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Mori's Program for the Moduli Space of Conics in Grassmannian

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Abstract. We complete Mori's program for Kontsevich's moduli space of degree 2 stable maps to the Grassmannian of lines. We describe all birational models in terms of moduli spaces (of curves and sheaves), incidence varieties, and Kirwan's partial desingularization.

1. Introduction and results

1.1. Rational curves in Grassmannian of lines

The space of rational curves in $\mathrm{Gr}(2,n)$ and its compactifications has been studied in various contexts. In the study of Fano manifolds, the space of lines in $\mathrm{Gr}(2,n)$ has been one of the main tools to study the geometry of the linear or quadratic sections of $\mathrm{Gr}(2,n)$ [31]. In fact, the codimension two linear section of $\mathrm{Gr}(2,5)$ is the answer for Hirzebruch's question in dimension 4: the classification of all minimal compactifications of $\mathbb{C}^4$.

On the other hand, a compactified moduli space of conics in $\mathrm{Gr}(2,n)$ for $n=5$ has been an essential ingredient in the construction of a new compact hyperkähler manifold. For example, in [18], by following the general construction of a symplectic two-form on the moduli space of sheaves or rational curves [11, 22], the authors proved that a certain contraction of the Hilbert scheme of conics in $\mathrm{Gr}(2,5)\cap H_1\cap H_2$ where H_d is a hypersurface of degree d, is a hyperkähler manifold discovered by O'Grady in [28].

In the study of homological mirror symmetry, it is important to present a pair of Calabi-Yau threefolds which are derived equivalent but not birationally equivalent. Only a few examples of such pairs have been known. In [15], the authors provided a new such pair by using the double cover (the so-called *double symmetroid*) of the determinantal symmetroid in the space of quadrics $\mathbb{P}(\mathrm{Sym}^2\,\mathbb{C}^{5*})$. One of the main steps of the construction is to find an explicit birational relation between the double symmetroid and the Hilbert scheme of conics in the Grassmannian $\mathrm{Gr}(3,5)\cong\mathrm{Gr}(2,5)$ of planes. This relation has been established in [16] in a broader setting, namely, for the space of quadrics in $\mathbb{P}(\mathrm{Sym}^2\,\mathbb{C}^{n+1*})$ and the Hilbert scheme of conics in $\mathrm{Gr}(n-1,n+1)\cong\mathrm{Gr}(2,n+1)$.

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1.2. Main results

The main result of this paper is the completion of the projective birational geometry of the space of conics in $\mathrm{Gr}(n-1, n+1)$ in the viewpoint of *Mori's program*. Mori's program, or the log minimal model program for a projective moduli space M aims for the classification of all rational contractions of M. If M is a Mori dream space, then for each effective divisor D, one can construct a projective model

$$M(D) := \mathrm{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(M, \mathcal{O}(mD))$$

and a rational contraction $M \dashrightarrow M(D)$. Provided by being a Mori dream space, there are only finitely many projective models.

For the moduli space $\overline{\mathrm{M}}_{0,0}(X, d)$ of stable maps, which is a compactification of rational curves in a projective variety X, Mori's program has been studied in [2, 4]. When $X = \mathrm{Gr}(k, V)$, the Grassmannian of subspaces in V and $d = 2, 3$, the stable base locus decomposition was obtained by Chen and Coskun in [3], as a first step toward Mori's program.

In this paper, we complete Mori's program for $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(n-1, V), 2)$. Furthermore, we describe all birational models in terms of moduli spaces, incidence varieties, and partial desingularizations [21].

Let V be a vector space of dimension $n + 1 \geq 5$. For the precise definition of the divisors in the statement below, see Definition 2.1.

Theorem 1.1. *For an effective divisor D on $\mathbf{M} := \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(n-1, V), 2)$,*

(1) *If $D = aH_{\sigma_{1,1}} + bH_{\sigma_2} + cT$ for $a, b, c > 0$, then $\mathbf{M}(D) \cong \mathbf{M}$.*

(2) *If $D = aH_{\sigma_{1,1}} + bH_{\sigma_2}$ for $a, b > 0$, then $\mathbf{M}(D) \cong \mathbf{C} := \mathrm{Chow}_{1,d}(\mathrm{Gr}(n-1, V))^{\nu}$, the normalization of the main component of the Chow variety.*

(3) *If $D = aH_{\sigma_{1,1}} + bH_{\sigma_2} + cP$ for $a, b, c > 0$, then $\mathbf{M}(D) \cong \mathbf{H} := \mathrm{Hilb}^{2m+1}(\mathrm{Gr}(n-1, V))$.*

(4) *If $D = aT + b\Delta$ for $a > 0$ and $b \geq 0$, then $\mathbf{M}(D) \cong \mathbf{U} := \overline{\mathrm{U}}_{0,0}(\mathrm{Gr}(n-1, V), 2)$, the normalization of the closure of the image of $\mathbf{M}$ in $\mathbb{P}(\wedge^{n-1} V^* \otimes \mathfrak{sl}_2) /\!/ \mathrm{SL}_2$ (Definition 3.2).*

(5) *If $D = aH_{\sigma_2} + bD_{\deg} + c\Delta$ for $a > 0$ and $b, c \geq 0$, then $\mathbf{M}(D) \cong \mathbf{K} := \mathbb{P}(V^* \otimes \mathfrak{gl}_2) /\!/ \mathrm{SL}_2 \times \mathrm{SL}_2$, which is isomorphic to a connected component of the moduli space $\mathrm{M}_{\mathbb{P}V}(v)$ of semistable sheaves with $v = 2\,\mathrm{ch}(\mathcal{O}_{\mathbb{P}^{n-1}})$ (Definition 3.2).*

(6) *If $D = aH_{\sigma_2} + bT + c\Delta$ for $a, b > 0$ and $c \geq 0$, then $\mathbf{M}(D) \cong \mathbf{X}^1 /\!/ \mathrm{SL}_2 \times \mathrm{SL}_2$, which is the first step of the partial desingularization of $\mathbf{K}$ (Section 5.2).*

(7) *If* $D = aH_{\sigma_2} + bP + cD_{\text{deg}}$ *for* $a, b > 0$ *and* $c \geq 0$*, then* $\mathbf{M}(D) = \widetilde{\mathbf{G}}$ *(Section* 3.2*).*

(8) *If* $D = aD_{\text{unb}} + bP + cD_{\text{deg}}$ *for* $a, b > 0$ *and* $c \geq 0$*, then* $\mathbf{M}(D) = \mathbf{G} := \mathrm{Gr}(3, \wedge^2 \mathcal{S})$ *where* $\mathcal{S}$ *is the universal subbundle over* $\mathrm{Gr}(4, V^*)$*.*

(9) *If* $D = aH_{\sigma_{1,1}} + bP + cD_{\text{unb}}$ *for* $a, b, c > 0$*, then* $\mathbf{M}(D) \cong \mathbf{B} := \mathrm{Bl}_{\mathrm{OG}(3, \wedge^2 \mathcal{S})_{\sigma_{(2)^*}}} \mathrm{Gr}(3, \wedge^2 \mathcal{S})$ *(Definition* 6.8*).*

(10) *If* $D = aH_{\sigma_{1,1}} + bD_{\text{unb}} + c\Delta$ *for* $a, b > 0$ *and* $c \geq 0$*, then* $\mathbf{M}(D) \cong \mathbf{K}_{\mathcal{S}} := \mathbb{P}(\mathcal{S}^* \otimes \mathfrak{gl}_2) /\!/ \mathrm{SL}_2 \times \mathrm{SL}_2 \cong \mathrm{M}_{\mathbb{P}\mathcal{S}}(m^2 + 3m + 2)$*, the relative moduli space of semistable sheaves (Definition* 6.4*).*

(11) *If* $D = aH_{\sigma_{1,1}} + bT + c\Delta$ *for* $a, b > 0$ *and* $c \geq 0$*, then* $\mathbf{M}(D)$ *is the normalization* $\mathbf{R}$ *of the incidence variety in* $\mathrm{M}_{\mathbb{P}V^*}(m^2 + 3m + 2) \times \mathbf{U}$*.*

(12) *If* $D = aH_{\sigma_{1,1}} + b\Delta$ *for* $a > 0$ *and* $b \geq 0$*, then* $\mathbf{M}(D) \cong \mathbf{L}$*, the normalization of the closure of the locus of sheaves supported on a smooth quadric surfaces in* $\mathrm{M}_{\mathbb{P}V^*}(m^2 + 3m + 2)$ *(Definition* 6.5*).*

(13) *If* $D = aP + bD_{\text{deg}}$ *for* $a > 0$ *and* $b \geq 0$*, then* $\mathbf{M}(D) \cong \overline{\mathbf{G}}$*, which is the normalization of the image of the envelope map* $\mathrm{env}\colon \mathbf{H} \to \mathrm{Gr}(3, \wedge^2 V^*)$ *(Definition* 3.3*).*

(14) *If* $D = aH_{\sigma_{1,1}} + bP$ *for* $a, b > 0$*, then* $\mathbf{M}(D)$ *is the blow-up* $\widehat{\mathbf{G}}$ *of* $\mathbf{G}$ *along a subvariety isomorphic to* $\mathrm{OG}(3, \wedge^2 \mathcal{S})$*.*

(15) *If* $D = a\Delta + bD_{\text{deg}}$ *for* $a, b \geq 0$*, then* $\mathbf{M}(D)$ *is a point.*

(16) *If* $D = aD_{\text{unb}} + b\Delta$ *for* $a > 0$ *and* $b \geq 0$*, or* $D = aD_{\text{unb}} + bD_{\text{deg}}$ *for* $a > 0$ *and* $b \geq 0$*, then* $\mathbf{M}(D)$ *is* $\mathrm{Gr}(4, V^*) \cong \mathrm{Gr}(n-3, V)$*.*

When $n = 3$, the description of birational models is simpler because of the self-dual map on $\mathrm{Gr}(2, 4)$. See Theorem 6.2 for the statement.

Note that there are only few examples of completed Mori's programs when the complexity of the moduli space is large. Except toric varieties and moduli spaces with Picard number two (for instance, [2, 25]), the completed examples are very rare (see [26] for such an example). Theorem 1.1 provides one additional example with Picard number three.

1.3. Application to the motivic invariants

Let us finish this section by mentioning some related works. One of the birational models of $\mathbf{M}$ turns out to be the moduli space of quiver representations with dimension vector $(2, 2)$ and $n+1$ arrows between them (Item (5) of Theorem 1.1). We call the moduli space of such quiver representations the moduli space of Kronecker modules of type $(n+1; 2, 2)$, or simply, the *Kronecker moduli space*. The Kronecker moduli space has been studied in the context of curve counting invariants (In particular, GW/Kronecker correspondence). For details, see [32].

The Kronecker moduli space of type $(6; 2, 2)$ is birational to the moduli space of semistable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $4m + 2$. The birational map can be explicitly described in terms of Bridgeland wall-crossing [1, Section 6]. Combining with our analysis, we obtain the virtual Poincaré polynomial of $\mathrm{M}_{\mathbb{P}^2}(4m+2)$ from that of $\mathbf{M}$. For a detail, see Section 7.

1.4. Structure of paper

This paper is organized as follows. In Section 2, for the reader's convenience, we recall the stable base locus decomposition of $\mathbf{M}$ obtained by Chen and Coskun. Section 3 introduces many birational models obtained in [3, 16]. In Section 4, we study geometric properties of the moduli space $\mathbf{K}$ of Kronecker modules, which is a key ingredient of the moduli theoretic interpretation of biratoinal models. In Section 5 we show that the partial desingularization of $\mathbf{K}$ is indeed $\mathbf{M}$ and investigate the geometry of the birational contraction. After introducing some more natural models, in Section 6 we prove Theorem 1.1. Finally, in the last section we compute topological invariants of some moduli spaces.

1.5. Notations

We work on $\mathbb{C}$. A projective space $\mathbb{P}V$ is the space of one-dimensional *subspaces* of V. For a partition λ, let Σ_λ be an associated Schubert variety in $\mathrm{Gr}(k, V)$. Its Poincaré dual is denoted by σ_λ. For a partition λ, λ^* is the dual partition. For a direct sum of sheaves, we will use additive notation. For instance, $2\mathcal{O}_X$ means $\mathcal{O}_X^{\oplus 2}$.

2. Stable base locus decomposition

In this section, we fix an integer $n \geq 3$. Let V be an $(n+1)$-dimensional vector space and let k be an integer such that $2 \leq k \leq n-1$.

When one runs Mori's program for a given moduli space M, the first step is the computation of the rank of Neron-Severi vector space $\mathrm{N}^1(M)$ and the effective cone $\mathrm{Eff}(M)$. For the moduli space $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k, V), 2)$, $\dim \mathrm{N}^1(\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k, V), 2)) = 3$ (see, [29, Theorem 1]). Its effective cone was computed by Coskun and Starr in [10]. To describe the result, we need to introduce several effective divisor classes on $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k, V), 2)$.

Definition 2.1. (1) Let Δ be the locus of stable maps with singular domains.

(2) Fix an $(n-1-k)$-dimensional subspace W of V. Let D_{deg} be the locus of stable maps f such that the projection of the smallest linear subspace in V generated by the image of f onto V/W is a proper subspace of V/W. If $k=n-1$, D_{deg} is the locus of stable maps whose image is in $\mathrm{Gr}(n-1,V')$ for some n-dimensional subspace $V'\subset V$.

(3) For a stable map $f\colon \mathbb{P}^1\to \mathrm{Gr}(k,V)$, let $0\to E\to \mathcal{O}_{\mathbb{P}^1}\otimes V$ be the induced subbundle of rank k of degree -2. If $k=2$, let D_{unb} be the closure of the locus of stable maps such that $E\neq 2\mathcal{O}_{\mathbb{P}^1}(-1)$. When $k>2$, for a general stable map f and its associated subbundle $0\to E\to \mathcal{O}_{\mathbb{P}^1}\otimes V$, there is a trivial subbundle $E' := (k-2)\mathcal{O}_{\mathbb{P}^1}\subset E$ which induces an $(k-2)$-dimensional sub vector space $V_{E'}\subset V$. Let D_{unb} be the closure of the locus of stable maps such that $V_{E'}\cap W\neq\{0\}$ for a fixed $(n+3-k)$-dimensional subspace $W\subset V$. In other words, $D_{\mathrm{unb}} = a^*(\mathcal{O}_{\mathrm{Gr}(n+3-k,V)}(1))$ for $a\colon \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k,V),2) \dashrightarrow \mathrm{Gr}(n+3-k,V)$.

(4) Let $H_{\sigma_{1,1}}$ (resp. H_{σ_2}) be the locus of stable maps whose image in $\mathrm{Gr}(k,V)$ intersects a fixed codimension two Schubert variety $\Sigma_{1,1}$ (resp. Σ_2).

(5) Let T be the locus of stable maps whose image in $\mathrm{Gr}(k,V)$ is tangent to a fixed hyperplane Σ_1.

(6) If we compose a general stable map $f\colon C\to \mathrm{Gr}(k,V)$ with the Plücker embedding $\mathrm{Gr}(k,V)\hookrightarrow \mathbb{P}(\wedge^k V)$, then we obtain a conic in $\mathbb{P}(\wedge^k V)$, which spans a two-dimensional subspace in $\mathbb{P}(\wedge^k V)$. Thus we obtain an element in $\mathrm{Gr}(3,\wedge^k V)$. Thus there is a rational map $p\colon \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k,V),2)\dashrightarrow \mathrm{Gr}(3,\wedge^k V)$. Let $P := p^*\mathcal{O}_{\mathrm{Gr}(3,\wedge^k V)}(1)$.

Theorem 2.2. [10, Section 2] *The effective cone of $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k,V),2)$ is generated by D_{unb}, D_{deg} and Δ. In particular, the effective cone is simplicial.*

The next step is the computation of the stable base locus decomposition, which is the first approximation of the Mori chamber decomposition of the effective cone. This was done by Chen and Coskun in [3, Theorem 3.6]. In particular, there are 8 open chambers as in Figure 2.1. Also the divisor classes of T, D_{deg}, D_{unb}, and P are calculated in terms of Δ, $H_{\sigma_{1,1}}$, and H_{σ_2} in [10, Sections 4, 5] and [3, Lemma 3.4].

For the computation of the stable base locus decomposition, Chen and Coskun introduced many curve classes on $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k,V),2)$ in [3, Section 3]. Since these curves will have a prominent role in the proof of Theorem 1.1, for the reader's convenience, we leave the definition.

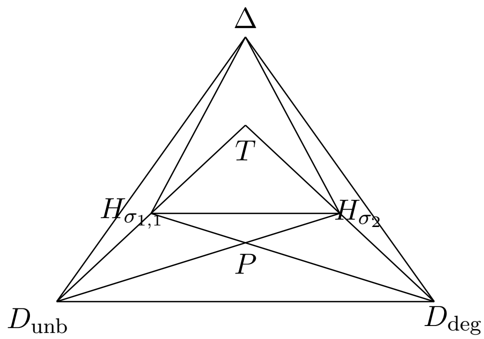

Figure 2.1: Stable base locus decomposition of $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k,V),2)$.

Definition 2.3. (1) Let C_1 (resp. C_2) be a general pencil of conics in a fixed plane of class $\Sigma_{(1,1)^*}$ (resp. $\Sigma_{(2)^*}$).

(2) Let C_5 be a one-parameter family of conics in a fixed $\Sigma_{(1,1)^*}$ tangent to four general lines.

(3) Let C_6 (resp. C_7) be a one-dimensional family of singular stable maps obtained by attaching a fixed line to the base point of a pencil of lines in a fixed $\Sigma_{(1,1)^*}$ (resp. $\Sigma_{(2)^*}$).

(4) Let C_8 be a one-parameter family of two-to-one covers of a fixed line.

The intersection numbers of curve classes with divisors are summarized in [3, Table 1].

3. Some birational models

From this section, we focus on the $k = n-1$ case, that is, $M = \mathbf{M} = \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(n-1,V),2)$.

After the computation of the stable base locus decomposition, the next step of Mori's program of $\mathbf{M}$ is to determine a birational model $\mathbf{M}(D)$ for each effective divisor D. Since $\mathbf{M}$ is a Mori dream space (see, [3, Corollary 1.2]), there are finitely many birational models for each cone in the stable base locus decomposition.

For $\mathbf{M}$, there have been two prior results. A family of birational models is obtained in [3] from the moduli theoretic viewpoint. On the other hand, by using multilinear algebra and incidence varieties, another family of birational models is obtained by Hosono and Takagi in [16]. In this section, we review these birational models.

3.1. Moduli theoretic models

Since $\mathbf{M}$ is a compactification of the moduli space of smooth conics in $\mathrm{Gr}(n-1,V)$, we obtain several natural birational models from different compactifications of moduli spaces of smooth conics. Here we review such birational models.

Definition 3.1. (1) Let $\mathbf{H} := \mathrm{Hilb}^{2m+1}(\mathrm{Gr}(n-1,V))$ be Hilbert scheme of conics in $\mathrm{Gr}(n-1,V)$.

(2) Let $\mathbf{C} := \mathrm{Chow}_{1,2}(\mathrm{Gr}(n-1,V))^{\nu}$ be the normalization of the main component of the Chow variety of dimension 1, degree 2 algebraic cycles in $\mathrm{Gr}(n-1,V)$.

Definition 3.2. For a projective space $\mathbb{P}W$, there is a divisorial contraction (see, [14, Section 11], [30, Theorem 0.1])

$$\overline{\mathrm{M}}_{0,0}(\mathbb{P}W,d) \to \mathbb{P}(W^* \otimes \mathrm{Sym}^d\,\mathbb{C}^2)/\!/\,\mathrm{SL}_2\,.$$

This map is indeed the partial desingularization when $d=2$ (see, [21], [19, Theorem 4.1]). Let $\mathbf{U} := \overline{\mathrm{U}}_{0,0}(\mathrm{Gr}(n-1,V),2)$ be the normalization of the image of the composition

$$\mathbf{M} \hookrightarrow \overline{\mathrm{M}}_{0,0}(\mathbb{P}(\wedge^{n-1}V),2) \to \mathbb{P}(\wedge^{n-1}V^* \otimes \mathfrak{sl}_2)/\!/\,\mathrm{SL}_2\,.$$

It is well-known that $\mathbf{H}$ is smooth (see, [8, Proposition 3.6]), and there is a diagram

$$\begin{tikzcd} \mathbf{M} \arrow[d] \arrow[dr] \arrow[rr, dashed, leftrightarrow] & & \mathbf{H} \arrow[dl] \\ \mathbf{U} & \mathbf{C} & \end{tikzcd}$$

Definition 3.3. For each conic $C \in \mathbf{H} = \mathrm{Hilb}^{2m+1}(\mathrm{Gr}(n-1,V)) \subset \mathrm{Hilb}^{2m+1}(\mathbb{P}(\wedge^{n-1}V))$, the smallest linear subspace $\langle C\rangle$ of $\mathbb{P}(\wedge^{n-1}V)$ containing C, the so-called *linear envelope* of C, is $\mathbb{P}^2$. Thus we have a regular morphism

$$\begin{aligned} \mathrm{env}\colon \mathbf{H} &\to \mathrm{Gr}(3,\wedge^{n-1}V) \\ C &\mapsto \langle C\rangle\,. \end{aligned} \tag{3.1}$$

For any conic C, $\langle C\rangle \cap \mathrm{Gr}(n-1,V) \subset \mathbb{P}(\wedge^{n-1}V)$ is either C or $\langle C\rangle \cong \mathbb{P}^2$. The second case happens precisely when $\langle C\rangle \subset \mathrm{Gr}(n-1,V)$. The moduli space of planes in $\mathrm{Gr}(n-1,V)$ has two connected components. One is the moduli space of Schubert planes $\Sigma_{(1,1)^*}$, which is isomorphic to a partial flag variety $\mathrm{Fl}(n-3,n,V)$. The other component is the moduli space of Schubert planes $\Sigma_{(2)^*}$, which is isomorphic to $\mathrm{Gr}(n-2,V)$.

When $n=3$, env is a birational morphism. Indeed, env is the blow-up along two disjoint orthogonal Grassmannians $\mathrm{OG}(3,\wedge^2V)$ which parametrize $\sigma_{(1,1)^*}$ (resp. $\sigma_{(2)^*}$) planes in $\mathrm{Gr}(2,V)$ [3, Lemma 3.9]. We denote them by $\mathrm{OG}(3,\wedge^2V)_{\sigma_{(1,1)^*}}$ (resp. $\mathrm{OG}(3,\wedge^2V)_{\sigma_{(2)^*}}$) whenever we want to distinguish them.

In summary,

$$\mathrm{Hilb}^{2m+1}(\mathrm{Gr}(2,V)) \cong \mathrm{Bl}_{2\,\mathrm{OG}(3,\wedge^2V)}\,\mathrm{Gr}(3,\wedge^2V). \tag{3.2}$$

3.2. Models from birational geometry of determinantal varieties

With a motivation in the context of homological projective duality, Hosono and Takagi studied birational geometry of T_r, which is a double cover of the space S_r of rank $\le r$ quadric hypersurfaces in $\mathbb{P}V$ (see, [16]). More precisely, for $1 \le r \le n+1$, let $S_r \subset \mathbb{P}(\mathrm{Sym}^2 V^*)$ be the locus of degree 2 polynomials such that the associated quadratic form has rank $\le r$. When r is even, there is a double cover T_r of S_r ramified along S_{r-1} (see, [16, Proposition 2.3]).

Set theoretically, $T_4 \setminus S_3$, which is an étale double cover of $S_4 \setminus S_3$, parametrizes pairs (Q, P) where Q is a rank 4 quadric hypersurface and P is a pencil of $\mathbb{P}^{n-2}$ in Q. Furthermore, they show that when $n \ge 3$, T_4 is birational to $\mathbf{H}$ and there is a contraction diagram:

$$\begin{tikzcd}
 & & \mathbf{H} \arrow[d] & \\
\mathbf{G} \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & \widetilde{\mathbf{G}} \arrow[dl] \arrow[dr] & \\
 & \overline{\mathbf{G}} & & T_4
\end{tikzcd}$$

where

(1) $\mathbf{G} := \mathrm{Gr}(3, \wedge^2 \mathcal{S})$ is the Grassmannian bundle where $\mathcal{S}$ is the rank 4 universal subbundle over $\mathrm{Gr}(4, V^*)$;

(2) $\overline{\mathbf{G}}$ is the normalization of the image of $\mathrm{env}\colon \mathbf{H} \to \mathrm{Gr}(3, \wedge^{n-1} V)$ (see, Definition 3.3);

(3) $\widetilde{\mathbf{G}}$ is the D-flip of $\mathbf{G}$ over $\overline{\mathbf{G}}$;

(4) $\widetilde{\mathbf{G}}$ is a divisorial contraction of $\mathbf{H}$, which contracts the curve class C_1, and hence $D_{\deg}$.

Furthermore, the following properties are studied in [16, Propositions 4.22, 4.11, and 4.5].

(1) The map $\widetilde{\mathbf{G}} \to T_4$ is a divisorial contraction which contracts the image of Δ;

(2) The normalization $\overline{\mathbf{G}} \to \mathrm{im}\,\mathrm{env}$ is bijective;

(3) If $n = 3$, then $\mathbf{G} \cong \overline{\mathbf{G}} \cong \mathrm{Gr}(3, \wedge^2 V)$. If $n > 3$, then $\mathbf{G} \to \overline{\mathbf{G}}$ is a small contraction whose exceptional fibers are $\mathbb{P}^{n-3}$;

(4) The map $\widetilde{\mathbf{G}} \to \overline{\mathbf{G}}$ is a contraction of the image of the locus of conics in a $\Sigma_{(2)^*}$-plane. So for an exceptional point, its fiber is $\mathbb{P}^5$. If $n = 3$, it is a blow-up, but if $n > 3$, this is a small contraction.

4. Moduli space of Kronecker modules

In this paper, the moduli space of Kronecker modules has a central role in connecting moduli spaces of sheaves and that of rational curves. In this section, we review its definition and basic properties.

4.1. Definitions and GIT stability

Fix two positive integers a, b and let W be a vector space. A *Kronecker W-module* is a quiver representation of an n-Kronecker quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r] \arrow[r, bend right] \arrow[r, phantom, shift left=2, "\vdots"] & \bullet \end{tikzcd}$$

with a dimension vector (a, b). Two Kronecker W-modules $\phi = (\phi_i)$ and $\psi = (\psi_i)$ are equivalent if there are $A \in \mathrm{SL}_a$ and $B \in \mathrm{SL}_b$ such that $\phi = B \circ \psi \circ A$. We may regard the GIT quotient

$$\mathbb{P}\,\mathrm{Hom}(W \otimes \mathbb{C}^a, \mathbb{C}^b) /\!/ \mathrm{SL}_a \times \mathrm{SL}_b$$

as the moduli space of *semistable* Kronecker W-modules. The GIT stability was obtained by Drézet (see, [12, Proposition 15]).

Theorem 4.1. *A closed point $M \in \mathbb{P}\,\mathrm{Hom}(W \otimes \mathbb{C}^a, \mathbb{C}^b)$ is (semi)stable with respect to the $\mathrm{SL}_a \times \mathrm{SL}_b$-action if and only if for every nonzero proper subspace $V_1 \subset \mathbb{C}^a$ and $V_2 \subset \mathbb{C}^b$ such that $M(W \otimes V_1) \subset V_2$,*

$$\frac{\dim V_2}{\dim V_1}\ (\geq) > \frac{b}{a}.$$

From now on, we restrict ourselves to a special case that $a = b = 2$.

Corollary 4.2. *A closed point $M \in \mathbb{P}\,\mathrm{Hom}(W \otimes \mathbb{C}^2, \mathbb{C}^2) \cong \mathbb{P}(W^* \otimes \mathfrak{gl}_2)$ is (semi)stable with respect to the $\mathrm{SL}_2 \times \mathrm{SL}_2$-action if and only if for every one-dimensional subspace $V_1 \subset \mathbb{C}^2$, $\dim \mathrm{im}\, M(W \otimes V_1)\ (\geq) > 1$.*

Let V be an $(n+1)$-dimensional vector space, as before. We may describe $M \in \mathbf{X} := \mathbb{P}\,\mathrm{Hom}(V \otimes \mathbb{C}^2, \mathbb{C}^2) = \mathbb{P}(V^* \otimes \mathfrak{gl}_2)$ as a 2×2 matrix of linear polynomials with $n+1$ variables $x_0, \dots, x_n$. Then M is semistable if and only if even after performing row/column operations, there is no zero row or column. M is stable if and only if M has no zero entry. In summary, we have the following result.

Lemma 4.3. *Let $M \in \mathbf{X} := \mathbb{P}(V^* \otimes \mathfrak{gl}_2)$.*

(1) *If M is unstable, then M is equivalent to*

$$\begin{bmatrix} g & h \\ 0 & 0 \end{bmatrix} \quad \textit{or} \quad \begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix}$$

for some $g, h \in V^*$.

(2) *If* M *is strictly semistable, then* M *is equivalent to*

$$\begin{bmatrix} g & k \\ 0 & h \end{bmatrix}$$

for some $g, h \in V^* \setminus \{0\}$ *and* $k \in V^*$.

(3) *If* M *is strictly semistable and has a closed orbit in the semistable locus, then* $k = 0$, *so* M *is equivalent to*

$$\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}$$

for some $g, h \in V^* \setminus \{0\}$. *If* g *is proportional to* h, *then* $\mathrm{Stab}\, M \cong \mathrm{SL}_2 \ltimes \mathbb{Z}_2$. *If* g *is not proportional to* h, *then* $\mathrm{Stab}\, M \cong \mathbb{C}^* \ltimes \mathbb{Z}_2$.

Remark 4.4. The description of stabilizers is different from that in [9, Lemma 6.4], because in this paper we are taking the $\mathrm{SL}_2 \times \mathrm{SL}_2$-quotient instead of the $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-quotient.

4.2. Moduli space of Kronecker modules as a moduli space of semistable sheaves

The moduli spaces of Kronecker modules can be understood as moduli spaces of semistable sheaves. Some explicit examples can be found in [12] and [23, Section 3]. In this section we investigate a generalization toward moduli spaces of sheaves on higher-dimensional projective spaces.

The following lemma is a direct generalization of [9, Lemma 5.2].

Lemma 4.5. *Let* $n \geq 2$. *Let* $F \in \mathrm{Coh}(\mathbb{P}^n)$ *have a resolution*

$$0 \to 2\mathcal{O}_{\mathbb{P}^n}(-1) \stackrel{M}{\to} 2\mathcal{O}_{\mathbb{P}^n} \to F \to 0 \tag{4.1}$$

such that M *is a semistable Kronecker module. Then* F *is isomorphic to either*

(1) $F = I_{\mathbb{P}^{n-2},Q}(1)$ *for some quadric hypersurface* Q *of rank* 3 *or* 4;

(2) *an extension of* $\mathcal{O}_H$ *by* $\mathcal{O}_{H'}$ *for two hyperplanes* H, H'.

In particular, F *is semistable. Furthermore, in the case of* (1), F *is stable.*

Proof. By composing M with an injective morphism $\mathcal{O}_{\mathbb{P}^n}(-1) \to 2\mathcal{O}_{\mathbb{P}^n}(-1)$, we obtain an injective morphism $0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to 2\mathcal{O}_{\mathbb{P}^n}$ whose cokernel is isomorphic to either $I_{L,\mathbb{P}^n}(1)$ for a linear subspace L of dimension $n-2$, or $\mathcal{O}_H \oplus \mathcal{O}_{\mathbb{P}^n}$.

Case 1. Suppose that the cokernel is isomorphic to $I_{L,\mathbb{P}^n}(1)$.

We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^n}(-1) \arrow[r] \arrow[d] & 2\mathcal{O}_{\mathbb{P}^n} \arrow[r] \arrow[d, equal] & I_{L,\mathbb{P}^n}(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathcal{O}_{\mathbb{P}^n}(-1) \arrow[r] & 2\mathcal{O}_{\mathbb{P}^n} \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

Applying the snake lemma, we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to I_{L,\mathbb{P}^n}(1) \to F \to 0.$$

From the sequence $0 \to I_{L,\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n}(1) \to \mathcal{O}_L(1) \to 0$ and the snake lemma again, we obtain $F \cong I_{L,Q}(1)$ for some quadric hypersurface Q. Here Q is the support of $\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}(1)$. Since the defining equation of Q is in $\mathrm{H}^0(I_L(2))$, it has rank at most 4.

If Q has rank 3 or 4, it is irreducible and reduced. Then every subsheaf of $I_{L,Q}(1)$ is of the form $I_{Z,Q}(1)$ for some subscheme $L \subset Z \subset Q$. If $\dim Z = \dim L$, then clearly $p(I_{Z,Q}(1)) < p(I_{L,Q}(1))$ where p is the reduced Hilbert polynomial. If $\dim Z = \dim Q$, since Q is irreducible and reduced, $Z = Q$ and $I_{Z,Q}(1) = 0$. Thus $I_{L,Q}(1)$ is stable.

If Q has rank ≤ 2, then $Q = H \cup H'$ or $2H$ for two hyperplanes H, H'. Suppose that $L \subset H'$. From $0 \to I_{H',H\cup H'}(1) \to I_{L,H\cup H'}(1) \to I_{L,H'}(1) \to 0$, we obtain

$$0 \to \mathcal{O}_{H'} \to I_{L,H\cup H'}(1) \to \mathcal{O}_H \to 0.$$

Thus $F = I_{L,H\cup H'}(1)$ is semistable. Furthermore, since $p(\mathcal{O}_{H'}) = p(I_{L,H\cup H'}(1))$, F is strictly semistable. $Q = 2H$ case is similar.

Case 2. Assume that the cokernel is $\mathcal{O}_H \oplus \mathcal{O}_{\mathbb{P}^n}$.

In this case, it is straightforward to see that M is represented by a matrix in item (2) or (3) in Lemma 4.3. Then $F = I_{L,H\cup H'}(1)$ (in the case of (2)) or $F = \mathcal{O}_H \oplus \mathcal{O}_{H'}$ (in the case of (3)) and F fits in an exact sequence $0 \to \mathcal{O}_{H'} \to F \to \mathcal{O}_H \to 0$. □

Proposition 4.6. *Let* $\mathrm{M}_{\mathbb{P}V}(v)$ *be the moduli space of semistable pure sheaves* F *with* $v := \mathrm{ch}(F) = 2\,\mathrm{ch}(\mathcal{O}_{\mathbb{P}^{n-1}})$. *Then* $\mathbf{K} := \mathbb{P}(V^* \otimes \mathfrak{gl}_2)/\!/\, \mathrm{SL}_2 \times \mathrm{SL}_2$ *is isomorphic to the connected component of* $\mathrm{M}_{\mathbb{P}V}(v)$ *containing* $I_{\mathbb{P}^{n-2},Q}(1)$.

Proof. Let $\mathrm{M}_{\mathbb{P}V}(v)^c$ (resp. $\mathrm{M}_{\mathbb{P}V}(v)^m$) be the connected (resp. irreducible) component of $\mathrm{M}_{\mathbb{P}V}(v)$ containing $I_{\mathbb{P}^{n-2},Q}(1)$. We will show that $\mathbf{K} \cong \mathrm{M}_{\mathbb{P}V}(v)^m \cong \mathrm{M}_{\mathbb{P}V}(v)^c$.

By Lemma 4.5, the universal family of quiver representations over $\mathbb{P}(V^* \otimes \mathfrak{gl}_2)^{ss}$ induces a morphism $f\colon \mathbb{P}(V^* \otimes \mathfrak{gl}_2)^{ss} \to \mathrm{M}_{\mathbb{P}V}(v)$ and f is $\mathrm{SL}_2 \times \mathrm{SL}_2$-invariant. Thus the map f descends to the quotient map

$$\overline{f}\colon \mathbf{K} := \mathbb{P}(V^* \otimes \mathfrak{gl}_2)/\!/\, \mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{M}_{\mathbb{P}V}(v).$$

From the description of cokernels in Lemma 4.5, it is clear that $\overline{f}$ is injective. Furthermore, at a general stable point $[F] \in \operatorname{im}\overline{f}$, $\dim T_{[F]}\mathrm{M}_{\mathbb{P}V}(v) = \operatorname{Ext}^1(F,F) = 4n-3 = \dim \mathbf{K}$. Therefore $\operatorname{im}\overline{f} = \mathrm{M}_{\mathbb{P}V}(v)^m$. Since $\mathbf{K}$ is normal and $\overline{f}$ is injective, $\mathbf{K}$ is isomorphic to the normalization of $\mathrm{M}_{\mathbb{P}V}(v)^m$.

Now it is sufficient to show that $\mathrm{M}_{\mathbb{P}V}(v)^c$ is irreducible and normal. From the standard construction of moduli spaces of semistable sheaves, $\mathrm{M}_{\mathbb{P}V}(v)$ is an $\mathrm{SL}_{P(m)}$-GIT quotient of the quot scheme $\operatorname{Quot}(\mathcal{H},P)$ where $\mathcal{H} = W \otimes \mathcal{O}_{\mathbb{P}V}(-m)$, $\dim W = P(m) = P(F(m))$ for some $m \gg 0$. From the resolution (4.1), it is straightforward to see that $\operatorname{Ext}^2(F,F) = 0$. Since F is m-regular for some $m \gg 0$, we may assume that $\operatorname{Ext}^1(\mathcal{H},F) = 0$. Thus if we denote the kernel of $\mathcal{H} \to F \to 0$ by K, then $\operatorname{Ext}^1(K,F) = 0$. This implies that the irreducible component of $\operatorname{Quot}(\mathcal{H},P)^{ss}$ containing $\mathcal{H} \to I_{\mathbb{P}^{n-2},Q}(1) \to 0$ is smooth and hence coincides with the connected component. In particular, its quotient, $\mathrm{M}_{\mathbb{P}V}(v)^c$, is irreducible and normal. Therefore $\mathrm{M}_{\mathbb{P}V}(v)^c = \mathrm{M}_{\mathbb{P}V}(v)^m = \mathbf{K}$. □

When $n = 3$, it was shown in [23, Proposition 3.6] that $\mathrm{M}_{\mathbb{P}V}(v)$ is indeed irreducible. Since $P(F)(m) = m^2+3m+2$ for a semistable sheaf F of class v, we will use the notation $\mathrm{M}_{\mathbb{P}V}(m^2+3m+2)$ for $\mathrm{M}_{\mathbb{P}V}(v)$ if it is better in the context.

Question 4.7. Is there any extra component of $\mathrm{M}_{\mathbb{P}V}(v)$ if $n > 3$?

4.3. Birational models of moduli spaces of rational curves in a Grassmannian

When the dimension vector is $(2,d)$ where $d < n+1 = \dim V$, the moduli space of Kronecker V-modules $\mathbb{P}\operatorname{Hom}(V \otimes \mathbb{C}^2, \mathbb{C}^d) /\!/ \mathrm{SL}_2 \times \mathrm{SL}_d$ provides a birational model of $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(d,V^*),d) \cong \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(n-d+1,V),d)$. This is a direct generalization of [9, Section 6.1].

Proposition 4.8. *There is a birational map*

$$\Phi\colon \mathbb{P}\operatorname{Hom}(V \otimes \mathbb{C}^2, \mathbb{C}^d) /\!/ \mathrm{SL}_2 \times \mathrm{SL}_d \dashrightarrow \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(d,V^*),d). \tag{4.2}$$

Proof. Let $S \to \mathbb{P}\operatorname{Hom}(V \otimes \mathbb{C}^2, \mathbb{C}^d) \cong \mathbb{P}(\operatorname{Hom}(\mathbb{C}^2,\mathbb{C}^d) \otimes V^*)$ be a morphism. It induces a family of sheaf morphisms

$$2\mathcal{O}_{S\times\mathbb{P}V}(-1) \xrightarrow{M} d\mathcal{O}_{S\times\mathbb{P}V}.$$

By taking the pull-back by the projection $q\colon S \times \mathbb{P}V \times \mathbb{P}^1 \to S \times \mathbb{P}V$, we obtain

$$2\mathcal{O}_{S\times\mathbb{P}V\times\mathbb{P}^1}(-1,0) \xrightarrow{q^*M} d\mathcal{O}_{S\times\mathbb{P}V\times\mathbb{P}^1}.$$

By composing with the tautological inclusion $\iota\colon \mathcal{O}_{S\times\mathbb{P}V\times\mathbb{P}^1}(-1,-1) \to 2\mathcal{O}_{S\times\mathbb{P}V\times\mathbb{P}^1}(-1,0)$, we have

$$\mathcal{O}_{S\times\mathbb{P}V\times\mathbb{P}^1}(-1,-1) \xrightarrow{q^*M\circ\iota} d\mathcal{O}_{S\times\mathbb{P}V\times\mathbb{P}^1}.$$

Take the dual

$$d\mathcal{O}_{S\times\mathbb{P}V\times\mathbb{P}^1} \stackrel{(q^*M\circ\iota)^*}{\longrightarrow} \mathcal{O}_{S\times\mathbb{P}V\times\mathbb{P}^1}(1,1),$$

take the push-forward p_* for $p\colon S\times\mathbb{P}V\times\mathbb{P}^1 \to S\times\mathbb{P}^1$, and finally take the tensor product with $\mathcal{O}_{S\times\mathbb{P}^1}(-1)$, we have

$$d\mathcal{O}_{S\times\mathbb{P}^1}(-1) \stackrel{p_*((q^*M\circ\iota)^*)\otimes\mathcal{O}_{\mathbb{P}^1}(-1)}{\longrightarrow} V^*\otimes\mathcal{O}_{S\times\mathbb{P}^1}. \tag{4.3}$$

For a general point, it defines a family of rank d, degree d subbundle of a trivial bundle $V^*\otimes\mathcal{O}_{\mathbb{P}^1}$. Thus we obtain a family of stable maps to $\mathrm{Gr}(d,V^*)$ (or equivalently, to $\mathrm{Gr}(n-d+1,V)$).

So we have a rational map $\mathbb{P}\operatorname{Hom}(V\otimes\mathbb{C}^2,\mathbb{C}^d) \dashrightarrow \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(d,V^*),d)$. This map is (on the domain) $\mathrm{SL}_2\times\mathrm{SL}_d$-equivariant since SL_2 acts as the change of coordinates of $\mathbb{P}^1$, and SL_d acts as the change of coordinates of $d\mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore the rational map induces the quotient map

$$\Phi\colon \mathbb{P}\operatorname{Hom}(V\otimes\mathbb{C}^2,\mathbb{C}^d)/\!/\,\mathrm{SL}_2\times\mathrm{SL}_d \dashrightarrow \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(d,V^*),d).$$

This map is birational since a general balanced, non-degenerate stable map can be obtained from a unique stable Kronecker module. □

In the next section, we will show that when $d=2$, the inverse of Φ is indeed a partial desingularization in the sense of Kirwan [21]. In particular, Φ^{-1} is regular. In general, Φ^{-1} is a rational contraction.

Question 4.9. For $d\geq 3$, is Φ^{-1} regular?

4.4. Moduli of Kronecker modules as a double cover

Fix a natural number $n\geq 3$. In this section, we show that the contraction T_4 of $\mathbf{H}$ in Section 3.2 is isomorphic to a moduli space of Kronecker modules.

Proposition 4.10. *Let V be an $(n+1)$-dimensional vector space. Then*

$$\mathbf{K} := \mathbb{P}(V^*\otimes\mathfrak{gl}_2)/\!/\,\mathrm{SL}_2\times\mathrm{SL}_2 \cong T_4.$$

Proof. There is the determinant map

$$\det\colon \mathbf{K}\to\mathbb{P}(\mathrm{Sym}^2 V^*),$$

which maps M to $\det M$. The image of det is exactly S_4. It is straightforward to check that

(1) det is finite, and generically two-to-one since $\det M = \det M^t$ and $M\not\equiv M^t$;

(2) it is ramified along S_3.

Let

$$U_4 := \{([\Pi],[Q]) \mid \mathbb{P}(\Pi) \subset Q\} \subset \mathrm{Gr}(n-1,V) \times \mathbb{P}(\mathrm{Sym}^2 V^*)$$

be the incidence variety. There is a morphism $\pi\colon U_4 \to T_4$ with connected fibers. Consider the $((\mathbb{C}^2 - \{0\})^2 \setminus \Delta)$-bundle

$$E_4 := \{(\ell_1,\ell_2,[\Pi],[Q]) \mid \ell_i|_\Pi = 0, \ell_i \neq 0, \mathbb{P}(\Pi) \subset Q\} \subset (V^*)^2 \times \mathrm{Gr}(n-1,V) \times \mathbb{P}(\mathrm{Sym}^2 V^*)$$

over U_4. Let $f \in \mathrm{Sym}^2 V^*$ be the defining equation of Q. Then $f = m_1\ell_1 + m_2\ell_2$ for $m_i \in V^*$ and m_i's are defined uniquely up to scalar multiple. There is a morphism $m\colon E_4 \to \mathbf{K}$ where

$$m(\ell_1,\ell_2,[\Pi],[Q]) = \begin{bmatrix} \ell_1 & -m_2 \\ \ell_2 & m_1 \end{bmatrix}.$$

Note that this map is well-defined since a scalar multiple of the second column defines the same point in the quotient.

Now it is straightforward to check that m descends to U_4 and to T_4, so we obtain a map $\overline{m}\colon T_4 \to \mathbf{K}$, which is an S_4-morphism. Since both T_4 and $\mathbf{K}$ are two-to-one to S_4, $\overline{m}$ is bijective. It is an isomorphism since it is a bijective morphism between two normal varieties. □

5. Partial desingularization

When $d = 2$, the birational map Φ^{-1} in (4.2) is indeed a regular contraction. Furthermore, it can be understood as the partial desingularization in the sense of [21]. In this section, we prove the following result. Let V be a fixed $(n+1)$-dimensional vector space.

Theorem 5.1. *The partial desingularization of* $\mathbf{K} := \mathbb{P}(V^* \otimes \mathfrak{gl}_2)/\!/ \mathrm{SL}_2 \times \mathrm{SL}_2$ *is isomorphic to* $\mathbf{M} := \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(n-1,V),2)$.

This result is a generalization of [9, Section 6] with minor modifications. For the reader's convenience, we give a detailed proof here.

5.1. GIT stratification on $\mathbf{X}^{ss}$

Let $\mathbf{X} := \mathbb{P}(V^* \otimes \mathfrak{gl}_2)$. By using the description of the semistable locus in Lemma 4.3, we can define a stratification

$$\mathbf{X}^{ss} = \mathbf{Y}_0 \sqcup \mathbf{Z}_0 \sqcup \mathbf{Y}_1 \sqcup \mathbf{Z}_1 \sqcup \mathbf{X}^s,$$

as follows. For notational simplicity, let $G = \mathrm{SL}_2 \times \mathrm{SL}_2$.

Let $\mathbf{Y}_0 \subset \mathbf{X}^{ss}$ be the locus of matrices equivalent to scalar matrices. More precisely, let $\mathbf{Y}_0'$ be the image of

$$\rho_0 \colon \mathbb{P}(\mathfrak{gl}_2) \times \mathbb{P}V^* \to \mathbf{X}, \quad (A, g) \mapsto A \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}$$

and let $\mathbf{Y}_0 := \mathbf{Y}_0' \cap \mathbf{X}^{ss}$. Then $\mathbf{Y}_0 = \rho_0(\mathrm{PGL}_2 \times \mathbb{P}V^*)$ and ρ_0 is an embedding on this locus. Thus $\mathbf{Y}_0$ is an $(n+3)$-dimensional smooth closed subvariety of $\mathbf{X}^{ss}$. At each closed point $M = g \cdot \mathrm{id} \in \mathbf{Y}_0$, the normal bundle $N_{\mathbf{Y}_0/\mathbf{X}^{ss}}|_M$ is naturally isomorphic to $H \otimes \mathfrak{sl}_2$, where $H \cong V^*/\langle g \rangle$ is an n-dimensional quotient of V^*.

Let $\mathbf{Z}_0 \subset \mathbf{X}^{ss}$ be the locus of matrices equivalent to upper triangular matrices whose diagonal entries are proportional to each other. Formally, we can define $\mathbf{Z}_0$ as follows. Let $\mathbf{Z}_0'$ be the image of

$$\tau_0 \colon G \times \mathbb{P}((V^*)^2) \to \mathbf{X}, \quad ((A, B), (g, k)) \mapsto A \begin{bmatrix} g & k \\ 0 & g \end{bmatrix} B^{-1}.$$

Then $\mathbf{Z}_0 = (\mathbf{Z}_0' \cap \mathbf{X}^{ss}) \setminus \mathbf{Y}_0$. Let $\overline{\mathbf{Z}}_0 := \mathbf{Z}_0 \sqcup \mathbf{Y}_0$, the closure of $\mathbf{Z}_0$ in $\mathbf{X}^{ss}$. A general fiber of τ_0 is 3-dimensional, so $\mathbf{Z}_0$ is a $(2n+4)$-dimensional subvariety. The normal cone $C_{\mathbf{Y}_0/\overline{\mathbf{Z}}_0}$ is an analytic locally trivial bundle, whose fiber at $M = g \cdot \mathrm{id} \in \mathbf{Y}_0$ is isomorphic to $\mathrm{Stab}\, M \cdot (H \otimes \langle e \rangle) = \mathrm{Stab}\, M \cdot (H \otimes \langle f \rangle)$. Here $\{h, e, f\}$ is the standard basis of $\mathfrak{sl}_2$. In $\mathbb{P}(N_{\mathbf{Y}_0/\overline{\mathbf{Z}}_0}|_M) \cong \mathbb{P}(H \otimes \mathfrak{sl}_2)$, $\mathbb{P}(C_{\mathbf{Y}_0/\overline{\mathbf{Z}}_0}|_M) \cong \mathbb{P}H \times \mathrm{PGL}_2 \cdot \mathbb{P}\langle e \rangle \cong \mathbb{P}H \times \mathbb{P}^1$.

Let $\mathbf{Y}_1 \subset \mathbf{X}^{ss}$ be the locus of matrices equivalent to non-scalar diagonal matrices. We may impose the scheme structure to $\mathbf{Y}_1$ in the following way. Let $\mathbf{Y}_1'$ be the image of

$$\rho_1 \colon G \times \mathbb{P}((V^*)^2) \to \mathbf{X}, \quad ((A, B), (g, k)) \mapsto A \begin{bmatrix} g & 0 \\ 0 & k \end{bmatrix} B^{-1}.$$

Let $\mathbf{Y}_1 = (\mathbf{Y}_1' \cap \mathbf{X}^{ss}) \setminus \mathbf{Y}_0$, and let $\overline{\mathbf{Y}}_1 := \mathbf{Y}_0 \sqcup \mathbf{Y}_1$ be the closure of $\mathbf{Y}_1$ in $\mathbf{X}^{ss}$. Then $\mathbf{Y}_1$ is an irreducible G-invariant smooth variety of dimension $2n+5$, and $\overline{\mathbf{Y}}_1$ is singular along $\mathbf{Y}_0$. At a closed point $M = \left[\begin{smallmatrix} g & 0 \\ 0 & k \end{smallmatrix}\right] \in \mathbf{Y}_1$, the normal bundle $N_{\overline{\mathbf{Y}}_1/\mathbf{X}^{ss}}|_M$ is isomorphic to $K \otimes \langle e, f \rangle$ where $K = V^*/\langle g, k \rangle$. For $M = g \cdot \mathrm{id} \in \mathbf{Y}_0$, the normal cone $C_{\mathbf{Y}_0/\overline{\mathbf{Y}}_1}|_M$ is isomorphic to $\mathrm{Stab}\, M \cdot (H \otimes \langle h \rangle) \subset H \otimes \mathfrak{sl}_2 \cong N_{\mathbf{Y}_0/\mathbf{X}^{ss}}|_M$ and its projectivization in $\mathbb{P}(H \otimes \mathfrak{sl}_2)$ is isomorphic to $\mathbb{P}H \times \overline{\mathrm{PGL}_2 \cdot \mathbb{P}\langle h \rangle} \cong \mathbb{P}H \times \mathbb{P}^2$. Note that although $\mathbf{Y}_1 \cap \mathbf{Z}_0 = \emptyset$, $\mathbb{P}(C_{\mathbf{Y}_0/\overline{\mathbf{Z}}_0}|_M) \subset \mathbb{P}(C_{\mathbf{Y}_0/\overline{\mathbf{Y}}_1}|_M)$ because the normal cone to $\overline{\mathbf{Z}}_0$ is tangent to the normal cone to $\overline{\mathbf{Y}}_1$.

Finally, let $\mathbf{Z}_1 \subset \mathbf{X}^{ss}$ be the locus of matrices equivalent to upper triangular matrices. If we denote the image of

$$\tau_1 \colon G \times \mathbb{P}((V^*)^3) \to \mathbf{X}, \quad ((A, B), (g, k, \ell)) \mapsto A \begin{bmatrix} g & k \\ 0 & \ell \end{bmatrix} B^{-1}$$

by $\mathbf{Z}_1'$, $\mathbf{Z}_1 = (\mathbf{Z}_1' \cap \mathbf{X}^{ss}) \setminus (\overline{\mathbf{Y}}_1 \sqcup \mathbf{Z}_0)$. Then $\overline{\mathbf{Z}}_1 = \mathbf{Z}_1 \sqcup \mathbf{Y}_1 \sqcup \mathbf{Z}_0 \sqcup \mathbf{Y}_0$. We may check that $\mathbf{Z}_1$ is an irreducible $(3n+4)$-dimensional G-invariant variety. The normal cone $C_{\overline{\mathbf{Y}}_1/\overline{\mathbf{Z}}_1}|_{\mathbf{Y}_1}$ is an analytic fiber bundle whose fiber is the union of two disjoint rank $n-1$ subbundles of $N_{\overline{\mathbf{Y}}_1/\mathbf{X}^{ss}}|_{\mathbf{Y}_1}$. At a closed point $M = \left[\begin{smallmatrix} g & 0 \\ 0 & k \end{smallmatrix}\right] \in \mathbf{Y}_1$, $C_{\overline{\mathbf{Y}}_1/\overline{\mathbf{Z}}_1}|_M \cong K \otimes \langle e \rangle \sqcup K \otimes \langle f \rangle \subset K \otimes \langle e, f \rangle \cong N_{\overline{\mathbf{Y}}_1/\mathbf{X}^{ss}}|_M$.

5.2. Kirwan's partial desingularization

In this section, we describe Kirwan's partial desingularization of $\mathbf{K} := \mathbf{X}/\!/G$. For the general construction and its proof, consult [21]. Let $\mathbf{X}^0 := \mathbf{X}^{ss}$. In $\mathbf{X}^0$, the deepest stratum with the largest stabilizer group is $\mathbf{Y}_0$. Let $\pi_1' \colon \mathbf{X}^{1'} \to \mathbf{X}^0$ be the blow-up of $\mathbf{X}^0$ along $\mathbf{Y}_0$. Let $\mathbf{Y}_0^1$ be the exceptional divisor, and let $\overline{\mathbf{Y}}_1^1$, $\overline{\mathbf{Z}}_i^1$ be the proper transform of $\overline{\mathbf{Y}}_1$, $\overline{\mathbf{Z}}_i$, respectively. Since the normal cone $C_{\mathbf{Y}_0/\overline{\mathbf{Y}}_1}$ is a cone over a smooth variety and $\mathbf{Y}_1$ is smooth, $\overline{\mathbf{Y}}_1^1$ is a smooth subvariety of $\mathbf{X}^{1'}$.

Since $\rho(\mathbf{X}) = 1$, there is a unique linearization L_0 on $\mathbf{X}$, up to scaling. Let $L_1 := \pi_1'^*(L_0) \otimes \mathcal{O}(-\epsilon_1 \mathbf{Y}_0^1)$ for some $0 < \epsilon_1 \ll 1$. Then L_1 is an ample $\mathbb{Q}$-line bundle with a linearized G-action. With respect to this linearization, $\overline{\mathbf{Z}}_0^1$ is unstable since any orbit in $\mathbf{Z}_0$ is not closed in $\mathbf{X}^0$ (see, [21, Lemma 6.6]). Let $\mathbf{X}^1 := \mathbf{X}^{1'} \setminus \overline{\mathbf{Z}}_0^1$ and let $\pi_1 \colon \mathbf{X}^1 \to \mathbf{X}^0$ be the restriction of π_1'.

Similarly, let $\pi_2' \colon \mathbf{X}^{2'} \to \mathbf{X}^1$ be the blow-up of $\mathbf{X}^1$ along $\overline{\mathbf{Y}}_1^1 \cap \mathbf{X}^1$. Since $\overline{\mathbf{Y}}_1^1 \cap \mathbf{X}^1$ is smooth, $\mathbf{X}^{2'}$ is also smooth. Let $\overline{\mathbf{Y}}_1^2$ be the exceptional divisor. And let $\mathbf{Y}_0^2$, $\overline{\mathbf{Z}}_1^2$ be the proper transform of $\mathbf{Y}_0^2 \cap \mathbf{X}^1$, $\overline{\mathbf{Z}}_1^1 \cap \mathbf{X}^1$, respectively. Let $L_2 := \pi_2^*(L_1) \otimes \mathcal{O}(-\epsilon_2 \overline{\mathbf{Y}}_1^2)$ for some $0 < \epsilon_2 \ll \epsilon_1$. Then L_2 is ample. Furthermore, since $\overline{\mathbf{Y}}_1^1 \cap \mathbf{X}^1$ is a G-invariant subvariety, L_2 inherits a linearized G-action, too. With respect to this G-action, $\overline{\mathbf{Z}}_1^2$ is unstable. Let $\mathbf{X}^2 := \mathbf{X}^{2'} \setminus \overline{\mathbf{Z}}_1^2$ and let $\pi_2 \colon \mathbf{X}^2 \to \mathbf{X}^1$ be the restriction of π_2'.

Note that on $\mathbf{X}^2$, every point has a finite stabilizer. Therefore $\mathbf{X}^2 = (\mathbf{X}^2)^{ss} = (\mathbf{X}^2)^s$. The partial desingularization of $\mathbf{X}/\!/G$ is $\mathbf{X}^2/\!/_{L_2}G = \mathbf{X}^2/G$. The blow-up morphisms π_1, π_2 induce quotient maps $\overline{\pi}_1$ and $\overline{\pi}_2$. In summary, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathbf{X}^2 \arrow[r, "\pi_2", "\overline{\mathbf{Y}}_1^1 \cap \mathbf{X}^1"'] \arrow[d, "/G"] & \mathbf{X}^1 \arrow[r, "\pi_1", "\mathbf{Y}_0"'] \arrow[d, "/G"] & \mathbf{X}^0 \arrow[d, "/G"] \\
\mathbf{X}^2/G \arrow[r, "\overline{\pi}_2"] & \mathbf{X}^1/\!/G \arrow[r, "\overline{\pi}_1"] & \mathbf{X}/\!/G
\end{tikzcd}$$

Let $\pi := \pi_1 \circ \pi_2$, and let $\overline{\pi} := \overline{\pi}_1 \circ \overline{\pi}_2$ be its quotient map. Note that the partial desingularization $\mathbf{X}^2/G$ has only finite quotient singularities, since every point on $\mathbf{X}^2$ has a finite stabilizer.

During the desingularization process, we can keep track of the change of the GIT quotient. For $M \in \mathbf{Y}_0$, $\pi_1'^{-1}(M) \cong \mathbb{P}(H \otimes \mathfrak{sl}_2)$ where H is an n-dimensional quotient of

V^*, because it is the projectivized normal cone. On the fiber ${\pi'_1}^{-1}(M)$, there is an induced $\operatorname{Stab} M \cong \mathrm{SL}_2 \ltimes \mathbb{Z}_2$-action, which is induced by a trivial action on H and the standard SL_2-action on $\mathfrak{sl}_2$. Also the $\mathbb{Z}_2$ acts trivially. The unstable locus is precisely $\mathbb{P}(C_{\mathbf{Y}_0/\overline{\mathbf{Z}}_0}|_M) \cong \mathbb{P}H \times \mathbb{P}^1$. Therefore in $\mathbf{X}^1 = \mathbf{X}^{1\prime} \setminus \overline{\mathbf{Z}}_0^1$, the inverse image of M is $\mathbb{P}(H \otimes \mathfrak{sl}_2)^{ss}$. If we denote the image of M in $\mathbf{X}/\!/G$ by $\overline{M}$, then

$$\overline{\pi}_1^{-1}(\overline{M}) \cong \mathbb{P}(H \otimes \mathfrak{sl}_2)^{ss} / \operatorname{Stab} M \cong \mathbb{P}(H \otimes \mathfrak{sl}_2) /\!/ \mathrm{SL}_2 .$$

The locus of strictly semistable points with closed orbits on $\mathbb{P}(H \otimes \mathfrak{sl}_2)^{ss}$ is isomorphic to $(\mathbb{P}H \times \mathbb{P}\mathfrak{sl}_2)^{ss}$, which is precisely $\mathbb{P}(C_{\mathbf{Y}_0/\overline{\mathbf{Y}}_1})^{ss}$. (Indeed the strictly semistable locus is $\mathbb{P}(C_{\mathbf{Y}_0/\overline{\mathbf{Z}}_1})^{ss}$.) Thus on the fiber of $\overline{M}$, the second blow-up $\overline{\pi}_2\colon \mathbf{X}^2/\!/G \to \mathbf{X}^1/\!/G$ is the partial desingularization of the fiber $\mathbb{P}(H \otimes \mathfrak{sl}_2)/\!/ \mathrm{SL}_2$. In [19, Theorem 4.1], it was shown that the partial resolution is isomorphic to the moduli space $\overline{\mathrm{M}}_{0,0}(\mathbb{P}H, 2)$ of degree two stable maps to $\mathbb{P}H$.

For $M \in \mathbf{Y}_1^1$, the inverse image ${\pi'_2}^{-1}(M)$ is isomorphic to $\mathbb{P}(K \otimes \langle e, f \rangle)$. On this normal bundle, there is an induced $\operatorname{Stab} M \cong \mathbb{C}^* \ltimes \mathbb{Z}_2$-action. The unstable locus is precisely $\mathbb{P}(K \otimes \langle e \rangle) \sqcup \mathbb{P}(K \otimes \langle f \rangle)$ and there is no strictly semistable point. Thus on $\mathbf{X}^2$, $\pi_2^{-1}(M) \cong \mathbb{P}(K \otimes \langle e, f \rangle)^s$. Therefore in $\mathbf{X}^2/G$,

$$\overline{\pi}_2^{-1}(\overline{M}) \cong \mathbb{P}(K \otimes \langle e, f \rangle)/\!/\mathbb{C}^* \ltimes \mathbb{Z}_2 \cong \mathbb{P}^{n-2} \times \mathbb{P}^{n-2}/\!/\mathbb{Z}_2 \cong (\mathbb{P}^{n-2})^2.$$

Note that $\mathbb{Z}_2$ acts trivially on the projectivized normal cone.

5.3. Elementary modification of maps

In this section, we prove Theorem 5.1.

Proof of Theorem 5.1. Let $\mathbf{X}^0 := \mathbb{P}(V^* \otimes \mathfrak{gl}_2)^{ss}$. By taking the dual of (4.3), we have

$$V \otimes \mathcal{O}_{\mathbf{X}^0 \times \mathbb{P}^1} \overset{\wedge^2(p_*((q^*M\circ\iota)^*)\otimes\mathcal{O}_{\mathbb{P}^1}(-1))^*}{\longrightarrow} 2\mathcal{O}_{\mathbf{X}^0 \times \mathbb{P}^1}(1).$$

It induces a bundle morphism

$$\wedge^2 V \otimes \mathcal{O}_{\mathbf{X}^0 \times \mathbb{P}^1} \overset{\wedge^2(p_*((q^*M\circ\iota)^*)\otimes\mathcal{O}_{\mathbb{P}^1}(-1))^*}{\longrightarrow} \mathcal{O}_{\mathbf{X}^0 \times \mathbb{P}^1}(2).$$

Since this map is surjective on $\mathbf{X}^s \times \mathbb{P}^1$, we obtain a rational map

$$f_0\colon \mathbf{X}^0 \times \mathbb{P}^1 \dashrightarrow \mathrm{Gr}(n-1, V) \hookrightarrow \mathbb{P}(\wedge^{n-1} V) \cong \mathbb{P}(\wedge^2 V^*),$$

which is regular on $\mathbf{X}^s \times \mathbb{P}^1$.

Let $F_0 := \wedge^2(p_*((q^*M \circ \iota)^*) \otimes \mathcal{O}_{\mathbb{P}^1}(-1))^*$. Let $\pi_1 \times \mathrm{id}\colon \mathbf{X}^1 \times \mathbb{P}^1 \to \mathbf{X}^0 \times \mathbb{P}^1$ be the blow-up morphism. All sections giving F_0 simultaneously vanish along $\mathbf{Y}_0^1 \times \mathbb{P}^1$. So f_0 is not defined on $\mathbf{Y}_0^1 \times \mathbb{P}^1$. But this implies that the pull-back morphism $(\pi_1 \times \mathrm{id})^* F_0$ factors through

$$\wedge^2 V \otimes \mathcal{O}_{\mathbf{X}^0 \times \mathbb{P}^1} \stackrel{F_1 := (\pi_1 \times \mathrm{id})^* F_0}{\longrightarrow} \mathcal{O}_{\mathbf{X}^1 \times \mathbb{P}^1}(2)(-\mathbf{Y}_0^1).$$

Therefore we obtain an extended family f_1 of rational maps over $\mathbf{X}^1$

$$\begin{tikzcd} \mathbf{X}^1 \times \mathbb{P}^1 \arrow[d, "\pi_1 \times \mathbb{P}^1"] \arrow[drr, dashed, "f_1"] & & \\ \mathbf{X}^0 \times \mathbb{P}^1 \arrow[r, dashed, "f_0"] & \mathrm{Gr}(n-1, V) \arrow[r] & \mathbb{P}(\wedge^2 V^*) \end{tikzcd}$$

whose undefined locus is precisely a two-to-one étale cover of $\overline{\mathbf{Y}}_1^1$ because for each point $M \in \overline{\mathbf{Y}}_1^1$, the undefined locus of f_1 restricted to $\{M\} \times \mathbb{P}^1$ is two distinct points.

Let $\pi_2 \times \mathrm{id} \colon \mathbf{X}^2 \times \mathbb{P}^1 \to \mathbf{X}^1 \times \mathbb{P}^1$ be the second blow-up morphism. The base locus $\mathbf{B}$ of $(\pi_2 \times \mathrm{id})^* f_1$ is a two-to-one étale cover of $\overline{\mathbf{Y}}_1^2$, so it is a smooth codimension two subvariety of $\mathbf{X}^2 \times \mathbb{P}^1$. Let $\sigma \colon \Gamma \to \mathbf{X}^2 \times \mathbb{P}^1$ be the blow-up along $\mathbf{B}$. Let $\mathbf{E}$ be the exceptional divisor. The composition $s \colon \Gamma \to \mathbf{X}^2 \times \mathbb{P}^1 \to \mathbf{X}^2$ is a flat family of rational curves. Moreover, the pull-back morphism $\sigma^*(\pi_2 \times \mathrm{id})^* F_1$

$$\wedge^2 V \otimes \mathcal{O}_\Gamma \stackrel{\sigma^*(\pi_2 \times \mathrm{id})^* F_1}{\longrightarrow} \sigma^* \mathcal{O}_{\mathbf{X}^2 \times \mathbb{P}^1}(2)(-\mathbf{Y}_0^2)$$

factors through

$$\wedge^2 V \otimes \mathcal{O}_\Gamma \stackrel{F_2}{\longrightarrow} \sigma^* \mathcal{O}_{\mathbf{X}^2 \times \mathbb{P}^1}(2)(-\mathbf{Y}_0^2 - \mathbf{E}).$$

Now F_2 is surjective and we obtain a regular morphism $f_2 \colon \Gamma \to \mathbb{P}(\wedge^2 V^*)$.

$$\begin{tikzcd} \Gamma \arrow[d, "\sigma"] \arrow[ddddrr, "f_2"] & & \\ \mathbf{X}^2 \times \mathbb{P}^1 \arrow[d, "\pi_2 \times \mathrm{id}"] & & \\ \mathbf{X}^1 \times \mathbb{P}^1 \arrow[d, "\pi_1 \times \mathrm{id}"] \arrow[drr, dashed, "f_1"] & & \\ \mathbf{X}^0 \times \mathbb{P}^1 \arrow[r, dashed, "f_0"] & \mathrm{Gr}(n-1, V) \arrow[r] & \mathbb{P}(\wedge^2 V^*) \end{tikzcd}$$

So we have a flat family of maps $(s \colon \Gamma \to \mathbf{X}^2, f_2 \colon \Gamma \to \mathbb{P}(\wedge^2 V^*))$ over $\mathbf{X}^2$. By stabilizing, we obtain a family of stable maps $(\overline{s} \colon \overline{\Gamma} \to \mathbf{X}^2, \overline{f}_2 \colon \overline{\Gamma} \to \mathbb{P}(\wedge^2 V^*))$. Clearly $\overline{f}_2$ factors through $\mathrm{Gr}(n-1, V)$ since it does on an open dense subset. Therefore we have a map $\Psi \colon \mathbf{X}^2 \to \mathbf{M}$, which is G-invariant from the construction. Thus we obtain the quotient map $\overline{\Psi} \colon \mathbf{X}^2/G \to \mathbf{M}$. This is an isomorphism since it is a birational morphism between two $\mathbb{Q}$-factorial normal varieties with the same Picard number, which is 3. $\square$

Here we leave two explicit examples of the elementary modification of a family of maps over a curve.

Example 5.2. Let S be a small disk in $\mathbb{C}$ containing 0 (or the Spec of a discrete valuation ring) and let

$$\begin{aligned} g\colon S &\to \mathbb{P}(V^* \otimes \mathfrak{gl}_2) \\ \lambda &\mapsto \begin{bmatrix} x_0 & \lambda(\sum_{i=1}^n a_i x_i) \\ \lambda(\sum_{i=1}^n b_i x_i) & x_0 \end{bmatrix}. \end{aligned}$$

Then the associated map

$$V \otimes \mathcal{O}_{S\times\mathbb{P}^1} \stackrel{G}{\to} 2\mathcal{O}_{S\times\mathbb{P}^1}(1)$$

is given by a matrix

$$G = \begin{bmatrix} s & \lambda a_1 t & \lambda a_2 t & \cdots & \lambda a_n t \\ t & \lambda b_1 s & \lambda b_2 s & \cdots & \lambda b_n s \end{bmatrix}$$

where $[s:t]$ is the homogeneous coordinate of $\mathbb{P}^1$. Note that this family of maps is not surjective when $\lambda = 0$. By taking the wedge product, we obtain a family of maps $\wedge^2 V \otimes \mathcal{O}_{S\times\mathbb{P}^1} \stackrel{F_0:=\wedge^2 G}{\longrightarrow} \mathcal{O}_{S\times\mathbb{P}^1}(2)$ where

$$[F_0]_I = \begin{cases} \lambda(b_i s^2 - a_i t^2) & I = \{0,i\}, \\ \lambda^2(a_i b_j - a_j b_i)st & I = \{i,j\},\ 0 \notin I. \end{cases}$$

Thus if we take the map

$$\lambda^2 V \otimes \mathcal{O}_{S\times\mathbb{P}^1} \stackrel{F_1}{\to} \mathcal{O}_{S\times\mathbb{P}^1}(2)(-\mathbf{Y}_0^1),$$

it is given by

$$[F_1]_I = \begin{cases} (b_i s^2 - a_i t^2) & I = \{0,i\}, \\ \lambda(a_i b_j - a_j b_i)st & I = \{i,j\},\ 0 \notin I. \end{cases}$$

When $\lambda = 0$, we could recover the modified map $G'(0)$ so that $\wedge^2 G'(0) = F_1(0)$, which is

$$V \otimes \mathcal{O}_{S(0)\times\mathbb{P}^1} \stackrel{G'(0)}{\longrightarrow} \mathcal{O}_{S(0)\times\mathbb{P}^1} \oplus \mathcal{O}_{S(0)\times\mathbb{P}^1}(2),$$

and

$$G'(0) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & b_1 s^2 - a_1 t^2 & b_2 s^2 - a_2 t^2 & \cdots & b_n s^2 - a_n t^2 \end{bmatrix}.$$

Since the image has an $\mathcal{O}_{S(0)\times\mathbb{P}^1}$ factor, the subbundle $\ker G'(0)$ is degenerated. Therefore the modified map is in $D_{\deg}$.

Example 5.3. Let

$$h\colon S \to \mathbb{P}(V^* \otimes \mathfrak{gl}_2)$$
$$\lambda \mapsto \begin{bmatrix} x_0 & \lambda(\sum_{i=2}^n a_i x_i) \\ \lambda(\sum_{i=2}^n b_i x_i) & x_1 \end{bmatrix}$$

be a family over a small disk S. The associated map $V \otimes \mathcal{O}_{S\times\mathbb{P}^1} \xrightarrow{H} 2\mathcal{O}_{S\times\mathbb{P}^1}(1)$ is given by

$$H = \begin{bmatrix} s & 0 & \lambda a_2 t & \cdots & \lambda a_n t \\ 0 & t & \lambda b_2 s & \cdots & \lambda b_n s \end{bmatrix}.$$

For $F_0 := \wedge^2 H$,

$$[F_0]_I = \begin{cases} st & I = \{0,1\}, \\ \lambda b_i s^2 & I = \{0,i\},\ i \geq 2, \\ -\lambda a_i t^2 & I = \{1,i\},\ i \geq 2, \\ \lambda^2(a_i b_j - a_j b_i) & I = \{i,j\},\ i,j \geq 2. \end{cases}$$

When $\lambda = 0$, H is not surjective at two points $[0:1]$ and $[1:0]$, and except for those two points, the map is constant.

At $\lambda = s = 0$, take the blow-up and let $E_1 \cong \mathbb{P}^1$ be the exceptional divisor. On E_1 (with homogeneous coordinate $[s:\lambda]$), we obtain an extended degree one map $F_2^1(0)$ given by

$$[F_2^1(0)]_I = \begin{cases} s & I = \{0,1\}, \\ -\lambda a_i & I = \{1,i\},\ i \geq 2, \\ 0 & \text{otherwise}. \end{cases}$$

Thus the map $H^1(0)\colon V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$ so that $\wedge^2 H^1(0) = F_2^1(0)$, is given by

$$H^1(0) = \begin{bmatrix} s & 0 & a_2\lambda & \cdots & a_n\lambda \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Similarly, at $\lambda = t = 0$, we can compute the map $H^2(0)$ on the second exceptional divisor E_2:

$$H^2(0) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & t & b_2\lambda & \cdots & b_n\lambda \end{bmatrix}.$$

The stabilization at $\lambda = 0$ contracts the central constant component. Therefore we obtain a limit $\lambda \to 0$ which is a stable map from a nodal curve.

The proof of the theorem and the above two examples show the following corollary.

Corollary 5.4. (1) *The partial desingularization map* $d\colon \mathbf{M} \to \mathbf{K}$ *contracts* Δ *and* D_{deg}. *For a point* $[2H]$ *of* $d(D_{\mathrm{deg}}) \cong \mathbb{P}V^*$, *its fiber is isomorphic to the moduli space of stable maps* $\overline{\mathrm{M}}_{0,0}(\mathbb{P}H, 2)$.

(2) *The second step of the partial desingularization* $c\colon \mathbf{M} \to \mathbf{X}^1/\!/G$ *contracts* Δ. *This morphism maps a singular stable map to the union of two hyperplanes in* $\mathbb{P}V$ *where each of them is the envelope of the irreducible component of the stable map. For a general point* $[H \cup H']$ *of* $c(\Delta)$, *the fiber is* $(\mathbb{P}(H \cap H')^*)^2$.

Remark 5.5. Note that in item (2), the image $c(f)$ of a stable map $f\colon C_1 \cup C_2 \to \mathrm{Gr}(n-1, V)$ remembers not only $f(C_1 \cap C_2)$, but also $\langle f(C_i)\rangle \subset \mathbb{P}V$. On the other hand, the morphism $\mathbf{M} \to \mathbf{U}$ in Definition 3.2 maps a singular stable map $f\colon C_1 \cup C_2 \to \mathrm{Gr}(n-1, V)$ to a point $C_1 \cap C_2 \in \mathrm{Gr}(n-1, V)$. By the rigidity lemma, we obtain a morphism $\mathbf{X}^1/\!/G \to \mathbf{U}$.

6. Mori's program

In this section, we prove our main theorem (Theorem 1.1) and complete Mori's program for $\mathbf{M} := \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(n-1, V), 2)$.

We start with a simple but useful observation.

Lemma 6.1. *Let* $\Phi\colon \mathbf{M} := \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(n-1, V), 2) \to \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2, V^*), 2)$ *be an isomorphism induced by* $\phi\colon \mathrm{Gr}(n-1, V) \cong \mathrm{Gr}(2, V^*)$. *Then* Φ_* *induces a reflection along the vertical line connecting* Δ *and* P *in Figure* 2.1. *In other words,* $\Phi_*(D_{\mathrm{deg}}) = D_{\mathrm{unb}}$, $\Phi_*(D_{\mathrm{unb}}) = D_{\mathrm{deg}}$, $\Phi_*(H_{\sigma_{1,1}}) = H_{\sigma_2}$, $\Phi_*(H_{\sigma_2}) = H_{\sigma_{1,1}}$, *and* $\Phi_*(\Delta) = \Delta$.

Proof. It follows from the induced isomorphism $\phi_*\colon \mathrm{A}^*(\mathrm{Gr}(n-1, V)) \to \mathrm{A}^*(\mathrm{Gr}(2, V^*))$ such that $\phi_*(\sigma_{1,1}) = \sigma_2$, $\phi_*(\sigma_2) = \sigma_{1,1}$, $\phi_*(\sigma_{(1,1)^*}) = \sigma_{(2)^*}$, and $\phi_*(\sigma_{(2)^*}) = \sigma_{(1,1)^*}$. $\Phi_*(\Delta) = \Delta$ is clear. □

6.1. $n = 3$ case

The first nontrivial case is $n = 3$, where $\mathrm{Gr}(n-1, V) = \mathrm{Gr}(2, 4)$. In this case, because of the self-duality of $\mathrm{Gr}(2, 4)$, the complete description is particularly clear. Essentially all of the birational models in this case have been described in [3, 9]. For the reader's convenience, we leave the statement and references.

Theorem 6.2. *Let* V *be a vector space of dimension* 4. *For an effective divisor* D *on* $\mathbf{M} := \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2, V), 2)$,

(1) *If* $D = aH_{\sigma_{1,1}} + bH_{\sigma_2} + cT$ *for* $a, b, c > 0$, *then* $\mathbf{M}(D) \cong \mathbf{M}$.

(2) *If* $D = aH_{\sigma_{1,1}} + bH_{\sigma_2} + cP$ *for* $a, b, c > 0$, *then* $\mathbf{M}(D) \cong \mathbf{H} := \mathrm{Hilb}^{2m+1}(\mathrm{Gr}(2, V))$.

(3) *If* $D = aH_{\sigma_2} + bD_{\mathrm{deg}} + c\Delta$ *for* $a > 0$ *and* $b, c \geq 0$*, then* $\mathbf{M}(D) \cong \mathbf{K} := \mathbb{P}(V^* \otimes \mathfrak{gl}_2)/\!/ \mathrm{SL}_2 \times \mathrm{SL}_2 = \mathrm{M}_{\mathbb{P}V}(m^2 + 3m + 2) \cong T_4$*.*

(4) *If* $D = aH_{\sigma_2} + bT + c\Delta$ *for* $a, b > 0$ *and* $c \geq 0$*, then* $\mathbf{M}(D) \cong \mathbf{X}^1 /\!/ \mathrm{SL}_2 \times \mathrm{SL}_2$*, the intermediate space of the partial desingularization of* $\mathbf{K}$*.*

(5) *If* $D = aH_{\sigma_2} + bP + cD_{\mathrm{deg}}$ *for* $a, b > 0$ *and* $c \geq 0$*, then* $\mathbf{M}(D) \cong \mathrm{Bl}_{\mathrm{OG}(3,\wedge^2 V)_{\sigma_{(1,1)^*}}} \mathrm{Gr}(3, \wedge^2 V)$*.*

(6) *If* $D = aP + bD_{\mathrm{unb}} + cD_{\mathrm{deg}}$ *for* $a > 0$ *and* $b, c \geq 0$*, then* $\mathbf{M}(D) \cong \mathrm{Gr}(3, \wedge^2 V) \cong \mathrm{Gr}(3, \wedge^2 V^*)$*.*

(7) *If* $D = aH_{\sigma_{1,1}} + bP + cD_{\mathrm{unb}}$ *for* $a, b > 0$ *and* $c \geq 0$*, then* $\mathbf{M}(D) \cong \mathrm{Bl}_{\mathrm{OG}(3,\wedge^2 V)_{\sigma_{(2)^*}}} \mathrm{Gr}(3, \wedge^2 V) \cong \mathrm{Bl}_{\mathrm{OG}(3,\wedge^2 V^*)_{\sigma(1,1)^*}} \mathrm{Gr}(3, \wedge^2 V^*)$*.*

(8) *If* $D = aH_{\sigma_{1,1}} + bD_{\mathrm{unb}} + c\Delta$ *for* $a > 0$ *and* $b, c \geq 0$*, then* $\mathbf{M}(D) \cong \mathbf{K}^* := \mathbb{P}(V \otimes \mathfrak{gl}_2)/\!/ \mathrm{SL}_2 \times \mathrm{SL}_2 = \mathrm{M}_{\mathbb{P}V^*}(m^2 + 3m + 2)$*.*

(9) *If* $D = aH_{\sigma_{1,1}} + bT + c\Delta$ *for* $a, b > 0$ *and* $c \geq 0$*, then* $\mathbf{M}(D) = (\mathbf{X}^1 /\!/ \mathrm{SL}_2 \times \mathrm{SL}_2)^*$*, the intermediate space of the partial desingularization of* $\mathbf{K}^*$*.*

(10) *If* $D = aT + b\Delta$ *for* $a > 0$ *and* $b \geq 0$*, then* $\mathbf{M}(D) \cong \mathbf{U} := \overline{\mathrm{U}}_{0,0}(\mathrm{Gr}(2, V), 2)$*.*

(11) *If* $D = aH_{\sigma_{1,1}} + bH_{\sigma_2}$ *for* $a, b > 0$*, then* $\mathbf{M}(D) \cong \mathbf{C} := \mathrm{Chow}_{1,2}(\mathrm{Gr}(2, V))^\nu$*.*

(12) *If* D *is on the boundary of* $\mathrm{Eff}(\mathbf{M})$*, then* $\mathbf{M}(D)$ *is a point.*

Proof. Items (1), (2), (5), (6), (7), (10) and (11) are from [3, Proposition 3.7, Theorems 3.8, 3.10]. In [3], the range of the divisors giving each model was not stated explicitly. However, since $\mathbf{M}(D) \cong \mathbf{M}(D + E)$ if E is an exceptional divisor of the rational contraction $\mathbf{M} \dashrightarrow \mathbf{M}(D)$, it is straightforward to extend the range of divisors. Items (3) and (4) are from [9, Remark 6.7]. Items (8) and (9) are obtained by the duality map Φ in Lemma 6.1. Note that for any divisor D on the boundary of $\mathrm{Eff}(\mathbf{M})$, $\mathbf{M}(D)$ is a contraction with positive dimensional fibers of one of the normal varieties $\mathbf{K}$, $\mathbf{K}^*$, and $\mathrm{Gr}(3, \wedge^2 V)$. Thus it has to be a point since those three varieties have Picard number one. $\square$

6.2. Relative moduli spaces and its contractions

When $n > 3$, Mori's program for $\mathbf{M}$ is more complicated. For instance, the movable cone is larger than the case of $n = 3$. To extend Theorem 6.2, we need to describe more birational models of $\mathbf{M}$. In this section, we introduce new models from the viewpoint of relative moduli spaces.

The construction of many moduli spaces in Sections 3 and 4 can be relativized. Let $\mathcal{S}$ be the rank 4 tautological subbundle over $\mathrm{Gr}(4, V^*)$. Consider the rank 2 Grassmannian bundle $\mathrm{Gr}(2, \mathcal{S})$ over $\mathrm{Gr}(4, V^*)$.

Definition 6.3. Let $\mathbf{M}_{\mathcal{S}} := \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2, \mathcal{S}), 2)$ be the relative moduli space of stable maps to the Grassmannian bundle. This is a Zariski locally trivial bundle over $\mathrm{Gr}(4, V^*)$ whose fiber over $S \in \mathrm{Gr}(4, V^*)$ is $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2, S), 2)$.

There is a functorial morphism

$$r_M \colon \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2, \mathcal{S}), 2) \to \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2, V^*), 2) \cong \mathbf{M}.$$

This map is surjective since any degree 2 stable map to $\mathrm{Gr}(2, V^*)$ factors through $\mathrm{Gr}(2, S)$ for some $S \subset V^*$ with $\dim S = 4$. Furthermore, r is not injective precisely on the locus of stable maps whose linear envelope is not 3 dimensional. Thus the exceptional set is the $D_{\deg}$-bundle over $\mathrm{Gr}(4, V^*)$, which is a divisor. We denote this divisor by $D_{\deg,\mathcal{S}}$. Then $D_{\deg,\mathcal{S}}$ is a $\mathbb{P}^{n-3}$-bundle over $r_M(D_{\deg,\mathcal{S}})$.

Definition 6.4. Let $\mathbf{K}_{\mathcal{S}} := \mathrm{M}_{\mathbb{P}\mathcal{S}}(m^2 + 3m + 2)$ be the relative moduli space of semistable sheaves over $\mathrm{Gr}(4, V^*)$. For each $S \in \mathrm{Gr}(4, V^*)$, the fiber is $\mathrm{M}_{\mathbb{P}S}(m^2 + 3m + 2) \cong \mathbb{P}(S^* \otimes \mathfrak{gl}_2) /\!/ \mathrm{SL}_2 \times \mathrm{SL}_2$.

This moduli space can be constructed as an $\mathrm{SL}_2 \times \mathrm{SL}_2$-GIT quotient of the projective space bundle $\mathbb{P}(\mathcal{S}^* \otimes \mathfrak{gl}_2)$ over $\mathrm{Gr}(4, V^*)$.

There is also a functorial morphism

$$r_K \colon \mathbf{K}_{\mathcal{S}} = \mathrm{M}_{\mathbb{P}\mathcal{S}}(m^2 + 3m + 2) \to \mathrm{M}_{\mathbb{P}V^*}(m^2 + 3m + 2).$$

For $S \in \mathrm{Gr}(4, V^*)$ and $[F] \in \mathrm{M}_{\mathbb{P}\mathcal{S}}(m^2 + 3m + 2)|_S \cong \mathrm{M}_{\mathbb{P}S}(m^2 + 3m + 2)$, $r_K([F]) = [i_* F]$ for $i \colon \mathbb{P}S \hookrightarrow \mathbb{P}V^*$. Then r_K contracts the locus of $[2\mathcal{O}_H]$ and the fiber of an exceptional point in $\mathrm{M}_{\mathbb{P}V^*}(m^2 + 3m + 2)$ is $\mathbb{P}^{n-3}$.

The map r_K is not surjective. For instance, $\mathrm{M}_{\mathbb{P}V^*}(m^2+3m+2)$ has an extra component isomorphic to $\mathrm{Sym}^2 \mathrm{Gr}(3, V^*)$ which parametrizes S-equivalent classes of $\mathcal{O}_H \oplus \mathcal{O}_{H'}$ for a pair of planes (H, H'). $r_K(\mathbf{K}_{\mathcal{S}})$ is the closure of the locus of semistable sheaves supported on a smooth quadric surface.

Definition 6.5. Let $\mathbf{L}$ be the normalization of the image of r_K in $\mathrm{M}_{\mathbb{P}V^*}(m^2 + 3m + 2)$. Since r_K has connected fibers, $\mathbf{L}$ is bijective to $r_K(\mathbf{K}_{\mathcal{S}})$.

We can relativize the partial desingularization process and obtain a morphism

$$d_{\mathcal{S}} \colon \mathbf{M}_{\mathcal{S}} \to \mathbf{K}_{\mathcal{S}}.$$

Let $\mathbf{X}^1_{\mathcal{S}} /\!/ \mathrm{SL}_2 \times \mathrm{SL}_2$ be the intermediate space of the relative partial desingularization.

Lemma 6.6. *There is a contraction* $\overline{d}\colon \mathbf{M} \cong \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2,V^*),2) \to \mathbf{L}$ *which contracts two curve classes* C_2 *and* C_7*. In particular,* $\overline{d}$ *contracts* D_{unb} *and* Δ*.*

Proof.

$$\begin{tikzcd} \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2,\mathcal{S}),2) \arrow[r, "d_{\mathcal{S}}"] \arrow[d, "r_M"'] & \mathrm{M}_{\mathbb{P}\mathcal{S}}(m^2+3m+2) \arrow[d, "r_K"] \\ \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2,V^*),2) \arrow[r, dashed, "\overline{d}"] & \mathrm{M}_{\mathbb{P}V^*}(m^2+3m+2) \end{tikzcd}$$

By Lemma 6.1, D_{unb} on $\mathbf{M}$ is identified with D_{deg} on $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2,V^*),2)$. For a point $f \in D_{\mathrm{deg}} \subset \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2,V^*),2)$, $r_M^{-1}(f)$ parametrizes pairs (f,S) where $S \in \mathrm{Gr}(4,V^*)$ such that the linear envelope of f is a $\mathbb{P}^2$ in $\mathbb{P}S$. Then $d_{\mathcal{S}}(r_M^{-1}(f))$ parametrizes pairs $([2\mathcal{O}_H],S)$ where H is a plane in $\mathbb{P}S$. Now $(r_K \circ d_{\mathcal{S}})(r_M^{-1}(f))$ is $\{[2\mathcal{O}_H]\}$. By the rigidity lemma, there is a morphism $\overline{d}\colon \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2,V^*),2) \to \mathrm{M}_{\mathbb{P}V^*}(m^2+3m+2)$. Clearly the image has $\mathbf{L}$ as its normalization.

From the description of the exceptional set above, we can see that the curve classes C_1 and C_6 on $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(2,V^*),2)$ are contracted. By duality, they correspond to C_2 and C_7 on $\mathbf{M}$. Since deformations of C_2 (resp. C_7) cover D_{unb} (resp. Δ), we obtain the result. □

The Hilbert scheme construction can also be relativized.

Definition 6.7. Let $\mathbf{H}_{\mathcal{S}} := \mathrm{Hilb}^{2m+1}(\mathrm{Gr}(2,\mathcal{S}))$ be the relative Hilbert scheme of conics over $\mathrm{Gr}(4,V^*)$. This is a Zariski locally trivial bundle over $\mathrm{Gr}(4,V^*)$ such that for $S \in \mathrm{Gr}(4,V^*)$, its fiber over S is $\mathrm{Hilb}^{2m+1}(\mathrm{Gr}(2,S))$.

Definition 6.8. Let $\mathbf{B} := \mathrm{Bl}_{\mathrm{OG}(3,\wedge^2\mathcal{S})_{\sigma_{(2)^*}}}\mathbf{G}$, the blow-up of the Grassmannian bundle $\mathbf{G} := \mathrm{Gr}(3,\wedge^2\mathcal{S})$ along the orthogonal Grassmannian bundle $\mathrm{OG}(3,\wedge^2\mathcal{S})_{\sigma_{(2)^*}}$ which parametrizes $\Sigma_{(2)^*}$-planes.

We have two birational contractions which are divisorial contractions of $D_{\mathrm{deg},\mathcal{S}}$, a divisor parametrizing pairs (C,S) where $C \in \mathrm{Hilb}^{2m+1}(\mathrm{Gr}(2,V^*))$ is a conic such that the span W of the union of the spaces parametrized by C is 3-dimensional and $W \subset S \in \mathrm{Gr}(4,V^*)$.

(6.1)
$$\begin{tikzcd} & \mathbf{H}_{\mathcal{S}} \arrow[dl, "s"'] \arrow[dr, "r_H"] & \\ \mathbf{B} \arrow[rr, dashed, leftrightarrow, "\phi"] & & \mathbf{H} \end{tikzcd}$$

The map r_H is a standard birational morphism

$$r_H\colon \mathbf{H}_{\mathcal{S}} \to \mathrm{Hilb}^{2m+1}(\mathrm{Gr}(2,V^*)) \cong \mathbf{H}.$$

r_H sends a pair (C,S) to C.

The map s is obtained from the identification

$$\mathbf{H}_{\mathcal{S}} \cong \mathrm{Bl}_{2\,\mathrm{OG}(3,\wedge^2\mathcal{S})}\,\mathrm{Gr}(3,\wedge^2\mathcal{S}),$$

which is the relativization of (3.2). Thus s is a blow-down, and it contracts the locus of conics in a $\Sigma_{(1,1)^*}$ (as a conic in $\mathrm{Gr}(2,V^*)$) to a point associated to the plane $\Sigma_{(1,1)^*}$.

Recall that there is a regular morphism env: $\mathrm{Hilb}^{2m+1}(\mathrm{Gr}(2,V^*)) \to \mathrm{Gr}(3,\wedge^2 V^*)$ which maps a conic $C \subset \mathrm{Gr}(2,V^*) \subset \mathbb{P}(\wedge^2 V^*)$ to a unique $\mathbb{P}^2$ containing C (Definition 3.3). In Section 3.2, the normalization of the image of env was called $\overline{\mathbf{G}}$.

Lemma 6.9. *The birational map ϕ in* (6.1) *is a flip over the blow-up $\widehat{\mathbf{G}}$ of $\overline{\mathbf{G}}$ along a subvariety isomorphic to* $\mathrm{OG}(3,\wedge^2\mathcal{S})_{\sigma(2)^*}$.

Proof. Let $\varphi\colon \mathbf{G} := \mathrm{Gr}(3,\wedge^2\mathcal{S}) \to \overline{\mathbf{G}}$ be the morphism obtained from the standard projection $\mathrm{Gr}(3,\wedge^2\mathcal{S}) \to \mathrm{Gr}(3,\wedge^2 V^*)$. An element of $\mathrm{OG}(3,\wedge^2\mathcal{S})_{\sigma_{(2)^*}} \subset \mathrm{Gr}(3,\wedge^2\mathcal{S})$ is a pair (U,W) where $W \in \mathrm{Gr}(4,V^*)$ and $U \in \mathrm{Gr}(3,\wedge^2 W)$. Since U is of type $\sigma_{(2)^*}$, U is generated by $v_i \wedge v$ for $1 \le i \le 3$ and a fixed $v \in W$. Thus W, which is the span of v_i and v, is uniquely determined by U. Therefore $\varphi(\mathrm{OG}(3,\wedge^2\mathcal{S})) \cong \mathrm{OG}(3,\wedge^2\mathcal{S})$.

Consider the following diagram:

$$\begin{tikzcd}
 & \mathbf{H}_{\mathcal{S}} \arrow[dl] \arrow[dr] & \\
\mathbf{B} \arrow[ddd] \arrow[dr, dashed, "\alpha"] \arrow[ddd, shift left=2, end anchor=north west] & & \mathbf{H} \arrow[dl, dashed, "\beta"'] \arrow[dddl] \\
 & \widehat{\mathbf{G}} \arrow[dd] & \\
 & & \\
\mathbf{G} \arrow[r, "\varphi"] & \overline{\mathbf{G}} &
\end{tikzcd}$$

The preimage of $\mathrm{OG}(3,\wedge^2\mathcal{S})_{\sigma_{(2)^*}}$ in each $\mathbf{B}$ and $\mathbf{H}$ is a divisor. From the universal property of blow-ups, we obtain two morphisms α and β.

From the construction, α,β are isomorphisms away from the image of $\mathrm{OG}(3,\wedge^2\mathcal{S})_{\sigma_{(1,1)^*}}$ in $\mathbf{H}_{\mathcal{S}}$. A point x in the exceptional set of $\overline{\mathbf{G}}$ represents a $\Sigma_{(1,1)^*} \subset \mathrm{Gr}(2,V^*)$. Then $\beta^{-1}(x) \cong \mathbb{P}^5$ is the space of conics in the $\Sigma_{(1,1)^*}$. $\alpha^{-1}(x)$ the space of $W \in \mathrm{Gr}(4,V^*)$'s such that $\Sigma_{(1,1)^*} \subset \mathbb{P}(\wedge^2 W)$. This fiber is isomorphic to $\mathbb{P}^{n-3}$. $\square$

Remark 6.10. The blow-up center $\mathrm{OG}(3,\wedge^2\mathcal{S})$ is isomorphic to the $\mathrm{Gr}(3,\mathcal{Q})$-bundle over $\mathbb{P}V^*$, where $\mathcal{Q}$ is the rank n universal quotient bundle over $\mathbb{P}V^*$.

6.3. General case

Now we are ready to prove the main result. From now on, let $n > 3$ and let V be an $(n+1)$-dimensional vector space. In the previous sections, we constructed new birational models of $\mathbf{M} := \overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(n-1,V),2)$. Once the associated model is constructed for each divisor class, as you will see, the proof is very straightforward and there is no technical difficulty since $\mathbf{M}$ is a Mori dream space (see, [3, Corollary 1.2]).

Proof of Theorem 1.1. Items (1) and (2) are [3, Proposition 3.6]. A special case $n = 3$ of Item (3) was proved in [3, Theorem 3.10], and the same idea can be used in the general case. Item (4) is [3, Theorem 3.8].

Items (5), (6). The partial desingularization $d\colon \mathbf{M} \to \mathbf{K}$ contracts two curve classes C_1 and C_6, and the second step $c\colon \mathbf{M} \to \mathbf{X}^1/\!/\,\mathrm{SL}_2 \times \mathrm{SL}_2$ contracts C_6 (see, Corollary 5.4). Thus $\mathbf{M}(aH_{\sigma_2}+bT) = \mathbf{X}^1/\!/\,\mathrm{SL}_2 \times \mathrm{SL}_2$ if $a, b > 0$ and $\mathbf{M}(H_{\sigma_2}) = \mathbf{K}$. By Corollary 5.4, Δ is in the exceptional locus of $\mathbf{M} \to \mathbf{X}^1/\!/\,\mathrm{SL}_2 \times \mathrm{SL}_2$. So $\mathbf{M}(aH_{\sigma_2}+bT+c\Delta) = \mathbf{M}(aH_{\sigma_2}+bT)$. The case of $\mathbf{K}$ is similar.

Item (7). Note that C_1 can be regarded as a curve in $\mathbf{H}$, too. By the contraction $\mathbf{H} \to \widetilde{\mathbf{G}}$, C_1 is contracted (see, [16, Proposition 4.13]). Since $aH_{\sigma_2}+bP$ for $a, b > 0$ are nef divisors on $\mathbf{H}$ with the property, $\mathbf{M}(aH_{\sigma_2} + bP) = \widetilde{\mathbf{E}}$. Again, D_{deg} is in the exceptional locus of $\mathbf{M} \dashrightarrow \widetilde{\mathbf{E}}$, $\mathbf{M}(aH_{\sigma_2} + bP + cD_{deg}) = \mathbf{M}(aH_{\sigma_2} + bP)$.

Item (13). By definition of the divisor class P, $\mathbf{M}(P) = \overline{\mathbf{G}}$. Since the rational contraction $\mathbf{M} \dashrightarrow \overline{\mathbf{G}}$ contracts D_{deg}, $\mathbf{M}(aP + bD_{\mathrm{deg}}) = \mathbf{M}(P)$.

Item (14). The curve class C_2 can be regarded as a curve in $\mathbf{H}$. From the proof of Lemma 6.9 and the duality, C_2 is contracted by $\mathbf{H} \to \widehat{\mathbf{G}}$. Since $aH_{\sigma_{1,1}}+bP$ are semiample divisors on $\mathbf{H}$ contracting C_2, (14) follows.

Item (12). By Lemma 6.6, there is a birational contraction $\overline{d}\colon \mathbf{M} \to \mathbf{L}$, which contracts D_{unb} and Δ. Note that $\overline{d}$ contracts two curve classes C_2 and C_7. Thus $\mathbf{M}(H_{\sigma_{1,1}}) = \mathbf{L}$. Δ is in the exceptional locus of $\overline{d}$ so we obtain the result.

Item (9). Let $D = aH_{\sigma_{1,1}} + bP + cD_{\mathrm{unb}}$ for $a, b, c > 0$. Let α, β be two contractions in the proof of Lemma 6.9. Then by the proof of Item (14), $-D$ is β-ample. Let B be the 1-parameter family of data $(\Sigma_{(1,1)^*}, W)$ such that $\Sigma_{(1,1)^*} \subset \mathbb{P}(\wedge^2 W)$ and W forms a line in $\mathrm{Gr}(4, V^*)$. Then B is contracted by α. Furthermore, the curve cone of the exceptional set is generated by B, since it is isomorphic to a projective space. Thus if $D \cdot B > 0$, then D is α-ample and the proof is completed. Note that $aH_{\sigma_{1,1}} + bP$ is the pull-back of an ample divisor on $\widehat{\mathbf{G}}$ by Item (14). Thus $B \cdot H_{\sigma_{1,1}} = B \cdot P = 0$. Also since W varies, the push-forward of B by $\mathbf{B} \to \mathbf{G} \to \mathrm{Gr}(4, V^*)$ is a curve. Now D_{unb} is the pull-back of an ample divisor on $\mathrm{Gr}(4, V^*) \cong \mathrm{Gr}(n-3, V)$ by Definition 2.1, $B \cdot D_{\mathrm{unb}} > 0$.

Item (8). Since the contraction $\mathbf{M}(aH_{\sigma_{1,1}} + bP + cD_{\mathrm{unb}}) = \mathbf{B} \to \overline{\mathbf{G}} = \mathbf{M}(P)$ factors through $\mathbf{G} = \mathrm{Gr}(3, \wedge^2 \mathcal{S})$, either $\mathbf{M}(aP + bD_{\mathrm{unb}})$ or $\mathbf{M}(aH_{\sigma_{1,1}} + bD_{\mathrm{unb}})$ for $a, b > 0$ is $\mathbf{G}$.

It is straightforward to check that the push-forward of the curve class C_1 is contracted by $\mathbf{B} \to \mathbf{G}$. Thus $\mathbf{M}(aP + bD_{\mathrm{unb}}) \cong \mathbf{G}$. D_{deg} is an exceptional divisor of $\mathbf{B} \to \mathbf{G}$. Thus we can obtain the result.

Item (10). The relative contraction $\mathbf{H}_{\mathcal{S}} \to \mathbf{K}_{\mathcal{S}}$ descends to $\mathbf{B} \to \mathbf{K}_{\mathcal{S}} \cong \mathrm{M}_{\mathbb{P}\mathcal{S}}(m^2 + 3m + 2)$ by the rigidity lemma. $\mathbf{K}$ admits two morphisms to $\mathrm{M}_{\mathbb{P}V^*}(m^2 + 3m + 2)$ and $\mathrm{Gr}(4, V^*)$. Thus $H_{\sigma_{1,1}}$ and D_{unb} are two semiample divisors on $\mathrm{M}_{\mathbb{P}\mathcal{S}}(m^2 + 3m + 2)$. The product morphism

$$\mathbf{K} = \mathrm{M}_{\mathbb{P}\mathcal{S}}(m^2 + 3m + 2) \to \mathrm{M}_{\mathbb{P}V^*}(m^2 + 3m + 2) \times \mathrm{Gr}(4, V^*)$$

is injective. Thus $aH_{\sigma_{1,1}} + bD_{\mathrm{unb}}$ with $a, b > 0$ is an ample divisor on $\mathrm{M}_{\mathbb{P}\mathcal{S}}(m^2 + 3m + 2)$. Therefore $\mathbf{M}(aH_{\sigma_{1,1}} + bD_{\mathrm{unb}}) \cong \mathbf{K}$. Finally, Δ is an exceptional divisor for $\mathbf{M} \dashrightarrow \mathrm{M}_{\mathbb{P}\mathcal{S}}(m^2 + 3m + 2)$, so we obtain the statement.

Item (11). Let $\mathbf{R}$ be the normalization of the image of the product map

$$\mathbf{M} \to \mathrm{M}_{\mathbb{P}V^*}(m^2 + 3m + 2) \times \mathbf{U}.$$

The first map is obtained from Lemma 6.6. Now it is clear that $\mathbf{R}$ has two birational morphisms to $\mathbf{L}$ and $\mathbf{U}$. Furthermore, since both $\mathbf{M} \to \mathbf{L}$ and $\mathbf{M} \to \mathbf{U}$ contracts C_7, so is $\mathbf{M} \to \mathbf{R}$. Thus $\mathbf{M}(aH_{\sigma_{1,1}} + bT) \cong \mathbf{R}$. Since Δ is contracted by $\mathbf{M} \to \mathbf{R}$, $\mathbf{M}(aH_{\sigma_{1,1}} + bT + c\Delta) \cong \mathbf{R}$.

Item (15). Since $\mathbf{M}$ is a Mori dream space, $\mathbf{M}(a\Delta + bD_{\mathrm{deg}})$ is a contraction of $\mathbf{M}(aH_{\sigma_2} + bD_{\mathrm{deg}} + c\Delta) \cong \mathbf{K}$. Since the Picard number of $\mathbf{K}$ is one, $\mathbf{M}(a\Delta + bD_{\mathrm{deg}})$ is a point.

Item (16). From Definition 2.1, we obtain $\mathbf{M}(D_{\mathrm{unb}}) = \mathrm{Gr}(4, V^*)$. Since Δ is in the exceptional locus of the rational contraction $\mathbf{M} \dashrightarrow \mathrm{Gr}(4, V^*)$, $\mathbf{M}(aD_{\mathrm{unb}} + b\Delta) \cong \mathbf{M}(D_{\mathrm{unb}}) = \mathrm{Gr}(4, V^*)$. The other case is similar. □

7. Applications

A quick application of describing birational morphisms between models in terms of explicit contractions is the computation of topological invariants. In this section, we leave computations of motivic invariants of the double symmetroid $T_4 \cong \mathbf{K}$ and that of the moduli space $\mathrm{M}_{\mathbb{P}^2}(4m + 2)$ of semistable torsion sheaves on $\mathbb{P}^2$.

7.1. Motivic invariants of the double symmetorid

An explicit description of the partial desingularization in Section 5.2 enables us to compute the virtual Poincaré polynomial of $T_4 \cong \mathbf{K}$. A nice summary of the definition and basic properties of the virtual Poincaré polynomial $P(X)$ of a projective variety X can be found in [27, Section 2]. The virtual Poincaré polynomial of $\mathbf{M}$ was calculated by A. López Martín by using the Bialynicki-Birula decomposition:

Proposition 7.1. [24, Theorem 3.1] *The virtual Poincaré polynomial of* $\mathbf{M}$ *is*

$$\frac{\left[(1+q^{n+1})(1+q^3)-q(1+q)(q^2+q^{n-1})\right](1-q^{n+1})(1-q^n)(1-q^{n-1})}{(1-q)^3(1-q^2)^2}.$$

Proposition 7.2. *The virtual Poincaré polynomial of* T_4 *is*

$$\begin{aligned}&P(\mathbf{M})-(P(\overline{\mathrm{M}}_{0,0}(\mathbb{P}^{n-1},2))-1)\left(\frac{1-q^n}{1-q}\right)\\ &-\left(P((\mathbb{P}^{n-2})^2)-1\right)\left(\frac{1}{2}\left(P(\mathbb{P}^n)^2+\frac{1-q^{2n+2}}{1-q^2}\right)-P(\mathbb{P}^n)\right).\end{aligned}\tag{7.1}$$

Proof. Let $d\colon \mathbf{M}\to T_4$ be the desingularization morphism in Theorem 5.1. On T_4, let N_i be the locally closed subvariety parametrizing rank i quadrics. By Corollary 5.4, for any closed point x on $\mathrm{N}_1\cong\mathbb{P}V^*$, $d^{-1}(x)\cong\overline{\mathrm{M}}_{0,0}(\mathbb{P}^{n-1},2)$ and for any closed point $y\in\mathrm{N}_2\cong(\mathbb{P}V^*\times\mathbb{P}V^*-\Delta)/\mathbb{Z}_2$ where Δ is the diagonal of $\mathbb{P}V^*\times\mathbb{P}V^*$, $d^{-1}(y)\cong(\mathbb{P}^{n-2})^2$. Thus we obtain

$$P(T_4)=P(\mathbf{M})-P(\mathrm{N}_1)P(\overline{\mathrm{M}}_{0,0}(\mathbb{P}^{n-1},2))+P(\mathrm{N}_1)-P(\mathrm{N}_2)P((\mathbb{P}^{n-2})^2)+P(\mathrm{N}_2).$$

By [27, Section 2], $P(\mathrm{N}_2)=\frac{1}{2}\left(P(\mathbb{P}^n)^2+\frac{1-q^{2n+2}}{1-q^2}\right)-P(\mathbb{P}^n)$. Thus we obtain (7.1). $\square$

Note that $P(\overline{\mathrm{M}}_{0,0}(\mathbb{P}^{n-1},2))=\frac{(1-q^{n+1})(1-q^n)(1-q^{n-1})}{(1-q)^2(1-q^2)}$ by [20, Theorem 1.3].

Remark 7.3. Let $T_4(n)$ be the double symmetroid for the $(n+1)$-dimensional vector space V. By using a computer algebra system, we are able to obtain a simpler expression of $P(T_4(n))$ for small n. For instance,

(1) $P(T_4(3))=(q^7+q^6+q^2+q+1)(q^2+1)$,

(2) $P(T_4(4))=(q^7-q^6+q^5-q^4+q^2-q+1)(q^4+q^3+q^2+q+1)(q^2+q+1)$,

(3) $P(T_4(5))=(q^{13}+q^{12}+q^{11}+q^{10}-q^8-q^7+q^4+q^3+q^2+q+1)(q^4+q^2+1)$ and

(4) $P(T_4(6))=(q^{15}+q^{13}+q^{11}-q^{10}-q^8+q^4+q^2+1)(q^6+q^5+q^4+q^3+q^2+q+1)$.

7.2. Motivic invariants of the pure sheaves on $\mathbb{P}^2$

Let $\mathrm{M}_{\mathbb{P}^2}(dm+\chi)$ be the moduli space of one-dimensional semistable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $dm+\chi$. In several papers, including [5–7,17,33], people have computed $P(\mathrm{M}_{\mathbb{P}^2}(dm+\chi))$ when d and χ are coprime. If d and χ are not coprime, because of the existence of the singular locus, the computation of the Poincaré polynomial seems to be hard. Proposition 7.2 and Bridgeland wall crossing [1, Section 6.1] enable us to compute $P(\mathrm{M}_{\mathbb{P}^2}(4m+2))$, which is the first nontrivial case with $(d,\chi)\neq 1$.

Proposition 7.4. *The virtual Poincaré polynomial of* $\mathrm{M}_{\mathbb{P}^2}(4m+2)$ *is given by*

$$\begin{aligned}P(\mathrm{M}_{\mathbb{P}^2}(4m+2)) = q^{17} + 2q^{16} + 5q^{15} + 9q^{14} + 11q^{13} + 11q^{12} + 10q^{11} + 10q^{10} \\ + 9q^9 + 10q^8 + 10q^7 + 12q^6 + 12q^5 + 12q^4 + 9q^3 + 5q^2 + 2q + 1.\end{aligned}$$

Proof. From [13, Section 4], we know that $T_4(5)$ is birational to $\mathrm{M}_{\mathbb{P}^2}(4m+2)$. Furthermore, as described in [1, Section 6.1], there are two wall-crossings from $\mathrm{M}_{\mathbb{P}^2}(4m+2)$ to $T_4(5)$. An object F in the exceptional locus in each wall-crossing is an extension of a particular type, described in Table 7.1. After the wall-crossing, we obtain new extensions F'. Here $\vee$ denotes the derived dual $\mathcal{R}\operatorname{Hom}(-, \mathcal{O}_{\mathbb{P}^2})$. Now by a simple calculation, one can see that

$$P(\mathrm{M}_{\mathbb{P}^2}(4m+2)) = \big(P(\mathbb{P}^{14}) - P(\mathbb{P}^2)\big) + P(\mathbb{P}^2 \times \mathbb{P}^2)\big(P(\mathbb{P}^{12}) - 1\big) + P(T_4(5)).$$

Combining these with Proposition 7.2, we obtain the result. □

The first wall	The second wall
$0 \to \mathcal{O}_{\mathbb{P}^2}(1) \to F \to \mathcal{O}_{\mathbb{P}^2}(-3)[1] \to 0$	$0 \to I_p(1) \to F \to I_q^\vee(-3)[1] \to 0$ for $p, q \in \mathbb{P}^2$
$0 \to \mathcal{O}_{\mathbb{P}^2}(-3)[1] \to F' \to \mathcal{O}_{\mathbb{P}^2}(1) \to 0$	$0 \to I_q^\vee(-3)[1] \to F' \to I_p(1) \to 0$ for $p, q \in \mathbb{P}^2$

Table 7.1: Bridgeland wall-crossings from $\mathrm{M}_{\mathbb{P}^2}(4m+2)$.

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