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$\delta^\sharp(2,2)$-Ideal Centroaffine Hypersurfaces of Dimension 5

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Abstract. The notion of an ideal submanifold was introduced by Chen at the end of the last century. A survey of recent results in this area can be found in his book [9]. Recently, in [10], an optimal collection of Chen's inequalities was obtained for Lagrangian submanifolds in complex space forms. As shown in [2], these inequalities have an immediate counterpart in centroaffine differential geometry. Centroaffine hypersurfaces realising the equality in one of these inequalities are called ideal centroaffine hypersurfaces.

So far, most results in this area have only been related with 3- and 4-dimensional $\delta^\sharp(2)$-ideal centroaffine hypersurfaces. The purpose of this paper is to classify $\delta^\sharp(2,2)$-ideal hypersurfaces of dimension 5 in centroaffine differential geometry.

1. Introduction

In early 1990s, Chen introduced new Riemannian invariants named δ-invariants for an n-dimensional Riemannian manifold M^n and used these invariants to determine an optimal lower bound for the mean curvature vector of submanifolds of real space forms. Submanifolds attaining this bound are called ideal submanifolds. Similar research has also been done for Lagrangian submanifolds of complex space forms, where an optimal inequality has been finally obtained in [10]. Although these invariants have been studied extensively and many examples have been obtained (see for instance [1, 4–13, 15, 16]), one is still very far from a complete classification.

Due to the similarity with those for Lagrangian submanifolds, such kind of invariants can be introduced for the submanifolds in centroaffine differential geometry as follows:

$$\delta^\sharp(n_1,\ldots,n_k)(p) = \widehat{\tau}(p) - \sup\left\{\widehat{\tau}(L_1) + \cdots + \widehat{\tau}(L_k)\right\},$$

where $L_1,\ldots,L_k$ run over all k mutually orthogonal subspaces of T_pM^n such that $\dim L_i = n_i$, $i = 1,\ldots,k$, satisfying $2 \leq n_1,\ldots,n_k < n$ and $n_1 + \cdots + n_k \leq n$. Invariant $\delta^\sharp(2)$ was

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first introduced in [21], where the first results about such submanifolds were described, including a lower bound for the length of Tchebychev vector field which is one of the main invariants of centroaffine differential geometry. Unfortunately, this lower bound turned out not to be optimal. The optimal bound in this case can be found in [2] and the general version is given in Theorem 3.1 of this paper or in [9].

Further results in this case include the classification of 3-dimensional $\delta^\sharp(2)$-ideal centroaffine hypersurfaces in $\mathbb{R}^4$ with a vanishing Tchebychev vector field. These hypersurfaces can be seen as the equiaffine hyperspheres realising the equality. Their classification was obtained in [17, 18]. In the case that the Tchebychev vector field does not vanish, it was shown that a $\delta^\sharp(2)$-ideal centroaffine hypersurface is necessarely of dimension 3 and a complete classification of such ideal hypersurfaces was obtained (see [2] for the details).

In this paper, we deal with ideal centroaffine hypersurfaces with respect to other $\delta^\sharp$-invariants. In particular, we study $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces of dimension 5. We consider different cases depending on whether the Tchebychev vector field vanishes.

2. Preliminaries

First, we recall some basic notions about centroaffine hypersurfaces. For more details, see [20] for instance.

Let M^n be an n-dimensional C^∞-manifold and let $F\colon M^n \to \mathbb{R}^{n+1}$ be a non-degenerate hypersurface whose position vector is nowhere tangent to M^n. Then, F is a transversal field along itself. We call $\xi = -F$ the centroaffine normal. Following [20], we call F together with this normalization a centroaffine hypersurface.

The centroaffine structure equations are given by

$$D_X F_*(Y) = F_*(\nabla_X Y) + h(X,Y)\xi, \tag{2.1}$$

$$D_X \xi = -F_*(X), \tag{2.2}$$

where D denotes the canonical flat connection of $\mathbb{R}^{n+1}$, ∇ is a torsion-free connection on M^n, called the induced centroaffine connection, and h is a non-degenerate symmetric $(0,2)$-tensor field, called the centroaffine metric. The corresponding equations of Gauss and Codazzi are given respectively by

$$R(X,Y)Z = h(Y,Z)X - h(X,Z)Y, \tag{2.3}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \tag{2.4}$$

where $X, Y, Z \in T(M^n)$. The totally symmetric $(0,3)$-tensor field $\mathfrak{C}(X,Y,Z) = (\nabla_X h)(Y,Z)$ is called the cubic form.

We assume that the centroaffine hypersurface is definite, i.e., h is definite. If h is negative definite, we replace $\xi = -F$ by $\xi = F$ for the centroaffine normal. Thus, the second fundamental form is always positive definite. In both cases, (2.1) and (2.4) hold whereas (2.2) and (2.3) change sign. In the case $\xi = -F$ (respectively, $\xi = F$), we say that M^n is positive definite (respectively, negative definite).

Denote by $\widehat{\nabla}$ the Levi-Civita connection of h and by $\widehat{R}$ (respectively, $\widehat{\tau}$) the curvature tensor (respectively, scalar curvature) of h. The difference tensor K is then defined by

$$K_X Y = K(X,Y) = \nabla_X Y - \widehat{\nabla}_X Y$$

which is a symmetric $(1,2)$-tensor field. The difference tensor K and the cubic form $\mathfrak{C}$ are related by

$$\mathfrak{C}(X,Y,Z) = -2h(K_X Y, Z).$$

Thus, for each X, K_X is self-adjoint with respect to h.

The Tchebychev form T and the Tchebychev vector field $T^\sharp$ of M^n are defined respectively by

$$T(X) = \frac{1}{n}\operatorname{trace} K_X,$$
$$h(T^\sharp, X) = T(X).$$

If $T = 0$ and M^n is a centroaffine hypersurface of the equiaffine space, then M^n is a so-called proper equiaffine hypersphere centered at the origin, in the sense of [20]. In particular, it is an elliptic (respectively, a hyperbolic) equiaffine hypersphere when it is positive (respectively, negative) definite. If the difference tensor K vanishes, then M^n is a hyperquadric centered at the origin. In particular, it is an ellipsoid (respectively, a two-sheeted hyperboloid) if it is positive (respectively, negative) definite.

It is well-known in centroaffine geometry that

$$\begin{aligned} h(K_X Y, Z) &= h(Y, K_X Z), \\ \widehat{R}(X,Y)Z &= K_Y K_X Z - K_X K_Y Z + \epsilon(h(Y,Z)X - h(X,Z)Y), \\ (\widehat{\nabla} K)(X,Y,Z) &= (\widehat{\nabla} K)(Y,Z,X) = (\widehat{\nabla} K)(Z,X,Y), \end{aligned} \tag{2.5}$$

where $\epsilon = 1$ (respectively, -1) when M^n is positive (respectively, negative) definite.

3. $\delta^\sharp$-invariants, inequalities and ideal immersions

Let M^n be an n-dimensional Riemannian manifold. For a plane section $\pi \subset T_p M^n$, $p \in M^n$, let $\kappa(\pi)$ be the sectional curvature of M^n associated with π. For an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_p M^n$, the scalar curvature $\widehat{\tau}$ at p is defined by

$$\widehat{\tau}(p) = \sum_{i<j} \kappa(e_i \wedge e_j).$$

Let L be a subspace of T_pM^n with dimension $r \geq 2$ and let $\{e_1, \ldots, e_r\}$ be an orthonormal basis of L. The scalar curvature $\widehat{\tau}(L)$ of L is defined by

$$\widehat{\tau}(L) = \sum_{\alpha<\beta} \kappa(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

Given integers $n \geq 3$ and $k \geq 1$, we denote by $\mathcal{S}(n,k)$ the finite set consisting of all k-tuples $(n_1, \ldots, n_k)$ of integers satisfying

$$2 \leq n_1, \ldots, n_k < n \quad \text{and} \quad n_1 + \cdots + n_k \leq n.$$

Moreover, we denote the union $\bigcup_{k\geq 1} \mathcal{S}(n,k)$ by $\mathcal{S}(n)$.

For each $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ and each $p \in M^n$, the invariant $\delta(n_1, \ldots, n_k)(p)$ is defined by

$$\delta(n_1, \ldots, n_k)(p) = \widehat{\tau}(p) - \inf\{\widehat{\tau}(L_1) + \cdots + \widehat{\tau}(L_k)\},$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM^n such that $\dim L_i = n_i$, $i = 1, \ldots, k$.

Chen gave in [6, 7] a sharp general relation between $\delta(n_1, \ldots, n_k)$ and the squared mean curvature $\mathcal{H}^2$ for submanifolds in real space forms. For Lagrangian submanifolds of a complex projective space, the sharp inequality was obtained finally in [10]. As explained in [2], this inequality can be adapted to centroaffine differential geometry by defining the following set of invariants:

$$\delta^\sharp(n_1, \ldots, n_k)(p) = \widehat{\tau}(p) - \sup\{\widehat{\tau}(L_1) + \cdots + \widehat{\tau}(L_k)\},$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM^n such that $\dim L_i = n_i$, $i = 1, \ldots, k$. The difference between this case and the Lagrangian case is due to the difference of sign in the Gauss equation.

Before stating the inequality in this case, we introduce some notations. For a given $\delta^\sharp$-invariant $\delta^\sharp(n_1, \ldots, n_k)$ on a Riemannian manifold M^n (with $2 \leq n_1 \leq \cdots \leq n_k \leq n-1$ and $n_1 + \cdots + n_k \leq n$) and a point $p \in M^n$, we consider mutually orthogonal subspaces $L_1, \ldots, L_k$ of T_pM^n with $\dim(L_i) = n_i$, maximizing the quantity $\widehat{\tau}(L_1) + \cdots + \widehat{\tau}(L_k)$. We then choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM^n such that

$$\begin{gathered} e_1, \ldots, e_{n_1} \in L_1,\ e_{n_1+1}, \ldots, e_{n_1+n_2} \in L_2,\ \ldots, \\ e_{n_1+\cdots+n_{k-1}+1}, \ldots, e_{n_1+\cdots+n_k} \in L_k \end{gathered}$$

and define

$$\begin{gathered} \Delta_1 := \{1, \ldots, n_1\},\ \Delta_2 := \{n_1+1, \ldots, n_1+n_2\},\ \ldots, \\ \Delta_k := \{n_1+\cdots+n_{k-1}+1, \ldots, n_1+\cdots+n_k\},\ \Delta_{k+1} := \{n_1+\cdots+n_k+1, \ldots, n\}. \end{gathered}$$

From now on, we will use the following conventions for the ranges of summation indices:

$$A, B, C \in \{1, \dots, n\}, \quad i, j \in \{1, \dots, k\}, \quad \alpha_i, \beta_i \in \Delta_i, \quad r, s \in \Delta_{k+1}.$$

Finally, we define $n_{k+1} := n - n_1 - \cdots - n_k$. Remark that this may eventually be zero, in which case Δ_{k+1} is empty. However, in the case that we treat in this paper, we always have $n_{k+1} > 0$. We denote the components of the second fundamental form by $K^C_{AB} = h(K(e_A, e_B), e_C)$. Due to the symmetry of the cubic form, it is symmetric with respect to the three indices A, B and C. Adapting the proof of [10], the following theorem follows in the centroaffine case:

Theorem 3.1. *Let M^n be an n-dimensional definite centroaffine hypersurface of $\mathbb{R}^{n+1}$. Take $\epsilon = 1$ (respectively, $\epsilon = -1$) if M^n is positive (respectively, negative) definite. Then, for each k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$ with $n_1 + \cdots + n_k < n$, we have*

$$\begin{aligned} \delta^\sharp(n_1, \dots, n_k) \geq & -\frac{n^2\left(n - \sum_{i=1}^k n_i + 3k - 1 - 6\sum_{i=1}^k \frac{1}{2+n_i}\right)}{2\left(n - \sum_{i=1}^k n_i + 3k + 2 - 6\sum_{i=1}^k \frac{1}{2+n_i}\right)} \left\|T^\sharp\right\|^2 \\ & + \frac{1}{2}\left(n(n-1) - \sum_{i=1}^k n_i(n_i - 1)\right)\epsilon. \end{aligned} \tag{3.1}$$

The equality case of inequality (3.1) *holds at a point $p \in M^n$ if and only if one has*

- *$K^A_{BC} = 0$ if A, B, C are mutually different and not all in the same Δ_i with $i \in \{1, \dots, k\}$,*
- *$K^{\alpha_i}_{\alpha_j \alpha_j} = K^{\alpha_i}_{rr} = \sum_{\beta_i \in \Delta_i} K^{\alpha_i}_{\beta_i \beta_i} = 0$ for $i \neq j$,*
- *$K^r_{rr} = 3K^r_{ss} = (n_i + 2)K^r_{\alpha_i \alpha_i}$ for $r \neq s$.*

A centroaffine immersion of M^n into R^{n+1} is called $\delta^\sharp(n_1, \dots, n_k)$-*ideal* if it satisfies the equality case of inequality (3.1) identically. Moreover, it is called *ideal* if it is $\delta^\sharp(n_1, \dots, n_k)$-ideal for the corresponding $(n_1, \dots, n_k) \in \mathcal{S}(n)$.

4. $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces with a vanishing Tchebychev vector field

In this section, we assume that M^5 is a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface with a vanishing Tchebychev vector field. We also suppose that the centroaffine normal is chosen such that the centroaffine metric h is positive definite. Note that this classification corresponds to the classification of Blaschke affine hyperspheres which realise the equality. Expressing the conditions of Theorem 3.1 in this case (and choosing an appropriate orthonormal basis in each Δ_i), the following lemma follows:

Lemma 4.1. *Let M^5 be a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface with a vanishing Tchebychev vector field. Then, at each point p of M^5, there exists an orthonormal frame $\{e_1, \ldots, e_5\}$ such that*

$$\begin{aligned} &K(e_1,e_1) = ae_1, \quad K(e_1,e_2) = -ae_2, \quad K(e_2,e_2) = -ae_1, \\ &K(e_3,e_3) = be_3, \quad K(e_3,e_4) = -be_4, \quad K(e_4,e_4) = -be_3, \\ &K(e_i,e_j) = 0, \quad \textit{otherwise}, \end{aligned}$$

where $a, b \in \mathbb{R}$.

Note that if at the point p, both numbers a and b vanish, then the difference tensor vanishes identically at that point. If this is the case on an open set, then the classical Berwald theorem states that the open set is congruent to an open part of an ellipsoid (or a hyperboloid) centered at the origin, (cf. [20]). A similar argument shows that if either a or b vanishes at a point, then M^5 is a $\delta^\sharp(2)$-ideal definite centroaffine hypersurface at that point. On the other hand, M^5 is said to be a $\delta^\sharp(2,2)$-proper ideal definite centroaffine hypersurface if and only if a and b are both non-vanishing. Note that in this case if necessary by changing the signs of e_1 and e_3, we may assume that $a > 0$ and $b > 0$.

Lemma 4.2. *Let M^5 be a $\delta^\sharp(2,2)$-proper ideal definite centroaffine hypersurface with a vanishing Tchebychev vector field. For any point p belonging to an open dense subset of M^5, there exists an orthonormal frame field which is denoted by $\{e_1, \ldots, e_5\}$ such that*

$$\begin{aligned} &K(e_1,e_1) = ae_1, \quad K(e_1,e_2) = -ae_2, \quad K(e_2,e_2) = -ae_1, \\ &K(e_3,e_3) = be_3, \quad K(e_3,e_4) = -be_4, \quad K(e_4,e_4) = -be_3, \\ &K(e_i,e_j) = 0, \quad \textit{otherwise}, \end{aligned}$$

where a and b are strictly positive functions.

Proof. Note that on the open set, where $a^2 - b^2 \neq 0$, the spaces Δ_i are well determined and differentiable as eigenspaces of the Ricci tensor. Applying a suitable rotation in each of these spaces yields the desired vector fields. Therefore, in order to complete the proof, we may assume that $a = b > 0$ on an open set. We consider the cubic function

$$\mathfrak{f}(v) = h(K(v,v),v)$$

defined on the unit tangent bundle. As $h(K(X,Y),Z)$ is totally symmetric, it follows that $\mathfrak{f}$ attains a critical value at v if and only if $h(K(v,v),w) = 0$ for any w orthogonal to v. This is equivalent to say that $K(v,v)$ is a multiple of v. If we write $v = y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4$, where $1 = y_1^2 + y_2^2 + y_3^2 + y_4^2$, we get

$$K(v,v) = a((y_1^2 - y_2^2)e_1 - 2y_1y_2e_2 + (y_3^2 - y_4^2)e_3 - 2y_3y_4e_4).$$

So, this is proportional to v if and only if

$$y_2(3y_1^2-y_2^2)=0,\quad y_4(3y_3^2-y_4^2)=0,\quad y_2y_4(y_1-y_3)=0,$$
$$y_3y_1(y_1-y_3)-y_3y_2^2+y_1y_4^2=0,\quad y_4(y_1^2-y_2^2+2y_1y_3)=0,\quad y_2(y_3^2-y_4^2+2y_1y_3)=0.$$

Hence, we obtain extremal vectors as

$$v=\pm e_1,\quad v=\pm e_3,\quad v=\pm\tfrac{1}{\sqrt{2}}(e_1+e_3),\quad v=\pm\left(-\tfrac{1}{2}e_1\pm\tfrac{\sqrt{3}}{2}e_2\right),$$
$$v=\pm\left(-\tfrac{1}{2}e_3\pm\tfrac{\sqrt{3}}{2}e_4\right),\quad v=\pm\tfrac{1}{\sqrt{2}}\left(e_1+\left(-\tfrac{1}{2}e_3\pm\tfrac{\sqrt{3}}{2}e_4\right)\right),$$
$$v=\pm\tfrac{1}{\sqrt{2}}\left(\left(-\tfrac{1}{2}e_1\pm\tfrac{\sqrt{3}}{2}e_2\right)+e_3\right),\quad v=\pm\tfrac{1}{\sqrt{2}}\left(\left(-\tfrac{1}{2}e_1\pm\tfrac{\sqrt{3}}{2}e_2\right)+\left(-\tfrac{1}{2}e_3\pm\tfrac{\sqrt{3}}{2}e_4\right)\right).$$

Consequently, we deduce critical values as $\pm a$ and $\pm a/\sqrt{2}$.

Now, we are interested in the differentiability. We take a point p in our open set and consider the corresponding orthonormal basis vectors at that point. We will show that we can extend these basis vectors to local differentiable vector fields denoted by $e_1,\dots,e_4$ which have the same expressions for the difference tensor.

In order to do so, we first take arbitrary differentiable extensions $F_1,\dots,F_4$ and consider

$$V=a_1F_1+\cdots+a_4F_4.$$

Then, we take into account the system of equations

$$b_j(q,a_1,\dots,a_4)=h(F_j,K(V,V)-aV),$$

where a is defined as before. Since

$$\left(\frac{\partial b_j}{\partial a_\ell}\right)\bigg|_{(p,1,0,0,0)}=2h(F_j,K(F_\ell,F_1))-a\delta_{j\ell}=\begin{cases}0 & \text{if } \ell\neq j,\\ a & \text{if } \ell=j=1,\\ -3a & \text{if } \ell=j=2,\\ -a & \text{if } \ell=j=3,4,\end{cases}$$

we can apply the implicit function theorem. Therefore, there exist differentiable functions $a_1,\dots,a_4$ in a neighborhood of p such that $V(p)=e_1$ and $K(V,V)=aV$. Taking $e_1=V/\|V\|$ gives us a unit vector field which we can associate a critical value. Due to the continuity and the fact that we have only 4 different critical values at each point, we must get $a(q)=a(p)/\|V\|$. Therefore, e_1 is the desired vector field. The vector field e_2 is then determined such that it spans the other 1-dimensional eigenspace of K_{e_1}. Finally, e_3 and e_4 can be determined by a rotation in $\{e_1,e_2\}^\perp$. □

From now on, we will always work on the open dense subset introduced in the previous lemma. We denote by $\widehat{\Gamma}_{ij}^k$ (respectively, $\widehat{\omega}_j^k(e_i)$), the Christoffel symbols (respectively, the connection forms) with respect to the Levi Civita connection of the affine metric.

Using the fact that K is totally symmetric, the following lemma follows in an elementary way:

Lemma 4.3. *Let M^5 be a $\delta^\sharp(2,2)$-proper ideal definite centroaffine hypersurface with a vanishing Tchebychev vector field. Then, we have*

$$e_1(a) = 3a\mu, \quad e_2(a) = 3a\nu, \quad e_3(a) = e_4(a) = 0, \quad e_5(a) = a\alpha,$$
$$e_1(b) = e_2(b) = 0, \quad e_3(b) = 3b\eta, \quad e_4(b) = 3b\varphi, \quad e_5(b) = b\beta,$$

where μ, ν, α, η, φ and β are defined respectively by

$$\mu = \widehat{\Gamma}^1_{22}\,(= \widehat{\omega}^1_2(e_2)), \quad \nu = \widehat{\Gamma}^2_{11}\,(= \widehat{\omega}^2_1(e_1)), \tag{4.1}$$
$$\alpha = \widehat{\Gamma}^5_{11} = \widehat{\Gamma}^5_{22}\,(= \widehat{\omega}^5_1(e_1) = \widehat{\omega}^5_2(e_2)), \tag{4.2}$$
$$\eta = \widehat{\Gamma}^3_{44}\,(= \widehat{\omega}^3_4(e_4)), \quad \varphi = \widehat{\Gamma}^4_{33}\,(= \widehat{\omega}^4_3(e_3)), \tag{4.3}$$
$$\beta = \widehat{\Gamma}^5_{33} = \widehat{\Gamma}^5_{44}\,(= \widehat{\omega}^5_3(e_3) = \widehat{\omega}^5_4(e_4)). \tag{4.4}$$

Moreover, we have $\widehat{\omega}^j_i(e_k) = 0$, where $1 \leq i, j, k \leq 5$ for the ones which do not appear in (4.1), (4.2), (4.3) *and* (4.4).

Lemma 4.4. *Under the hypothesis of Lemma* 4.3*, Levi Civita connection $\widehat{\nabla}$ of h satisfies*

$$\begin{aligned}
&\widehat{\nabla}_{e_1} e_1 = \nu e_2 + \alpha e_5, && \widehat{\nabla}_{e_1} e_2 = -\nu e_1 + \widehat{\Gamma}^5_{12} e_5, && \widehat{\nabla}_{e_1} e_5 = -\alpha e_1 - \widehat{\Gamma}^5_{12} e_2,\\
&\widehat{\nabla}_{e_2} e_1 = -\mu e_2 - \widehat{\Gamma}^5_{12} e_5, && \widehat{\nabla}_{e_2} e_2 = \mu e_1 + \alpha e_5, && \widehat{\nabla}_{e_2} e_5 = \widehat{\Gamma}^5_{12} e_1 - \alpha e_2,\\
&\widehat{\nabla}_{e_3} e_3 = \varphi e_4 + \beta e_5, && \widehat{\nabla}_{e_3} e_4 = -\varphi e_3 + \widehat{\Gamma}^5_{34} e_5, && \widehat{\nabla}_{e_3} e_5 = -\beta e_3 - \widehat{\Gamma}^5_{34} e_4,\\
&\widehat{\nabla}_{e_4} e_3 = -\eta e_4 - \widehat{\Gamma}^5_{34} e_5, && \widehat{\nabla}_{e_4} e_4 = \eta e_3 + \beta e_5, && \widehat{\nabla}_{e_4} e_5 = \widehat{\Gamma}^5_{34} e_3 - \beta e_4,\\
&\widehat{\nabla}_{e_5} e_1 = -\frac{1}{3}\widehat{\Gamma}^5_{12} e_2, && \widehat{\nabla}_{e_5} e_2 = \frac{1}{3}\widehat{\Gamma}^5_{12} e_1, && \widehat{\nabla}_{e_5} e_3 = -\frac{1}{3}\widehat{\Gamma}^5_{34} e_4,\\
&\widehat{\nabla}_{e_5} e_4 = \frac{1}{3}\widehat{\Gamma}^5_{34} e_3, && \widehat{\nabla}_{e_i} e_j = 0, \quad \textit{otherwise}.
\end{aligned}$$

Lemma 4.5. *Under the hypothesis of Lemma* 4.3*, the torsion-free connection ∇ on M^5 satisfies*

$$\begin{aligned}
&\nabla_{e_1} e_1 = a e_1 + \nu e_2 + \alpha e_5, && \nabla_{e_1} e_2 = -\nu e_1 - a e_2 + \widehat{\Gamma}^5_{12} e_5, && \nabla_{e_1} e_5 = -\alpha e_1 - \widehat{\Gamma}^5_{12} e_2,\\
&\nabla_{e_2} e_1 = -(a+\mu) e_2 - \widehat{\Gamma}^5_{12} e_5, && \nabla_{e_2} e_2 = (-a+\mu) e_1 + \alpha e_5, && \nabla_{e_2} e_5 = \widehat{\Gamma}^5_{12} e_1 - \alpha e_2,\\
&\nabla_{e_3} e_3 = b e_3 + \varphi e_4 + \beta e_5, && \nabla_{e_3} e_4 = -\varphi e_3 - b e_4 + \widehat{\Gamma}^5_{34} e_5, && \nabla_{e_3} e_5 = -\beta e_3 - \widehat{\Gamma}^5_{34} e_4,\\
&\nabla_{e_4} e_3 = -(b+\eta) e_4 - \widehat{\Gamma}^5_{34} e_5, && \nabla_{e_4} e_4 = (-b+\eta) e_3 + \beta e_5, && \nabla_{e_4} e_5 = \widehat{\Gamma}^5_{34} e_3 - \beta e_4,\\
&\nabla_{e_5} e_1 = -\frac{1}{3}\widehat{\Gamma}^5_{12} e_2, && \nabla_{e_5} e_2 = \frac{1}{3}\widehat{\Gamma}^5_{12} e_1, && \nabla_{e_5} e_3 = -\frac{1}{3}\widehat{\Gamma}^5_{34} e_4,\\
&\nabla_{e_5} e_4 = \frac{1}{3}\widehat{\Gamma}^5_{34} e_3, && \nabla_{e_i} e_j = 0, \quad \textit{otherwise}.
\end{aligned}$$

We deduce from Lemma 4.4 that

$$\begin{aligned}
\widehat{R}(e_1,e_4)e_4 &= \widehat{\nabla}_{e_1}\widehat{\nabla}_{e_4}e_4 - \widehat{\nabla}_{e_4}\widehat{\nabla}_{e_1}e_4 - \widehat{\nabla}_{[e_1,e_4]}e_4 \\
&= \widehat{\nabla}_{e_1}(\eta e_3 + \beta e_5) - \widehat{\nabla}_{(\widehat{\nabla}_{e_1}e_4 - \widehat{\nabla}_{e_4}e_1)}e_4 \\
&= e_1(\eta)e_3 + \eta\widehat{\nabla}_{e_1}e_3 + e_1(\beta)e_5 + \beta\widehat{\nabla}_{e_1}e_5 \\
&= e_1(\eta)e_3 + e_1(\beta)e_5 + \beta(-\alpha e_1 - \widehat{\Gamma}_{12}^5 e_2) \\
&= e_1(\eta)e_3 + e_1(\beta)e_5 - \alpha\beta e_1 - \widehat{\Gamma}_{12}^5\beta e_2 \\
&= -\alpha\beta e_1 - \widehat{\Gamma}_{12}^5\beta e_2 + e_1(\eta)e_3 + e_1(\beta)e_5.
\end{aligned}$$

So, we find

$$h(\widehat{R}(e_1,e_4)e_4,e_1) = -\alpha\beta \quad \text{and} \quad h(\widehat{R}(e_1,e_4)e_4,e_2) = -\widehat{\Gamma}_{12}^5\beta.$$

On the other hand, in terms of (2.5) and Lemma 4.2, we have

$$\begin{aligned}
\widehat{R}(e_1,e_4)e_4 &= \epsilon(h(e_4,e_4)e_1 - h(e_1,e_4)e_4) - [K_{e_1},K_{e_4}]e_4 \\
&= \epsilon e_1 - K_{e_1}(K_{e_4}e_4) + K_{e_4}(K_{e_1}e_4) \\
&= \epsilon e_1 + bK_{e_1}e_3 \\
&= \epsilon e_1.
\end{aligned}$$

Thus, we obtain $\epsilon = -\alpha\beta$ and therefore $\alpha \neq 0$ and $\beta \neq 0$. As a result, it follows that $\widehat{\Gamma}_{12}^5 = 0$. By similar considerations, we also get $\widehat{\Gamma}_{34}^5 = 0$. Moreover, we obtain that the functions α and β depend only on e_5 and their derivatives in that direction are respectively given by

$$e_5(\alpha) = \epsilon + \alpha^2 \quad \text{and} \quad e_5(\beta) = \epsilon + \beta^2.$$

We now consider the following distributions: $\mathcal{D}_1 = \{e_5\}$, $\mathcal{D}_2 = \{e_1,e_2\}$ and $\mathcal{D}_3 = \{e_3,e_4\}$. For this purpose, we remind some notions about distributions (see [19] for the details).

Let (M^n,h) be a Riemannian manifold and $\widehat{\nabla}$ its Levi-Civita connection. Then, a subbundle $E \subset TM^n$ is called autoparallel if $\widehat{\nabla}_X Y \in E$ holds for all $X,Y \in E$. On the other hand, a subbundle E is called totally umbilical if there exists a vector field $\mathcal{H} \in E^\perp$ such that $h(\widehat{\nabla}_X Y, Z) = h(X,Y)h(\mathcal{H},Z)$ for all $X,Y \in E$ and $Z \in E^\perp$. Here, we call $\mathcal{H}$ the mean curvature vector of E. If, moreover, $h(\widehat{\nabla}_X \mathcal{H}, Z) = 0$ holds, we say that E is spherical. We recall the following decomposition theorem of a Riemannian manifold:

Theorem 4.6. (cf. [19, Theorem 4]) *Let M^n be a Riemannian manifold, and let $TM^n = \bigoplus_{i=0}^k E_i$ be an orthogonal decomposition into non-trivial vector subbundles such that E_i is spherical and $E_i^\perp$ is autoparallel for $i = 1,\dots,k$. Then, we have the following:*

(a) *For every point* $\widetilde{p} \in M^n$, *there is an isometry* ψ *of a warped product* $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ *onto a neighborhood of* $\widetilde{p}$ *in* M^n *such that the following properties hold:*

$$\rho_1(\widetilde{p}_0) = \cdots = \rho_k(\widetilde{p}_0) = 1, \tag{4.5}$$

where $\widetilde{p}_0$ *is the component of* $\psi^{-1}(\widetilde{p})$ *in* M_0,

$$\begin{aligned} &\psi\left(\{p_0\} \times \cdots \times \{p_{i-1}\} \times M_i \times \{p_{i+1}\} \times \cdots \times \{p_k\}\right) \text{ is an integral} \\ &\quad \text{manifold of } E_i \text{ for } i = 0, \ldots, k \text{ and all } p_0 \in M_0, \ldots, p_k \in M_k, \end{aligned} \tag{4.6}$$

(b) *If* M^n *is simply connected and complete, then for every point* $\widetilde{p} \in M^n$, *there exists an isometry* ψ *of a warped product* $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ *onto all of* M^n *with the properties* (4.5) *and* (4.6).

As $\widehat{\Gamma}^5_{12} = 0 = \widehat{\Gamma}^5_{34}$, we can identify a neighborhood U of p with $U = I \times_{\rho_1} M_1^2 \times_{\rho_2} M_2^2$ from the previous theorem. The mean curvature normals of M_1^2 and M_2^2 in U are respectively given by $\mathcal{H}_1 = \alpha e_5 \in \mathcal{D}_1$ and $\mathcal{H}_2 = \beta e_5 \in \mathcal{D}_1$. We now choose a coordinate t tangent to the component I such that $\frac{\partial}{\partial t} = e_5$.

Solving the differential equations for α, β, a and b and also taking into account $\alpha\beta = -\epsilon$, we have the following possibilities (if necessary after a translation of the coordinate t):

$$\epsilon = 1, \quad \alpha = \tan t, \quad \beta = -\cot t, \quad a = \frac{c_1}{\cos t}, \quad b = \frac{c_2}{\sin t}, \tag{4.7}$$

$$\epsilon = 1, \quad \alpha = -\cot t, \quad \beta = \tan t, \quad a = \frac{c_1}{\sin t}, \quad b = \frac{c_2}{\cos t}, \tag{4.8}$$

$$\epsilon = -1, \quad \alpha = -\tanh t, \quad \beta = -\coth t, \quad a = \frac{c_1}{\cosh t}, \quad b = \frac{c_2}{\sinh t}, \tag{4.9}$$

$$\epsilon = -1, \quad \alpha = -\coth t, \quad \beta = -\tanh t, \quad a = \frac{c_1}{\sinh t}, \quad b = \frac{c_2}{\cosh t}, \tag{4.10}$$

$$\epsilon = -1, \quad \alpha = -1, \quad \beta = -1, \quad a = c_1 e^{-t}, \quad b = c_2 e^{-t}, \tag{4.11}$$

$$\epsilon = -1, \quad \alpha = 1, \quad \beta = 1, \quad a = c_1 e^{t}, \quad b = c_2 e^{t}, \tag{4.12}$$

where $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ such that they make a and b strictly positive functions. Note that the cases (4.7) and (4.8) can be interchanged by interchanging the roles played by the vector fields e_1, e_2 and e_3, e_4. The same is also valid for the cases (4.9) and (4.10).

We now determine the immersion which we denote by F explicitly. The Gauss formula states

$$D_X Y = \widehat{\nabla}_X Y + K(X,Y) - h(X,Y)\epsilon F. \tag{4.13}$$

From this equation, Lemmas 4.2 and 4.4, we obtain

$$D_{e_1} e_5 = -\alpha e_1, \quad D_{e_2} e_5 = -\alpha e_2, \quad D_{e_3} e_5 = -\beta e_3, \quad D_{e_4} e_5 = -\beta e_4. \tag{4.14}$$

On the other hand,

$$D_{e_1}F = e_1, \quad D_{e_2}F = e_2, \quad D_{e_3}F = e_3, \quad D_{e_4}F = e_4. \tag{4.15}$$

Now, we consider the following two maps:

$$G = \frac{1}{a}(e_5 + \alpha F), \quad H = \frac{1}{b}(e_5 + \beta F).$$

By means of G, (4.14) and (4.15), we find

$$D_{e_1}G = 0, \quad D_{e_2}G = 0, \quad D_{e_3}G = \frac{\alpha-\beta}{a}e_3, \quad D_{e_4}G = \frac{\alpha-\beta}{a}e_4, \quad D_{e_5}G = 0.$$

Hence, G depends only on M_1^2. Similarly, we obtain that H depends only on M_2^2. Also note that in the case $\alpha \neq \beta$, we can write F in terms of G and H by

$$F = \frac{1}{\alpha-\beta}(aG - bH).$$

4.1. Case 1

We use (4.7). Then, we have

$$\begin{aligned}
&D_{e_1}D_{e_3}G = 0, && D_{e_2}D_{e_3}G = 0,\\
&D_{e_3}D_{e_3}G = bD_{e_3}G + \varphi D_{e_4}G - \frac{1}{\sin^2 t}G, && D_{e_4}D_{e_3}G = -(\eta+b)D_{e_4}G,\\
&D_{e_5}D_{e_3}G = -\cot t\, D_{e_3}G, && D_{e_1}D_{e_4}G = 0,\\
&D_{e_2}D_{e_4}G = 0, && D_{e_3}D_{e_4}G = -\varphi D_{e_3}G - bD_{e_4}G,\\
&D_{e_4}D_{e_4}G = (\eta-b)D_{e_3}G - \frac{1}{\sin^2 t}G, && D_{e_5}D_{e_4}G = -\cot t\, D_{e_4}G.
\end{aligned}$$

These formulas imply that G is a surface contained in a 3-dimensional linear subspace of $\mathbb{R}^6$. A straightforward computation shows that we can consider it as a definite centroaffine surface with a vanishing Tchebychev vector field. Similar computations show that the same is also true for H and the subspaces containing G and H are complementary. Therefore, by a general linear transformation, we may assume that $G(u,v) = (g_1(u,v), g_2(u,v), g_3(u,v), 0, 0, 0)$ and $H(x,y) = (0,0,0,h_1(x,y),h_2(x,y),h_3(x,y))$. Consequently, we get

$$\begin{aligned}
F(t,u,v,x,y) = (&\sin t\, c_1\, g_1(u,v), \sin t\, c_1\, g_2(u,v), \sin t\, c_1\, g_3(u,v), \cos t\, c_2\, h_1(x,y),\\
&\cos t\, c_2\, h_2(x,y), \cos t\, c_2\, h_3(x,y))
\end{aligned}$$

which is centroaffine equivalent to

$$\begin{aligned}
F(t,u,v,x,y) = (&\sin t\, g_1(u,v), \sin t\, g_2(u,v), \sin t\, g_3(u,v), \cos t\, h_1(x,y),\\
&\cos t\, h_2(x,y), \cos t\, h_3(x,y)).
\end{aligned}$$

Conversely, a straightforward computation shows that if G and H are positive definite centroaffine surfaces with a vanishing Tchebychev vector field, then the immersion F described as above is a $\delta^\sharp(2,2)$-ideal centroaffine immersion with a vanishing Tchebychev vector field.

4.2. Case 2

We use (4.9). Then, by the similar computations as before, we obtain

$$F(t,u,v,x,y) = (\sinh t\, g_1(u,v), \sinh t\, g_2(u,v), \sinh t\, g_3(u,v), \cosh t\, h_1(x,y),$$
$$\cosh t\, h_2(x,y), \cosh t\, h_3(x,y)),$$

where G and H are negative definite centroaffine surfaces with a vanishing Tchebychev vector field.

4.3. Case 3

We use (4.11). Note that case (4.11) can be reduced to case (4.12) by replacing e_5 by $-e_5$. We now redefine

$$G = H = e^t(e_5 - F).$$

It is clear that $G = H$ is a constant vector. As e_5 and F are independent, it follows that $G = H$ is non-vanishing. Therefore, by a centroaffine transformation, we may suppose that $G = H = (0,0,0,0,0,-1)$. Thus, we can write

$$F_t - F = e^{-t}(0,0,0,0,0,-1).$$

Solving the above first order differential equation, we get

$$F(u_1,u_2,v_1,v_2,t) = \mathcal{C}(u_1,u_2,v_1,v_2)e^t + \frac{1}{2}e^{-t}(0,0,0,0,0,1).$$

Here, u_1, u_2 denote the coordinates on M_1^2 and v_1, v_2 denote the coordinates on M_2^2. So, in order to determine the immersion F, it is sufficient to determine $\mathcal{C}$. Note that

$$\mathcal{C}(u_1,u_2,v_1,v_2) = e^{-t}F(u_1,u_2,v_1,v_2,t) + \frac{1}{2}e^{-2t}(0,0,0,0,0,-1).$$

Consequently, we have

$$D_{e_5}\mathcal{C}(u_1,u_2,v_1,v_2) = 0, \quad D_{e_1}\mathcal{C}(u_1,u_2,v_1,v_2) = e^{-t}e_1, \quad D_{e_2}\mathcal{C}(u_1,u_2,v_1,v_2) = e^{-t}e_2,$$
$$D_{e_3}\mathcal{C}(u_1,u_2,v_1,v_2) = e^{-t}e_3, \quad D_{e_4}\mathcal{C}(u_1,u_2,v_1,v_2) = e^{-t}e_4.$$

On the other hand, it is clear from (4.15) that $D_{e_i}D_{e_j}F = D_{e_i}e_j$. Hence, in terms of (4.13), Lemmas 4.2 and 4.4, we deduce

$$
\begin{aligned}
&D_{e_1}D_{e_1}\mathcal{C} = e^{-t}(ae_1+\nu e_2-G), && D_{e_1}D_{e_2}\mathcal{C} = -e^{-t}(\nu e_1+ae_2),\\
&D_{e_1}D_{e_3}\mathcal{C} = 0, && D_{e_1}D_{e_4}\mathcal{C} = 0,\\
&D_{e_2}D_{e_1}\mathcal{C} = -e^{-t}(\mu+a)e_2, && D_{e_2}D_{e_2}\mathcal{C} = e^{-t}((\mu-a)e_1-G),\\
&D_{e_2}D_{e_3}\mathcal{C} = 0, && D_{e_2}D_{e_4}\mathcal{C} = 0,\\
&D_{e_3}D_{e_1}\mathcal{C} = 0, && D_{e_3}D_{e_2}\mathcal{C} = 0,\\
&D_{e_3}D_{e_3}\mathcal{C} = e^{-t}(be_3+\varphi e_4-G), && D_{e_3}D_{e_4}\mathcal{C} = -e^{-t}(\varphi e_3+be_4),\\
&D_{e_4}D_{e_1}\mathcal{C} = 0, && D_{e_4}D_{e_2}\mathcal{C} = 0,\\
&D_{e_4}D_{e_3}\mathcal{C} = -e^{-t}(\eta+b)e_4, && D_{e_4}D_{e_4}\mathcal{C} = e^{-t}((\eta-b)e_3-G).
\end{aligned}
$$

The above formulas immediately imply that we can write

$$\mathcal{C}(u_1,u_2,v_1,v_2) = \mathfrak{D}(u_1,u_2)+\mathcal{K}(v_1,v_2).$$

Moreover, it follows that both $\mathfrak{D}$ and $\mathcal{K}$ lie in a 3-dimensional affine subspace and both affine subspaces contain the constant vector G as only mutual direction. Therefore, by applying a centroaffine transformation, we may assume that

$$\mathcal{C}(u_1,u_2,v_1,v_2) = (d_1(u_1,u_2),d_2(u_1,u_2),k_1(v_1,v_2),k_2(v_1,v_2),c,d_3(u_1,u_2)+k_3(v_1,v_2)),$$

where c is a constant. As the immersion F is non-degenerate, it follows that c is non-vanishing and therefore, by a centroaffine transformation, we may suppose that $c=1$.

The formulas for the derivatives of $\mathcal{C}$ imply that the surfaces $(d_1(u_1,u_2),d_2(u_1,u_2),d_3(u_1,u_2))$ and $(k_1(v_1,v_2),k_2(v_1,v_2),k_3(v_1,v_2))$ are both positive definite improper equiaffine spheres with affine normal $(0,0,1)$, (cf. [20]). It is well-known that such a surface can be written as a graph in the direction of the affine normal and the graph function is a solution of the Monge Ampère equation. So, we can rewrite

$$\mathcal{C}(u_1,u_2,v_1,v_2) = (u_1,u_2,v_1,v_2,1,f(u_1,u_2)+g(v_1,v_2)),$$

where f is a solution of $\det\left(\frac{\partial^2 f}{\partial u_i\partial u_j}\right)=1$ and similarly g is a solution of $\det\left(\frac{\partial^2 g}{\partial v_i\partial v_j}\right)=1$. Therefore, F is as follows:

$$F(t,u_1,u_2,v_1,v_2) = \left(e^tu_1,e^tu_2,e^tv_1,e^tv_2,e^t,e^tf(u_1,u_2)+e^tg(v_1,v_2)+\frac{1}{2}e^{-t}\right).$$

Conversely, a straightforward computation shows that F given by the above formula is indeed a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface with a vanishing Tchebychev vector field.

Therefore, we have proven the following theorem:

Theorem 4.7. *Let M^5 be a $\delta^\sharp(2,2)$-proper ideal definite centroaffine hypersurface of $\mathbb{R}^6$ with a vanishing Tchebychev vector field. Then, M^5 is congruent to one of the following immersions:*

(1) $F(t,u,v,x,y) = (\sin t\, g_1(u,v), \sin t\, g_2(u,v), \sin t\, g_3(u,v), \cos t\, h_1(x,y), \cos t\, h_2(x,y), \cos t\, h_3(x,y))$, *where* $(g_1(u,v), g_2(u,v), g_3(u,v))$ *and* $(h_1(x,y), h_2(x,y), h_3(x,y))$ *are elliptic equiaffine spheres,*

(2) $F(t,u,v,x,y) = (\sinh t\, g_1(u,v), \sinh t\, g_2(u,v), \sinh t\, g_3(u,v), \cosh t\, h_1(x,y), \cosh t\, h_2(x,y), \cosh t\, h_3(x,y))$, *where* $(g_1(u,v), g_2(u,v), g_3(u,v))$ *and* $(h_1(x,y), h_2(x,y), h_3(x,y))$ *are hyperbolic equiaffine spheres,*

(3) $F(t,u_1,u_2,v_1,v_2) = \left(e^t u_1, e^t u_2, e^t v_1, e^t v_2, e^t, e^t f(u_1,u_2) + e^t g(v_1,v_2) + \frac{1}{2}e^{-t}\right)$, *where f and g are solutions of the Monge Ampère equations.*

As a corollary, we also obtain the following classification in equiaffine differential geometry:

Corollary 4.8. *Let M^5 be a proper equiaffine hypersphere of $\mathbb{R}^6$ (Blaschke geometry). Assume that at each point p of M^5, the difference tensor is given as in Lemma* 4.1, *where a and b are non-vanishing. Then, M^5 is congruent to one of the following immersions:*

(1) $F(t,u,v,x,y) = (\sin t\, g_1(u,v), \sin t\, g_2(u,v), \sin t\, g_3(u,v), \cos t\, h_1(x,y), \cos t\, h_2(x,y), \cos t\, h_3(x,y))$, *where* $(g_1(u,v), g_2(u,v), g_3(u,v))$ *and* $(h_1(x,y), h_2(x,y), h_3(x,y))$ *are elliptic equiaffine spheres,*

(2) $F(t,u,v,x,y) = (\sinh t\, g_1(u,v), \sinh t\, g_2(u,v), \sinh t\, g_3(u,v), \cosh t\, h_1(x,y), \cosh t\, h_2(x,y), \cosh t\, h_3(x,y))$, *where* $(g_1(u,v), g_2(u,v), g_3(u,v))$ *and* $(h_1(x,y), h_2(x,y), h_3(x,y))$ *are hyperbolic equiaffine spheres,*

(3) $F(t,u_1,u_2,v_1,v_2) = \left(e^t u_1, e^t u_2, e^t v_1, e^t v_2, e^t, e^t f(u_1,u_2) + e^t g(v_1,v_2) + \frac{1}{2}e^{-t}\right)$, *where f and g are solutions of the Monge Ampère equation.*

5. $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces with a non-vanishing Tchebychev vector field

In this section, we assume that M^5 is a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface with a non-vanishing Tchebychev vector field. Moreover, we again assume that the centroaffine normal is chosen such that the centroaffine metric h is positive definite. Expressing the conditions of Theorem 3.1 in this case (and choosing an appropriate orthonormal basis in each Δ_i), we have the following lemma:

Lemma 5.1. *Let M^5 be a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface with a non-vanishing Tchebychev vector field. Then, at each point p of M^5, there exists an orthonormal frame $\{e_1, u_1, u_2, v_1, v_2\}$ such that*

$$\begin{aligned}
&K(u_i, v_j) = 0, && K(e_1, u_i) = \mu u_i, && K(e_1, v_i) = \mu v_i,\\
&K(u_1, u_1) = a u_1 + \mu e_1, && K(u_1, u_2) = -a u_2, && K(u_2, u_2) = -a u_1 + \mu e_1,\\
&K(v_1, v_1) = b v_1 + \mu e_1, && K(v_1, v_2) = -b v_2, && K(v_2, v_2) = -b v_1 + \mu e_1,\\
&K(e_1, e_1) = 4\mu e_1,
\end{aligned}$$

where $a, b, \mu \in \mathbb{R}$ with $\mu \neq 0$.

Note that as the Tchebychev vector field is non-vanishing, e_1 is a globally defined vector field on the centroaffine hypersurface. Working on the orthogonal complement of e_1, a similar argument as in the previous case shows that on an open dense subset of M^5, the above vector fields can be locally extended in a differentiable way to the vector fields which we denote again by $\{e_1, u_1, u_2, v_1, v_2\}$ such that

$$\begin{aligned}
&K(u_i, v_j) = 0, && K(e_1, u_i) = \mu u_i, && K(e_1, v_i) = \mu v_i,\\
&K(u_1, u_1) = a u_1 + \mu e_1, && K(u_1, u_2) = -a u_2, && K(u_2, u_2) = -a u_1 + \mu e_1,\\
&K(v_1, v_1) = b v_1 + \mu e_1, && K(v_1, v_2) = -b v_2, && K(v_2, v_2) = -b v_1 + \mu e_1,\\
&K(e_1, e_1) = 4\mu e_1.
\end{aligned}$$

We now call such a $\delta^\sharp(2,2)$-ideal centroaffine hypersurface "of type 1" if $a \neq 0 \neq b$, "of type 2" if $a \neq 0 = b$ and "of type 3" if $a = 0 = b$. Following the same type of argument as in [3, Lemmas 2.1, 2.2 and 2.3], it follows that

$$\begin{aligned}
u_i(\mu) &= \mu h(\widehat{\nabla}_{e_1} e_1, u_i),\\
v_i(\mu) &= \mu h(\widehat{\nabla}_{e_1} e_1, v_i),\\
h(\widehat{\nabla}_{u_i} e_1, v_j) &= h(\widehat{\nabla}_{v_i} e_1, u_j) = 0,\\
2\mu h(\widehat{\nabla}_{u_i} e_1, u_j) &= e_1(\mu) h(u_i, u_j) + h(K(u_i, u_j), \widehat{\nabla}_{e_1} e_1),\\
2\mu h(\widehat{\nabla}_{v_i} e_1, v_j) &= e_1(\mu) h(v_i, v_j) + h(K(v_i, v_j), \widehat{\nabla}_{e_1} e_1),
\end{aligned}$$

$$0 = h(K(u_i, u_k), \widehat{\nabla}_{e_1} v_j) = -v_j(\mu) h(u_i, u_k) + h(K(u_i, u_k), \widehat{\nabla}_{v_j} e_1), \tag{5.1}$$

$$0 = h(K(v_i, v_k), \widehat{\nabla}_{e_1} u_j) = -u_j(\mu) h(v_i, v_k) + h(K(v_i, v_k), \widehat{\nabla}_{u_j} e_1). \tag{5.2}$$

5.1. $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces of type 1

(5.1) implies that $\widehat{\nabla}_{e_1} v_j$ is orthogonal to e_1. A similar conclusion follows from (5.2). So, by combining them, we find $\widehat{\nabla}_{e_1} e_1 = 0$. Hence, $u_i(\mu) = v_i(\mu) = 0$. The other equations now reduce to

$$
\begin{aligned}
\widehat{\nabla}_{e_1} v_1 &= \alpha_1 v_2, & \widehat{\nabla}_{e_1} v_2 &= -\alpha_1 v_1,\\
\widehat{\nabla}_{e_1} u_1 &= \alpha_2 u_2, & \widehat{\nabla}_{e_1} u_2 &= -\alpha_2 u_1,\\
h(\widehat{\nabla}_{u_i} u_j, e_1) &= -\frac{e_1(\mu)}{2\mu} h(u_i, u_j), & h(\widehat{\nabla}_{v_i} v_j, e_1) &= -\frac{e_1(\mu)}{2\mu} h(v_i, v_j),\\
\widehat{\nabla}_{v_1} e_1 &= \frac{e_1(\mu)}{2\mu} v_1, & \widehat{\nabla}_{v_2} e_1 &= \frac{e_1(\mu)}{2\mu} v_2,\\
\widehat{\nabla}_{u_1} e_1 &= \frac{e_1(\mu)}{2\mu} u_1, & \widehat{\nabla}_{u_2} e_1 &= \frac{e_1(\mu)}{2\mu} u_2.
\end{aligned}
$$

Specifying Lemma 2.4(i) and (ii) of [3] in this case, we deduce that $\alpha_1 = \alpha_2 = 0$, $e_1(a) = -\frac{a}{2\mu} e_1(\mu)$ and $e_1(b) = -\frac{b}{2\mu} e_1(\mu)$. Exploiting the remaining Codazzi equations in a similar way after a straightforward computation, we get $v_1(a) = v_2(a) = u_1(b) = u_2(b) = 0$,

$$
\begin{aligned}
u_1(a) &= -3ah(\widehat{\nabla}_{u_2} u_1, u_2), & u_2(a) &= 3ah(\widehat{\nabla}_{u_1} u_1, u_2),\\
v_1(b) &= -3bh(\widehat{\nabla}_{v_2} v_1, v_2), & v_2(b) &= 3bh(\widehat{\nabla}_{v_1} v_1, v_2)
\end{aligned}
$$

and

$$
\begin{gathered}
\widehat{\nabla}_{e_1} e_1 = \widehat{\nabla}_{e_1} v_i = \widehat{\nabla}_{e_1} u_i = \widehat{\nabla}_{u_i} v_j = \widehat{\nabla}_{v_i} u_j = 0, \quad \widehat{\nabla}_{u_1} u_1 = -\frac{e_1(\mu)}{2\mu} e_1 + a_1 u_2,\\
\widehat{\nabla}_{u_1} u_2 = -a_1 u_1, \quad \widehat{\nabla}_{u_2} u_1 = -a_2 u_2, \quad \widehat{\nabla}_{u_2} u_2 = -\frac{e_1(\mu)}{2\mu} e_1 + a_2 u_1, \quad \widehat{\nabla}_{u_i} e_1 = \frac{e_1(\mu)}{2\mu} u_i,\\
\widehat{\nabla}_{v_1} v_1 = -\frac{e_1(\mu)}{2\mu} e_1 + b_1 v_2, \quad \widehat{\nabla}_{v_1} v_2 = -b_1 v_1, \quad \widehat{\nabla}_{v_2} v_1 = -b_2 v_2,\\
\widehat{\nabla}_{v_2} v_2 = -\frac{e_1(\mu)}{2\mu} e_1 + b_2 v_1, \quad \widehat{\nabla}_{v_i} e_1 = \frac{e_1(\mu)}{2\mu} v_i.
\end{gathered}
$$

We take $\wp = e_1(\mu)/(2\mu)$. Applying Gauss equation, it follows that $\wp$ depends only on e_1 and

$$\mu^2 - \wp^2 = \epsilon, \quad e_1(\mu) = 2\wp\mu, \quad e_1(\wp) = 3\mu^2 - \wp^2 - \epsilon.$$

We now consider the following distributions: $\mathcal{D}_1 = \{e_1\}$, $\mathcal{D}_2 = \{u_1, u_2\}$ and $\mathcal{D}_3 = \{v_1, v_2\}$. Applying [19], we can identify a neighborhood U of p with $U = I \times_{\rho_1} M_1^2 \times_{\rho_2} M_2^2$. The mean curvature normals of M_1^2 and M_2^2 in U are given by $\mathcal{H}_1 = \mathcal{H}_2 = -\wp e_1 \in \mathcal{D}_1$. We choose a coordinate t tangent to the component I such that $\frac{\partial}{\partial t} = e_1$. Therefore, we obtain that $\wp$ and μ depend only on the variable t which satisfies the following system of differential equations:

$$\mu^2 - \wp^2 = \epsilon, \quad \frac{\partial}{\partial t}(\mu) = 2\wp\mu, \quad \frac{\partial}{\partial t}(\wp) = 3\mu^2 - \wp^2 - \epsilon.$$

Note that this system implies that if $\mu^2 - \wp^2 - \epsilon = 0$ at a point, it vanishes at every point. So, it is sufficient to pick initial conditions for μ and $\wp$ satisfying $\mu^2 - \wp^2 = \epsilon$ at that point and then to take the corresponding solution of the differential equation. We consider λ depending only on t as a solution of

$$\frac{\partial}{\partial t}(\lambda) = \mu - \wp.$$

It then follows by a straightforward computation that the vector

$$\mathcal{C} = e^{\lambda}(-\epsilon F + (\mu - \wp)e_1)$$

is a constant vector in $\mathbb{R}^6$, where F denotes the immersion. As e_1 and F are independent vectors, $\mathcal{C}$ must be non-vanishing and so by a centroaffine transformation, we may assume that $\mathcal{C} = (0,0,0,0,0,1)$. Moreover, we deduce that F is determined by the following differential equation:

$$F_t - \frac{\epsilon}{\mu - \wp}F = e^{-\lambda}\frac{1}{\mu - \wp}\mathcal{C}$$

which can be rewritten as

$$F_t - (\mu + \wp)F = e^{-\lambda}\epsilon(\mu + \wp)\mathcal{C}.$$

Denote a solution of the homogeneous equation by γ_1 and a solution of the non-homogeneous equation by γ_2. Then, we have

$$F(t, x_1, y_1, x_2, y_2) = \gamma_1(t)G(x_1, y_1, x_2, y_2) + \gamma_2(t)\mathcal{C},$$

where x_1, y_1 are coordinates on M_1^2 and x_2, y_2 are coordinates on M_2^2. As in the case of a vanishing Tchebychev vector field, we can again interpret G as the sum of two improper equiaffine spheres with affine normal $\mathcal{C}$, (cf. [20]). Consequently, it follows that

$$\begin{aligned} F(t, x_1, y_1, x_2, y_2) = (&\gamma_1(t), \gamma_1(t)x_1, \gamma_1(t)y_1, \gamma_1(t)x_2, \gamma_1(t)y_2, \\ &\gamma_1(t)(g_1(x_1, y_1) + g_2(x_2, y_2)) + \gamma_2(t)), \end{aligned}$$

where g_1 and g_2 are solutions of the Monge Ampère equation. Conversely, a straightforward computation shows that such a hypersurface is indeed a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface.

5.2. $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces of type 2

Exploring the Codazzi equations as in the previous case, we deduce

$$\widehat{\nabla}_{e_1}e_1 = \widehat{\nabla}_{e_1}u_i = 0, \qquad \widehat{\nabla}_{v_i}e_1 = \frac{e_1(\mu)}{2\mu}v_i,$$

$$\widehat{\nabla}_{u_i}e_1 = \frac{e_1(\mu)}{2\mu}u_i, \qquad \widehat{\nabla}_{e_1}v_1 = c_1v_2,$$

$$\widehat{\nabla}_{e_1} v_2 = -c_1 v_1, \qquad \widehat{\nabla}_{u_1} u_1 = -\frac{e_1(\mu)}{2\mu} e_1 + a_1 u_2 + c_2 v_1 + c_3 v_2,$$
$$\widehat{\nabla}_{u_1} u_2 = -a_1 u_1 + c_4 v_1 + c_5 v_2, \qquad \widehat{\nabla}_{u_2} u_1 = -a_2 u_2 - c_4 v_1 - c_5 v_2,$$
$$\widehat{\nabla}_{u_2} u_2 = -\frac{e_1(\mu)}{2\mu} e_1 + a_2 u_1 + c_2 v_1 + c_3 v_2, \qquad \widehat{\nabla}_{v_1} v_1 = -\frac{e_1(\mu)}{2\mu} e_1 + b_1 v_2,$$
$$\widehat{\nabla}_{v_1} v_2 = -b_1 v_1, \qquad \widehat{\nabla}_{v_2} v_1 = -b_2 v_2,$$
$$\widehat{\nabla}_{v_2} v_2 = -\frac{e_1(\mu)}{2\mu} e_1 + b_2 v_1, \qquad \widehat{\nabla}_{u_1} v_1 = -c_2 u_1 - c_4 u_2 + c_6 v_2,$$
$$\widehat{\nabla}_{u_1} v_2 = -c_3 u_1 - c_5 u_2 - c_6 v_1, \qquad \widehat{\nabla}_{u_2} v_1 = c_4 u_1 - c_2 u_2 + c_7 v_2,$$
$$\widehat{\nabla}_{u_2} v_2 = c_5 u_1 - c_3 u_2 - c_7 v_1, \qquad \widehat{\nabla}_{v_1} u_1 = -\frac{1}{3} c_4 u_2,$$
$$\widehat{\nabla}_{v_1} u_2 = \frac{1}{3} c_4 u_1, \qquad \widehat{\nabla}_{v_2} u_1 = -\frac{1}{3} c_5 u_2,$$
$$\widehat{\nabla}_{v_2} u_2 = \frac{1}{3} c_4 u_1,$$

where c_1, c_2, c_3, c_4, c_5, c_6, c_7, a_1, a_2, b_1 and b_2 are functions. Moreover, we get that μ depends only on e_1 and $e_1(a) = -\frac{1}{2} a \frac{e_1(\mu)}{\mu}$. The derivatives of a in the other directions are given by

$$u_1(a) = 3aa_2, \quad u_2(a) = 3aa_1, \quad v_1(a) = ac_2, \quad v_2(a) = ac_3.$$

We write $\wp = e_1(\mu)/(2\mu)$. Applying the Gauss equation, it follows that $\wp$ depends only on e_1 and satisfies

$$e_1(\wp) = 3\mu^2 - \wp^2 - \epsilon.$$

We now consider the following distributions: $\mathcal{D}_1 = \{e_1\}$ and $\mathcal{D}_2 = \{u_1, u_2, v_1, v_2\}$. Applying [19], we can identify a neighborhood U of p with $U = I \times_{\rho_1} M_1^4$. The mean curvature normal of M_1^4 in U is given by $\mathcal{H}_1 = -\wp e_1 \in \mathcal{D}_1$. We now choose a coordinate t tangent to the component I such that $\frac{\partial}{\partial t} = e_1$. Therefore, we obtain that $\wp$ and μ depend only on the variable t which satisfies the following system of differential equations:

$$\frac{\partial}{\partial t}(\mu) = 2p\mu, \quad \frac{\partial}{\partial t}(\wp) = 3\mu^2 - \wp^2 - \epsilon.$$

Note that this system implies that if $\mu^2 - \wp^2 - \epsilon = 0$ at a point, it vanishes at every point. This will lead to different subcases.

First, we deal with again the case $\mu^2 - \wp^2 - \epsilon = 0$. This can be treated in a similar way as in the previous case. We take λ depending only on t as a solution of

$$\frac{\partial}{\partial t}(\lambda) = \mu - \wp.$$

Then, it follows by a straightforward computation that the vector

$$\mathcal{C} = e^{\lambda}(-\epsilon F + (\mu - \wp) e_1)$$

is a constant vector in $\mathbb{R}^6$, where F denotes the immersion. As e_1 and F are independent vectors, $\mathcal{C}$ must be non-vanishing and so, by a centroaffine transformation, we may assume that $\mathcal{C} = (0,0,0,0,0,1)$. Moreover, we have that F is determined by the following differential equation:

$$F_t - \frac{\epsilon}{\mu - \wp} F = e^{-\lambda} \frac{1}{\mu - \wp} \mathcal{C}$$

which can be rewritten as

$$F_t - (\mu + \wp) F = e^{-\lambda} \epsilon (\mu + \wp) \mathcal{C}.$$

Denote a solution of the homogeneous equation by γ_1 and a solution of the non-homogeneous equation by γ_2. Then, we deduce

$$F(t, x_1, x_2, x_3, x_4) = \gamma_1(t) G(x_1, x_2, x_3, x_4) + \gamma_2(t)\mathcal{C},$$

where x_1, x_2, x_3, x_4 are coordinates on M_1. Now, we can interpret G as an improper equiaffine hypersphere with affine normal $\mathcal{C}$ contained in an affine hyperplane, (cf. [20]). Moreover, it has the additional property that the difference tensor admits a basis $\{f_1, \ldots, f_4\}$ at every point such that

$$K(f_1, f_1) = af_1, \quad K(f_1, f_2) = -af_2, \quad K(f_2, f_2) = -af_1, \quad K(f_i, f_j) = 0, \text{ otherwise.}$$

Such improper equiaffine hyperspheres can be classified as described in [14]. And therefore, we get

$$\begin{aligned} F(t, x_1, y_1, x_2, y_2) = (&\gamma_1(t), \gamma_1(t)x_1, \gamma_1(t)x_2, \gamma_1(t)x_3, \gamma_1(t)x_4, \\ &\gamma_1(t)(g(x_1, x_2, x_3, x_4)) + \gamma_2(t)), \end{aligned}$$

where g is a solution of the Monge Ampère equation and the associate improper equiaffine hypersphere satisfies the previously mentioned additional condition. Conversely, a straightforward computation shows that such a hypersurface is indeed a $\delta^\sharp(2,2)$-ideal centroaffine hypersurface.

Next, we deal with the case $\mu^2 - \wp^2 - \epsilon < 0$. We take λ_1 depending only on t as a solution of

$$\frac{\partial}{\partial t}(\lambda_1) = -(\mu - \wp).$$

Then, it follows by a straightforward computation that the vector

$$G_1 = e^{\lambda_1}(-\epsilon F + (\mu - \wp) e_1)$$

depends only on the component M_1 and the vector

$$G_2 = e^{\lambda_2}((\mu + \wp) F - e_1)$$

is a constant vector in $\mathbb{R}^6$, where F denotes the immersion. Here, λ_2 depending only on t is a solution of

$$\frac{\partial}{\partial t}(\lambda_2) = -(3\mu - \wp).$$

Since e_1 and F are independent vectors, G_2 must be non-vanishing and therefore by a centroaffine transformation, we may assume that $G_2 = (0,0,0,0,0,1)$. Thus, F can be written by means of G_1 and G_2 as follows:

$$F = \frac{1}{-\epsilon + \mu^2 - \wp^2}(G_1 + (\mu - \wp)G_2).$$

A straightforward computation shows that G_1 is a 4-dimensional positive definite centroaffine hypersurface with a vanishing Tchebychev vector field (elliptic equiaffine Blaschke hypersphere) which is $\delta^\sharp(2)$-ideal contained in a complimentary affine subspace to G_2. Again, the converse can be verified by a direct computation.

Finally, we deal with the case $\mu^2 - \wp^2 - \epsilon > 0$. This case can be treated precisely in the same way. The only difference is that G_1 would need to be a negative definite centroaffine hypersurface with a vanishing Tchebychev vector field (hyperbolic equiaffine Blaschke hypersphere) which is $\delta^\sharp(2)$-ideal contained in a complimentary affine subspace to G_2.

5.3. $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces of type 3

We consider the distributions $\mathcal{D}_1 = \{e_1\}$ and $\mathcal{D}_2 = \{u_1, u_2, v_1, v_2\}$. We still get sufficient information from the Codazzi equation and [19] in order to identify an open neighborhood U of p with $U = I \times_{\rho_1} M_1^4$. The mean curvature normal of M_1^4 in U is given by $\mathcal{H}_1 = -\wp e_1 \in \mathcal{D}_1$, where p is defined as before. Therefore, this case can be treated precisely as the previous one. The only difference is that due to the very special form of the difference tensor, instead of having a special improper (respectively, an elliptic or a hyperbolic) equiaffine hypersphere, we obtain a paraboloid (respectively, an ellipsoid or a hyperboloid.)

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