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Relation Between the Class of M. Sama and the Class of ℓ-stable Functions

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Abstract. The aim of this paper is to show the equivalence of two classes of nonsmooth functions. We also compare optimality conditions which have been stated for these classes.

1. Introduction and preliminaries

Some second-order optimality conditions were stated for the class of $C^{1,1}$ functions, it means for the functions having a locally Lipschitz gradient. We note that the functions with $C^{1,1}$ property appear for example in the augmented Lagrange method, the penalty function method and the proximal point method.

M. Sama generalized [18] the $C^{1,1}$ property with respect to the calmness of the Clarke gradient and defined the class $\mathcal{F}$ of functions with certain property. Let us recall the definition of the class $\mathcal{F}$ gradually (Definition 1.5).

Throughout this work X is a real Banach space and X^* is its dual, i.e., the space of all bounded linear operators from X to $\mathbb{R}$. By $\langle \cdot , \cdot \rangle$ we mean the canonical dual pairing between X and X^*.

$B(x;r) := \{y \in X; \|y-x\| \leq r\}$ denotes the closed ball with a center $x \in X$ and a radius $r > 0$ and $S_X := \{y \in X; \|y\| = 1\}$ is a unit sphere of X.

A function $f\colon X \to \mathbb{R}$ is said to be continuous at $x \in X$ if there exists a neighbourhood U of x such that f is continuous on U.

A function $f\colon X \to \mathbb{R}$ is said to be Lipschitz at $x \in X$ if there exist $r > 0$ and $K > 0$ such that $|f(y) - f(z)| \leq K \|y - z\|$ for every $y, z \in B(x;r)$.

A function $f\colon X \to \mathbb{R}$ is said to be strictly differentiable at $x \in X$ if there exists an element $f'_s(x) \in X^*$ such that

$$\lim_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y+th) - f(y)}{t} = \left\langle f'_s(x), h \right\rangle, \quad \forall\, h \in X,$$

where the convergence is assumed to be uniform for h in compact sets. Then $f'_s(x)$ is called the strict derivative of f at x.

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Proposition 1.1. [9, Proposition 2.1.1] *If a function* $f\colon X \to \mathbb{R}$ *is strictly differentiable at* $x \in X$*, then* f *is Lipschitz at* x*.*

By $f^\circ(x;h)$, $f_\circ(x;h)$ we denote the Clarke upper and lower generalized derivatives of f at x in the direction h respectively, i.e.,

$$f^\circ(x;h) = \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{f(y+th)-f(y)}{t},$$
$$f_\circ(x;h) = \liminf_{\substack{y\to x\\ t\downarrow 0}} \frac{f(y+th)-f(y)}{t},$$

and the Clarke generalized gradient of f at x is defined as

$$\partial f(x) = \{x^* \in X^*; \langle x^*, h\rangle \le f^\circ(x;h), \forall\, h \in X\}\,.$$

The following lemma follows immediately from [9, Proposition 2.1.2] and from the convexity of the Clarke generalized gradient.

Lemma 1.2. *Let* $f\colon X \to \mathbb{R}$ *be Lipschitz at* $x \in X$ *and* $h \in X$*. Then*

$$f^\circ(x;h) = \max\left\{\langle x^*, h\rangle\,; x^* \in \partial f(x)\right\},$$
$$f_\circ(x;h) = \min\left\{\langle x^*, h\rangle\,; x^* \in \partial f(x)\right\}.$$

Moreover, for every $h \in X$*, the convexity of* $\partial f(x)$ *yields the existence of* $x^* \in \partial f(x)$ *satisfying*

$$\langle x^*, h\rangle = \gamma$$

provided

$$f_\circ(x;h) \le \gamma \le f^\circ(x;h).$$

We recall [9] that the Clarke generalized gradient of f at x consisting of the single point if and only if f is strictly differentiable at x. In this case $\partial f(x) = \{f_s'(x)\}$.

The mean value theorem in terms of the Clarke gradient has been an important tool in proving many assertions.

Lemma 1.3. [9, Theorem 2.3.7] *Let* $f\colon X \to \mathbb{R}$ *be Lipschitz on a neighbourhood* U *and* $a, b \in U$*. Then there exists a* $t \in (0,1)$ *such that*

$$f(b) - f(a) = \langle x^*, b-a\rangle\,,$$

where $x = a + t(b-a)$ *and* $x^* \in \partial f(x)$*.*

By $F\colon X \rightsquigarrow Y$ we denote a set-valued map from X to Y, and by graph(F) we mean a set

$$\{(x,y) \in X \times Y; y \in F(x)\}.$$

The following Definitions 1.4 and 1.5 were introduced in [18]. By $F\colon X \rightsquigarrow Y$ we denote a set-valued map from X to Y and by graph(F) we mean its graph, i.e., the set $\{(x,y) \in X \times Y; y \in F(x)\}$.

Definition 1.4. Let $M > 0$. F is said to be M-calm at $(x_0, y_0) \in \operatorname{graph}(F)$ if there exists $\varepsilon > 0$ such that

$$F(x) \subset \{y_0\} + M\,\|x - x_0\|\, B(0;1)$$

for every $x \in B(x_0;\varepsilon) \setminus \{x_0\}$.

Definition 1.5. Let $x \in X$. Then

$$\mathcal{F}(x) = \big\{ f\colon X \to \mathbb{R};\ f \text{ is strictly differentiable at } x \text{ and there exists } M > 0 \\ \text{such that } \partial f \text{ is } M\text{-calm at } (x, f'_s(x)) \big\}.$$

The concept of ℓ-stability means another generalization of $C^{1,1}$-property. It was introduced in [3], where the authors weakened the $C^{1,1}$-property of functions in such a way that the unconstrained scalar optimality condition presented in [12] remains true. Let us recall the definition of scalar ℓ-stability.

Definition 1.6. The lower directional derivative of the function $f\colon X \to \mathbb{R}$ at the point $x \in X$ in the direction $h \in X$ is defined by

$$f^{\ell}(x;h) = \liminf_{t \downarrow 0} \frac{f(x+th) - f(x)}{t}.$$

The function f is called ℓ-stable at x if there exist a neighbourhood U of x and $K > 0$ such that

$$\left| f^{\ell}(y;h) - f^{\ell}(x;h) \right| \le K\,\|y - x\|\,, \quad \forall\, y \in U,\ \forall\, h \in S_X.$$

The properties of ℓ-stable functions were then studied e.g. in [2,4–8,10,11,13–17]. We note only that the concept of ℓ-stability was broadened to vector functions from X into Y and that the class of ℓ-stable functions was used in vector optimization.

Theorem 1.7. [17, Theorem 3.1] *Let X be a normed linear space, and let $f\colon X \to \mathbb{R}$ be a continuous function at $x \in X$. If f is ℓ-stable at x, then f is strictly differentiable at x.*

Having in mind the previous theorem, it arises a question what is relation between the class of ℓ-stable at the point x functions and the class $\mathcal{F}(x)$. In [2], the authors solved this problem in finite-dimensional setting.

Theorem 1.8. [2] *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function and $x \in \mathbb{R}^n$. Then f is ℓ-stable at x if and only if $f \in \mathcal{F}(x)$.*

2. Infinite dimension

We will show that Theorem 1.8 is true also in the case when we replace $\mathbb{R}^n$ by an arbitrary Banach space.

Lemma 2.1. [4, Lemma 1] *Let $f\colon X \to \mathbb{R}$ be a continuous function on an open set $U \subset X$, and let $a, b \in U$. Then there exist*

$$\xi_1, \xi_2 \in (a,b) := \{z; z = at + (1-t)b, t \in (0,1)\}$$

such that

$$f^\ell(\xi_1; b-a) \leq f(b) - f(a) \leq f^\ell(\xi_2; b-a).$$

Theorem 2.2. *Let $f\colon X \to \mathbb{R}$ be a continuous function at $x \in X$. Then the function f is ℓ-stable at x if and only if $f \in \mathcal{F}(x)$.*

Proof. We will show the equivalence in the following two steps.

Step 1. We suppose at first that f is ℓ-stable at $x \in X$. By Theorem 1.7 the function f is strictly differentiable at x. We will show that the set-valued mapping $\partial f\colon X \rightsquigarrow X^*$ is calm at x.

On the contrary, we assume that the mapping $\partial f\colon X \rightsquigarrow X^*$ is not calm at x. Then there exist sequences $\{y_n\}_{n=1}^{+\infty} \subset X$, $\lim_{n\to\infty} y_n = x$, $\{y_n^*\}_{n=1}^{+\infty} \subset X^*$, $y_n^* \in \partial f(y_n)$, and $\{h_n\}_{n=1}^{+\infty} \subset S_X$ such that

$$|\langle y_n^*, h_n\rangle - f^\circ(x; h_n)| \geq n \|y_n - x\|, \quad \forall\, n \in \mathbb{N}. \tag{2.1}$$

We recall that due to the strict differentiability of f at x we have

$$\left\langle f_s'(x), h\right\rangle = f^\ell(x;h) = f_\circ(x;h) = f^\circ(x;h), \quad \forall\, h \in S_X. \tag{2.2}$$

By Lemma 1.2 it holds

$$f_\circ(y_n; h_n) \leq \langle y_n^*, h_n\rangle \leq f^\circ(y_n; h_n), \quad \forall\, n \in \mathbb{N}. \tag{2.3}$$

Considering formulas (2.1), (2.2) and (2.3), for every $n \in \mathbb{N}$ it holds either

$$f^\circ(y_n; h_n) - f^\circ(x; h_n) \geq n \|y_n - x\| \tag{2.4}$$

or

$$f_\circ(x; h_n) - f_\circ(y_n; h_n) \geq n \|y_n - x\|. \tag{2.5}$$

Then we can find sequences $\{z_n\}_{n=1}^{+\infty}$ and $\{t_n\}_{n=1}^{+\infty}$ such that for every $n \in \mathbb{N}$ we have either (in case (2.4))

$$\frac{f(z_n + t_n h_n) - f(z_n)}{t_n} - f^\ell(x; h_n) \geq (n-1) \|y_n - x\| \tag{2.6}$$

or (in case (2.5))

$$f^{\ell}(x;h_n) - \frac{f(z_n+t_nh_n)-f(z_n)}{t_n} \geq (n-1)\left\|y_n - x\right\|. \tag{2.7}$$

Since for every $n \in \mathbb{N}$ we can choose z_n arbitrarily close to y_n and we can also choose t_n arbitrarily small, we can assume (in both cases (2.6) and (2.7)) without loss of generality that

$$\max_{\alpha\in[0,1]} \left\|z_n + \alpha t_n h_n - x\right\| \leq 2\left\|y_n - x\right\|, \quad \forall\, n \in \mathbb{N}. \tag{2.8}$$

Now, using Lemma 2.1 and formulas (2.6) and (2.7), we can find for every $n \in \mathbb{N}$ (in both cases) a point $\xi_n \in (z_n, z_n + t_nh_n)$ such that we have

$$\left|f^{\ell}(\xi_n;h_n) - f^{\ell}(x;h_n)\right| \geq (n-1)\left\|y_n - x\right\|.$$

Since f is ℓ-stable at x, there exists $K \geq 0$ such that

$$K\left\|\xi_n - x\right\| \geq \left|f^{\ell}(\xi_n;h_n) - f^{\ell}(x;h_n)\right| \geq (n-1)\left\|y_n - x\right\|. \tag{2.9}$$

The inequalities (2.8) and (2.9) imply

$$2K\left\|y_n - x\right\| \geq (n-1)\left\|y_n - x\right\|, \quad \forall\, n \in \mathbb{N},$$

a contradiction. Therefore the set-valued mapping $\partial f\colon X \rightsquigarrow X^*$ is calm at x, and thus $f \in \mathcal{F}(x)$.

Step 2. On the other hand, we suppose that $f \in \mathcal{F}(x)$. Assuming, on the contrary, that f is not ℓ-stable at x, there exist sequences $\{x_n\}_{n=1}^{+\infty}$ and $\{h_n\}_{n=1}^{+\infty} \subset S_X$ such that $\lim_{n\to+\infty} x_n = x$ and

$$\left|f^{\ell}(x_n;h_n) - \left\langle f_s'(x), h_n\right\rangle\right| \geq n\left\|x_n - x\right\|.$$

Since

$$f_\circ(x_n;h_n) \leq f^{\ell}(x_n;h_n) \leq f^\circ(x_n;h_n), \quad \forall\, n \in \mathbb{N},$$

using Lemma 1.2 we can find $x_n^* \in \partial f(x_n)$ satisfying

$$\left|\langle x_n^*, h_n\rangle - \left\langle f_s'(x), h_n\right\rangle\right| \geq n\left\|x_n - x\right\|, \quad \forall\, n \in \mathbb{N},$$

but it means that the set-valued mapping $\partial f\colon X^* \rightsquigarrow X$ is not calm at x, a contradiction. □

For $f \in \mathcal{F}(x)$ strict differentiability at x implies that f is continuous at x. Then Theorem 2.2 gives the following corollary.

Corollary 2.3. *Let $x \in X$, and let $f\colon X \to \mathbb{R}$ be a function. If $f \in \mathcal{F}(x)$, then f is ℓ-stable at x.*

3. Optimization results

In this section, we study the unconstrained problem

$$\text{optimize}\,\{f(x); x \in X\}\,,$$

where $f \in \mathcal{F}(x)$ (due to Theorem 2.2 it means that f is ℓ-stable at x_0). First, we remind the sufficient optimality conditions obtained for finite dimension in [18] and [3] respectively.

As obviously, we say that x_0 is a strict local minimum of $f\colon X \to \mathbb{R}$ if there exists a neighbourhood U of x_0 such that $f(x) > f(x_0)$ for every $x \in U$.

Further, we say that x_0 is an isolated minimizer of second-order if there are a neighbourhood U of x_0 and an $A > 0$ satisfying

$$f(x) \geq f(x_0) + A\,\|x - x_0\|^2\,.$$

In the result given by M. Sama the notion of contingent derivative of set-valued maps was used. Assuming that $F\colon X \rightsquigarrow Y$ and $(x_0, y_0) \in \operatorname{graph}(F)$, the contingent derivative $D_cF(x_0, y_0)$ of F at (x_0, y_0) is the set-valued map from X to Y defined by

$$\operatorname{graph}(D_cF(x_0, y_0)) = T(\operatorname{graph}(F), (x_0, y_0)),$$

where $T(\operatorname{graph}(F), (x_0, y_0))$ denotes the contingent cone to $\operatorname{graph}(F)$ at (x_0, y_0) [1]. By $\mathbb{R}_{++}$ we denote the set $\{t \in \mathbb{R}; t > 0\}$.

Theorem 3.1. [18, Proposition 6.3] *Let $f\colon \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $f \in \mathcal{F}(x_0)$. If $f_s'(x_0) = 0$ and*

$$D_c\,\langle \partial f, h\rangle\,(x_0, 0)(h) \subset \mathbb{R}_{++}, \quad \forall\, h \in \mathbb{R}^n,$$

then x_0 is a strict local minimizer of f.

For the result presented in [3] the following directional derivatives were used. We suppose that $f\colon X \to \mathbb{R}$ is a function and $x, h \in X$.

$$f'(x; h) = \lim_{t \downarrow 0} \frac{f(x + th) - f(x)}{t},$$
$$f_P'^{\ell}(x; h) = \liminf_{t \downarrow 0} \frac{f(x + th) - f(x) - tf'(x; h)}{t^2/2}.$$

We notice that for $f \in \mathcal{F}(x)$ we can write

$$f_P'^{\ell}(x; h) = \liminf_{t \downarrow 0} \frac{f(x + th) - f(x) - t\,\langle f_s'(x), h\rangle}{t^2/2}.$$

The result given in [3] we will formulate in terms of $\mathcal{F}(x)$ (with respect to its equivalence with ℓ-stability at x).

Theorem 3.2. [3, Theorem 6] *Let* $f\colon \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$ *and* $f \in \mathcal{F}(x_0)$. *If* $f'_s(x_0) = 0$ *and*

$$f''^{\ell}_P(x;h) > 0, \quad \forall\, h \in \mathbb{R}^n,$$

then x_0 *is an isolated minimizer of second-order.*

We would like to compare Theorems 3.1 and 3.2.

Proposition 3.3. *Let* $f\colon \mathbb{R}^n \to \mathbb{R}$, $x_0, h \in \mathbb{R}^n$ *and* $f \in \mathcal{F}(x_0)$. *If* $f'_s(x_0) = 0$ *and*

$$D_c \left\langle \partial f, h \right\rangle (x_0, 0)(h) \subset \mathbb{R}_{++},$$

then

$$f''^{\ell}_P(x;h) > 0.$$

Proof. Assume that

$$\liminf_{t \downarrow 0} \frac{f(x_0 + th) - f(x_0)}{t^2/2} \leq 0.$$

Then there exist sequences $\{t_n\}_{n=1}^{+\infty}$, $\{\varepsilon_n\}_{n=1}^{+\infty}$ such that $t_n > 0$, $\varepsilon_n > 0$ for every $n \in \mathbb{N}$, $\lim_{n\to+\infty} t_n = 0$, $\lim_{n\to+\infty} \varepsilon_n = 0$, and

$$\frac{f(x_0 + t_n h) - f(x_0)}{t_n^2} \leq \varepsilon_n, \quad \forall\, n \in \mathbb{N}. \tag{3.1}$$

By Lemma 1.3 we can find sequences $\{\theta_n\}_{n=1}^{+\infty}$, $\{x_n\}_{n=1}^{+\infty}$, and $\{x_n^*\}_{n=1}^{+\infty}$ such that

$$\begin{gathered} \theta_n \in (0,1), \quad \forall\, n \in \mathbb{N}, \\ x_n = x_0 + t_n \theta_n h, \quad \forall\, n \in \mathbb{N}, \\ x_n^* \in \partial f(x_n), \end{gathered} \tag{3.2}$$

and

$$f(z_n) - f(x_0) = \left\langle x_n^*, t_n h \right\rangle, \quad \forall\, n \in \mathbb{N}, \tag{3.3}$$

where $z_n = x_0 + t_n h$, $\forall\, n \in \mathbb{N}$. It follows from formulas (3.1) and (3.3) that

$$\frac{\langle x_n^*, h\rangle}{t_n} \leq \varepsilon_n, \quad \forall\, n \in \mathbb{N}. \tag{3.4}$$

Now, because ∂f is M-calm at x_0 and $\partial f(x_0) = \{0\}$, we have that

$$\|x_n^*\| \leq M t_n \theta_n, \quad \forall\, n \in \mathbb{N}. \tag{3.5}$$

We denote

$$s_n = t_n \theta_n, \quad \forall\, n \in \mathbb{N}. \tag{3.6}$$

Passing to a subsequence if necessary, we have that

$$\lim_{n\to+\infty} \frac{x_n^*}{s_n} = x^* \in (\mathbb{R}^n)^*. \tag{3.7}$$

We notice that

$$(x_n, \langle x_n^*, h\rangle) \in \operatorname{graph} \langle \partial f, h\rangle .$$

Using (3.2), (3.6) and (3.7), we obtain

$$\lim_{n\to+\infty} \frac{(x_n, \langle x_n^*, h\rangle) - (x_0, 0)}{s_n} = (h, \langle x^*, h\rangle).$$

The assumption

$$D_c \langle \partial f, h\rangle (x_0, 0)(h) \subset \mathbb{R}_{++}$$

gives that

$$\langle x^*, h\rangle > 0. \tag{3.8}$$

On the other hand, since $s_n < t_n$, formula (3.5) implies (passing to a subsequence again if necessary) that

$$\lim_{n\to+\infty} \frac{x_n^*}{t_n} = y^* \in (\mathbb{R}^n)^*.$$

Due to (3.4) we have

$$\langle y^*, h\rangle \le 0. \tag{3.9}$$

We denote

$$K := \frac{\|y^*\|}{\|x^*\|}.$$

Since

$$\frac{x_n^*}{s_n}, \frac{x_n^*}{t_n} \in \{\alpha x_n^*; \alpha > 0\}, \quad \forall\, n \in \mathbb{N},$$

we obtain that $y^* = Kx^*$, and thus inequality (3.9) implies

$$\langle x^*, h\rangle \le 0,$$

what is a contradiction with inequality (3.8). □

On account of Proposition 3.3 and Theorem 3.2 we are able to strengthen the assertion of Theorem 3.1.

Corollary 3.4. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $f \in \mathcal{F}(x_0)$. If $f_s'(x_0) = 0$ and*

$$D_c \langle \partial f, h\rangle (x_0, 0)(h) \subset \mathbb{R}_{++}, \quad \forall\, h \in \mathbb{R}^n,$$

then x_0 is an isolated minimizer of second-order.

Continuing in the comparison of Theorems 3.1 and 3.2, we present an example illustrating that Theorem 3.2 overcomes Theorem 3.1 (and Corollary 3.4).

Example 3.5. Consider a sequence $a_n = 1/n$, $n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} \frac{a_{n+1} + a_n^2}{a_{n+1} + a_n} = \frac{1}{2} > 0.$$

Let us define a function $\varphi\colon [0, +\infty) \to \mathbb{R}$ as follows (see Figure 3.1):

$$\varphi(u) = \begin{cases} a_1 & \text{if } u > a_1, \\ \frac{a_n^2 - a_{n+1}}{a_n - a_{n+1}}(u - a_{n+1}) + a_{n+1} & \text{if } u \in (a_{n+1}, a_n], \\ 0 & \text{if } u = 0. \end{cases}$$

Next, we will define a function $f\colon \mathbb{R} \to \mathbb{R}$ via the Riemann integral:

$$f(x) := \int_0^{|x|} \varphi(u)\, du, \quad x \in \mathbb{R}.$$

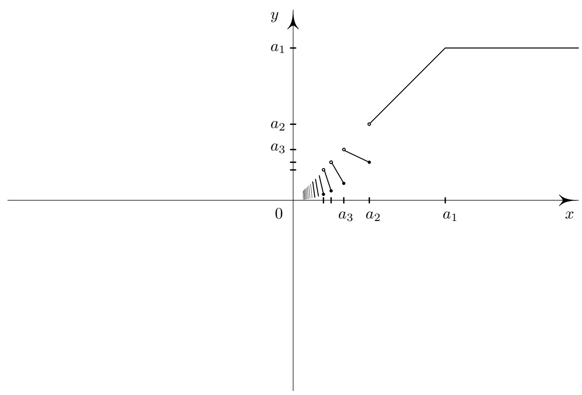

Figure 3.1: function φ.

It is easy to see that $f(0) = 0$ and it was shown in [3, Example 2] that $f_s'(0) = 0$, $f \in \mathcal{F}(0)$ and

$$\liminf_{t\downarrow 0} \frac{f(t)}{2/t^2} > \varepsilon$$

for some $\varepsilon > 0$. Therefore, due to Theorem 3.2 the considered function f attains an isolated minimizer of second-order at 0.

On the other hand, since

$$\varphi(a_n) = a_n^2 = \frac{1}{n^2} \in \partial f(a_n)$$

for every $n \in \mathbb{N}$, we have

$$0 \in D_c \langle \partial f, 1 \rangle (0, 0)(1).$$

But it means that we cannot use Corollary 3.4 (Theorem 3.1) to show that the function f attains an isolated minimizer of second-order (a strict local minimizer) at 0.

In the rest of this section we will deal with necessary optimality conditions in infinite dimension generally. We again compare the results given in terms of $D_c \langle \partial f, h\rangle (x_0, 0)$ and $f'^{\ell}_P(x_0; h)$.

M. Sama stated the following necessary optimality condition.

Theorem 3.6. [18, Proposition 6.1] *Let $f\colon X \to \mathbb{R}$, $x_0 \in X$, and $f \in \mathcal{F}(x_0)$. If x_0 is a local minimum of f, then $f'_s(x_0) = 0$ and*

$$D_c \langle \partial f, h\rangle (x_0, 0)(h) \cap \mathbb{R}_+ \neq \emptyset, \quad \forall\, h \in X.$$

We notice that if x_0 is a local minimum of $f \in \mathcal{F}(x_0)$, then by [9, Proposition 2.3.2] $\partial f(x_0) = f'_s(x_0) = 0$ and the definitional property of local minimum gives $f'^{\ell}_P(x_0; h) \geq 0$ for every $h \in X$. Thus, we can state the following theorem.

Theorem 3.7. *Let $f\colon X \to \mathbb{R}$, $x_0 \in X$, and $f \in \mathcal{F}(x_0)$. If x_0 is a local minimum of f, then $f'_s(x_0) = 0$ and*

$$f'^{\ell}_P(x_0; h) \geq 0, \quad \forall\, h \in X.$$

We will show gradually that the assertion of Theorem 3.6 follows from the assertion of Theorem 3.7, but not conversely.

Proposition 3.8. *Let $f\colon X \to \mathbb{R}$, $x_0, h \in X$, and $f \in \mathcal{F}(x_0)$. If $f'_s(x_0) = 0$ and*

$$f'^{u}_P(x_0; h) := \limsup_{t \downarrow 0} \frac{f(x_0 + th) - f(x_0) - t\langle f'_s(x_0), h\rangle}{t^2/2} \geq 0,$$

then

$$D_c \langle \partial f, h\rangle (x_0, 0)(h) \cap \mathbb{R}_+ \neq \emptyset.$$

Proof. There exist sequences $\{t_n\}_{n=1}^{+\infty}$, $t_n > 0$, $\lim_{n\to+\infty} t_n = 0$, and $\{\varepsilon_n\}_{n=1}^{+\infty}$, $\lim_{n\to+\infty} \varepsilon_n = 0$, such that

$$\frac{f(x_0 + t_n h) - f(x_0)}{t_n^2/2} \geq \varepsilon_n, \quad \forall\, n \in \mathbb{N}. \tag{3.10}$$

Since f is locally Lipschitz, we can find $\alpha_n \in (0, t_n)$ and $x_n^* \in \partial f(x_0 + \alpha_n h)$ such that

$$\langle x_n^*, t_n h\rangle = f(x_0 + t_n h) - f(x_0). \tag{3.11}$$

Due to the M-calmness property we have

$$\left\| \frac{x_n^*}{\alpha_n} \right\| \leq \frac{M}{\alpha_n} \alpha_n \|h\| = M \|h\|, \quad \forall\, n \in \mathbb{N}.$$

Then

$$\left\langle \frac{x_n^*}{\alpha_n}, h \right\rangle \leq M \|h\|^2, \quad \forall n \in \mathbb{N}.$$

Now, we can suppose without loss of generality that

$$\lim_{n\to+\infty} \left\langle \frac{x_n^*}{\alpha_n}, h \right\rangle = \lambda,$$

and it follows from formulas (3.10), (3.11) that

$$\left\langle \frac{x_n^*}{\alpha_n}, h \right\rangle \geq \left\langle \frac{x_n^*}{t_n}, h \right\rangle \geq \frac{\varepsilon_n}{2},$$

and thus

$$\lambda \in \mathbb{R}_+. \tag{3.12}$$

On the other hand, we notice that

$$(x_0 + \alpha_n h, \langle x_n^*, h\rangle) \in \operatorname{graph} \langle \partial f, h\rangle,$$

and thus

$$\lim_{n\to+\infty} \frac{(x_0 + \alpha_n h, \langle x_n^*, h\rangle) - (x_0, 0)}{\alpha_n} = (h, \lambda).$$

Therefore

$$\lambda \in D_c \langle \partial f, h\rangle (x_0, 0)(h). \tag{3.13}$$

It follows from formulas (3.12) and (3.13) that

$$(D_c \langle \partial f, h\rangle (x_0, 0)(h)) \cap \mathbb{R}_+ \neq \emptyset.$$ □

Since $f_P'^{\ell}(x_0; h) \leq f_P'^{u}(x_0; h)$, Proposition 3.8 implies immediately the following assertion.

Proposition 3.9. *Let $f\colon X \to \mathbb{R}$, $x_0, h \in X$, and $f \in \mathcal{F}(x_0)$. If $f_s'(x_0) = 0$ and $f_P'^{\ell}(x_0; h) \geq 0$, then*

$$D_c \langle \partial f, h\rangle (x_0, 0)(h) \cap \mathbb{R}_+ \neq \emptyset.$$

Thus, with respect to Proposition 3.9, Theorem 3.6 can be considered for the consequence of Theorem 3.7.

On the other hand, we will show an example, where we can use Theorem 3.7 to reject a certain point to be a local minimum in contrast to Theorem 3.6.

Example 3.10. Let us define a function $\varphi\colon [0,+\infty) \to \mathbb{R}$ as follows:

$$\varphi(u) = \begin{cases} u & \text{if } u \in (\frac{1}{n}, \frac{1}{n-1}] \text{ and } n \text{ is odd,} \\ -2u & \text{if } u \in (\frac{1}{n}, \frac{1}{n-1}] \text{ and } n \text{ is even.} \end{cases}$$

Now, we will define a function $f\colon \mathbb{R} \to \mathbb{R}$ via the Riemann integral:

$$f(x) = \int_0^{|x|} \varphi(u)\, du, \quad x \in \mathbb{R}.$$

It follows from the definition of φ that f is Lipschitz at 0 and, for example due to [9, Theorem 2.5.1], we can express ∂f by the following way. Since f is even, it holds $\partial f(-x) = -\partial f(x)$ for every $x \in \mathbb{R}$, and therefore we express $\partial f(x)$ explicitly only for $x \geq 0$.

$$\partial f(x) = \begin{cases} \{x\} & \text{if } x \in (\frac{1}{n}, \frac{1}{n-1}] \text{ and } n \in \mathbb{N} \text{ is odd,} \\ \{-2x\} & \text{if } x \in (\frac{1}{n}, \frac{1}{n-1}] \text{ and } n \in \mathbb{N} \text{ is even,} \\ [-2x, x] & \text{if } x = \frac{1}{n} \text{ and } n \in \mathbb{N}, \\ \{0\} & \text{if } x = 0. \end{cases}$$

Thus, $f'_s(0) = 0$ and $f \in \mathcal{F}(x_0)$. We notice that

$$D_c \langle \partial f, h\rangle (0,0) = \begin{cases} [-2h, h] & \text{if } h > 0, \\ 0 & \text{if } h = 0, \\ [2h, -h] & \text{if } h < 0. \end{cases}$$

Therefore $D_c \langle \partial f, h\rangle (x_0, 0)(h) \cap \mathbb{R}_+ \neq \emptyset$ for every $h \in \mathbb{R}$, and we cannot use Theorem 3.6 to reject the point 0 to be a local minimum of f.

On the other hand, since $f'^{\ell}_P(0;1) < 0$, Theorem 3.7 can be used to reject the point 0 to be a local minimum of f.

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