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Stable and Unstable Periodic Solutions of the Forced Pendulum of Variable Length

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Abstract. In this paper, we study the existence of stable and unstable periodic solutions of the forced pendulum of variable length. The proof is based on a stability criterion which was obtained in [11] by using the third order approximation method and a generalized version of the Poincaré-Birkhoff fixed point theorem.

1. Introduction

During the last few decades, the existence and stability of periodic solutions of the forced pendulum

$$\ddot{x} + a\sin x = p(t) \tag{1.1}$$

has been studied by many researchers, and many interesting results have been obtained in literature, where $a > 0$ is a parameter and $p \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ is an external force. We refer the reader to the classical monograph [14] for a very complete survey on this problem. Recently, it was proved in [16] that if $a \leq 1/4$, then (1.1) has a stable 2π-periodic solution for almost every forcing p satisfying

$$\overline{p} = \frac{1}{2\pi}\int_0^{2\pi} p(t)\,\mathrm{d}t = 0. \tag{1.2}$$

In [17], it was proved that if $a > 1/9$, then there exists a real analytic 2π-periodic function p satisfying (1.2) and $p(-t) = -p(t)$, $t \in \mathbb{R}$, such that (1.1) has no stable 2π-periodic solution.

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If the parameter a in (1.1) is replaced by a function $h(t)$, then we obtain the equation of the swing or the forced pendulum of variable length

$$\ddot{x} + h(t)\sin x = p(t), \tag{1.3}$$

where $h\colon \mathbb{R} \to \mathbb{R}$ is a continuous and 2π-periodic function related with the variable length. Equation (1.3) is a very famous physical mode which can describe a lot of physical phenomena, besides the swing and the forced pendulum of variable length. For example,

- if $h(t) = gr^3(t)$, then (1.3) is the equation of a particle moving on a pulsating circle under the action of gravity, where $r\colon \mathbb{R} \to \mathbb{R}$ is a continuous and 2π-periodic function which is regarded as the radius of the pulsating circle;
- if $h(t) = \alpha\sqrt{1 + 3\sin^2 t}$, then (1.3) can describe the motion of a satellite in orbit around the earth [22], where α is a positive parameter related with magnetic intensity.

Therefore, it is interesting to study the dynamical behavior of (1.3). But as far as we know, there are no any stability results about this problem up to now. The purpose of this paper is to fill this gap. In this paper, we study the existence of stable and unstable periodic solutions of (1.3).

In Section 3, we study the existence of twist periodic solutions of (1.3). Such twist periodic solutions are stable in the sense of Lyapunov. We prove that (1.3) has a twist periodic solution if h is a continuous and positive 2π-periodic function with $\overline{h} \le 1/64$ and $\max_{t\in[0,2\pi]} \lvert p(t)\rvert / h(t)$ is not too large in some sense. We remark that we don't need the assumption (1.2). The proof is based on a stability criterion which was obtained in [11] by using the third order approximation method. The method of third order approximation was developed by Ortega [15] and Zhang [23] to study the Lyapunov stability for Lagrangian equations, and has been applied in [3–6, 10–12, 20, 21] for different kinds of equations.

In Section 4, by the use of a generalized version of the Poincaré-Birkhoff fixed point theorem [8, 9], we prove that (1.3) has at least two geometrically distinct 2π-periodic solutions if (1.2) holds, and at least one of them is unstable. Furthermore, we also study the existence of periodic and subharmonic solutions with winding number of (1.3). Here, we say that (1.3) has at least two geometrically distinct 2π-periodic solutions, if such solutions are not differing by a multiple of 2π. The Poincaré-Birkhoff fixed point theorem was originally conjectured by Poincaré [18] in 1912 when he studied the restricted three body problems, and was first proved by Birkhoff [1, 2] in 1913. During the last century, different proofs and developments were given. We refer the reader to [7, Section 1] for a short brief on the Poincaré-Birkhoff fixed point theorem and its modified versions.

2. Preliminaries

2.1. A stability criterion

Consider the scalar equation

$$\ddot{x} + f(t,x) = 0, \tag{2.1}$$

where $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is 2π-periodic in t and of class $\mathbb{C}^{0,4}$ in (t,x). Let $\psi(t)$ be a 2π-periodic solution of (2.1). By translating the periodic solution $\psi(t)$ of (2.1) to the origin, we obtain the third order approximation

$$\ddot{x} + a(t)x + b(t)x^2 + c(t)x^3 + o(x^3) = 0, \tag{2.2}$$

where

$$a(t) = f_x(t,\psi(t)), \quad b(t) = \frac{1}{2} f_{xx}(t,\psi(t)), \quad c(t) = \frac{1}{6} f_{xxx}(t,\psi(t)).$$

Suppose that $a \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ and its mean value $\overline{a} = \frac{1}{2\pi}\int_0^{2\pi} a(t)\,\mathrm{d}t$ is positive. Define positive constants σ and δ by

$$\sigma^2 = \overline{a}, \quad \delta = \int_0^{2\pi} \left| a(t) - \sigma^2 \right| \mathrm{d}t.$$

Let

$$\sigma_1 = \sigma - \frac{\delta}{4\pi\sigma}, \quad \sigma_2 = \sigma + \frac{\delta}{4\pi\sigma}. \tag{2.3}$$

In addition, we denote $b_+(t) = \max\{b(t),0\}$ and $b_-(t) = \max\{-b(t),0\}$ the positive and the negative parts of a given function $b(t)$. Based on the method of third order approximation, Lei and Torres in [11] proved the following stability criterion.

Theorem 2.1. [11, Theorem 2.1] *Assume that σ_1, σ_2 defined by* (2.3) *satisfy $\sigma_1, \sigma_2 \in (0,1/4)$. Then the zero solution $x = 0$ of* (2.2) *is of twist type if the following condition holds:*

$$\left(\frac{\tan 2\pi\sigma_1}{\tan 2\pi\sigma_2}\right)^2 \|c_-\|_1 - \|c_+\|_1 > \frac{7}{3\sigma}\left(\frac{\tan 2\pi\sigma_2}{\tan 2\pi\sigma_1}\right)^{1/2} \|b_+\|_1 \, \|b_-\|_1$$

for any $b, c \in L^1(\mathbb{R}/2\pi\mathbb{Z})$.

2.2. The Poincaré-Birkhoff fixed point theorem

Consider two strips $A = \mathbb{R}\times[-\alpha,\alpha]$ and $B = \mathbb{R}\times[-\beta,\beta]$, where $\beta > \alpha > 0$. We will work with a C^k-diffeomorphism $f\colon A \to B$ defined by

$$f(\theta,r) = (Q(\theta,r), P(\theta,r)),$$

where $Q, P\colon \mathbb{R}^2 \to \mathbb{R}$ are functions of class C^k satisfying the periodicity conditions

$$Q(\theta + 2\pi, r) = Q(\theta, r) + 2\pi, \quad P(\theta + 2\pi, r) = P(\theta, r).$$

Such generalized periodicity conditions tell us that the map is the lift to $\mathbb{R}^2$ of the corresponding map $\overline{f}\colon \overline{A} \to \overline{B}$, where $\overline{A} = \mathbb{R}/2\pi\mathbb{Z}\times[-\alpha,\alpha]$ and $\overline{B} = \mathbb{R}/2\pi\mathbb{Z}\times[-\beta,\beta]$. After the identification $\theta + 2\pi = \theta$, the domain of f can be interpreted as an annulus or a cylinder. We shall think that it is a cylinder with vertical coordinator r and the variable θ as an angle. We say that f is isotopic to the inclusion, if there exists a function $H\colon A \times [0,1] \to B$ such that for every $\lambda \in [0,1]$, $H_\lambda(x) = H(x,\lambda)$ is a homeomorphism with $H_0(x) = f(x)$ and $H_1(x) = x$. The class of the maps satisfying the above characteristics will be indicated by $\varepsilon^k(A)$.

We say that $f \in \varepsilon^k(A)$ is exact symplectic if there exists a smooth function $V = V(\theta, r)$ with $V(\theta + 2\pi, r) = V(\theta, r)$ and such that

$$\mathrm{d}V = P\,\mathrm{d}Q - r\,\mathrm{d}\theta. \tag{2.4}$$

The following theorem is a slight modified version of the Poincaré-Birkhoff fixed point theorem proved by Franks in [8, 9] and the statement on the instability was proved by Marò in [13]. Here we say that a fixed point p_1 of the one-to-one map $f\colon U \subset \mathbb{R}^N \to \mathbb{R}^N$ is said to be stable in the sense of Lyapunov if for every neighbourhood U_{p_1} of p_1 there exists another neighbourhood $U^* \subset U_{p_1}$ such that, for each $n > 0$, $f^n(U^*)$ is well defined and $f^n(U^*) \subset U_{p_1}$.

Theorem 2.2. [8, 13] *Let $f\colon A \to B$ be an exact symplectic diffeomorphism belonging to $\varepsilon^2(A)$ such that $f(A) \subset \operatorname{int}(B)$. Suppose that there exists $\epsilon > 0$ such that*

$$\begin{aligned} &Q(\theta,\alpha) - \theta > \epsilon, && \forall\,\theta \in [0,2\pi),\\ &Q(\theta,-\alpha) - \theta < -\epsilon, && \forall\,\theta \in [0,2\pi). \end{aligned} \tag{2.5}$$

Then f has at least two distinct fixed points p_1 and p_2 in A such that $p_1 - p_2 \neq (2k\pi, 0)$ for every $k \in \mathbb{Z}$. Moreover, at least one of the fixed points is unstable if f is analytic.

3. Stable periodic solutions

In this section, we first prove that (1.3) has a unique 2π-periodic solution $x(t)$ such that the L^∞ normal $\|x\|_\infty$ is the smallest among all 2π-periodic solutions of (1.3). Such a periodic solution is called the least amplitude periodic solution. Finally, we prove that the least amplitude periodic solution of (1.3) is of twist type.

Consider the Hill equation

$$x'' + h(t)x = 0 \tag{3.1}$$

with periodic boundary conditions

$$x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \tag{3.2}$$

where $h(t) \succ 0$ and $h \in L^1(\mathbb{R}/2\pi\mathbb{Z})$. It follows from [19, Corollary 2.3] that if $\|h\|_1 \le 2/\pi$, then the Green function $G(t,s)$ of the problem (3.1)–(3.2) is positive for all $(t,s) \in [0,2\pi] \times [0,2\pi]$. In this case, the unique T-periodic solution of the nonhomogeneous equation

$$x'' + h(t)x = f(t)$$

can be written as

$$x(t) = \int_0^{2\pi} G(t,s) f(s)\,\mathrm{d}s.$$

Lemma 3.1. [10, Lemma 2.1] *Let γ and η be positive parameters. Then the cubic equation*

$$\gamma x^3 + \eta = x$$

has a positive root if and only if $27\gamma\eta^2 < 4$. In this case, the minimal positive root is given by

$$x = \Phi(\gamma,\eta) = \frac{2}{\sqrt{3\gamma}} \cos\frac{\pi + y}{y}, \quad \left(y = \arccos\left(\frac{3\eta\sqrt{3\gamma}}{2}\right)\right),$$

which satisfies

$$\Phi(\gamma,\eta) \le \frac{3\eta}{2}. \tag{3.3}$$

Now we are in a position to prove the existence of the least amplitude periodic solution of (1.3).

Theorem 3.2. *Assume that h is a continuous and positive 2π-periodic function with $\|h\|_1 \le 2/\pi$ and*

$$\omega < \frac{2\sqrt{2}}{3}, \tag{3.4}$$

where $\omega = \max_{t\in[0,2\pi]} |p(t)|/h(t)$. Then (1.3) has a unique 2π-periodic solution x such that $\|x\|_\infty$ is the smallest among all 2π-periodic solutions of (1.3). Moreover, x satisfies

$$\|x\|_\infty \le \Phi\left(\frac{1}{6}, \omega\right) \le \frac{3\omega}{2}, \tag{3.5}$$

where $\Phi(\cdot\,,\cdot)$ is defined as in Lemma 3.1.

Proof. We know that x is a 2π-periodic solution of (1.3) if and only if $x \in \mathbb{C}(\mathbb{R}/2\pi\mathbb{Z})$ satisfies

$$x(t) = \int_0^{2\pi} G(t,s)\left[h(s)(x(s) - \sin x(s)) + p(s)\right]\mathrm{d}s := (\mathbb{T}x)(t).$$

Obviously, the operator $\mathbb{T}\colon \mathbb{C}(\mathbb{R}/2\pi\mathbb{Z}) \to \mathbb{C}(\mathbb{R}/2\pi\mathbb{Z})$ is a completely continuous operator (with the uniform norm $\|\cdot\|_\infty$). We define the closed ball

$$\mathbb{B} = \left\{ x \in \mathbb{C}(\mathbb{R}/2\pi\mathbb{Z}) : \|x\|_\infty \le \Phi\left(\frac{1}{6}, \omega\right) \right\}. \tag{3.6}$$

We will prove that the operator $\mathbb{T}$ has a unique fixed point x in $\mathbb{B}$ by using Banach contraction mapping theorem.

First, we prove that $\mathbb{T}$ maps $\mathbb{B}$ into itself. For any $x \in \mathbb{B}$, it follows from the basic estimate $|x - \sin x| \le |x|^3/6$ that

$$\begin{aligned}
|(\mathbb{T}x)(t)| &= \left| \int_0^{2\pi} G(t,s)\left[h(s)(x(s) - \sin x(s)) + p(s)\right] \mathrm{d}s \right| \\
&\le \int_0^{2\pi} G(t,s)h(s)\left|x(s) - \sin x(s)\right| \mathrm{d}s + \int_0^{T} G(t,s)\left|p(s)\right| \mathrm{d}s \\
&\le \frac{\|x\|_\infty^3}{6} \int_0^{2\pi} G(t,s)h(s)\,\mathrm{d}s + \int_0^{2\pi} G(t,s)h(s)\frac{|p(s)|}{h(s)}\,\mathrm{d}s \\
&\le \frac{\|x\|_\infty^3}{6} \int_0^{2\pi} G(t,s)h(s)\,\mathrm{d}s + \max_{t\in[0,2\pi]} \frac{|p(t)|}{h(t)} \int_0^{2\pi} G(t,s)h(s)\,\mathrm{d}s \\
&= \frac{\|x\|_\infty^3}{6} + \omega,
\end{aligned}$$

where $\omega = \max_{t\in[0,2\pi]} |p(t)|/h(t)$, and we have used the fact that

$$\int_0^{2\pi} G(t,s)h(s)\,\mathrm{d}s = 1.$$

If (3.4) holds, then by Lemma 3.1, we have

$$\|\mathbb{T}x\|_\infty \le \frac{\|x\|_\infty^3}{6} + \omega \le \frac{\Phi^3\left(\frac{1}{6},\omega\right)}{6} + \omega = \Phi\left(\frac{1}{6},\omega\right),$$

which implies that $\mathbb{T}$ maps $\mathbb{B}$ into itself.

Next, we prove that $\mathbb{T}\colon \mathbb{B} \to \mathbb{B}$ is a strict contraction map. Let $x, y \in \mathbb{B}$, then

$$\begin{aligned}
|(\mathbb{T}x)(t) - (\mathbb{T}y)(t)| &= \left| \int_0^{2\pi} G(t,s)h(s)\left[(x(s) - \sin x(s)) - (y(s) - \sin y(s))\right] \mathrm{d}s \right| \\
&\le \int_0^{2\pi} G(t,s)h(s)\left|(x(s) - \sin x(s)) - (y(s) - \sin y(s))\right| \mathrm{d}s.
\end{aligned} \tag{3.7}$$

Using the estimate (3.3) and (3.6), we have

$$\begin{aligned}
|(x(s) - \sin x(s)) - (y(s) - \sin y(s))| &\le \frac{1}{2}\Phi^2\left(\frac{1}{6},\omega\right)|x(s) - y(s)| \\
&\le \frac{9}{8}\omega^2 \left|x(s) - y(s)\right|.
\end{aligned} \tag{3.8}$$

By (3.4), (3.7) and (3.8), we have

$$\begin{aligned}\|\mathbb{T}x - \mathbb{T}y\|_\infty &\le \frac{9}{8}\omega^2 \|x-y\|_\infty \int_0^{2\pi} G(t,s)h(s)\,\mathrm{d}s \\ &\le \|x-y\|_\infty\end{aligned}$$

for all $x, y \in \mathbb{B}$. Thus, if the strict inequality in condition (3.4) is satisfied, then $\mathbb{T}\colon \mathbb{B} \to \mathbb{B}$ is a strict contraction map.

Finally, by using the Banach contraction mapping theorem, we know that $\mathbb{T}$ has a unique fixed point x in $\mathbb{B}$ if the strict inequality in condition (3.4) is satisfied.

Note that if $\omega = 2\sqrt{2}/3$, one can also obtain the uniqueness from the proof above, although $\mathbb{T}$ may not be a strict contraction map.

By the uniqueness of the 2π-periodic solution of (1.3) in $\mathbb{B}$, we know that $\|x\|_\infty$ is smaller than other possible 2π-periodic solutions of (1.3). Moreover, (3.5) holds. □

Our main result reads as follows.

Theorem 3.3. *Assume that h is a continuous and positive 2π-periodic function with $\overline{h} \le 1/64$. Then there exists a constant $\rho \in (0, 2\sqrt{2}/3]$, such that the least amplitude 2π-periodic solution x of* (1.3) *obtained in Theorem* 3.2 *is of twist type if* $0 \le \omega < \rho$.

Proof. From Theorem 3.2, we know that

$$\|x\|_\infty \le \frac{3}{2}\omega \le \sqrt{2} < \frac{\pi}{2}. \tag{3.9}$$

A computation of the coefficients of the expansion (2.2) gives

$$a(t) = h(t)\cos x(t), \quad b(t) = -\frac{h(t)\sin x(t)}{2}, \quad c(t) = -\frac{h(t)\cos x(t)}{6}.$$

By (3.9), we know that $a(t) = h(t)\cos x(t) > 0$. Then, we have

$$\delta = \int_0^{2\pi} \left|a(t) - \sigma^2\right| \mathrm{d}t < 4\pi\overline{a} = 4\pi\sigma^2,$$

which implies that $\sigma_1 = \sigma - \delta/(4\pi\sigma) > 0$. Furthermore, it follows from $\overline{h} \le 1/64$ that

$$\sigma_2 = \sigma + \frac{\delta}{4\pi\sigma} < 2\sigma = 2\sqrt{\frac{1}{2\pi}\int_0^{2\pi} h(t)\cos x(t)\,\mathrm{d}t} \le 2\sqrt{\overline{h}} \le \frac{1}{4}.$$

Therefore, $\sigma_1, \sigma_2 \in (0, 1/4)$. It is obvious that

$$c_-(t) = \frac{h(t)\cos x(t)}{6} \quad \text{and} \quad c_+(t) = 0,$$

then we have

$$\left(\frac{\tan 2\pi\sigma_1}{\tan 2\pi\sigma_2}\right)^2 \|c_-\|_1 - \|c_+\|_1 \geq \frac{\pi\overline{h}}{3}\left(\frac{\tan 2\pi\sigma_1}{\tan 2\pi\sigma_2}\right)^2 \cos\left(\frac{3\omega}{2}\right)$$
$$:= G_1(\omega).$$

Note that

$$\begin{aligned}\frac{7}{3\sigma}\left(\frac{\tan 2\pi\sigma_2}{\tan 2\pi\sigma_1}\right)^{1/2} \|b_+\|_1 \|b_-\|_1 &\leq \frac{7}{3\sigma}\left(\frac{\tan 2\pi\sigma_2}{\tan 2\pi\sigma_1}\right)^{1/2} (\|b\|_1)^2 \\ &\leq \frac{7(\pi\overline{h})^2}{3\sigma}\left(\frac{\tan 2\pi\sigma_2}{\tan 2\pi\sigma_1}\right)^{1/2} \sin^2\left(\frac{3\omega}{2}\right) \\ &:= G_2(\omega).\end{aligned}$$

Moreover, by straightforward computations, we obtain

$$G_1(0) - G_2(0) = \frac{\pi\overline{h}}{3}\left(\frac{\tan 2\pi\sigma_1}{\tan 2\pi\sigma_2}\right)^2 > 0.$$

Therefore, by continuity there exists a constant $\rho \in (0, 2\sqrt{2}/3]$ such that

$$\left(\frac{\tan 2\pi\sigma_1}{\tan 2\pi\sigma_2}\right)^2 \|c_-\|_1 - \|c_+\|_1 > \frac{7}{3\sigma}\left(\frac{\tan 2\pi\sigma_2}{\tan 2\pi\sigma_1}\right)^{1/2} \|b_+\|_1 \|b_-\|_1$$

holds whenever $0 \leq \omega < \rho$. Then, the proof is finished by using Theorem 2.1. □

4. Unstable periodic solutions

In this section, we prove that (1.3) has at least two geometrically distinct 2π-periodic solutions, and at least one of them is unstable. Furthermore, we also study the existence of periodic and subharmonic solutions with winding number of (1.3). Let x be a periodic solution of (1.3) and take its minimal period $\tau > 0$. Since 2π is the minimal period of p, then $\tau = 2k\pi$, $k \geq 1$. Therefore, if $k = 1$, x is a harmonic (periodic) solution of (1.3); if $k > 1$, x is a subharmonic solution of (1.3).

It is obvious that (1.3) can be written as the following Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x'}\right) - \frac{\partial L}{\partial x} = 0,$$

where $'$ means $\frac{\mathrm{d}}{\mathrm{d}t}$ and

$$L(x, x', t) = \frac{(x')^2}{2} + h(t)\cos x + p(t)x.$$

Now we perform the change of variables given by the Legendre transform

$$x = x, \quad y = \frac{\partial L}{\partial x'} = x'.$$

Then we can get the Hamiltonian

$$H(x,y,t)=\frac{y^2}{2}-h(t)\cos x-p(t)x,$$

and the new Hamiltonian system

$$\begin{cases} x'=\dfrac{\partial H}{\partial y}=y,\\ y'=-\dfrac{\partial H}{\partial x}=-h(t)\sin x+p(t). \end{cases} \tag{4.1}$$

Denote by $(x(t),y(t))^\top=(x(t,\theta,r),y(t,\theta,r))^\top$ the solution of the system (4.1) satisfying the initial conditions

$$x(0)=\theta,\quad y(0)=r. \tag{4.2}$$

It is obvious that the solution $(x,y)^\top$ of the problem (4.1)–(4.2) is unique and globally defined. Then we can define the Poincaré map associated to the system (4.1) as

$$S(\theta,r)=(Q(\theta,r),P(\theta,r))=(x(2\pi,\theta,r),y(2\pi,\theta,r)).$$

Clearly, the fixed points of the Poincaré map S correspond to the 2π-periodic solutions of (4.1). It follows from 2π-periodicity of the function $\sin x$ and the uniqueness of $(x(t,\theta,r),y(t,\theta,r))^\top$ that

$$x(t,\theta+2\pi,r)=x(t,\theta,r)+2\pi,\quad y(t,\theta+2\pi,r)=y(t,\theta,r). \tag{4.3}$$

Thus

$$Q(\theta+2\pi,r)=Q(\theta,r)+2\pi,\quad P(\theta+2\pi,r)=P(\theta,r),$$

which implies that the Poincaré map S is defined on the cylinder.

It is obvious that all the partial derivatives of $H(x,y,t)$ with respect to the variables (x,y) are of order equal to 2 are continuous in the variables (x,y,t). Based on the theorem of differentiability with respect to the initial conditions, we know that the Poincaré map $S\in C^2(A)$. Since $(x(t,\theta,r),y(t,\theta,r))^\top$ is unique and globally defined, so the Poincaré map S is a diffeomorphism of A. The isotopy to the identity is given by the flow

$$\begin{aligned}\Psi_\lambda(\theta,r)&=\Psi(2(1-\lambda)\pi,\theta,r)\\ &=(x(2(1-\lambda)\pi,\theta,r),y(2(1-\lambda)\pi,\theta,r)),\quad \lambda\in[0,1].\end{aligned}$$

Notice that $\Psi_0(\theta,r)=S(\theta,r)$, $\Psi_1(\theta,r)=(\theta,r)$ and this isotopy is valid on the cylinder. Then we can assert that the Poincaré map $S\in\varepsilon^2(A)$.

Now, we state and prove the main result of this section.

Theorem 4.1. *Assume that* (1.2) *holds. Then* (1.3) *has at least two geometrically distinct* 2π*-periodic solutions, and at least one of them is unstable.*

Proof. First, we claim that the Poincaré map $S(\theta, r)$ is exact symplectic. Consider the C^1 function

$$\begin{aligned} V(\theta, r) &= \int_0^{2\pi} L(x(t,\theta,r), x'(t,\theta,r), t)\,\mathrm{d}t \\ &= \int_0^{2\pi} \left[\frac{x'^2(t,\theta,r)}{2} + h(t)\cos x(t,\theta,r) + p(t)x(t,\theta,r)\right] \mathrm{d}t. \end{aligned}$$

By (1.2) and the fact (4.3), we obtain

$$\begin{aligned} V(\theta+2\pi, r) &= \int_0^{2\pi} \left[\frac{x'^2(t,\theta+2\pi,r)}{2} + h(t)\cos x(t,\theta+2\pi,r) + p(t)x(t,\theta+2\pi,r)\right] \mathrm{d}t \\ &= \int_0^{2\pi} \left[\frac{x'^2(t,\theta,r)}{2} + h(t)\cos x(t,\theta,r) + p(t)x(t,\theta,r) + 2\pi p(t)\right] \mathrm{d}t \\ &= V(\theta,r) + 2\pi \int_0^{2\pi} p(t)\,\mathrm{d}t \\ &= V(\theta,r). \end{aligned}$$

The partial derivatives of $V(\theta, r)$ is computed as

$$\begin{aligned} V_\theta(\theta,r) = &\int_0^{2\pi} \left[x'(t,\theta,r)\frac{\partial x'(t,\theta,r)}{\partial\theta} - h(t)\sin x(t,\theta,r)\frac{\partial x(t,\theta,r)}{\partial\theta}\right] \mathrm{d}t \\ &+ \int_0^{2\pi} p(t)\frac{\partial x(t,\theta,r)}{\partial\theta}\,\mathrm{d}t, \end{aligned}$$

which follows from the second equation of (4.1) that

$$V_\theta(\theta,r) = \int_0^{2\pi} \left[x'(t,\theta,r)\frac{\partial x'(t,\theta,r)}{\partial\theta} + y'(t,\theta,r)\frac{\partial x(t,\theta,r)}{\partial\theta}\right] \mathrm{d}t. \tag{4.4}$$

Integrating by parts and using the first equation of (4.1), we get

$$\begin{aligned} \int_0^{2\pi} y'(t,\theta,r)\frac{\partial x(t,\theta,r)}{\partial\theta}\,\mathrm{d}t &= \left(\frac{\partial x(t,\theta,r)}{\partial\theta}y(t,\theta,r)\right)\Big|_0^{2\pi} - \int_0^{2\pi} \frac{\partial x'(t,\theta,r)}{\partial\theta}y(t,\theta,r)\,\mathrm{d}t \\ &= y(2\pi,\theta,r)\frac{\partial x(2\pi,\theta,r)}{\partial\theta} - y(0,\theta,r)\frac{\partial x(0,\theta,r)}{\partial\theta} \\ &\quad - \int_0^{2\pi} \frac{\partial x'(t,\theta,r)}{\partial\theta}x'(t,\theta,r)\,\mathrm{d}t. \end{aligned}$$

Substituting the above equality into (4.4) gives

$$V_\theta(\theta,r) = y(2\pi,\theta,r)\frac{\partial x(2\pi,\theta,r)}{\partial\theta} - y(0,\theta,r)\frac{\partial x(0,\theta,r)}{\partial\theta}. \tag{4.5}$$

Similarly, we have

$$V_r(\theta,r) = y(2\pi,\theta,r)\frac{\partial x(2\pi,\theta,r)}{\partial r} - y(0,\theta,r)\frac{\partial x(0,\theta,r)}{\partial r}. \tag{4.6}$$

By (4.5) and (4.6), we get

$$\begin{aligned}
\mathrm{d}V &= V_\theta(\theta,r)\,\mathrm{d}\theta + V_r(\theta,r)\,\mathrm{d}r \\
&= \left[y(2\pi,\theta,r)\frac{\partial x(2\pi,\theta,r)}{\partial \theta} - y(0,\theta,r)\frac{\partial x(0,\theta,r)}{\partial \theta}\right]\mathrm{d}\theta \\
&\quad + \left[y(2\pi,\theta,r)\frac{\partial x(2\pi,\theta,r)}{\partial r} - y(0,\theta,r)\frac{\partial x(0,\theta,r)}{\partial r}\right]\mathrm{d}r \\
&= y(2\pi,\theta,r)\left[\frac{\partial x(2\pi,\theta,r)}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial x(2\pi,\theta,r)}{\partial r}\,\mathrm{d}r\right] \\
&\quad - y(0,\theta,r)\left[\frac{\partial x(0,\theta,r)}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial x(0,\theta,r)}{\partial r}\,\mathrm{d}r\right] \\
&= y(2\pi,\theta,r)\,\mathrm{d}x(2\pi,\theta,r) - y(0,\theta,r)\,\mathrm{d}x(0,\theta,r) \\
&= P\,\mathrm{d}Q - r\,\mathrm{d}\theta,
\end{aligned}$$

which means that the Poincaré map $S(\theta,r)$ is exact symplectic.

Secondly, we prove that the Poincaré map S satisfies the boundary twist condition (2.5), that is, there exist constants $\rho > 0$ and $\epsilon > 0$ such that

$$Q(\theta,\rho) - \theta > \epsilon, \quad Q(\theta,-\rho) - \theta < -\epsilon, \quad \theta \in [0,2\pi).$$

Integrating the second equation of (4.1) from 0 to $t \in [0,2\pi]$, we get

$$\begin{aligned}
y(t) &= r + \int_0^t [-h(s)\sin x + p(s)]\,\mathrm{d}s \\
&\geq r - \int_0^{2\pi} (|h(s)| + |p(s)|)\,\mathrm{d}s \\
&= r - 2\pi\left(\overline{|h|} + \overline{|p|}\right),
\end{aligned}$$

where $\overline{|h|} = \int_0^{2\pi} |h(s)|\,\mathrm{d}s$. Then we can find a positive constant $\rho_1 \geq 2\pi\left(\overline{|h|} + \overline{|p|}\right) > 0$ such that $y(t) > 0$ if $r > \rho_1$, $\forall\, t \in [0,2\pi]$. It follows from the first equation of (4.1) that

$$x'(t) = y(t) > 0,$$

which means that x is increasing for $t \in [0,2\pi]$. So we can choose constant $\rho = \rho_1 + 1$, then we have

$$Q(\theta,\rho) - \theta = x(2\pi,\theta,\rho) - x(0,\theta,\rho) > 0.$$

By a standard compactness argument, we can deduce that there exists $\epsilon > 0$ such that

$$Q(\theta,\rho) - \theta > \epsilon, \quad \theta \in [0,2\pi).$$

Analogously, we can conclude that

$$Q(\theta, -\rho) - \theta < -\epsilon, \quad \theta \in [0, 2\pi).$$

Finally, in order to apply Theorem 2.2, we take $A = \mathbb{R} \times [-\rho, \rho]$. Because the solutions of (4.1) are globally defined, one can find a larger B such that $S(A) \subset \operatorname{int} B$. Since the right-hand side of (4.1) is analytic with respect to the variables (x, y), so it follows from the analytic dependence on initial conditions that the Poincaré map S is also analytic.

Up to now, all conditions of Theorem 2.2 are established, thus we get that the Poincaré map $S(\theta, r) = (Q(\theta, r), P(\theta, r)) = (x(2\pi, \theta, r), y(2\pi, \theta, r))$ has at least two fixed points, and at least one of them is unstable. That is, (4.1) has at least two geometrically distinct 2π-periodic solutions and at least one of them is unstable. □

In this section, we also consider the existence of the so-called 2π-periodic solutions with winding number of (1.3), i.e., solutions x such that

$$x(t + 2\pi) = x(t) + 2N\pi, \quad N \in \mathbb{Z},\ \forall\, t \in \mathbb{R}.$$

Such solutions are also called the running solutions. Clearly, we get the usual 2π-periodic solutions when the winding number is zero.

Let x be a 2π-periodic solution of (1.3) with winding number N. Taking the change of variables

$$u(t) = x(t) - Nt,$$

then we obtain

$$u(t + 2\pi) = x(t + 2\pi) - N(t + 2\pi) = x(t) - Nt = u(t),$$

which implies that 2π-periodic solutions with winding number N of (1.3) correspond to 2π-periodic solutions of the equation

$$u'' + h(t)\sin(u + Nt) = p(t).$$

Proceeding as in the proof of Theorem 4.1, we can prove the following result.

Theorem 4.2. *Assume that* (1.2) *holds. Then for every integer N,* (1.3) *has at least two geometrically distinct 2π-periodic solutions with winding number N, and at least one of them is unstable.*

Finally, we study the existence of k-order subharmonic solutions with winding number N of (1.3), that is,

$$x(t + 2k\pi) = x(t) + 2N\pi, \quad \forall\, t \in \mathbb{R}. \tag{4.7}$$

Theorem 4.3. *Assume that* (1.2) *holds. Then for each N, k being relatively prime,* (1.3) *has at least two geometrically distinct k-order subharmonic solutions with winding number N and $2k\pi$ is the minimal period. Moreover, at least one of them is unstable.*

Proof. By Theorem 4.2, with 2π replaced by $2k\pi$, we get that (1.3) has at least two geometrically distinct solutions satisfying (4.7). Moreover, at least one of them is unstable. It remains to verify that $2k\pi$ is the minimal period of periodic solutions with winding number N of (1.3).

Assume by contradiction that $2l\pi$ is the minimal period, where $l \in \{1, 2, \ldots, k-1\}$, which means that there exists a nonzero integer j such that

$$x(t + 2l\pi) = x(t) + 2j\pi, \quad \forall\, t \in \mathbb{R}. \tag{4.8}$$

Obviously, there exist two positive integers n_1 and n_2 such that

$$n_1 l = n_2 k. \tag{4.9}$$

By (4.7), we have

$$x(t + 2n_2 k\pi) = x(t) + 2n_2 N\pi, \quad \forall\, t \in \mathbb{R}.$$

By (4.8), we have

$$x(t + 2n_1 l\pi) = x(t) + 2n_1 j\pi, \quad \forall\, t \in \mathbb{R}.$$

It follows from the above two equalities and the uniqueness of the solution x that

$$n_2 N = n_1 j,$$

i.e.,

$$\frac{n_2}{n_1} = \frac{j}{N}.$$

From (4.9), we know that

$$\frac{n_2}{n_1} = \frac{l}{k}.$$

Then we have

$$\frac{N}{k} = \frac{j}{l},$$

which is impossible because N and k are relatively prime and j is a nonzero integer and $l \in \{1, 2, \ldots, k-1\}$. □

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