathcal{E}, \Phi)$ is any co-Higgs bundle of rank two on X, then Φ is nilpotent. Moreover, if $(\mathcal{E}, \Phi)$ is stable and $\kappa(X) \geq 0$, then we have $\Phi = 0$.*

Proof. Note that $H^0(T_X) = 0$ implies that Φ is trace-free. The first assertion is from the proof of [13, Theorem 7.1, pp. 148–149]. For the second, if $\Phi \neq 0$, then by [4, Theorem 1.1] $(\mathcal{E}, \Phi)$ is strictly semistable, a contradiction. □

In [4] the classification of smooth surfaces with semistable co-Higgs bundles of rank two with nilpotent co-Higgs fields is done. This together with Lemma 4.1 motivates to investigate the criterion for the vanishing $H^0(T_X) = H^0(S^2T_X) = 0$.

Remark 4.2. Let X be a smooth projective surface. The space $H^0(T_X)$ of global tangent vector fields, is the tangent space of the algebraic group $\mathrm{Aut}^0(X)$ at the identity map $X \to X$. So if $\pi\colon X \to Y$ is the blow-up at $p \in Y$, then we get $H^0(X, T_X) \cong H^0(Y, \mathcal{I}_p \otimes T_Y)$.

Iterating the observation in Remark 4.2, we get information on $H^0(T_X)$, with respect to $\kappa(X)$. Recall that any smooth rational surface has as its minimal model either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_e$ for $e \in \mathbb{N} \setminus \{1\}$.

(1) Assume that X is rational and that there is a birational morphism $X \to Y$ with Y the blow-up of $\mathbb{P}^2$ at four points of $\mathbb{P}^2$, no three of them collinear. Then we get $H^0(T_X) = 0$.

(2) Let $\pi_i\colon \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ for $i = 1, 2$, denote the projection on the i-th factor. Let Y be the blow-up of $\mathbb{F}_0$ at a finite set of points $S \subset \mathbb{F}_0$ such that $\sharp(\pi_i(S)) \geq 3$ for each i. Then we get $H^0(T_Y) = 0$. Thus if there is a birational morphism $X \to Y$, then we also get $H^0(T_X) = 0$.

(3) We have $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ and every automorphism of $\mathbb{F}_e$ preserves the ruling $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \to \mathbb{P}^1$. Let us take $e > 0$. If S is a general subset of $\mathbb{F}_e$ with $\sharp(S) \geq \max\{3, e+1\}$, then we get $h^0(\mathcal{I}_S \otimes T_{\mathbb{F}_e}) = 0$ from $h^0(\mathrm{End}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) = 3+e$. If we assume the existence of a birational morphism $X \to Y$ with Y the blow-up of $\mathbb{F}_e$ at S, then we get $h^0(T_X) \leq h^0(T_Y) = 0$.

(4) Let us assume X is a non-rational surface with $\kappa(X) = -\infty$. Then we have $q := h^1(\mathcal{O}_X) > 0$ and the Albanese variety $f\colon X \to C$ of X has as target a smooth curve C of genus q and $\mathbb{P}^1$ as its general fiber. There exists a vector bundle $\mathcal{G}$ of rank two on C and a finite sequence $u\colon X \to Y$ of blowing ups such that $\pi\colon Y = \mathbb{P}(\mathcal{G}) \to C$ is $\mathbb{P}^1$-bundle over C and $f = \pi \circ u$. Here we may assume that $\mathcal{G}$ is initialized, i.e., $h^0(\mathcal{G}) > 0$ and $h^0(\mathcal{G} \otimes \mathcal{L}^\vee) = 0$ for all $\mathcal{L} \in \mathrm{Pic}^d(C)$ with $d \geq 1$. If $q \geq 2$ and $\mathcal{G}$ is simple, then we get $H^0(T_Y) = 0$ and so $H^0(T_X) = 0$. If $\mathcal{G}$ is not simple, then we get an upper bound for $h^0(T_Y)$ in terms of $h^0(\mathrm{End}(\mathcal{G}))$ and so, as in part (c), we get many X with minimal model Y and $h^0(T_X) = 0$. This observation also applies in the case $q = 1$, when $\mathcal{G}$ is decomposable. If $q = 1$ and $\mathcal{G}$ is indecomposable, then we get $h^0(\mathrm{End}(\mathcal{G})) = 2$ by Atiyah's classification of vector bundles on elliptic curves.

(5) Assume $\kappa(X) = 0$ and let $u\colon X \to Y$ be the morphism to its minimal model. From the list of possible surface Y in [1, p. 244] we have $H^0(T_Y) = 0$, unless Y is an Abelian surface or it has an Abelian surface as a finite unramified covering. If Y is an Abelian surface, then we have $H^0(\mathcal{I}_p \otimes T_Y) = 0$ for all $p \in Y$. Hence we get $H^0(T_X) \neq 0$ if and only if X is either an Abelian surface or a bi-elliptic surface. In particular, we have $H^0(T_X) = 0$ if X is not a minimal model.

(6) Let Y be a projective surface with $\kappa(Y) = 1$ and $h^0(T_Y) > 0$. Since Y has only finitely many rational curves and only a one-dimensional family of elliptic curves, we have $h^0(T_Y) = 1$ and so $h^0(\mathcal{I}_p \otimes T_Y) = 0$ for a general $p \in Y$. Thus we have $H^0(T_X) = 0$ if X has Y as its minimal model, but it factors through a blow-up of Y at a general point of Y. We saw that $H^0(T_X) = 0$ if the minimal model Y' of X satisfies $H^0(T_{Y'}) = 0$.

(7) In case of $\kappa(X) = 2$, we have $H^0(T_X) = 0$, because X has neither a positive dimensional family of elliptic curves nor a positive dimensional family of rational curves.

Remark 4.3. Let X be a smooth projective surface. Assume the existence of a finite unramified covering $u \colon Y \to X$ with $H^0(S^2T_Y) = 0$. Since u is unramified, we get $u^*(S^2T_X) = S^2T_Y$ and so $H^0(S^2T_X) = 0$. In particular [13, Lemma 7.1] shows that $H^0(S^2T_X) = 0$, if X is an Enriques surface.

Let $u \colon X \to Y$ be the blow-up at one point $p \in Y$ and set $D := u^{-1}(p)$. We have $D \cong \mathbb{P}^1$ and $\mathcal{O}_D(D)$ is the line bundle on D of degree -1. The natural map $H^0(\Omega^1_Y) \to H^0(\Omega^1_X)$ is an isomorphism. Hence $H^0(S^2\Omega^1_X)$ contains a three-dimensional linear subspace spanning $S^2\Omega^1_X$ outside the exceptional locus D of the map $u \colon X \to Y$. We have an exact sequence

$$0 \to T_X \to u^*(T_Y) \to \mathcal{O}_D(-D) \to 0 \tag{4.1}$$

(see [5, Lemma 15.4(iv)]), in which $\mathcal{O}_D(-D)$ is the line bundle on $D \cong \mathbb{P}^1$ of degree 1. From (4.1) we also get a map $S^2T_X \to u^*(S^2T_Y)$, which is an isomorphism outside D. Applying u_*, we get a map $u_*(S^2T_X) \to S^2T_Y$ which is an isomorphism on $Y \setminus \{p\}$, and in particular it is injective as a map of sheaves.

Remark 4.4. Let $u \colon X \to Y$ be a birational morphism of smooth surfaces. Since u is a finite sequence of blow-ups of points, we have $u_*(\mathcal{O}_X) = \mathcal{O}_Y$ and $R^iu_*(\mathcal{O}_X) = 0$ for all $i > 0$. The natural map $S^2T_X \to u^*(S^2T_Y)$ is injective, because it is an isomorphism outside finitely many divisors of X. Applying u_* and the projection formula, we get an inclusion $u_*(H^0(S^2T_X)) \subseteq H^0(S^2T_Y)$.

Lemma 4.5. *Let $u \colon X \to Y$ be a blow-up at a point p on a smooth surface Y. If $j \colon H^0(S^2T_X) \to H^0(S^2T_Y)$ is the map induced by u_*, then* $\mathrm{Im}(j) \subseteq H^0(\mathcal{I}_p \otimes S^2T_Y)$.

Proof. It is sufficient to prove that $\mathrm{Im}(j_1) \subseteq H^0(\mathcal{I}_p \otimes T_Y^{\otimes 2})$, where $j_1 \colon H^0(T_X^{\otimes 2}) \to H^0(T_Y^{\otimes 2})$ is the map induced by u_*. The corresponding map $u_*(T_X) \to T_Y$ has $\mathcal{I}_p \otimes T_Y$ as its image and $u_*(\mathcal{O}_D(-D)) = \mathbb{C}_p^{\oplus 2}$. The sequence (4.1) tensored with $u^*(T_Y)$ gives an inclusion $T_X \otimes u^*(T_Y) \to u^*(T_Y^{\otimes 2})$. Applying u_* and the projection formula, we get an injective map $j_2 \colon H^0(\mathcal{I}_p \otimes T_Y^{\otimes 2}) \to H^0(T_Y^{\otimes 2})$. Now j_1 factors through j_2, implying the assertion. □

Lemma 4.6. *Let Y be a smooth projective surface such that $H^0(S^2T_Y) = 0$. For a birational morphism $u \colon X \to Y$, we get $H^0(S^2T_X) = 0$.*

Proof. Since u is the composition of finitely many blow-ups, it is sufficient to prove it when u is the blow-up at a point $p \in Y$. Denote by $E := u^{-1}(p)$ the exceptional divisor and set $s' := s_{|X \setminus E}$ for a section $s \in H^0(S^2T_X)$. Then s' induces $s_1 \in H^0(Y \setminus \{p\}, (S^2T_Y)_{|Y\setminus\{p\}})$ and it extends to $\sigma \in H^0(S^2T_Y)$ by Hartogs' theorem, because S^2T_Y is locally free. By assumption we have $\sigma = 0$. It implies that $s' = 0$, which also implies $s = 0$. □

Lemma 4.7. *Let X be a smooth and compact complex surface whose minimal model $Y \ncong X$ is either a complex torus or $C \times W$ with C an elliptic curve and W a smooth curve of genus at least two. Then we have $H^0(S^2T_X) = 0$.*

Proof. Since $X \ncong Y$, there is a blow-up $B \to Y$ at a point $p \in Y$ and a birational morphism $X \to B$. Remark 4.2 and Lemma 4.5 give $H^0(T_B) = H^0(S^2T_B) = 0$ and then Remark 4.4 gives $H^0(T_X) = H^0(S^2T_X) = 0$. □

Remark 4.8. Let $f\colon X \to W$ be an unramified covering between smooth surfaces. If $H^0(T_W) \neq 0$ (resp. $H^0(S^2T_W) \neq 0$), then we have $H^0(T_X) \neq 0$ (resp. $H^0(S^2T_W) \neq 0$). Now assume that W is not minimal and take any $D \subset W$ with $D \cong \mathbb{P}^1$ and $D^2 = -1$. Let $v\colon W \to W'$ be the blow-down of D. Since D has an open simply connected neighborhood in the Euclidean topology, there is an unramified covering $f'\colon X' \to W'$ and a map $v'\colon X \to X'$ such that v' is the blow-down of $\deg(f)$ disjoint exceptional curves and $f' \circ v' = v \circ f$.

Proposition 4.9. *Let Y be a smooth surface and $S \subset Y$ be any finite subset such that $H^0(\mathcal{I}_S \otimes T_Y) = H^0(\mathcal{I}_S \otimes S^2T_Y) = 0$. If X is any smooth surface with a birational morphism $u\colon X \to Y$ with S contained in the image of divisors of X contracted by u, then we have $h^0(T_X) = h^0(S^2T_X) = 0$. In particular, Φ is trivial for any co-Higgs bundle of rank two on X.*

Proof. By Remarks 4.2, 4.4 and Lemma 4.5, we have $H^0(T_X) = H^0(S^2T_X) = 0$. Now we may apply Lemma 4.1 for the second assertion. □

Remark 4.10. Let Y be a minimal surface with $\kappa(Y) = 0$. We have $H^0(T_Y) = H^0(S^2T_Y) = 0$, unless either Y is an Abelian surface or a bielliptic surface. Thus in Proposition 4.9 with $\kappa(Y) = 0$, we may take either $S = \emptyset$ or, in the two exceptional cases, as S any point of Y by Remark 4.8.

Remark 4.11. The surface $Y = C \times W$ described in Lemma 4.7 and Proposition 4.13 is the only one, up to a finite unramified covering, with $H^0(T_Y) \neq 0$ and $\kappa(Y) = 1$. This can be obtained by Lemma 4.5 and Remark 4.8 for these surfaces and their unramified coverings.

Proposition 4.12. *Let $(\mathcal{E}, \Phi)$ be a co-Higgs bundle of rank two on an Abelian surface X.*

(i) *If $\mathcal{E}$ is simple, then either $\Phi = 0$ or Φ is injective. In particular, $(\mathcal{E}, \Phi)$ is stable, semistable, strictly semistable or unstable if and only if $\mathcal{E}$ has the same property.*

(ii) *If $\mathcal{E}$ is not semistable, then $(\mathcal{E}, \Phi)$ is not semistable.*

(iii) *If $\mathcal{E}$ is strictly semistable and indecomposable, but not simple, then $(\mathcal{E}, \Phi)$ is strictly semistable.*

Proof. Note that we have $T_X \cong \mathcal{O}_X^{\oplus 2}$ and so $\Phi\colon \mathcal{E} \to \mathcal{E}^{\oplus 2}$. If $\mathcal{E}$ is simple, then there are $c_1, c_2 \in \mathbb{C}$ such that $\Phi = (c_1 \operatorname{id}_{\mathcal{E}}, c_2 \operatorname{id}_{\mathcal{E}})$. So for any sheaf $\mathcal{B} \subset \mathcal{E}$, we have $\Phi(\mathcal{B}) \subset \mathcal{B} \otimes T_X$, implying (i). The assertion (ii) is a special case of Theorem 2.9.

Now assume that $\mathcal{E}$ is strictly semistable, but not simple. Since $\mathcal{E}$ is strictly semistable, then it fits in an exact sequence (2.2) with $\deg \mathcal{A} = \deg \mathcal{L}$. Since $\mathcal{E}$ is not simple, either (2.2) is not the unique destabilizing sequence of $\mathcal{E}$ or $\operatorname{Hom}(\mathcal{I}_Z \otimes \mathcal{A}, \mathcal{L})$ is not trivial. In the former case we have $Z = \emptyset$ and $\mathcal{E} \cong \mathcal{L} \oplus \mathcal{A}$; in this case we get

$$\dim \operatorname{End}(\mathcal{E}) = \begin{cases} 2 & \text{if } \mathcal{A} \ncong \mathcal{L}, \\ 4 & \text{if } \mathcal{A} \cong \mathcal{L}. \end{cases}$$

In the latter case the condition $\deg \mathcal{A} = \deg \mathcal{L}$ implies $\mathcal{A} \cong \mathcal{L}$. So we get either either $Z = \emptyset$ and (2.2) splits or $\dim \operatorname{End}(\mathcal{E}) = 3$. If $\mathcal{E}$ is indecomposable, then we have $\mathcal{A} \cong \mathcal{L}$ and any endomorphism of $\mathcal{E}$ sends $\mathcal{L}$ into itself. Thus we get $\Phi(\mathcal{L}) \subset \mathcal{L} \otimes T_X$. □

Proposition 4.13. *Let $X = C \times W$, where C and W are smooth curves of genus one and $g \geq 2$, respectively.*

(i) *If $\mathcal{E}$ is simple, then either $\Phi = 0$ or Φ is injective. In particular, $(\mathcal{E}, \Phi)$ is stable, semistable, strictly semistable or unstable if and only if $\mathcal{E}$ has the same property.*

(ii) *If $\mathcal{E}$ is not semistable, then $(\mathcal{E}, \Phi)$ is not semistable.*

(iii) *If $\mathcal{E}$ is strictly semistable and indecomposable, but not simple, then $(\mathcal{E}, \Phi)$ is strictly semistable.*

Proof. Let $\pi_2\colon X \to W$ denote the projection and then we have $T_X \cong \mathcal{O}_X \oplus \mathcal{R}$ with $\mathcal{R}^\vee \cong \pi_2^*(\omega_W)$. Here $\mathcal{R}^\vee$ is spanned, but not trivial. If $\mathcal{E}$ is either simple or semistable, then Φ is associated to an endomorphism of $\mathcal{E}$, because there is no non-zero map $\mathcal{E} \to \mathcal{E} \otimes \mathcal{R}$. If $\mathcal{E}$ is simple, then we get $\Phi(\mathcal{B}) \subset \mathcal{B} \otimes \mathcal{O}_X \oplus \{0\} \subset \mathcal{B} \otimes T_X$ and so the part (i). Parts (ii) and (iii) are proved as in Proposition 4.12. □

Remark 4.14. Let $\mathbb{F}_e$ for $e \geq 0$, be a Hirzebruch surface with $\mathrm{Pic}(\mathbb{F}_e) \cong \mathbb{Z}^{\oplus 2} \cong \mathbb{Z}\langle h, f\rangle$, where f is a ruling $\pi\colon \mathbb{F}_e \to \mathbb{P}^1$ and h is a section of this ruling with $h^2 = -e$. We have

$$T_{\mathbb{F}_e} \cong \mathcal{O}_{\mathbb{F}_e}(2f) \oplus \mathcal{O}_{\mathbb{F}_e}(2h + ef),$$
$$S^2T_{\mathbb{F}_e} \cong \mathcal{O}_{\mathbb{F}_e}(4f) \oplus \mathcal{O}_{\mathbb{F}_e}(2h + (e+2)f) \oplus \mathcal{O}_{\mathbb{F}_e}(4h + 2ef)).$$

Now fix a finite subset $S \subset \mathbb{F}_e$.

(a) Assume $e = 0$ and we may consider $\mathbb{F}_0$ as a smooth quadric surface in $\mathbb{P}^3$. Letting $\eta\colon \mathbb{F}_0 \to \mathbb{P}^1$ be the other projection, we have $h^0(\mathcal{I}_S \otimes T_{\mathbb{F}_0}) = 0$ if and only if $\sharp(\pi(S)) \geq 3$ and $\sharp(\eta(S)) \geq 3$. We have $h^0(\mathcal{I}_S \otimes S^2T_{\mathbb{F}_0}) = 0$ if and only if $\sharp(\pi(S)) \geq 5$, $\sharp(\eta(S)) \geq 5$ and S is not contained in other quadric surfaces in $\mathbb{P}^3$. If $h^0(\mathcal{I}_S \otimes S^2T_{\mathbb{F}_0}) = 0$, then we have $\sharp(S) \geq 9$. Conversely, if $\sharp(S) \geq 9$ and S is general in $\mathbb{F}_0$, then we get $h^0(\mathcal{I}_S \otimes S^2T_{\mathbb{F}_0}) = 0$.

(b) Assume $e = 1$ and let $u\colon \mathbb{F}_1 \to \mathbb{P}^2$ denote the blow-down of h. The linear system $|\mathcal{O}_{\mathbb{F}_1}(2h + f)|$ has h as its base locus and u is induced by $|h + f|$. Then the linear system $|\mathcal{O}_{\mathbb{F}_1}(4h + 2f)|$ has $2h$ as its base locus and $H^0(\mathcal{O}_{\mathbb{F}_1}(2h+2f)) = u^*(H^0(\mathcal{O}_{\mathbb{P}^2}(2))$. We have $h^0(\mathcal{I}_S \otimes T_{\mathbb{F}_1}) = 0$ if and only if $\sharp(\pi(S)) \geq 3$. We have $H^0(\mathcal{I}_S \otimes S^2T_{\mathbb{F}_1}) = 0$ if and only if $h^0(\mathbb{P}^2, \mathcal{I}_{u(S)}(2)) = 0$.

(c) Assume $e \geq 2$ and that S is general. We have

$$h^0(\mathcal{O}_{\mathbb{F}_e}(2h + ef)) = h^0(\mathcal{O}_{\mathbb{F}_e}(h + ef)) = e + 2,$$
$$h^0(\mathcal{O}_{\mathbb{F}_e}(4h + 2ef)) = h^0(\mathcal{O}_{\mathbb{F}_e}(2h + 2ef)) = 3e + 3,$$

and this implies that $H^0(\mathcal{I}_S \otimes T_{\mathbb{F}_e}) = 0$ if $\sharp(S) \geq e + 2$ and $H^0(\mathcal{I}_S \otimes S^2T_{\mathbb{F}_e}) = 0$ if $\sharp(S) \geq 3e + 3$.

Remark 4.15. Take a finite subset $S \subset \mathbb{P}^2$ and assume the existence of $p \in S$ such that $h^0(\mathcal{I}_{S\setminus\{p\}}(2)) = 0$. By part (b) of Remark 4.14 we have $h^0(\mathcal{I}_S \otimes T_{\mathbb{P}^2}) = h^0(\mathcal{I}_S \otimes S^2T_{\mathbb{P}^2}) = 0$.

Lemma 4.1 with Remarks 4.14 and 4.15 give the following.

Corollary 4.16. *Let X be a rational surface with a relative minimal model $u\colon X \to Y$ and fix a finite subset $S \subset Y$ such that $u^{-1}(o)$ contains a curve for $o \in S$. Assume that*

- *if $Y = \mathbb{P}^2$, then there exists $p \in S$ such that $H^0(\mathcal{I}_{S\setminus\{p\}}(2)) = 0$;*
- *if $Y = \mathbb{F}_e$, then S is general in $\mathbb{F}_e$ and*

$$\sharp(S) \geq \begin{cases} 9 - 3e & \text{if } e \leq 1, \\ 3e + 3 & \text{if } e \geq 2. \end{cases}$$

Then Φ is nilpotent for any co-Higgs bundle $(\mathcal{E}, \Phi)$ of rank two on X.

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