TAIWANESE JOURNAL OF MATHEMATICS
Vol. 21, No. 2, pp. 489–503, April 2017
DOI: 10.11650/tjm/7256
This paper is available online at http://journal.tms.org.tw

Duality Theorems for Quasiconvex Programming with a Reverse Quasiconvex Constraint

Satoshi Suzuki

Abstract. In this paper, we study duality theorems for quasiconvex programming with a reverse quasiconvex constraint. We introduce quasiaffine transformation methods for a reverse quasiconvex constraint. Especially, we show a linear characterization of a reverse quasiconvex constraint. By using transformation methods, we show necessary optimality conditions in terms of Greenberg-Pierskalla subdifferential. In addition, we introduce a surrogate duality theorem for quasiconvex programming with a reverse quasiconvex constraint.

1. Introduction

In mathematical programming, a constraint set is usually defined by inequality or equality constraints. Since equality constraints can be characterized by inequality constraints, in this paper, we study the following constraint set:

$$\{x \in \mathbb{R}^n \mid \forall\, i \in I, g_i(x) \leq 0, \forall\, j \in J, h_j(x) \geq 0\}. \tag{1.1}$$

In linear programming, g_i and h_j are linear functions. Since $-h_j$ is also linear, the constraint set is always convex. However, in nonlinear programming, the constraint set is not convex and also is not connected in many cases. Especially, in convex programming, "$h_j(x) \geq 0$" is called a reverse convex constraint and has been widely studied. Many researchers have introduced useful results, for example, characterizations of the constraint set, duality theorems, optimality conditions and so on, see [4, 13–15, 30, 31] and references therein. In addition, reverse convex constraints are important research aspects in set containment characterization, see [5, 8, 16, 17]. In the research of reverse convex constraints, characterizations of the constraint set in terms of Fenchel conjugate play a central role. Actually, if h is real-valued convex, then h coincides with the Fenchel biconjugate of h. Hence

$$\{x \in \mathbb{R}^n \mid h(x) \geq 0\} = \bigcup_{v \in \mathbb{R}^n} \{x \in \mathbb{R}^n \mid \langle v, x\rangle - h^*(v) \geq 0\}$$

Received January 21, 2016; Accepted October 17, 2016.
Communicated by Ruey-Lin Sheu.
2010 *Mathematics Subject Classification*. Primary: 90C26; Secondary: 26B25, 49N15.
Key words and phrases. Quasiconvex programming, Reverse quasiconvex constraint, Optimality condition, Surrogate duality.

where $h^*(v)$ is the Fenchel conjugate of h at $v \in \mathbb{R}^n$. Most of results in reverse convex constraints have been shown by using the linear characterization. By the linear characterization, we can treat the reverse convex programming problem in terms of convex subproblems with affine inequality constraints. Even if the constraint set of the reverse convex programming problem is not convex, the constraint sets of subproblems are convex, and we can solve subproblems by using useful duality results and optimality conditions in convex analysis.

In quasiconvex programming, the constraint set in (1.1) is not convex in most cases. However, there are not many results for reverse quasiconvex constraints as far as we know. It is expected to study reverse quasiconvex constraints in terms of recent advances in quasiconvex analysis. In quasiconvex analysis, linear transformation methods for quasiconvex inequality systems play an important role. In [18], we define the notion of generators of quasiconvex functions, and show a linear transformation method for quasiconvex inequality systems. By generators and the transformation method, we show some fruitful results. For example, the sum of quasiconvex functions is not quasiconvex in general. This causes theoretical and numerical difficulties in quasiconvex programming. By the linear transformation method, quasiconvex inequality systems can be treated as affine inequality systems which are closed under addition, and we show Lagrange-type duality and its constraint qualifications, see [18, 28]. Additionally, we study optimality conditions, subdifferential calculus, a sandwich theorem, duality theorems for vector-valued constraints, and so on, see [18–22, 24, 28]. Some types of conjugate functions are closely related to the generators and linear transformation methods for quasiconvex inequality systems, see [3, 9, 10, 12, 25].

In this paper, we study duality theorems for quasiconvex programming with a reverse quasiconvex constraint. We introduce quasiaffine transformation methods for a reverse quasiconvex constraint. Especially, we show a linear characterization of a reverse quasiconvex constraint in terms of Q-conjugate. By using transformation methods, we show necessary optimality conditions in terms of Greenberg-Pierskalla subdifferential. In addition, we introduce a surrogate duality theorem for quasiconvex programming with a reverse quasiconvex constraint.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and previous results. In Section 3, we study transformation methods for a reverse quasiconvex constraint in terms of Q-conjugate. In Section 4, we show necessary optimality conditions in terms of Greenberg-Pierskalla subdifferential. In Section 5, we introduce a surrogate duality theorem for quasiconvex programming with a reverse quasiconvex constraint.

2. Preliminaries

Let $\langle v, x\rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. Given nonempty sets $A, B \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}$, we define $A+B$ and ΛA as follows:

$$A + B = \{x + y \in \mathbb{R}^n \mid x \in A, y \in B\},$$
$$\Lambda A = \{\lambda x \in \mathbb{R}^n \mid \lambda \in \Lambda, x \in A\}.$$

Also, we define $A + \emptyset = \Lambda\emptyset = \emptyset A = \emptyset$. We denote the closure, the convex hull, and the conical hull, generated by A, by $\operatorname{cl} A$, $\operatorname{co} A$ and $\operatorname{cone} A$, respectively. By convention, we define $\operatorname{cone}\emptyset = \{0\}$. We denote the unit sphere of $\mathbb{R}^n$ by $S_{\mathbb{R}^n} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. The normal cone of A at $x \in A$ is denoted by $N_A(x) = \{v \in \mathbb{R}^n \mid \forall y \in A, \langle v, y - x\rangle \leq 0\}$. A set A is said to be evenly convex if it is the intersection of some family of open halfspaces. Note that the whole space and the empty set are evenly convex. Clearly, every evenly convex set is convex. Furthermore, any open convex set and any closed convex set are evenly convex.

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. We denote the domain of f by $\operatorname{dom} f$, that is, $\operatorname{dom} f = \{x \in \mathbb{R}^n \mid f(x) < \infty\}$. The epigraph of f is defined as $\operatorname{epi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq r\}$, and f is said to be convex if $\operatorname{epi} f$ is convex. The subdifferential of f at x is defined as $\partial f(x) = \{v \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n, f(y) \geq f(x) + \langle v, y - x\rangle\}$. Fenchel conjugate of f, $f^*\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined as $f^*(v) = \sup_{x\in\mathbb{R}^n} \{\langle v, x\rangle - f(x)\}$. Define level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \beta) = \{x \in \mathbb{R}^n \mid f(x) \diamond \beta\}$$

for any $\beta \in \mathbb{R}$. A function f is said to be quasiconvex if for all $\beta \in \mathbb{R}$, $L(f, \leq, \beta)$ is a convex set. Any convex function is quasiconvex, but the opposite is not true. A function f is said to be quasiaffine if it is quasiconvex and quasiconcave.

A function f is said to be evenly quasiconvex (strictly evenly quasiconvex) if $L(f, \leq, \beta)$ ($L(f, <, \beta)$, respectively) is evenly convex for each $\beta \in \mathbb{R}$. A function f is said to be evenly quasiaffine if f is evenly quasiconvex and quasiconcave. It is clear that every evenly quasiconvex function is quasiconvex. It is easy to show that every strictly evenly quasiconvex function is evenly quasiconvex. However, converse implications are not generally true, in detail, see [12,17]. It is known that every lower semicontinuous (lsc) quasiconvex function is evenly quasiconvex, and every upper semicontinuous (usc) quasiconvex function is strictly evenly quasiconvex. It is important to notice that f is evenly quasiaffine if and only if there exist an extended real-valued nondecreasing function k on $\mathbb{R}$ and $w \in \mathbb{R}^n$ such that $f = k \circ w$. In addition, each level set of an evenly quasiaffine function is open or closed halfspace, and each level set of a lsc quasiaffine function is closed halfspace, see [9,12] for more details. These results are closely related to the generators of quasiconvex functions, see [18, 28].

A function f is said to be essentially quasiconvex if f is quasiconvex and each local minimizer $x \in \operatorname{dom} f$ of f in $\mathbb{R}^n$ is a global minimizer of f in $\mathbb{R}^n$. Clearly, all convex functions are essentially quasiconvex. A real-valued continuous quasiconvex function is essentially quasiconvex if and only if it is semistrictly quasiconvex, see [1, Theorem 3.37].

The following function h^{-1} is said to be the hypo-epi-inverse of a nondecreasing function h:

$$h^{-1}(a) = \inf \{b \in \mathbb{R} \mid a < h(b)\} = \sup \{b \in \mathbb{R} \mid h(b) \leq a\}.$$

In [12], it is shown that if h has the inverse function, then the inverse and the hypo-epi-inverse of h are the same. In the present paper, we denote the hypo-epi-inverse of h by h^{-1}.

In quasiconvex analysis, various types of conjugates and subdifferentials have been investigated. In this paper, Greenberg-Pierskalla subdifferential and Q-conjugate play an important role. In [3], Greenberg and Pierskalla introduced the Greenberg-Pierskalla subdifferential of f at $x_0 \in \mathbb{R}^n$ as follows:

$$\partial^{GP} f(x_0) = \{v \in \mathbb{R}^n \mid \langle v, x\rangle \geq \langle v, x_0\rangle \text{ implies } f(x) \geq f(x_0)\}.$$

The Q-conjugate of f, $f^Q \colon \mathbb{R}^{n+1} \to \overline{\mathbb{R}}$, is defined as follows: for each $(v,t) \in \mathbb{R}^{n+1}$,

$$f^Q(v,t) = -\inf \{f(x) \mid \langle v, x\rangle \geq t\}.$$

In addition, the Q-conjugate of $g \colon \mathbb{R}^{n+1} \to \overline{\mathbb{R}}$ is the function $g^{\underline{Q}} \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ such that for each $x \in \mathbb{R}^n$

$$g^{\underline{Q}}(x) = -\inf \{g(v,t) \mid \langle v, x\rangle \geq t\},$$

and the Q-biconjugate of f is the function $f^{QQ} \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ such that for each $x \in \mathbb{R}^n$

$$f^{QQ}(x) = (f^Q)^{\underline{Q}}(x) = -\inf \left\{f^Q(v,t) \mid \langle v, x\rangle \geq t\right\}.$$

Greenberg-Pierskalla subdifferential and Q-conjugate are special cases of Moreau's generalized conjugation, in detail, see [9–11, 25]. We introduce the following previous results.

Theorem 2.1. [9–11, 25] *Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and $x \in \mathbb{R}^n$. Then, the following statements hold:*

(i) $f \geq f^{QQ}$,

(ii) $f = f^{QQ}$ *if and only if f is evenly quasiconvex,*

(iii) $\partial f(x) \subset \partial^{GP} f(x)$,

(iv) *for each* $v \in \partial^{GP} f(x)$ *and* $\lambda > 0$, $\lambda v \in \partial^{GP} f(x)$.

In [25], we studied the following necessary and sufficient optimality condition for essentially quasiconvex programming.

Theorem 2.2. [25] *Let f be an usc essentially quasiconvex function, F a nonempty convex subset of $\mathbb{R}^n$ and $x_0 \in F$. Then, the following statements are equivalent:*

(i) $f(x_0) = \min_{x \in F} f(x)$,

(ii) $0 \in \partial^{GP} f(x_0) + N_F(x_0)$.

In [23, 26, 29], we studied surrogate duality for quasiconvex programming. The assumption (i) in the following theorem is called the closed cone constraint qualification for surrogate duality (S-CCCQ). S-CCCQ is a necessary and sufficient constraint qualification for surrogate duality via quasiconvex programming.

Theorem 2.3. [23] *Let I be an index set and g_i a real-valued continuous convex function on $\mathbb{R}^n$ for each $i \in I$. Assume that $C = \{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \leq 0\} \neq \emptyset$. Then, the following conditions are equivalent:*

(i)

$$\bigcup_{\lambda \in \mathbb{R}_+^{(I)}} \operatorname{cl} \operatorname{cone} \operatorname{epi} \left(\sum_{i \in I} \lambda_i g_i \right)^*$$

is closed,

(ii) *for each usc quasiconvex function f from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$,*

$$\inf_{x \in C} f(x) = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf \left\{ f(x) \;\middle|\; \sum_{i \in I} \lambda_i g_i(x) \leq 0 \right\},$$

where $\mathbb{R}_+^{(I)} = \left\{ \lambda \in \mathbb{R}^I \mid \forall i \in I, \lambda_i \geq 0, \{i \in I \mid \lambda_i \neq 0\} \text{ is finite} \right\}$.

3. A reverse quasiconvex constraint

In this section, we study a reverse quasiconvex constraint. We characterize the constraint set by quasiaffine inequality constraints in terms of Q-conjugate. As a special case of the characterization, we show a linear characterization of a reverse quasiconvex constraint.

Let I be an index set, g_i an extended real-valued lsc quasiconvex function on $\mathbb{R}^n$, h an extended real-valued strictly evenly quasiconvex function on $\mathbb{R}^n$, f an extended real-valued quasiconvex function on $\mathbb{R}^n$ and $C = \{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \leq 0\}$. Assume that

$A = \{x \in C \mid h(x) \geq 0\}$ is nonempty. We consider the following quasiconvex programming problem with a reverse quasiconvex constraint:

$$\text{(P)} \qquad \text{minimize } f(x), \text{ subject to } x \in A.$$

At first, we show the following lemma.

Lemma 3.1. *Let h be an extended real-valued strictly evenly quasiconvex function on $\mathbb{R}^n$, $v \in \mathbb{R}^n$ and h_v be the following function on $\mathbb{R}$:*

$$h_v(t) = \inf\{h(x) \mid \langle v, x\rangle \geq t\}.$$

Then, the following statements hold:

(i) *$h^Q(v, \langle v, \cdot\rangle)$ is an evenly quasiaffine function on $\mathbb{R}^n$,*

(ii) *if h is real-valued convex, then $h_v \circ v$ is a convex function on $\mathbb{R}^n$,*

(iii) *if h is real-valued, and bounded from below, then $h_v \circ v$ is real-valued on $\mathbb{R}^n$,*

(iv) *if h is real-valued convex and bounded from below, then $h_v \circ v$ is continuous on $\mathbb{R}^n$.*

Proof. (i) We can check that

$$\begin{aligned} h^Q(v, \langle v, x\rangle) &= -\inf\{h(y) \mid \langle v, y\rangle \geq \langle v, x\rangle\} \\ &= -h_v(\langle v, x\rangle). \end{aligned}$$

It is clear that h_v is nondecreasing. Hence $h^Q(v, \langle v, \cdot\rangle)$ is an evenly quasiaffine function as a composite function of the non-increasing function $-h_v$ and $v \in \mathbb{R}^n$.

(ii) Let $x, y \in \mathbb{R}^n$ and $\alpha \in (0,1)$. For each $z_1, z_2 \in \mathbb{R}^n$ with $\langle v, z_1\rangle \geq \langle v, x\rangle$ and $\langle v, z_2\rangle \geq \langle v, y\rangle$,

$$\begin{aligned} (1-\alpha)h(z_1) + \alpha h(z_2) &\geq h((1-\alpha)z_1 + \alpha z_2) \\ &\geq \inf\{h(z) \mid \langle v, z\rangle \geq \langle v, (1-\alpha)x + \alpha y\rangle\} \\ &= h_v(\langle v, (1-\alpha)x + \alpha y\rangle). \end{aligned}$$

Hence,

$$\begin{aligned} h_v(\langle v, (1-\alpha)x + \alpha y\rangle) \leq (1-\alpha) &\inf\{h(z_1) \mid \langle v, z_1\rangle \geq \langle v, x\rangle\} \\ &+ \alpha \inf\{h(z_2) \mid \langle v, z_2\rangle \geq \langle v, y\rangle\}, \end{aligned}$$

that is, $h_v \circ v((1-\alpha)x + \alpha y) \leq (1-\alpha)h_v \circ v(x) + \alpha h_v \circ v(y)$.

(iii) Assume that h is bounded from below. Then there exists $m \in \mathbb{R}$ such that $h \geq m$. For each $x \in \mathbb{R}^n$,

$$-\infty < m \leq \inf \{h(z) \mid \langle v, z \rangle \geq \langle v, x \rangle\} \leq h(x) < \infty.$$

Hence, $h_v \circ v(x) = \inf \{h(z) \mid \langle v, z \rangle \geq \langle v, x \rangle\} \in \mathbb{R}$.

(iv) By the statements (ii) and (iii), if h is real-valued convex and bounded from below, then $h_v \circ v$ is continuous since $h_v \circ v$ is a real-valued convex function. □

In the following theorem, we show characterizations of the constraint set A by quasi-affine or affine inequality constraints in terms of Q-conjugate.

Theorem 3.2. *Let h be an extended real-valued strictly evenly quasiconvex function on $\mathbb{R}^n$ and C be a closed convex subset of $\mathbb{R}^n$. Assume that $A = \{x \in C \mid h(x) \geq 0\}$ is nonempty. Then, the following sets are equal:*

(i) $A = \{x \in C \mid h(x) \geq 0\}$,

(ii) $\bigcup_{v \in \mathbb{R}^n} \left\{x \in C \mid h^Q(v, \langle v, x \rangle) \leq 0\right\}$,

(iii) $\bigcup_{v \in S_{\mathbb{R}^n}} \left\{x \in C \mid h^Q(v, \langle v, x \rangle) \leq 0\right\}$.

Furthermore, if $h^Q(v, \langle v, \cdot \rangle)$ is lsc, then

$$A = \bigcup_{v \in \mathbb{R}^n} \left\{x \in C \mid \langle -v, x \rangle - (\overline{h}_v)^{-1}(0) \leq 0\right\},$$

where $\overline{h}_v$ is the following nondecreasing function, $\overline{h}_v(t) = -h_v(-t) = -\inf\{h(x) \mid \langle v, x \rangle \geq -t\}$ for each $t \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}^n$. By the strictly evenly quasiconvexity of h,

$$h(x) = h^{QQ}(x) = \max_{v \in \mathbb{R}^n} (-h^Q(v, \langle v, x \rangle)).$$

Actually, by Theorem 2.1, $h(x) = h^{QQ}(x) = -\inf \left\{h^Q(v, t) \mid \langle v, x \rangle \geq t\right\}$. If $L(h, <, h(x)) = \emptyset$, then

$$h(x) = \inf_{y \in \mathbb{R}^n} h(y) = -h^Q(0, \langle 0, x \rangle).$$

Assume that $L(h, <, h(x)) \neq \emptyset$. Then $L(h, <, h(x))$ is a nonempty evenly convex set and $x \notin L(h, <, h(x))$. By the definition of evenly convexity, there exist $v_0 \in \mathbb{R}^n \setminus \{0\}$ and $t \in \mathbb{R}$ such that for each $y \in L(h, <, h(x))$,

$$\langle v_0, x \rangle \geq t > \langle v_0, y \rangle.$$

This shows that

$$\langle v_0, y\rangle \geq \langle v_0, x\rangle \implies y \notin L(h, <, h(x)) \implies h(y) \geq h(x),$$

that is,

$$\begin{aligned}
-h^Q(v_0, \langle v_0, x\rangle) &\geq h(x) \\
&= h^{QQ}(x) \\
&= -\inf\left\{h^Q(v,t) \mid \langle v, x\rangle \geq t\right\} \\
&\geq \sup_{v\in\mathbb{R}^n}(-h^Q(v, \langle v, x\rangle)).
\end{aligned}$$

Hence $h(x) = -h^Q(v_0, \langle v_0, x\rangle) = \max_{v\in\mathbb{R}^n}(-h^Q(v, \langle v, x\rangle))$. We can check that

$$\begin{aligned}
A &= \{x \in C \mid h(x) \geq 0\} \\
&= \left\{x \in C \;\middle|\; \max_{v\in\mathbb{R}^n}(-h^Q(v, \langle v, x\rangle)) \geq 0\right\} \\
&= \left\{x \in C \;\middle|\; \min_{v\in\mathbb{R}^n} h^Q(v, \langle v, x\rangle) \leq 0\right\} \\
&= \bigcup_{v\in\mathbb{R}^n}\left\{x \in C \;\middle|\; h^Q(v, \langle v, x\rangle) \leq 0\right\}.
\end{aligned}$$

Since $v_0 \neq 0$, by the similar way in the first half of the proof, we show that $h(x) = \max_{v\in S_{\mathbb{R}^n}}(-h^Q(v, \langle v, x\rangle))$. Hence

$$A = \bigcup_{v\in S_{\mathbb{R}^n}}\left\{x \in C \mid h^Q(v, \langle v, x\rangle) \leq 0\right\}.$$

If $h^Q(v, \langle v, \cdot\rangle)$ is lsc, by Lemma 3.1, $h^Q(v, \langle v, \cdot\rangle)$ is lsc quasiaffine. Hence $\{x \in \mathbb{R}^n \mid h^Q(v, \langle v, x\rangle) \leq 0\}$ is a closed halfspace. By the closedness of the halfspace $\{x \in \mathbb{R}^n \mid h^Q(v, \langle v, x\rangle) \leq 0\}$,

$$\begin{aligned}
\left\{x \in \mathbb{R}^n \mid h^Q(v, \langle v, x\rangle) \leq 0\right\} &= \{x \in \mathbb{R}^n \mid -h_v(\langle v, x\rangle) \leq 0\} \\
&= \left\{x \in \mathbb{R}^n \mid \overline{h}_v(\langle -v, x\rangle) \leq 0\right\} \\
&= \left\{x \in \mathbb{R}^n \mid \langle -v, x\rangle \leq (\overline{h}_v)^{-1}(0)\right\} \\
&= \left\{x \in \mathbb{R}^n \mid \langle -v, x\rangle - (\overline{h}_v)^{-1}(0) \leq 0\right\}.
\end{aligned}$$

This completes the proof. □

We define the following problem (P_v) for each $v \in \mathbb{R}^n$:

(P_v) minimize $f(x)$, subject to $\forall\, i \in I$, $g_i(x) \leq 0$, $h^Q(v, \langle v, x\rangle) \leq 0$.

The constraint set $A_v = \left\{x \in C \mid h^Q(v, \langle v, x\rangle) \leq 0\right\}$ of (P_v) is an evenly convex set as the intersection of the closed convex set $C = \{x \in \mathbb{R}^n \mid \forall\, i \in I, g_i(x) \leq 0\}$ and the open or closed halfspace $\left\{x \in \mathbb{R}^n \mid h^Q(v, \langle v, x\rangle) \leq 0\right\}$. If $h^Q(v, \langle v, \cdot\rangle)$ is lsc, especially if h is real-valued convex and bounded from below, then A_v is closed convex.

By Theorem 3.2, we show the following theorem.

Theorem 3.3. *Let I be an index set, g_i an extended real-valued lsc quasiconvex function on $\mathbb{R}^n$, h an extended real-valued strictly evenly quasiconvex function on $\mathbb{R}^n$, f an extended real-valued quasiconvex function on $\mathbb{R}^n$ and $C = \{x \in \mathbb{R}^n \mid \forall\, i \in I, g_i(x) \leq 0\}$. Assume that $A = \{x \in C \mid h(x) \geq 0\}$ is nonempty. Then, the following equation holds:*

$$\inf_{x \in A} f(x) = \inf_{v \in \mathbb{R}^n} \inf \left\{ f(x) \mid x \in C, h^Q(v, \langle v, x\rangle) \leq 0 \right\}.$$

4. Optimality condition

In this section, we study optimality conditions for quasiconvex programming with a reverse quasiconvex constraint. We show a necessary optimality condition in terms of Greenberg-Pierskalla subdifferential. Additionally, we investigate a necessary optimality condition in terms of constraint qualifications.

At first, in Theorem 4.1, we show a necessary optimality condition under the assumption "$\inf_{x\in A} f(x) > \inf_{x \in C} f(x)$." If the assumption holds, then we can characterize a global minimizer of f in A by the necessary optimality condition in Theorem 4.1. If $\inf_{x\in A} f(x) = \inf_{x\in C} f(x)$, then we can characterize a global minimizer of f in A by the following necessary and sufficient optimality condition: let $x_0 \in A$, then x_0 is a global minimizer of f in A if and only if

$$0 \in \partial^{GP} f(x_0) + N_C(x_0).$$

By Theorem 2.2, we can prove the above statement. The proof is easy and omitted.

Theorem 4.1. *Let h be an extended real-valued strictly evenly quasiconvex function on $\mathbb{R}^n$, f an extended real-valued usc essentially quasiconvex function on $\mathbb{R}^n$, C a closed convex subset of $\mathbb{R}^n$, $x_0 \in A = \{x \in C \mid h(x) \geq 0\}$ and $A_v = \left\{x \in C \mid h^Q(v, \langle v, x\rangle) \leq 0\right\}$ for each $v \in \mathbb{R}^n$. Assume that $\inf_{x\in A} f(x) > \inf_{x\in C} f(x)$ and for each $v \in \mathbb{R}^n$,*

$$N_{C \cap L(v, \geq, \langle v, x_0\rangle)} = N_C(x_0) + \mathbb{R}_+ \{-v\}, \tag{4.1}$$

where $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$.

If x_0 is a global minimizer of f in A, then

$$\partial^{GP} h(x_0) \subset \partial^{GP} f(x_0) + N_C(x_0).$$

Proof. Assume that x_0 is a global minimizer of f in A and let $v \in \partial^{GP} h(x_0)$. For each $x \in L(v, \geq, \langle v, x_0 \rangle)$,

$$\begin{aligned} h^Q(v, \langle v, x \rangle) &= -\inf \{h(y) \mid \langle v, y \rangle \geq \langle v, x \rangle\} \\ &\leq -\inf \{h(y) \mid \langle v, y \rangle \geq \langle v, x_0 \rangle\} \\ &= h^Q(v, \langle v, x_0 \rangle) \\ &= -h(x_0) \\ &\leq 0. \end{aligned}$$

This shows that $x_0 \in C \cap L(v, \geq, \langle v, x_0 \rangle) \subset A_v \subset A$. Since x_0 is a global minimizer of f in A,

$$f(x_0) = \inf_{x \in A} f(x) \leq \inf_{x \in C \cap L(v, \geq, \langle v, x_0 \rangle)} f(x) \leq f(x_0).$$

This shows that x_0 is a global minimizer of f in $C \cap L(v, \geq, \langle v, x_0 \rangle)$.

By Theorem 2.2,

$$0 \in \partial^{GP} f(x_0) + N_{C \cap L(v, \geq, \langle v, x_0 \rangle)}(x_0).$$

Hence there exists $w \in \partial^{GP} f(x_0)$ such that $-w \in N_{C \cap L(v, \geq, \langle v, x_0 \rangle)}(x_0)$. By the assumption (4.1), $-w \in N_C(x_0) + \mathbb{R}_+ \{-v\}$, that is, there exist $u \in N_C(x_0)$ and $\lambda \geq 0$ such that $-w = u - \lambda v$. We show that $\lambda > 0$. Actually, if $\lambda = 0$,

$$0 = w + u \in \partial^{GP} f(x_0) + N_C(x_0).$$

By Theorem 2.2, x_0 is a global minimizer of f in C. Hence $f(x_0) = \inf_{x \in A} f(x) = \inf_{x \in C} f(x)$. This is a contradiction.

Since the statement (iv) of Theorem 2.1 holds and $N_C(x_0)$ is a cone,

$$v = \frac{1}{\lambda} w + \frac{1}{\lambda} u \in \partial^{GP} f(x) + N_C(x_0).$$

This completes the proof. □

Remark 4.2. The assumption (4.1) of Theorem 4.1 is called strong conical hull intersection property (strong CHIP). We can check easily that strong CHIP holds if and only if

$$N_{C \cap L(v, \geq, \langle v, x_0 \rangle)} \subset N_C(x_0) + \mathbb{R}_+ \{-v\}.$$

The strong CHIP is closely related to the subdifferential sum formula, and is widely studied as a constraint qualification for convex programming. Furthermore, it is known that the assumption (4.1) of Theorem 4.1 holds if C is polyhedral set, see [2, 6, 7].

Remark 4.3. By using the notion of generators of quasiconvex functions, we study constraint qualifications for quasiconvex programming in [18,19,22,24,27,28].

A set $G = \{(k_j, w_j) \mid j \in J\} \subset Q \times \mathbb{R}^n$ is said to be a generator of f if $f = \sup_{j \in J} k_j \circ w_j$, where $Q = \left\{h \colon \mathbb{R} \to \overline{\mathbb{R}} \mid h \text{ is lsc and nondecreasing}\right\}$. All lsc quasiconvex functions have at least one generator, see [9,12,18].

Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions on $\mathbb{R}^n$, $\left\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\right\} \subset Q \times \mathbb{R}^n$ a generator of g_i for each $i \in I$, $T = \{t = (i,j) \mid i \in I, j \in J_i\}$ and $C = \{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \le 0\}$. The inequality system $\{g_i(x) \le 0 \mid i \in I\}$ is said to satisfy the basic constraint qualification for quasiconvex programming (Q-BCQ) with respect to $\{(k_t, w_t) \mid t \in T\}$ at $x \in C$ if

$$N_C(x) = \operatorname{cone} \operatorname{co} \bigcup_{t \in T(x)} \{w_t\},$$

where $T(x) = \left\{t \in T \mid \langle w_t, x\rangle = k_t^{-1}(0)\right\}$. Q-BCQ is a necessary and sufficient constraint qualification for Lagrange-type min-max duality, see [19,28] for more details.

If the Q-BCQ at $x_0 \in C$ and strong CHIP for each $v \in \mathbb{R}^n$ hold, then we can show the following necessary optimality condition: if x_0 is a global minimizer of an usc essential quasiconvex function f in $A = \{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \le 0, h(x) \ge 0\}$,

$$\partial^{GP} h(x_0) \subset \partial^{GP} f(x_0) + \operatorname{cone} \operatorname{co} \bigcup_{t \in T(x)} \{w_t\}.$$

The proof is similar to the proof of Theorem 4.1 and omitted.

5. Surrogate duality

In the following theorem, we study surrogate duality for quasiconvex programming with a reverse quasiconvex constraint.

Theorem 5.1. *Let I be an index set, g_i a real-valued convex function on $\mathbb{R}^n$, h an extended real-valued strictly evenly quasiconvex function on $\mathbb{R}^n$, f an usc extended real-valued quasiconvex function on $\mathbb{R}^n$ and $C = \{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \le 0\}$. Assume that $A = \{x \in C \mid h(x) \ge 0\}$ is nonempty, $h^Q(v, \langle v, \cdot\rangle)$ is lsc and*

$$\bigcup_{(\mu,\lambda) \in \mathbb{R}_+ \times \mathbb{R}_+^{(I)}} \operatorname{cl} \operatorname{cone} \operatorname{epi} \left(\mu(-v - (\overline{h}_v)^{-1}(0)) + \sum_{i \in I} \lambda_i g_i\right)^*$$

is closed for each $v \in \mathbb{R}^n$. Then,

$$\inf_{x \in A} f(x) = \inf_{v \in \mathbb{R}^n} \max_{(\mu,\lambda) \in \mathbb{R}_+ \times \mathbb{R}_+^{(I)}} \inf \left\{ f(x) \;\middle|\; \mu(\langle -v, x\rangle - (\overline{h}_v)^{-1}(0)) + \sum_{i \in I} \lambda_i g_i(x) \le 0 \right\}.$$

Proof. By Theorem 3.2,

$$A = \bigcup_{v\in\mathbb{R}^n} \left\{x \in C \mid \langle -v, x\rangle - (\overline{h}_v)^{-1}(0) \le 0\right\}.$$

By the assumption, $\left\{g_i(x) \le 0, \langle -v, x\rangle - (\overline{h}_v)^{-1}(0) \le 0 \mid i \in I\right\}$ satisfies the S-CCCQ for each $v \in \mathbb{R}^n$. Hence, by Theorem 2.3,

$$\inf_{x\in A_v} f(x) = \max_{(\mu,\lambda)\in\mathbb{R}_+\times\mathbb{R}_+^{(I)}} \inf \left\{ f(x) \;\middle|\; \mu(\langle -v, x\rangle - (\overline{h}_v)^{-1}(0)) + \sum_{i\in I} \lambda_i g_i(x) \le 0 \right\}.$$

By Theorem 3.3,

$$\begin{aligned}\inf_{x\in A} f(x) &= \inf_{v\in\mathbb{R}^n} \inf_{x\in A_v} f(x) \\ &= \inf_{v\in\mathbb{R}^n} \max_{(\mu,\lambda)\in\mathbb{R}_+\times\mathbb{R}_+^{(I)}} \inf \left\{ f(x) \;\middle|\; \mu(\langle -v, x\rangle - (\overline{h}_v)^{-1}(0)) + \sum_{i\in I} \lambda_i g_i(x) \le 0 \right\}.\end{aligned}$$

This completes the proof. □

Remark 5.2. By Lemma 3.1, if h is real-valued convex and bounded from below, then $h^Q(v, \langle v, \cdot\rangle)$ is lsc. Hence in this case, we can transform the reverse convex constraint "$h(x) \ge 0$" to affine constraints in terms of Q-conjugate as follows:

$$A = \bigcup_{v\in\mathbb{R}^n} \left\{x \in C \mid \langle -v, x\rangle - (\overline{h}_v)^{-1}(0) \le 0\right\}.$$

For a quasiconvex inequality system $\{g_i(x) \le 0 \mid i \in I\}$, $\sum_{i\in I} \lambda_i g_i$ is not always quasiconvex. Hence in Theorem 5.1, we assume that g_i is convex. However, even if g_i is quasiconvex, we can transform the quasiconvex inequality system to an affine inequality system by using the notion of generators. Actually, by the definition of the generator, we show the following linear characterization of quasiconvex inequality constraints:

$$\begin{aligned}C &= \{x \in \mathbb{R}^n \mid \forall\, i \in I, g_i(x) \le 0\} \\ &= \{x \in \mathbb{R}^n \mid \forall\, t \in T, k_t \circ w_t(x) \le 0\} \\ &= \left\{x \in \mathbb{R}^n \mid \forall\, t \in T, \langle w_t, x\rangle - k_t^{-1}(0) \le 0\right\}.\end{aligned}$$

We can regard C as a closed convex set defined by (possibly infinitely many) affine inequalities, see [18, 19, 22, 24, 27, 28] for more details.

Summarizing the above mentioned, if h is real-valued convex and bounded from below, and $\left\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\right\} \subset Q \times \mathbb{R}^n$ is a generator of a lsc quasiconvex function g_i for each $i \in I$, then the following equation holds:

$$A = \bigcup_{v\in\mathbb{R}^n} \left\{x \in \mathbb{R}^n \mid \forall\, t \in T, \langle w_t, x\rangle - k_t^{-1}(0) \le 0, \langle -v, x\rangle - (\overline{h}_v)^{-1}(0) \le 0\right\}.$$

Since these constraint functions are affine, we can consider a surrogate duality theorem with its constraint qualification, S-CCCQ.

Acknowledgments

This work was partially supported by JSPS KAKENHI Grant Number 15K17588. The author is grateful to the anonymous referee for many comments and suggestions improved the quality of the paper.

References

[1] M. Avriel, W. E. Diewert, S. Schaible and I. Zang, *Generalized Concavity*, Mathematical Concepts and Methods in Science and Engineering **36**, Plenum Press, New York, 1988. https://doi.org/10.1007/978-1-4684-7600-2

[2] H. H. Bauschke, J. M. Borwein and W. Li, *Strong conical hull intersection property, bounded linear regularity, Jameson's property (G), and error bounds in convex optimization*, Math. Program. Ser. A **86** (1999), no. 1, 135–160. https://doi.org/10.1007/s101070050083

[3] H. J. Greenberg and W. P. Pierskalla, *Quasi-conjugate functions and surrogate duality*, Cahiers Centre Études Recherche Opér. **15** (1973), 437–448.

[4] R. J. Hillestad and S. E. Jacobsen, *Reverse convex programming*, Appl. Math. Optim. **6** (1980), no. 1, 63–78. https://doi.org/10.1007/bf01442883

[5] V. Jeyakumar, *Characterizing set containments involving infinite convex constraints and reverse-convex constraints*, SIAM J. Optim. **13** (2003), no. 4, 947–959. https://doi.org/10.1137/s1052623402401944

[6] ______, *The strong conical hull intersection property for convex programming*, Math. Program. Ser. A **106** (2006), no. 1, 81–92. https://doi.org/10.1007/s10107-005-0605-4

[7] C. Li, K. F. Ng and T. K. Pong, *Constraint qualifications for convex inequality systems with applications in constrained optimization*, SIAM J. Optim. **19** (2008), no. 1, 163–187. https://doi.org/10.1137/060676982

[8] O. L. Mangasarian, *Set containment characterization*, J. Global Optim. **24** (2002), no. 4, 473–480. https://doi.org/10.1023/A:1021207718605

[9] J.-E. Martínez-Legaz, *Quasiconvex duality theory by generalized conjugation methods*, Optimization **19** (1988), no. 5, 603–652. https://doi.org/10.1080/02331938808843379

[10] J.-J. Moreau, *Inf-convolution, sous-additivité, convexité des fonctions numériques*, J. Math. Pures Appl. (9) **49** (1970), 109–154.

[11] J.-P. Penot, *Projective dualities for quasiconvex problems*, J. Global Optim. **62** (2015), no. 3, 411–430. https://doi.org/10.1007/s10898-014-0261-4

[12] J.-P. Penot and M. Volle, *On quasi-convex duality*, Math. Oper. Res. **15** (1990), no. 4, 597–625. https://doi.org/10.1287/moor.15.4.597

[13] Y. Saeki and D. Kuroiwa, *Optimality conditions for DC programming problems with reverse convex constraints*, Nonlinear Anal. **80** (2013), 18–27. https://doi.org/10.1016/j.na.2012.11.025

[14] I. Singer, *Some further duality theorems for optimization problems with reverse convex constraint sets*, J. Math. Anal. Appl. **171** (1992), no. 1, 205–219. https://doi.org/10.1016/0022-247x(92)90385-q

[15] ______, *Lagrangian duality theorems for reverse convex infimization*, Numer. Funct. Anal. Optim. **21** (2000), no. 7-8, 933–944. https://doi.org/10.1080/01630560008816995

[16] S. Suzuki, *Set containment characterization with strict and weak quasiconvex inequalities*, J. Global Optim. **47** (2010), no. 2, 273–285. https://doi.org/10.1007/s10898-009-9473-4

[17] S. Suzuki and D. Kuroiwa, *Set containment characterization for quasiconvex programming*, J. Global Optim. **45** (2009), no. 4, 551–563. https://doi.org/10.1007/s10898-008-9389-4

[18] ______, *On set containment characterization and constraint qualification for quasiconvex programming*, J. Optim. Theory Appl. **149** (2011), no. 3, 554–563. https://doi.org/10.1007/s10957-011-9804-8

[19] ______, *Optimality conditions and the basic constraint qualification for quasiconvex programming*, Nonlinear Anal. **74** (2011), 1279–1285.

[20] ______, *Sandwich theorem for quasiconvex functions and its applications*, J. Math. Anal. Appl. **379** (2011), no. 2, 649–655. https://doi.org/10.1016/j.jmaa.2011.01.061

[21] ______, *Subdifferential calculus for a quasiconvex function with generator*, J. Math. Anal. Appl. **384** (2011), no. 2, 677–682. https://doi.org/10.1016/j.jmaa.2011.06.015

[22] ———, *Necessary and sufficient conditions for some constraint qualifications in quasiconvex programming*, Nonlinear Anal. **75** (2012), no. 5, 2851–2858. https://doi.org/10.1016/j.na.2011.11.025

[23] ———, *Necessary and sufficient constraint qualification for surrogate duality*, J. Optim. Theory Appl. **152** (2012), no. 2, 366–377. https://doi.org/10.1007/s10957-011-9893-4

[24] ———, *Some constraint qualifications for quasiconvex vector-valued systems*, J. Global Optim. **55** (2013), no. 3, 539–548. https://doi.org/10.1007/s10898-011-9807-x

[25] ———, *Characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential*, J. Global Optim. **62** (2015), no. 3, 431–441. https://doi.org/10.1007/s10898-014-0255-2

[26] ———, *A constraint qualification characterizing surrogate duality for quasiconvex programming*, Pac. J. Optim. **12** (2016), no. 1, 87–100.

[27] ———, *Nonlinear error bounds for quasiconvex inequality systems*, to appear in Optimization Letters. https://doi.org/10.1007/s11590-015-0992-2

[28] ———, *Generators and constraint qualifications for quasiconvex inequality systems*, to appear in Journal of Nonlinear and Convex Analysis.

[29] S. Suzuki, D. Kuroiwa and G. M. Lee, *Surrogate duality for robust optimization*, European J. Oper. Res. **231** (2013), no. 2, 257–262. https://doi.org/10.1016/j.ejor.2013.02.050

[30] P. T. Thach, *Quasiconjugates of functions, duality relationship between quasiconvex minimization under a reverse convex constraint and quasiconvex maximization under a convex constraint, and applications*, J. Math. Anal. Appl. **159** (1991), no. 2, 299–322. https://doi.org/10.1016/0022-247x(91)90197-8

[31] S. Yamada, T. Tanino and M. Inuiguchi, *Inner approximation method for a reverse convex programming problem*, J. Optim. Theory Appl. **107** (2000), no. 2, 355–389. https://doi.org/10.1023/a:1026456730792

Satoshi Suzuki
Department of Mathematics, Shimane University, 1060, Nishikawatsu, Matsue, Shimane 690-8504, Japan
E-mail address: suzuki@riko.shimane-u.ac.jp