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New Stability Criteria for Linear Volterra Time-varying Integro-differential Equations

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Abstract. Using a novel approach, we get some new explicit criteria for uniform asymptotic stability and exponential asymptotic stability of linear Volterra time-varying integro-differential equations of non-convolution type. Some examples are given to illustrate the obtained results.

1. Introduction

Motivated by many applications in Biology, Economics, Physics, Engineering, and other Applied Sciences, problems of stability of Volterra integro-differential equations have attracted much attention from researchers during the past decades, see [1–20, 22, 23] and the references therein. Stability analysis of time-varying integro-differential equations is, in general, difficult. The traditional approach to stability of time-varying Volterra integro-differential equations is the Lyapunov function method and most of existing stability criteria for such equations are abstract and not easy to use, see e.g. [10–12, 22].

In particular, problems of stability of linear time-varying Volterra integro-differential equations of non-convolution type

$$\dot{x}(t) = A(t)x(t) + \int_0^t B(t,s)x(s)\,ds, \quad t \geq \sigma \geq 0, \tag{1.1}$$

$$x(t) = \varphi(t), \quad t \in [0,\sigma], \tag{1.2}$$

has been studied intensively. Some characterisations of the uniform asymptotic stability and the exponential asymptotic stability of the linear time-varying integro-differential equation (1.1) have been reported in [11, 12, 23]. By the direct Lyapunov method, some abstract criteria for asymptotic stability of (1.1) have been given in [5, 10, 22, 23]. To the best of our knowledge, there are not many explicit stability criteria for (1.1). There are

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a few explicit stability criteria for scalar linear Volterra time-varying integro-differential equations of non-convolution type, see e.g. [2, 10, 23].

Obviously, problems of stability of linear Volterra time-invariant integro-differential equations of the convolution type

$$\dot{y}(t) = A_0 y(t) + \int_0^t B_0(t-s)y(s)\,ds, \quad t \geq \sigma \geq 0, \tag{1.3}$$

are much easier. There have been many works dedicated to the uniform asymptotic stability and the exponential stability of the linear time-invariant integro-differential equation of the convolution type (1.3). Several explicit stability criteria for (1.3) are given in [1, 3, 5, 17–20].

In this paper, we present a new approach to the uniform asymptotic stability and the exponential asymptotic stability of the linear Volterra time-varying integro-differential equations of non-convolution type (1.1). Our approach relies upon the spectral properties of Metzler matrices and the comparison principle. Consequently, we get some new explicit criteria for the uniform asymptotic stability and the exponential asymptotic stability of the linear time-varying integro-differential equation of non-convolution type (1.1). Furthermore, we derive two explicit stability bounds for (1.1) subject to time-varying structured perturbations and time-varying affine perturbations. Some examples are given to illustrative the obtained results. To the best of our knowledge, most of the obtained results of this paper are new.

2. Preliminaries

Let $\mathbb{N}$ be the set of all natural numbers. For given $m \in \mathbb{N}$, let us denote $\underline{m} := \{1, 2, \ldots, m\}$ and $\underline{m_0} := \{0, 1, 2, \ldots, m\}$. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ where $\mathbb{C}$ and $\mathbb{R}$ denote the sets of all complex and all real numbers, respectively. Define $\mathbb{R}_+ := \{s \in \mathbb{R} : s \geq 0\}$. For given integers $l, q \geq 1$, $\mathbb{K}^l$ denotes the l-dimensional vector space over $\mathbb{K}$ and $\mathbb{K}^{l\times q}$ stands for the set of all $l \times q$-matrices with entries in $\mathbb{K}$. Inequalities between real matrices or vectors will be understood componentwise, i.e., for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l\times q}$, we write $A \geq B$ if $a_{ij} \geq b_{ij}$ for $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, q$. In particular, if $a_{ij} > b_{ij}$ for $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, q$, then we write $A \gg B$ instead of $A \geq B$. We denote by $\mathbb{R}_+^{l\times q}$ the set of all nonnegative matrices $A \geq 0$. Similar notations are adopted for vectors. For $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l\times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. Then one has $|PQ| \leq |P|\,|Q|$, for all $P \in \mathbb{R}^{l\times q}$, $Q \in \mathbb{R}^{q\times r}$. A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *monotonic* if $\|x\| \leq \|y\|$ whenever $x, y \in \mathbb{K}^n$, $|x| \leq |y|$. Every p-norm on $\mathbb{K}^n$ ($\|x\|_p = (|x|_1^p + |x|_2^p + \cdots + |x|_n^p)^{1/p}$, $1 \leq p < \infty$) and $\|x\|_\infty = \max_{i=1,2,\ldots,n} |x_i|$, is monotonic. Throughout the paper, if otherwise not stated, the norm of a matrix $P \in \mathbb{K}^{l\times q}$

is understood as its operator norm associated with a given pair of monotonic vector norms on $\mathbb{K}^l$ and $\mathbb{K}^p$, that is $\|P\| = \max_{\|y\|=1} \|Py\|$. Note that, one has

$$P \in \mathbb{K}^{l\times q},\ Q \in \mathbb{R}_+^{l\times q},\ |P| \le Q \implies \|P\| \le \||P|\| \le \|Q\|\,,$$

see, e.g. [21]. In particular, if $\mathbb{R}^n$ is endowed with $\|\cdot\|_1$ or $\|\cdot\|_\infty$ then $\|A\| = \||A|\|$ for any $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. More precisely, one has $\|A\|_1 = \||A|\|_1 = \max_{1\le j\le n}\sum_{i=1}^n |a_{ij}|$; $\|A\|_\infty = \||A|\|_\infty = \max_{1\le i\le n}\sum_{j=1}^n |a_{ij}|$.

For any matrix $M \in \mathbb{K}^{n\times n}$ the *spectral abscissa* of M is denoted by $s(M) = \max\{\Re\lambda : \lambda \in \sigma(M)\}$, where $\sigma(M) := \{z \in \mathbb{C} : \det(zI_n - M) = 0\}$ is the spectrum of M. A matrix $M \in \mathbb{R}^{n\times n}$ is said to be *Hurwitz stable* if, and only if, $s(M) < 0$. A matrix $M \in \mathbb{R}^{n\times n}$ is called a *Metzler matrix* if all off-diagonal elements of M are nonnegative. We now summarize in the following theorem some properties of Metzler matrices which will be used in that follows.

Theorem 2.1. [21] *Suppose $M \in \mathbb{R}^{n\times n}$ is a Metzler matrix. Then*

(i) *(Perron-Frobenius) $s(M)$ is an eigenvalue of M and there exists a nonnegative eigenvector $x \ne 0$ such that $Mx = s(M)x$.*

(ii) *Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \ge 0$ such that $Mx \ge \alpha x$ if and only if $s(M) \ge \alpha$.*

(iii) *$(tI_n - M)^{-1}$ exists and is nonnegative if and only if $t > s(M)$.*

(iv) *Given $B \in \mathbb{R}_+^{n\times n}$, $C \in \mathbb{C}^{n\times n}$. Then*

$$|C| \le B \implies s(M + C) \le s(M + B).$$

The following is immediate from Theorem 2.1.

Theorem 2.2. *Let $M \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Then the following statements are equivalent:*

(i) $s(M) < 0$*;*

(ii) *$Mp \ll 0$ for some $p \in \mathbb{R}_+^n$;*

(iii) *M is invertible and $M^{-1} \le 0$;*

(iv) *For given $b \in \mathbb{R}^n$, $b \gg 0$ there exists $x \in \mathbb{R}_+^n$, such that $Mx + b = 0$;*

(v) *For any $x \in \mathbb{R}_+^n \setminus \{0\}$, the row vector x^TM has at least one negative entry.*

Let $\mathbb{K}^{m\times n}$ be endowed with the norm $\|\cdot\|$ and let J be an interval of $\mathbb{R}$. As usual, $L^1(\mathbb{R}_+,\mathbb{K}^{m\times n})$ denotes the Banach space of L^1-integrable matrix functions on $\mathbb{R}_+$ with values in $\mathbb{K}^{m\times n}$ and endowed with the L^1-norm. Let $C(J,\mathbb{K}^{m\times n})$ be the vector space of all continuous functions on J with values in $\mathbb{K}^{m\times n}$. In particular, for each $\sigma\in\mathbb{R}_+$, $C([0,\sigma],\mathbb{K}^{m\times n})$ is a Banach space endowed with the norm

$$\|\varphi\| := \max\left\{\|\varphi(\theta)\| : \theta\in[0,\sigma]\right\}.$$

For given $\varphi\in C([0,\sigma],\mathbb{K}^{m\times n})$, $|\varphi|\in C([0,\sigma],\mathbb{K}^{m\times n})$ is defined by $|\varphi|(t) := |\varphi(t)|$, $t\in[0,\sigma]$.

3. Explicit criteria for uniformly asymptotic stability

Consider the linear time-varying Volterra integro-differential equation of nonconvolution type (1.1), where $A(\cdot)\colon\mathbb{R}_+\to\mathbb{R}^{n\times n}$ and $B(\cdot\,,\cdot)\colon\Delta=\{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+ : t\geq s\}\to\mathbb{R}^{n\times n}$ are given matrix-valued continuous functions.

Definition 3.1. Let $\sigma\in\mathbb{R}_+$ and $\varphi\in C([0,\sigma],\mathbb{R}^n)$ be given. A function $x(\cdot)\colon\mathbb{R}_+\to\mathbb{R}^n$ is said to be a solution of (1.1)–(1.2) if $x(\cdot)$ satisfies (1.1) for any $t\geq\sigma$ and fullfils the initial condition (1.2).

It is well-known that for a fixed $\sigma\geq 0$ and a given $\varphi\in C([0,\sigma],\mathbb{R}^n)$, the initial value problem (1.1)–(1.2) always has a unique solution, see e.g. [4]. We denote it by $x(\cdot\,;\sigma,\varphi)$.

Definition 3.2. (i) The zero solution of (1.1) is said to be uniformly stable (shortly, US) if for any $\epsilon>0$ there exists $\delta=\delta(\epsilon)>0$ such that

$$\sigma\geq 0,\ \varphi\in C([0,\sigma],\mathbb{R}^n),\ \|\varphi\|<\delta \implies \|x(t;\sigma,\varphi)\|<\epsilon,\ \forall\, t\geq\sigma.$$

(ii) The zero solution of (1.1) is said to be uniformly asymptotically stable (shortly, UAS) if it is US and there exists $\delta>0$ with the property that, for each $\epsilon>0$ there is a number $T=T(\epsilon)>0$ such that

$$\sigma\geq 0,\ \varphi\in C([0,\sigma],\mathbb{R}^n),\ \|\varphi\|<\delta \implies \|x(t;\sigma,\varphi)\|<\epsilon,\ \forall\, t\geq\sigma+T.$$

If the zero solution of (1.1) is US (UAS) then we also say that (1.1) is US (UAS), respectively.

The following theorem gives a characterisation of the uniform asymptotic stability of (1.1).

Theorem 3.3. [22] *Assume that*

(i)
$$\sup_{t\ge 0}\left\{\|A(t)\| + \int_0^t \|B(t,s)\|\,ds\right\} < \infty;$$

(ii) *for any* $\epsilon > 0$*, there exists* $S = S(\epsilon)$ *such that*
$$\int_0^{t-S} \|B(t,s)\|\,ds < \epsilon, \quad t \ge S;$$

(iii) $A(t)$ *and* $B(t, t+s)$ *are bounded and uniformly continuous in* $(t,s) \in \{(t,s) \in \mathbb{R}_+ \times K : -t \le s \le 0\}$ *for any compact set* $K \subset (-\infty, 0]$*.*

Then the zero solution of (1.1) *is UAS if and only if*
$$\sup_{t\ge 0}\int_0^t \|R(t,s)\|\,ds < \infty,$$

where $R(t,s)$ *is the resolvent equation of* (1.1) *defined by*
$$\frac{\partial R(t,s)}{\partial s} = -R(t,s)A(s) - \int_s^t R(t,u)B(u,s)\,du, \quad R(t,t) = I_n,\ t \ge s \ge 0.$$

In general, the problem of the uniform asymptotic stability of the linear time-varying Volterra integro-differential equation of nonconvolution type (1.1) is difficult and most of existing stability criteria in the literature are often complicated and hard to apply in practice, see e.g. [13, 22]. Thus, it is interesting to find simple criteria for the uniform asymptotic stability of (1.1).

For given $A := (a_{ij}) \in \mathbb{R}^{n\times n}$, let
$$M(A) := \operatorname{diag}(a_{11}, a_{22}, \dots, a_{nn}) + |A - \operatorname{diag}(a_{11}, a_{22}, \dots, a_{nn})|\,.$$

Theorem 3.4. *Suppose there exist* $A_0 \in \mathbb{R}^{n\times n}$ *and a continuous matrix-valued function* $B_0(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ *such that*

$$M(A(t)) \le A_0,\ \forall\, t \ge 0 \quad \text{and} \quad |B(t,s)| \le B_0(t-s),\ \forall\, t \ge s \ge 0. \tag{3.1}$$

If $B_0(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ *and*

$$s\left(A_0 + \int_0^\infty B_0(s)\,ds\right) < 0, \tag{3.2}$$

then (1.1) *is UAS.*

Remark 3.5. Roughly speaking, Theorem 3.4 says that if (1.1) is bounded above (in some sense) by (1.3) and (1.3) is UAS then (1.1) is UAS. More precisely, (3.1) means that (1.1) is bounded above by (1.3) and (3.2) ensures that (1.3) is UAS. Then, (1.1) is UAS too.

Proof. Consider the linear time-invariant integro-differential equation of the convolution type (1.3) where $A_0 \in \mathbb{R}^{n\times n}$ and $B_0(\cdot)$ satisfy (3.1)–(3.2). It is well-known that (1.3) is UAS if and only if its characteristic equation has no roots in the right-hand complex plane, that is,

$$\det(\lambda I_n - A_0 - \widehat{B}_0(\lambda)) \neq 0, \quad \forall\, \lambda \in \mathbb{C}^+ := \{\lambda \in \mathbb{C} : \Re\lambda \geq 0\}\,, \tag{3.3}$$

see e.g. [4], where

$$\widehat{B}_0(\lambda) := \int_0^\infty e^{-\lambda s} B_0(s)\, ds$$

is the Laplace transform of $B_0(\cdot)$. We now show that (3.3) holds and thus, (1.3) is UAS. Assume that on the contrary that

$$\det\left(\lambda I_n - A_0 - \int_0^\infty e^{-\lambda s} B_0(s)\, ds\right) = 0$$

for some $\lambda \in \mathbb{C}^+$. It follows that

$$0 \leq \Re\lambda \leq s\left(A_0 + \int_0^\infty e^{-\lambda s} B_0(s)\, ds\right).$$

Since A_0 is a Metzler matrix, Theorem 2.1(iv) implies

$$0 \leq s\left(A_0 + \int_0^\infty e^{-\lambda s} B_0(s)\, ds\right) \leq s\left(A_0 + \int_0^\infty B_0(s)\, ds\right),$$

which conflicts with (3.2).

Fix $\sigma \in \mathbb{R}_+$ and let $\varphi \in C([0,\sigma],\mathbb{R}^n)$. Denote by $x(\cdot) := x(\cdot\,;\sigma,\varphi)$ the solution of (1.1)–(1.2). Let $y(\cdot) := y(\cdot\,;\sigma,|\varphi|)$ be the solution of (1.3) satisfying the initial condition $y(t) = |\varphi|\,(t)$, $t \in [0,\sigma]$. By Theorem 2.2, (3.2) implies

$$\left(A_0 + \int_0^\infty B_0(s)\, ds\right) p \ll 0 \tag{3.4}$$

for some $p \in \mathbb{R}^n$, $p \gg 0$. For a given $\epsilon > 0$, we claim that $|x(t)| \leq y(t) + \epsilon p$, $\forall\, t \in [0,+\infty)$. Clearly, $|x(t)| = |\varphi(t)| = y(t) \ll y(t) + \epsilon p$, $\forall\, t \in [0,\sigma]$. We show that $|x(t)| \leq y(t) + \epsilon p$, $\forall\, t \in [\sigma,\infty]$. Assume on the contrary that there exists $t_1 > \sigma$ such that $|x(t_1)| \not\leq y(t_1)+\epsilon p$. Set $t^* := \inf\{t \in (\sigma,+\infty) : |x(t)| \not\leq y(t) + \epsilon p\}$. By continuity, $t^* > \sigma$ and there is an index $i_0 \in \underline{n}$ such that

$$|x(t)| \leq y(t)+\epsilon p,\ \forall\, t \in [0,t^*); \quad |x_{i_0}(t^*)| = y_{i_0}(t^*)+\epsilon p_{i_0}, \quad |x_{i_0}(\tau_k)| > y_{i_0}(\tau_k)+\epsilon p_{i_0} \tag{3.5}$$

for some $\tau_k \in (t^*, t^*+1/k)$, $k \in \mathbb{N}$. Let $A(t) := (a_{ij}(t)) \in \mathbb{R}^{n\times n}$, $\forall\, t \in \mathbb{R}_+$; $A_0 := (a_{ij}^{(0)}) \in \mathbb{R}^{n\times n}$ and let $B(t,s) := (b_{ij}(t,s)) \in \mathbb{R}^{n\times n}$, $t \geq s \geq 0$; $B_0(t) := (b_{ij}^{(0)}(t)) \in \mathbb{R}^{n\times n}$, $t \in \mathbb{R}_+$. For every $i \in \underline{n}$, we have

$$\frac{d}{dt}|x_i(t)| = \operatorname{sgn}(x_i(t))\dot{x}_i(t) \leq a_{ii}(t)\,|x_i(t)| + \sum_{\substack{j=1\\ j\neq i}}^n |a_{ij}(t)|\,|x_j(t)| + \int_0^t \sum_{j=1}^n |b_{ij}(t,s)|\,|x_j(s)|\, ds$$

for almost any $t \in [\sigma, +\infty)$. It follows that for any $t \in [\sigma, \infty)$

$$
\begin{aligned}
\mathfrak{D}^+ |x_i(t)| &:= \limsup_{h\to 0^+} \frac{|x_i(t+h)| - |x_i(t)|}{h} = \limsup_{h\to 0^+} \frac{1}{h}\int_t^{t+h} \frac{d}{ds}|x_i(s)|\,ds \\
&\le a_{ii}(t)\,|x_i(t)| + \sum_{\substack{j=1\\ j\ne i}}^n |a_{ij}(t)|\,|x_j(t)| + \sum_{j=1}^n \int_0^t |b_{ij}(t,s)|\,|x_j(s)|\,ds,
\end{aligned} \tag{3.6}
$$

where $\mathfrak{D}^+$ denotes the Dini upper-right derivative. In particular, it follows from (3.5) and (3.6) that

$$
\begin{aligned}
\mathfrak{D}^+ |x_{i_0}(t^*)| \quad &\le \quad a_{i_0 i_0}(t^*)\,|x_{i_0}(t^*)| + \sum_{\substack{j=1\\ j\ne i_0}}^n |a_{i_0 j}(t^*)|\,|x_j(t^*)| \\
&\quad + \sum_{j=1}^n \int_0^{t^*} |b_{i_0 j}(t^*,s)|\,|x_j(t^*)|\,ds \\
&\overset{(3.5)}{\le} \quad a_{i_0 i_0}(t^*)(y_{i_0}(t^*) + \epsilon p_{i_0}) + \sum_{\substack{j=1\\ j\ne i_0}}^n |a_{i_0 j}(t^*)|\,(y_j(t^*) + \epsilon p_j) \\
&\quad + \sum_{j=1}^n \int_0^{t^*} |b_{i_0 j}(t^*,s)|\,(y_j(t^*) + \epsilon p_j)\,ds \\
&\overset{(3.1)}{\le} \quad a^{(0)}_{i_0 i_0}(y_{i_0}(t^*) + \epsilon p_{i_0}) + \sum_{\substack{j=1\\ j\ne i_0}}^n a^{(0)}_{i_0 j}(y_j(t^*) + \epsilon p_j) \\
&\quad + \sum_{j=1}^n \int_0^{t^*} b^{(0)}_{i_0 j}(t^* - s)(y_j(t^*) + \epsilon p_j)\,ds \\
&= \quad \sum_{j=1}^n a^{(0)}_{i_0 j}(y_j(t^*) + \epsilon p_j) + \sum_{j=1}^n \int_0^{t^*} b^{(0)}_{i_0 j}(t^* - s)(y_j(t^*) + \epsilon p_j)\,ds \\
&= \quad \dot{y}_{i_0}(t^*) + \epsilon\left([A_0 p]_{i_0} + \left[\left(\int_0^{t^*} B_0(t^*-s)\,ds\right)p\right]_{i_0}\right) \\
&\le \quad \dot{y}_{i_0}(t^*) + \epsilon\left(\left[\left(A_0 + \int_0^{+\infty} B_0(t^*-s)\,ds\right)p\right]_{i_0}\right) \\
&\overset{(3.4)}{<} \quad \dot{y}_{i_0}(t^*) = \mathfrak{D}^+ y_{i_0}(t^*).
\end{aligned}
$$

On the other hand, (3.5) implies that

$$
\begin{aligned}
\mathfrak{D}^+ |x_{i_0}(t^*)| &= \limsup_{t\to t^{*+}} \frac{|x_{i_0}(t)| - |x_{i_0}(t^*)|}{t - t^*} \ge \overline{\lim_{k\to\infty}} \frac{|x_{i_0}(\tau_k)| - |x_{i_0}(t^*)|}{\tau_k - t^*} \\
&\ge \overline{\lim_{k\to\infty}} \frac{y_{i_0}(\tau_k) - y_{i_0}(t^*)}{\tau_k - t^*} = \lim_{k\to\infty} \frac{y_{i_0}(\tau_k) - y_{i_0}(t^*)}{\tau_k - t^*} = \mathfrak{D}^+ y_{i_0}(t^*).
\end{aligned}
$$

This is a contradiction. Hence, $|x(t)| \ll y(t) + \epsilon p$, $\forall\, t \in [\sigma, +\infty)$. Since $\epsilon > 0$ is arbitrary, it follows that

$$|x(t;\sigma,\varphi)| \leq y(t;\sigma,|\varphi|), \quad \forall\, \varphi \in C([0,\sigma],\mathbb{R}^n),\ t \geq 0.$$

By the monotonicity of vector norms, this yields

$$\|x(t;\sigma,\varphi)\| \leq \|y(t;\sigma,|\varphi|)\|, \quad \forall\, t \geq 0,\ \forall\, \varphi \in C([0,\sigma],\mathbb{R}^n). \tag{3.7}$$

As (1.3) is UAS, (3.7) implies that (1.1) is UAS. This completes the proof. □

For given $A \in \mathbb{R}^{n\times n}$, let

$$\mu(A) := \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

be the matrix measure of A, see e.g. [6]. The following theorem is a scalar version of Theorem 3.4.

Theorem 3.6. *Suppose there exist $\gamma > 0$ and a continuous function $b(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\mu(A(t)) \leq -\gamma,\ \forall\, t \geq 0 \quad \textit{and} \quad \|B(t,s)\| \leq b(t-s),\ \forall\, t \geq s \geq 0. \tag{3.8}$$

If $b(\cdot) \in L^1(\mathbb{R}_+,\mathbb{R})$ and

$$-\gamma + \int_0^\infty b(s)\, ds < 0, \tag{3.9}$$

then (1.1) *is UAS.*

Proof. The proof is similar to that of Theorem 3.4. So we omit the proof. □

Example 3.7. Consider the linear time-varying Volterra integro-differential equation in $\mathbb{R}^2$ defined by

$$\dot{x}(t) = A(t)x(t) + \int_0^t B(t,s)x(s)\, ds, \quad t \geq \sigma \geq 0, \tag{3.10}$$

where

$$A(t) := \begin{pmatrix} -4 - t^2 & 0 \\ -e^{-t} & -4 \end{pmatrix},\ t \geq 0; \quad B(t,s) := \begin{pmatrix} -e^{-(t-s)}\sin s & \frac{\cos(st)}{(1+t-s)^2} \\ 0 & e^{-(t-s)}\cos s \end{pmatrix},\ t \geq s \geq 0.$$

Let

$$A_0 := \begin{pmatrix} -4 & 0 \\ 1 & -4 \end{pmatrix}; \quad B_0(t) := \begin{pmatrix} e^{-t} & \frac{1}{(1+t)^2} \\ 0 & e^{-t} \end{pmatrix},\ t \geq 0.$$

Then A_0 and $B_0(\cdot)$ satisfy (3.1) for $t \geq s \geq 0$. By simple computation, we have

$$A_0 + \int_0^\infty B_0(t)\, dt = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}; \quad s\left(A_0 + \int_0^\infty B_0(t)\, dt\right) = -2 < 0.$$

Thus (3.10) is UAS by Theorem 3.4.

4. Explicit criteria for exponential asymptotic stability

In this section, we deal with the exponential asymptotic stability of (1.1). In contrast to ordinary differential equations and time-delay differential equations, the uniform asymptotic stability and the exponential asymptotic stability of linear integro-differential equations are, in general, not equivalent. For example, the exponential asymptotic stability of (1.3) implies its uniformly asymptotic stability, but the converse statement does not hold. Actually, they are equivalent provided the kernel of (1.3) $B_0(\cdot)$, exponentially decays, see Theorem 4.2 below.

Definition 4.1. The zero solution of (1.1) is said to be exponentially asymptotically stable (shortly, EAS), if there exists $K, \beta > 0$ such that

$$\|x(t;\sigma,\varphi)\| \le K e^{-\beta(t-\sigma)} \|\varphi\|, \quad \forall\, t \ge \sigma,\ \forall\, \varphi \in C([0,\sigma],\mathbb{R}^n).$$

When the zero solution of (1.1) is EAS, we also say that (1.1) is EAS.

Theorem 4.2. [18] *Assume that* (1.3) *is UAS and there exists* $\alpha > 0$ *such that*

$$\int_0^\infty \|B_0(s)\| e^{\alpha s}\, ds < \infty. \tag{4.1}$$

Then (1.3) *is EAS.*

We are now in the position to state the main results of this section.

Theorem 4.3. *Suppose all hypotheses of Theorem* 3.4 *are satisfied. If* (4.1) *holds for some* $\alpha > 0$ *then* (1.1) *is EAS.*

Proof. Fix $\sigma \in \mathbb{R}_+$ and let $\varphi \in C([0,\sigma],\mathbb{R}^n)$. Let $x(\cdot) := x(\cdot\,;\sigma,\varphi)$ be the solution of (1.1)–(1.2) and let $y(\cdot) := y(\cdot\,;\sigma,|\varphi|)$ be the solution of (1.3) satisfying $y(t) = |\varphi|(t)$, $t \in [0,\sigma]$. We already showed in the proof of Theorem 3.4 that

$$\|x(t;\sigma,\varphi)\| \le \|y(t;\sigma,|\varphi|)\|, \quad \forall\, t \ge 0,\ \forall\, \varphi \in C([0,\sigma],\mathbb{R}^n). \tag{4.2}$$

Note that (1.3) is UAS. Then (4.1) ensures that (1.3) is EAS by Theorem 4.2. Finally, (4.2) implies that (1.3) is EAS. This completes the proof. □

Theorem 4.4. *Suppose all hypotheses of Theorem* 3.6 *are satisfied. If*

$$\int_0^\infty b(s) e^{\alpha s}\, ds < \infty \tag{4.3}$$

for some $\alpha > 0$ *then* (1.1) *is EAS.*

Proof. The proof is similar to that of Theorem 4.3 and it is omitted here. □

The following theorem gives new criteria for the exponential asymptotic stability of (1.1). As far as we know, a result like Theorem 4.5 below cannot be found in the literature.

Theorem 4.5. *The zero solution of* (1.1) *is EAS provided one of the following conditions holds:*

(i) *There exist* $\beta_1 > 0$ *and* $p \in \mathbb{R}^n$, $p \gg 0$ *so that*

$$\left(M(A(t)) + \int_0^t |B(t,s)|\, e^{\beta_1(t-s)}\, ds\right) p \ll -\beta_1 p, \quad \forall\, t \geq 0. \tag{4.4}$$

(ii) *There exist* $\beta_2 > 0$ *and* $M \in \mathbb{R}^{n\times n}$, $s(M) < 0$ *so that*

$$\left(M(A(t)) + \int_0^t |B(t,s)|\, e^{\beta_2(t-s)}\, ds\right) \leq M, \quad \forall\, t \geq 0. \tag{4.5}$$

Proof. Assume that (i) holds. Let $\varphi \in C([0,\sigma],\mathbb{R}^n)$, $\|\varphi\| \leq 1$. Since $p \gg 0$, there exists a positive number K (K is independent of σ) such that

$$|\varphi(t)| \ll Kp, \quad \forall\, t \in [0,\sigma],\ \forall\, \varphi \in C([0,\sigma],\mathbb{R}^n),\ \|\varphi\| \leq 1.$$

Set $u(t) := Ke^{-\beta_1(t-\sigma)}p$, $t \in [0,\infty)$ and let $x(t) := x(t;\sigma,\varphi)$, $t \in [0,\infty)$, $\|\varphi\| \leq 1$. Clearly,

$$|x(t)| = |\varphi(t)| \ll Kp \leq Ke^{-\beta_1(t-\sigma)}p = u(t), \quad \forall\, t \in [0,\sigma].$$

We claim that $|x(t)| \leq u(t)$, $\forall\, t \in [\sigma,+\infty)$. Assume on the contrary that there exists $t_1 > \sigma$ such that $|x(t_1)| \nleq u(t_1)$. Set $t^* := \inf\{t \in (\sigma,+\infty) : |x(t)| \nleq u(t)\}$. By continuity, $t^* > \sigma$ and there is $i_0 \in \underline{n}$ such that

$$|x(t)| \leq u(t),\ \forall\, t \in [\sigma,t^*]; \quad |x_{i_0}(t^*)| = u_{i_0}(t^*), \quad |x_{i_0}(\xi_k)| > u_{i_0}(\xi_k) \tag{4.6}$$

for some $\xi_k \in (t^*, t^*+1/k)$, $k \in \mathbb{N}$. Let $A(t) := (a_{ij}(t)) \in \mathbb{R}^{n\times n}$, $\forall\, t \in \mathbb{R}_+$ and let $B(t,s) := (b_{ij}(t,s)) \in \mathbb{R}^{n\times n}$, $t \in \mathbb{R}$, $s \in \mathbb{R}_+$. For every $i \in \underline{n}$, we have

$$\begin{aligned}\frac{d}{dt}|x_i(t)| = \operatorname{sgn}(x_i(t))\dot{x}_i(t) &\leq a_{ii}(t)\,|x_i(t)| + \sum_{\substack{j=1\\ j\neq i}}^n |a_{ij}(t)|\,|x_j(t)| \\ &\quad + \int_0^t \sum_{j=1}^n |b_{ij}(t,s)|\,|x_j(s)|\, ds\end{aligned}$$

for almost any $t \in [\sigma,+\infty)$. It follows that for any $t \in [\sigma,+\infty)$,

$$\begin{aligned}\mathfrak{D}^+|x_i(t)| &:= \limsup_{h\to 0^+} \frac{|x_i(t+h)| - |x_i(t)|}{h} = \limsup_{h\to 0^+} \frac{1}{h}\int_t^{t+h} \frac{d}{ds}|x_i(s)|\, ds \\ &\leq a_{ii}(t)\,|x_i(t)| + \sum_{\substack{j=1\\ j\neq i}}^n |a_{ij}(t)|\,|x_j(t)| + \sum_{j=1}^n \int_0^t |b_{ij}(t,s)|\,|x_j(s)|\, ds,\end{aligned}$$

where $\mathfrak{D}^+$ denotes the Dini upper-right derivative. In particular, it follows from (4.4) and (4.6) that

$$\begin{aligned}
\mathfrak{D}^+ |x_{i_0}(t^*)| &\overset{(4.6)}{\leq} a_{i_0 i_0}(t) K e^{-\beta_1(t^*-\sigma)}\alpha_{i_0} + \sum_{\substack{j=1\\ j\neq i_0}}^{n} |a_{i_0 j}(t)|\, K e^{-\beta_1(t^*-\sigma)}\alpha_j \\
&\quad + \sum_{j=1}^{n}\int_0^{t^*} |b_{i_0 j}(t^*, s)|\, K e^{-\beta_1(s-\sigma)}\alpha_j\, ds \\
&= K e^{-\beta_1(t^*-\sigma)} \left[a_{i_0 i_0}(t)\alpha_{i_0} + \sum_{\substack{j=1\\ j\neq i_0}}^{n} |a_{i_0 j}(t)|\,\alpha_j + \sum_{j=1}^{n}\int_0^{t^*} |b_{i_0 j}(t^*, s)|\,\alpha_j\, ds \right] \\
&\overset{(4.4)}{<} -\beta_1 K e^{-\beta_1(t^*-\sigma)}\alpha_{i_0} = \mathfrak{D}^+ u_{i_0}(t^*).
\end{aligned}$$

However, this conflicts with (4.6). Therefore

$$|x(t;\sigma,\varphi)| \leq u(t) = K e^{-\beta_1(t-\sigma)} p, \quad \forall\, t \geq \sigma \geq 0,\ \forall\, \varphi \in C([0,\sigma],\mathbb{R}^n),\ \|\varphi\|_\sigma \leq 1.$$

By the monotonicity of vector norms, this yields

$$\|x(t;\sigma,\varphi)\| \leq K_1 e^{-\beta_1(t-\sigma)}, \quad \forall\, t \geq \sigma \geq 0,\ \forall\, \varphi \in C([0,\sigma],\mathbb{R}^n),\ \|\varphi\|_\sigma \leq 1$$

for some $K_1 > 0$. Since (1.1) is linear, it follows that

$$\|x(t;\sigma,\varphi)\| \leq K_1 e^{-\beta_1(t-\sigma)} \|\varphi\|_\sigma, \quad \forall\, t \geq \sigma \geq 0;\ \forall\, \varphi \in C([0,\sigma],\mathbb{R}^n).$$

Hence (1.1) is EAS.

Next, we show that (ii) implies (i). Since M is a Metzler matrix and $s(M) < 0$, there is a vector $p \in \mathbb{R}^n$, $p \gg 0$ such that $Mp \ll p$ by Theorem 2.2. By continuity,

$$Mp \ll -\eta p \tag{4.7}$$

for some $\eta > 0$. Let β be as in (ii) and let $\beta_2 := \min\{\beta, \eta\}$. Clearly, $\beta_2 > 0$ and

$$\left(M(A(t)) + \int_0^t |B(t,s)|\, e^{\beta_2(t-s)}\, ds \right) \leq \left(M(A(t)) + \int_0^t |B(t,s)|\, e^{\beta(t-s)}\, ds \right) \overset{(4.5)}{\leq} M.$$

Therefore,

$$\left(M(A(t)) + \int_0^t |B(t,s)|\, e^{\beta_2(t-s)}\, ds \right) p \leq Mp \overset{(4.7)}{\ll} -\eta p \leq -\beta_2 p.$$

Thus, (i) holds. This completes the proof. □

Remark 4.6. By a slight modification of the proof of Theorem 4.5, it is easy to show that Theorem 4.5 still holds, if (4.4) (or (4.5)) holds for any $t \in [t_0, \infty)$ for some $t_0 \geq 0$. That is, (1.1) is EAS provided one of the following conditions holds:

(i') There exist $\beta_1 > 0$ and $p \in \mathbb{R}^n$, $p \gg 0$ so that

$$\left(M(A(t)) + \int_0^t |B(t,s)|\, e^{\beta_1(t-s)}\, ds \right) p \ll -\beta_1 p, \quad \forall\, t \geq t_0. \tag{4.8}$$

(ii') There exist $\beta_2 > 0$ and $M \in \mathbb{R}^{n\times n}$, $s(M) < 0$ so that

$$\left(M(A(t)) + \int_0^t |B(t,s)|\, e^{\beta_2(t-s)}\, ds \right) \leq M, \quad \forall\, t \geq t_0. \tag{4.9}$$

Consequently, the scalar linear time-varying Volterra integro-differential equation

$$\dot{x}(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\, ds$$

is EAS if

$$\liminf_{t\to\infty} \left(a(t) - \int_0^t |b(t,s)|\, e^{\beta(t-s)} \right) ds < 0$$

for some $\beta > 0$, see e.g. [10, Theorem 3.1].

Consider the linear Volterra time-varying integro-differential equation

$$\dot{x}(t) = -a(t)x(t) + \int_0^t b(t-s)x(s)\, ds, \quad t \geq 0, \tag{4.10}$$

where $a(\cdot), b(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}$ are continuous functions and $b(t) \geq 0$ for all $t \geq 0$.

It has been shown in [2] that if there exists an $\alpha > 0$ such that

$$\int_0^\infty b(s)e^{\alpha s}\, ds < \infty$$

and

$$-a(t) + \int_0^{t-t_0} e^{\int_{t-s}^t a(r)\, dr} b(s)\, ds \leq -\alpha, \quad \forall\, t \geq t_0 \geq 0, \tag{4.11}$$

then the zero solution of (4.10) is EAS.

Since $b(t) \geq 0$ for all $t \geq 0$ and (4.11) implies that $-a(t) \leq -\alpha$, $t \geq t_0 \geq 0$. We have

$$-a(t) + \int_0^{t-t_0} e^{\int_{t-s}^t \alpha\, dr} b(s)\, ds \leq -a(t) + \int_0^{t-t_0} e^{\int_{t-s}^t a(r)\, dr} b(s)\, ds \leq -\alpha, \quad t \geq t_0 \geq 0.$$

Thus,

$$-a(t) + \int_0^t e^{\alpha s} b(s)\, ds \leq -\alpha, \quad t \geq 0.$$

Therefore, (i) of Theorem 4.5 holds.

We illustrate the obtained results by a couple of examples.

Example 4.7. Consider the linear time-varying Volterra integro-differential equation in $\mathbb{R}^2$ defined by

$$\dot{x}(t) = A(t)x(t) + \int_0^t B(t,s)x(s)\,ds, \quad t \geq \sigma \geq 0, \tag{4.12}$$

where

$$A(t) := \begin{pmatrix} -4-t & 1 \\ -e^{-t} & -4 \end{pmatrix}, \; t \geq 0; \quad B(t,s) := \begin{pmatrix} -e^{-(t-s)}\sin s & e^{-\frac{t-s}{2}} \\ 0 & e^{-(t-s)}\cos s \end{pmatrix}, \; t \geq s \geq 0.$$

Let

$$A_0 := \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}; \quad B_0(t) := \begin{pmatrix} e^{-t} & e^{-\frac{t}{2}} \\ 0 & e^{-t} \end{pmatrix}, \; t \geq 0.$$

Then A_0 and $B_0(\cdot)$ satisfy (3.1) for $t \geq s \geq 0$ and

$$\int_0^\infty \|B_0(s)\|\, e^{\alpha s}\,ds < \infty$$

for any $\alpha \in (0, 1/2)$. By simple computation, we have

$$A_0 + \int_0^{+\infty} B_0(t)\,dt = \begin{pmatrix} -3 & 3 \\ 1 & -3 \end{pmatrix}; \quad s\left(A_0 + \int_0^{+\infty} B_0(t)\,dt\right) = -3 + \sqrt{3} < 0.$$

Thus (4.12) is EAS by Theorem 4.3.

Example 4.8. Consider the linear time-varying Volterra integro-differential equation

$$\dot{x}(t) = A(t)x(t) + \int_0^t B(t,s)x(s)\,ds, \quad t \geq \sigma \geq 0, \tag{4.13}$$

where

$$A(t) := \begin{pmatrix} -2-6t & -1 \\ -1 & -7t-3 \end{pmatrix}, \; t \geq 0; \quad B(t,s) := \begin{pmatrix} -e^{-(t-s)}s & e^{-(t-s)} \\ 0 & e^{-\frac{1}{2}(t-s)}s \end{pmatrix}, \; t \geq s \geq 0.$$

Note that Theorem 4.3 may not be applied to (4.13). Clearly,

$$M(A(t)) = \begin{pmatrix} -2-6t & 1 \\ 1 & -7t-3 \end{pmatrix}, \quad t \geq 0,$$

and we have for $\beta > 0$, $\beta \neq 1/2$ and $\beta \neq 1$,

$$\int_0^t |B(t,s)|\, e^{\beta(t-s)}\,ds = \begin{pmatrix} \frac{t}{1-\beta} + \frac{e^{(\beta-1)t}-1}{(1-\beta)^2} & \frac{1-e^{(\beta-1)t}}{1-\beta} \\ 0 & \frac{t}{\frac{1}{2}-\beta} + \frac{e^{(\beta-\frac{1}{2})t}-1}{(\frac{1}{2}-\beta)^2} \end{pmatrix}, \quad t \geq 0.$$

Let $p := (1,1)^T \in \mathbb{R}^2$, $p \gg 0$. By Theorem 4.5, (4.13) is EAS, if there exists $\beta > 0$ such that

$$\left(M(A(t)) + \int_0^t |B(t,s)|\, e^{\beta(t-s)}\, ds\right) p \ll -\beta p, \quad \forall\, t \geq 0. \tag{4.14}$$

For a fixed $\beta \in (0, 1/4)$, (4.14) is equivalent to

$$-1 - 6t + \frac{t}{1-\beta} + \frac{e^{(\beta-1)t} - 1}{(1-\beta)^2} + \frac{1 - e^{(\beta-1)t}}{1-\beta} < -\beta, \quad \forall\, t \geq 0, \tag{4.15}$$

$$-2 - 7t + \frac{t}{\frac{1}{2} - \beta} + \frac{e^{(\beta-\frac{1}{2})t} - 1}{(\frac{1}{2} - \beta)^2} < -\beta, \quad \forall\, t \geq 0. \tag{4.16}$$

Let us define

$$f(t) := -1 - 6t + \frac{t}{1-\beta} + \frac{e^{(\beta-1)t} - 1}{(1-\beta)^2} + \frac{1 - e^{(\beta-1)t}}{1-\beta}, \quad t \geq 0$$

and

$$g(t) := -2 - 7t + \frac{t}{\frac{1}{2} - \beta} + \frac{e^{(\beta-\frac{1}{2})t} - 1}{(\frac{1}{2} - \beta)^2}, \quad t \geq 0.$$

It is easy to show that $f'(t) < 0$, $g'(t) < 0$, $\forall\, t \geq 0$. This gives $f(t) < f(0) = -1 < -\beta$ and $g'(t) < g(0) = -2 < -\beta$ for any $t \geq 0$. Therefore, (4.15) and (4.16) hold and then (4.13) is EAS.

5. Stability of perturbed equations

We are now concerned with stability of the linear time-varying Volterra integro-differential equation (1.1) subject to time-varying structured perturbations and affine perturbations. The classes of structured perturbations and affine perturbations are well-known in the theory of robust stability of dynamical systems, see e.g. [14, 15]. They are different and complement each other and describe a large class of parameter perturbations, see [15] for detailed information.

To the best of our knowledge, problems of stability of the linear time-varying Volterra integro-differential equation (1.1) subject to time-varying structured perturbations and affine perturbations have not been studied yet in the literature.

Throughout this section, we suppose all of the hypotheses of Theorem 3.4 are satisfied. Thus, (1.1) is UAS.

5.1. Time-varying structured perturbations

Consider a perturbed equation of the form

$$\begin{aligned} \dot{x}(t) &= (A(t) + D_0(t)\Delta(t)E_0(t))x(t) \\ &\quad + \int_0^t (B(t,s) + D(t,s)\delta(t,s)E(t,s))x(s)\, ds, \quad t \geq 0, \end{aligned} \tag{5.1}$$

where

(i) $D_0(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}^{n\times l_0}$, $E_0(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}^{q_0\times n}$, $D(\cdot\,,\cdot)\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{n\times l_1}$, $E(\cdot\,,\cdot)\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{q_1\times n}$ are given matrix-valued continuous functions;

(i) $\Delta(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}^{l_0\times q_0}$ and $\delta(\cdot\,,\cdot)\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{l_1\times q_1}$ are *unknown* matrix-valued continuous functions.

The main problem here is to find a positive number r such that the perturbed equation (5.1) *remains UAS whenever the size of perturbations is less than r.*

Theorem 5.1. *Assume that all of the hypotheses of Theorem* 3.4 *are satisfied. Suppose there exist $D_0 \in \mathbb{R}_+^{n\times l_0}$, $D_1 \in \mathbb{R}_+^{n\times l_1}$, $E_0 \in \mathbb{R}_+^{q_0\times n}$, $E_1 \in \mathbb{R}_+^{q_1\times n}$, $\Delta_0 \in \mathbb{R}_+^{l_0\times q_0}$, and $\delta_1(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+^{l_1\times q_1}$ such that*

(i) $|D_0(t)| \le D_0$, $|E_0(t)| \le E_0$, $|\Delta(t)| \le \Delta_0$, $\forall\, t \ge 0$*;*

(ii) $|D(t,s)| \le D_1$, $|E(t,s)| \le E_1$, $|\delta(t,s)| \le \delta_1(t-s)$, $\forall\, t \ge s \ge 0$*;*

(iii) $\int_0^\infty \|\delta_1(s)\|\, ds < \infty$.

Then the perturbed equation (5.1) *remains UAS if*

$$\|\Delta_0\| + \int_0^\infty \|\delta_1(s)\|\, ds < \frac{1}{\max_{P\in\{E_1,E_2\},Q\in\{D_1,D_2\}} \left\| P\left(A_0 + \int_0^{+\infty} B_0(s)\, ds\right)^{-1} Q\right\|}. \tag{5.2}$$

In addition, if

$$\int_0^\infty \|B_0(s)\|\, e^{\alpha s}\, ds < \infty; \quad \int_0^\infty \|\delta_1(s)\|\, e^{\alpha s}\, ds < +\infty \tag{5.3}$$

for some $\alpha > 0$, then (5.1) *is EAS.*

Proof. From (i), it follows that

$$|D_0(t)\Delta(t)E_0(t)| \le D_0\Delta_0E_0, \quad \forall\, t \ge 0. \tag{5.4}$$

Furthermore, (3.1), (ii) and (iii) imply

$$|B(t,s) + D(t,s)\delta(t,s)E(t,s)| \le B_0(t-s) + D_1\delta_1(t-s)E_1, \quad t \ge s \ge 0$$

and

$$\int_0^\infty \|B_0(s) + D_1\delta_1(s)E_1\|\, ds < \int_0^\infty \|B_0(s)\|\, ds + \|D_1\|\,\|E_1\| \int_0^\infty \|\delta_1(s)\|\, ds < +\infty.$$

Let $D_0\Delta_0E_0 := (m_{ij}^{(0)}) \in \mathbb{R}^{n\times n}$, and let $D_0(t)\Delta(t)E_0(t) := (m_{ij}(t)) \in \mathbb{R}^{n\times n}$, and $A(t) := (a_{ij}(t)) \in \mathbb{R}^{n\times n}$, $t \geq 0$. Thus $A(t) + D_0(t)\Delta(t)E_0(t) = (a_{ij}(t) + m_{ij}(t))$. From (3.1) and (5.4), it follows that

$$(a_{ii}(t) + m_{ii}(t)) \leq (a_{ii}^{(0)} + m_{ii}^{(0)}), \quad \forall\, t \geq 0,\, \forall\, i \in \underline{n}$$

and

$$|(a_{ij}(t) + m_{ij}(t))| \leq (a_{ij}^{(0)} + m_{ij}^{(0)}), \quad \forall\, t \geq 0,\, \forall\, i,j \in \underline{n},\ i \neq j.$$

By Theorem 3.4, the zero solution of (5.1) is UAS if

$$M_1 = A_0 + D_0\Delta_0E_0 + \int_0^\infty (B_0(s) + D_1\delta_1(s)E_1)\, ds$$

is Hurwitz stable.

Assume on the contrary that $s_1 := s(M_1) > 0$. By the Perron-Frobenius theorem (Theorem 2.1(i)), there exists $x_1 \in \mathbb{R}^n_+$, $x_1 \neq 0$ such that

$$\left(A_0 + D_0\Delta_0E_0 + \int_0^\infty (B_0(s) + D_1\delta_1(s)E_1)\, ds\right) x_1 = s_1x_1.$$

By assumption, $s\left(A_0 + \int_0^\infty B_0(s)\, ds\right) < 0$, thus $\left(s_1I_n - A_0 - \int_0^\infty B_0(s)\, ds\right)$ is invertible and this implies

$$\left(s_1I_n - A_0 - \int_0^\infty B_0(s)\, ds\right)^{-1} \left(D_0\Delta_0E_0x_1 + D_1\int_0^\infty \delta_1(s)\, dsE_1x_1\right) = x_1. \tag{5.5}$$

Let $i_0 \in \{0,1\}$ be the index such that $\|E_{i_0}x_1\| := \max\{\|E_0x_1\|, \|E_1x_1\|\}$. Then (5.5) implies $\|E_{i_0}x_1\| > 0$. Multiply both sides of (5.5) from the left by E_{i_0}, to get

$$E_{i_0}\left(s_1I_n - A_0 - \int_0^\infty B_0(s)\, ds\right)^{-1} \left(D_0\Delta_0E_0x_1 + D_1\int_0^\infty \delta_1(s)\, dsE_1x_1\right) = E_{i_0}x_1.$$

Taking norms, we get

$$\begin{aligned} &\left\|E_{i_0}\left(s_1I_n - A_0 - \int_0^\infty B_0(s)\, ds\right)^{-1} D_0\right\| \|\Delta_0\| \|E_0x_1\| \\ &+ \left\|E_{i_0}\left(s_1I_n - A_0 - \int_0^\infty B_0(s)\, ds\right)^{-1} D_1\right\| \int_0^\infty \|\delta_1(s)\|\, ds\, \|E_1x_1\| \geq \|E_{i_0}x_1\|. \end{aligned}$$

It follows that

$$\left(\max_{\substack{P\in\{E_0,E_1\}\\ Q\in\{D_0,D_1\}}} \left\|P\left(s_1I_n - A_0 - \int_0^\infty B_0(s)\, ds\right)^{-1} Q\right\|\right)\left(\|\Delta_0\| + \int_0^\infty \|\delta_1(s)\|\, ds\right) \geq 1,$$

or equivalently,

$$\|\Delta_0\| + \int_0^\infty \|\delta_1(s)\|\,ds \tag{5.6}$$
$$\geq \frac{1}{\max_{P\in\{E_0,E_1\},Q\in\{D_0,D_1\}} \left\| P\left(s_1 I_n - A_0 - \int_0^\infty B_0(s)\,ds\right)^{-1} Q \right\|}.$$

On the other hand, the resolvent identity gives

$$\left(0I_n - A_0 - \int_0^\infty B_0(s)\,ds\right)^{-1} - \left(s_1 I_n - A_0 - \int_0^\infty B_0(s)\,ds\right)^{-1}$$
$$= (s_1 - 0)\left(0I_n - A_0 - \int_0^\infty B_0(s)\,ds\right)^{-1}\left(s_1 I_n - A_0 - \int_0^\infty B_0(s)\,ds\right)^{-1}.$$

Since $s_1 \geq 0$, Theorem 2.1(iii) implies that

$$\left(-A_0 - \int_0^\infty B_0(s)\,ds\right)^{-1} \geq \left(s_1 I_n - A_0 - \int_0^\infty B_0(s)\,ds\right)^{-1} \geq 0.$$

Therefore,

$$P\left(-A - \int_0^\infty B_0(s)\,ds\right)^{-1} Q \geq P\left(s_1 I_n - A_0 - \int_0^\infty B_0(s)\,ds\right)^{-1} Q \geq 0$$

for any $P \in \{E_0, E_1\}$, $Q \in \{D_0, D_1\}$. By monotonicity of the operator norm,

$$\left\| P\left(-A_0 - \int_0^\infty B_0(s)\,ds\right)^{-1} Q \right\| \geq \left\| P\left(s_1 I_n - A_0 - \int_0^\infty B_0(s)\,ds\right)^{-1} Q \right\| \tag{5.7}$$

for any $P \in \{E_0, E_1\}$, $Q \in \{D_0, D_1\}$. Finally, (5.6) and (5.7) imply that

$$\|\Delta_0\| + \int_0^\infty \|\delta_1(s)\|\,ds \geq \frac{1}{\max_{P\in\{E_0,E_1\},Q\in\{D_0,D_1\}} \left\| P\left(A_0 + \int_0^\infty B_0(s)\,ds\right)^{-1} Q \right\|}.$$

However, this conflicts with (5.2). Thus, (5.1) is UAS.

The remainder of Theorem 5.1 follows from Theorem 4.3. This completes the proof. □

5.2. Time-varying affine perturbations

In contrast to the previous subsection, we now assume that (1.1) is subject to affine perturbations, that is,

$$\dot{x}(t) = \left(A(t) + \sum_{j=1}^N \alpha_j A_j(t)\right) x(t) + \int_0^t \left(B(t,s) + \sum_{j=1}^N \beta_j B_j(t,s)\right) x(s)\,ds, \quad t \geq 0, \tag{5.8}$$

where

(i) $A_j(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, $B_j(\cdot\,,\cdot)\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, $j \in \underline{N}$, are given matrix-valued continuous functions;

(ii) $(\alpha_j)_{j\in\underline{N}} \in \mathbb{R}^N$ and $(\beta_j)_{j\in\underline{N}} \in \mathbb{R}^N$ are *unknown.*

Theorem 5.2. *Assume that all of the hypotheses of Theorem* 3.4 *are satisfied. Suppose there exist* $A_{0j} \in {\mathbb{R}_+}^{n\times n}$ *and* $B_{0j}(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+^{n\times n}$, $j \in \underline{N}$ *such that*

(i) $|A_j(t)| \le A_{0j}$, $\forall\, t \ge 0$, $j \in \underline{N}$; $|B_j(t,s)| \le B_{0j}(t-s)$, $\forall\, t \ge s \ge 0$, $j \in \underline{N}$;

(ii) $\int_0^\infty \|B_{0j}(s)\|\, ds < \infty$, $\forall\, j \in \underline{N}$.

Then the perturbed equation (5.8) *remains UAS provided*

$$\max\left\{\max_{j\in\underline{N}} |\alpha_j|\,, \max_{j\in\underline{N}} |\beta_j|\right\} < \frac{1}{s\left(\left(-A_0 - \int_0^{+\infty} B_0(s)\, ds\right)^{-1}\left(\sum_{j=1}^N A_{0j} + \sum_{j=1}^N \int_0^\infty B_{0j}(s)\, ds\right)\right)}. \tag{5.9}$$

In addition, if

$$\int_0^\infty \|B_0(s)\|\, e^{\alpha s}\, ds < \infty; \quad \int_0^\infty \|B_{0j}(s)\|\, e^{\alpha s}\, ds < \infty,\ j \in \underline{N} \tag{5.10}$$

for some $\alpha > 0$ *then the perturbed equation* (5.8) *is EAS.*

Proof. From the assumption of Theorem 3.4 and (ii), it follows that

$$\int_0^\infty \left\| B_0(s) + \sum_{j=1}^N \beta_j B_{0j}(s)\right\| ds \le \int_0^{+\infty} \|B_0(s)\|\, ds + \sum_{j=1}^N \int_0^\infty |\beta_j|\, \|B_{0j}(s)\|\, ds < \infty.$$

By Theorem 3.4, it remains to show that

$$s_0 := s\left(A_0 + \sum_{j=1}^N |\alpha_j|\, A_{0j} + \int_0^\infty \left(B_0(s) + \sum_{j=1}^N |\beta_j|\, B_{0j}(s)\right) ds\right) < 0.$$

Assume on the contrary that $s_0 \ge 0$. Since $A_0 + \sum_{j=1}^N |\alpha_j|\, A_{0j} + \int_0^\infty (B_0(s) + \sum_{j=1}^N |\beta_j|\, B_{0j}(s))\, ds$ is a Metzler matrix,

$$\left(A_0 + \sum_{j=1}^N |\alpha_j|\, A_{0j} + \int_0^\infty \left(B_0(s) + \sum_{j=1}^N |\beta_j|\, B_{0j}(s)\right) ds\right) x_0 = s_0 x_0$$

for some $x_0 \ge 0$, $x_0 \ne 0$, by the Perron-Frobenius theorem (Theorem 2.1(i)). This implies

$$\left(s_0 I_n - A_0 - \int_0^\infty B_0(s)\, ds\right)^{-1}\left(\sum_{j=1}^N |\alpha_j|\, A_{0j} + \sum_{j=1}^N |\beta_j|\, B_{0j}(s)\, ds\right) x_0 = x_0.$$

Thus,

$$
\begin{aligned}
|x_0| &= \left| \left(s_0 I_n - A_0 - \int_0^{\infty} B_0(s)\, ds \right)^{-1} \left(\sum_{j=1}^{N} |\alpha_j|\, A_{0j} + \sum_{j=1}^{N} |\beta_j|\, B_{0j}(s)\, ds \right) x_0 \right| \\
&\le \left(s_0 I_n - A_0 - \int_0^{\infty} B_0(s)\, ds \right)^{-1} \left| \left(\sum_{j=1}^{N} |\alpha_j|\, A_{0j} + \sum_{j=1}^{N} |\beta_j|\, B_{0j}(s)\, ds \right) x_0 \right| \\
&\le \left(-A_0 - \int_0^{+\infty} B_0(s)\, ds \right)^{-1} \left| \left(\sum_{j=1}^{N} |\alpha_j|\, A_{0j} + \sum_{j=1}^{N} \int_0^{\infty} |\beta_j|\, B_{0j}(s)\, ds \right) x_0 \right| \\
&\le \left(-A_0 - \int_0^{+\infty} B_0(s)\, ds \right)^{-1} \left(\sum_{j=1}^{N} |\alpha_j|\, A_{0j} + \sum_{j=1}^{N} \int_0^{\infty} e^{-\lambda s}\, |\beta_j|\, B_{0j}(s)\, ds \right) |x_0| \\
&\le \max \left\{ \max_{j \in \underline{N}} |\alpha_j|\, , \max_{j \in \underline{N}} |\beta_j| \right\} \\
&\quad \times \left(\left(-A_0 - \int_0^{\infty} B_0(s)\, ds \right)^{-1} \left(\sum_{j=1}^{N} A_{0j} + \sum_{j=1}^{N} \int_0^{+\infty} B_{0j}(s)\, ds \right) \right) |x_0|\, .
\end{aligned}
$$

Since the matrix $\left(-A_0 - \int_0^{\infty} B_0(s)\, ds\right)^{-1} \left(\sum_{j=1}^{N} A_{0j} + \sum_{j=1}^{N} \int_0^{\infty} e^{-\lambda s} B_{0j}(s)\, ds\right)$ is non-negative, it follows from Theorem 2.1(iii) that

$$
\begin{aligned}
& s\left(\left(-A_0 - \int_0^{\infty} B_0(s)\, ds \right)^{-1} \left(\sum_{j=1}^{N} A_{0j} + \int_0^{\infty} \sum_{j=1}^{N} B_{0j}(s)\, ds \right) \right) \\
& \ge \left(\max \left\{ \max_{j \in \underline{N}} |\alpha_j|\, , \max_{j \in \underline{N}} |\beta_j| \right\} \right)^{-1} > 0.
\end{aligned}
$$

However, this conflicts with (5.9).

The remainder of Theorem 5.2 follows from Theorem 4.3. This completes the proof. □

5.3. Illustrative examples

We now reconsider (4.12) given in Example 4.7. As shown in Example 4.7, (4.12) is EAS. Consider the perturbed equation

$$
\dot{x}(t) = (A(t) + D_0(t)\Delta(t)E_0(t))x(t) + \int_0^t (B(t,s) + D(t,s)\delta(t,s)E(t,s))x(s)\, ds, \tag{5.11}
$$

where

$$
A(t) := \begin{pmatrix} -4-t & 1 \\ -e^{-t} & -4 \end{pmatrix},\ t \ge 0; \quad B(t,s) := \begin{pmatrix} -e^{-(t-s)} \sin s & e^{-\frac{t-s}{2}} \\ 0 & e^{-(t-s)} \cos s \end{pmatrix},\ t \ge s \ge 0,
$$

$$D_0(t) := \begin{pmatrix} \frac{\cos(t^2)}{5} \\ 0 \end{pmatrix}; \quad \Delta(t) := \begin{pmatrix} \frac{1}{2}\sin t & \frac{1}{2}e^{-t}\cos t \end{pmatrix}; \quad E_0(t) := \begin{pmatrix} e^{-t} & 0 \\ 0 & \frac{1}{1+2t^2} \end{pmatrix}, \quad t \geq 0,$$

$$D(t,s) := \begin{pmatrix} 0 \\ \frac{1}{5}e^{-t}\sin s \end{pmatrix}; \quad \delta(t,s) := \begin{pmatrix} \frac{1}{2}e^{-(t-s)}\sin t & \frac{1}{2}e^{-(t-s)} \end{pmatrix};$$

and

$$E(t,s) := \begin{pmatrix} -\sin^2 t & 0 \\ 0 & -\ln(1+e^{-s}) \end{pmatrix}, \quad t \geq s \geq 0.$$

In Example 4.7, it is shown that

$$s\left(A_0 + \int_0^\infty B_0(t)\,dt\right) = -3 + \sqrt{3} < 0,$$

where

$$A_0 := \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}; \quad B_0(t) := \begin{pmatrix} e^{-t} & e^{-\frac{t}{2}} \\ 0 & e^{-t} \end{pmatrix}, \quad t \geq 0.$$

Note that

$$|D_0(t)| \leq D_0 := \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix}; \quad |\Delta(t)| \leq \Delta_0 := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}; \quad |E_0(t)| \leq E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for any $t \geq 0$ and

$$|D(t,s)| \leq D_1 := \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix}; \quad |\delta(t,s)| \leq \delta_1(t-s) := \begin{pmatrix} \frac{1}{2}e^{-(t-s)} & \frac{1}{2}e^{-(t-s)} \end{pmatrix};$$

$$|E(t,s)| \leq E_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for any $t \geq s \geq 0$. Let $\mathbb{R}^2$ be endowed with 1-norm. It is easy to check that

$$\|\Delta_0\| + \int_0^\infty \|\delta_1(s)\|\,ds = 1 < \frac{1}{\max_{P\in\{E_0,E_1\},Q\in\{D_0,D_1\}} \left\| P\left(A_0 + \int_0^{+\infty} B_0(s)\,ds\right)^{-1} Q \right\|} = 5.$$

Thus (5.11) is UAS by Theorem 5.1. Furthermore, since $B_0(\cdot)$ and $\delta_1(\cdot)$ satisfy (5.3) for any $\alpha \in (0, 1/2)$ then (5.11) is EAS.

Next, we consider the perturbed equation given by

$$\dot{x}(t) = (A(t) + \alpha_1 A_1(t))\,x(t) + \int_0^t (B(t,s) + \beta_1 B_1(t,s))x(s)\,ds, \tag{5.12}$$

where

$$A_1(t) := \begin{pmatrix} 2\cos^3 t & 0 \\ \frac{2}{1+2t^4} & -4 \end{pmatrix}; \quad B_1(t,s) := \begin{pmatrix} -4e^{-(t-s)}\sin t & 0 \\ 4e^{-(t-s)} & -2e^{-(t-s)}\frac{1}{1+t^2} \end{pmatrix}, \quad t \ge s \ge 0$$

and $\alpha_1, \beta_1 \in \mathbb{R}$ are unknown. Clearly,

$$|A_1(t)| \le A_1 := \begin{pmatrix} 2 & 0 \\ 2 & 4 \end{pmatrix}; \quad |B_1(t,s)| \le B_1(t-s) = \begin{pmatrix} 4e^{-(t-s)} & 0 \\ 4e^{-(t-s)} & 2e^{-(t-s)} \end{pmatrix}$$

for any $t \ge s \ge 0$ and

$$\left(-A_0 - \int_0^{+\infty} B_0(s)\,ds\right)^{-1} \left(A_1 + \int_0^{+\infty} B_1(s)\,ds\right) = \begin{pmatrix} 6 & 3 \\ 4 & 3 \end{pmatrix}.$$

By Theorem 5.2, (5.12) is still EAS if

$$\max\{|\alpha_1|, |\beta_1|\} < \frac{1}{s\left(\left(-A_0 - \int_0^{+\infty} B_0(s)\,ds\right)^{-1}\left(A_1 + \int_0^{+\infty} B_1(s)\,ds\right)\right)} = \frac{2}{9+\sqrt{51}}.$$

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