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# The Influence of Conjugacy Class Sizes on the Structure of Finite Groups

Ruifang Chen* and Xianhe Zhao

Abstract. Let $G$ be a group. The question of how certain arithmetical conditions on the sizes of the conjugacy classes of $G$ influence the group structure has been studied by many authors. In this paper, we investigate the influence of conjugacy class sizes of primary and biprimary elements on the structure of $G$. A criterion for a group to have abelian Sylow subgroups is given and some well-known results on Baer-groups are generalized.

## 1. Introduction

All groups considered in this paper are finite. Let $G$ be a group and $x$ be an element in $G$. We use $\left|x^G\right|$ to denote the conjugacy class size of $x$ in $G$ and $x$ is said to be primary or biprimary if the order of $x$ is a prime power or is divisible by exactly two distinct primes, respectively. Furthermore, we set $\rho'(G) = \big\{p \mid p$ is a prime and $p$ divides $\left|g^G\right|$ for some primary or biprimary element $g$ in $G\big\}$ and $\pi(G) = \{p \mid p$ is a prime and $p$ divides $|G|\}$. If $\pi$ is a set of primes contained in $\pi(G)$, we denote by $\rho c_\pi(G) = \big|\{p \mid p$ is a prime and $p$ divides $\left|g^G\right|$ for some primary or biprimary $\pi$-element $g$ in $G\}\big|$. All other notations are standard.

In finite group theory, a classic problem is to study how the set of conjugacy class sizes may determine properties of a group. For example, Burnside [3] showed that a group $G$ can not be simple if there is a non-central element $x$ in $G$ such that $\left|x^G\right|$ is a prime power and Itô [8] showed that if the conjugacy class sizes of elements in $G$ are 1 and $m$, where $m$ is an integer, then $G$ is nilpotent, $m = p^a$ for some prime $p$ and $G = P \times A$, with $P$ a Sylow $p$-subgroup of $G$ and $A \leq Z(G)$.

Recently, many authors attempt to obtain some properties of a group by replacing conditions for all conjugacy class sizes by conditions referring to conjugacy class sizes of some elements, and some useful results have been already obtained (see, e.g., [7, 9, 10]).

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*Corresponding author.

In this paper, we first attempt to investigate the relation between a group and the conjugacy class sizes of its primary or biprimary elements, and we get the following theorem, which generalizes some well-known results in [2, 5, 8].

**Theorem 1.1.** *Let $G$ be a $\pi$-separable group, where $\pi = \{p, q\}$ with $p$ and $q$ two distinct primes. If $p, q \in \rho'(G)$ and $pq$ does not divide $\left|x^G\right|$ for any primary or biprimary element $x$ in $G$, then $G$ has abelian Sylow $p$- and $q$-subgroups.*

On the other hand, inspired by a result of R. Baer [1], we investigate the structure of a group, the conjugacy class size of whose every primary or biprimary $\pi$-element is a prime power, where $\pi \subseteq \pi(G)$. In fact, we have the following theorem.

**Theorem 1.2.** *Let $G$ be a group and $\pi \subseteq \pi(G)$. Suppose that the conjugacy class size of every primary or biprimary $\pi$-element in $G$ is a prime power. Then $G$ is $\pi$-separable and $\rho c_\pi(G) \leq 2$. Furthermore, one of the following two possibilities occurs:*

(1) *the conjugacy class size of every primary or biprimary $\pi$-element in $G$ is a power of $q$, where $q$ is a prime and $q \notin \pi$. In this case, $G$ has abelian Hall $\pi$-subgroups.*

(2) *$G = H \times K$ with $H$ and $K$ a Hall $\pi$-subgroup and a $\pi$-complement of $G$, respectively. Moreover, $H$ has the following possibilities:*

(2a) *$H \leq Z(G)$;*

(2b) *$H = H_1 \times H_2$, where $H_1$ is a non-abelian Sylow subgroup of $H$ and $H_2 \leq Z(G)$;*

(2c) *$H = PQ \times T$, with $P$ and $Q$ abelian Sylow $p$- and $q$-subgroups of $G$ respectively, $Q \trianglelefteq H$, $P \cap P^h = O_p(H)$ for every $h \in H - N_H(P)$ and $T \leq Z(G)$.*

## 2. Preliminaries

**Lemma 2.1.** [2, Lemma 7] *Let $G$ be a $\pi$-separable group. Then $\left|x^G\right|$ is a $\pi'$-number for every $\pi$-element $x$ in $G$ if and only if $G$ has abelian Hall $\pi$-subgroups.*

**Lemma 2.2.** [6, Theorem 1] *Let $G$ be a group acting transitively on a set $\Omega$ with $|\Omega| > 1$. Then there exists a prime $p$ and a $p$-element $x$ in $G$ such that $x$ acts without fixed points on $\Omega$.*

**Lemma 2.3.** *Let $G$ be a group and $H$ be a subgroup of $G$. If $\bigcup_{g \in G} H^g$ contains all primary elements of $G$, then $G = H$.*

*Proof.* Set $\Omega = \{H^g \mid g \in G\}$. Then the conjugacy action of $G$ on $\Omega$ is transitive. Suppose to the contrary that $H < G$, then $|\Omega| > 1$. Therefore, by Lemma 2.2, there exists a prime $p$ and a $p$-element $x$ in $G$ such that $x$ acts without fixed points on $\Omega$, whence $H^{gx} \neq H^g$

for any $g \in G$. However, by hypothesis, there exists $H^h$ such that $x \in H^h$, and thus $H^{hx} = H^h$, which is a contradiction. □

**Lemma 2.4.** [5, Lemma 1.1] *Let $G$ be a group and $N$ be a normal subgroup of $G$. If $x \in N$ and $y \in G$, then*

(1) $\left|x^N\right| \mid \left|x^G\right|$;

(2) $\left|(yN)^{G/N}\right| \mid \left|y^G\right|$.

## 3. Proof of Theorem 1.1

In [8], the following theorem is given.

**Theorem 3.1.** [8, Proposition 5.1] *Let $G$ be a group and $p$, $q$ be two distinct primes. If $pq$ does not divide $\left|x^G\right|$ for any $x \in G$, then $G$ has abelian Sylow $p$- or $q$-subgroups.*

By just investigating $\{p, q\}$-elements of $G$, we have the following theorem.

**Theorem 3.2.** *Let $G$ be a $\pi$-separable group, where $\pi = \{p, q\}$ with $p$ and $q$ two distinct primes. If $pq$ does not divide $\left|x^G\right|$ for any $\pi$-element $x$ in $G$, then $G$ has abelian Sylow $p$- or $q$-subgroups.*

*Proof.* If $G$ is a $\pi$-group, then the theorem is true by Theorem 3.1. So we may assume that $G$ is not a $\pi$-group. Since $\pi = \{p, q\}$ and $G$ is $\pi$-separable, a Hall $\pi$-subgroup of $G$ is solvable. Therefore, $G$ is $p$-solvable and $q$-solvable. Furthermore, we may assume that $p \mid \left|x^G\right|$ for some $p$-element $x$ and $q \mid \left|y^G\right|$ for some $q$-element $y$. For otherwise, without loss of generality, we can assume that $p$ does not divide the conjugacy class size of any $p$-element in $G$, then $G$ has abelian Sylow $p$-subgroups by Lemma 2.1, and the theorem is true.

From the above paragraph, we may choose $x$ to be a $p$-element in $G$ such that $p$ divides $\left|x^G\right|$. Then $q$ does not divide $\left|x^G\right|$ by the hypothesis. Therefore, there exists a Sylow $q$-subgroup $Q$ of $G$ such that $Q \leq C_G(x)$. For every element $y \in Q$, we have $C_G(xy) = C_G(x) \cap C_G(y)$, and thus both $\left|x^G\right|$ and $\left|y^G\right|$ divide $\left|(xy)^G\right|$. So $p$ divides $\left|(xy)^G\right|$. The hypothesis implies that $q$ does not divide $\left|y^G\right|$. Again by Lemma 2.1, we see that the Sylow $q$-subgroups of $G$ are abelian and the proof of this theorem completes. □

**Corollary 3.3.** *Let $G$ be a $\pi$-separable group, where $\pi = \{p, q\}$ with $p$, $q$ two distinct primes. If $pq$ does not divide $\left|x^G\right|$ for any primary or biprimary element $x$ in $G$, then $G$ has abelian Sylow $p$- or $q$-subgroups.*

Now, we come to the proof of Theorem 1.1.

*Proof of Theorem* 1.1. It is obvious that $G$ is $p$-solvable and $q$-solvable.

By Corollary 3.3, we may assume that the Sylow $q$-subgroups of $G$ are abelian. Suppose that $G$ is a minimal counterexample with the Sylow $p$-subgroups non-abelian and we will provide a contradiction by the following two steps. Let $P$, $Q$ and $H$ be a Sylow $p$-subgroup, a Sylow $q$-subgroup and a $q$-complement of $G$, respectively.

*Step* 1: $O_{q'}(G) \neq 1$.

Suppose to the contrary that $O_{q'}(G) = 1$, then $O_q(G) \neq 1$ since $G$ is $q$-solvable. Therefore, $Q \leq C_G(O_q(G)) \leq O_q(G)$ since $Q$ is abelian, and thus $Q = O_q(G) \trianglelefteq G$. This follows that $C_G(Q) = Q$. For every primary element $x$ in $G$, since $pq$ does not divide $\left|x^G\right|$, we see that $C_G(x)$ contains a Sylow $p$-subgroup or a Sylow $q$-subgroup of $G$. Therefore, there exists $g \in G$ such that $x \in C_G(P^g) \cup C_G(Q) = C_G(P)^g \cup Q \subseteq C_G(P)^g Q = (C_G(P)Q)^g$. It follows that every primary element of $G$ is contained in $\bigcup_{g\in G}(C_G(P)Q)^g$. So $G = C_G(P)Q$ by Lemma 2.3. It is easy to get that $P \leq C_G(P)$, and thus $P$ is abelian, which is a contradiction.

*Step* 2: The contradiction.

Since $O_{q'}(G) \neq 1$ by Step 1, we can choose $N$ to be a minimal normal subgroup of $G$ which is contained in $H$. We use $\overline{G}$ to denote the group $G/N$. Then $pq$ does not divide the conjugacy class size of any primary or biprimary element in $\overline{G}$ by Lemma 2.4. If $p \notin \rho'(\overline{G})$, then $\overline{G}$ has abelian Sylow $p$-subgroups by Lemma 2.1, and so does $G$, which is a contradiction. If $q \in \rho'(\overline{G})$, then $\overline{G}$ has abelian Sylow $p$-subgroups since $G$ is a minimal counterexample, which also yields to the contradiction that $G$ has abelian Sylow $p$-subgroups. Therefore, $q \notin \rho'(\overline{G})$. It follows that the Sylow $q$-subgroup of $\overline{G}$, say $\overline{Q}$ is contained in $Z(\overline{G})$, and $\overline{G} = \overline{Q} \times \overline{H}$. Therefore, $H$ is normal in $G$ and $[G, QN] \leq N$. Since $[G, Q] > 1$, we see that $1 < [G, QN]$ and $[G, QN]$ is a normal subgroup of $G$. Notice that $N$ is a minimal normal subgroup of $G$, we have that $[G, QN] = N$. Let $C = \bigcup_{n\in N} C_N(Q)^n$. Since $QN \trianglelefteq G$, for every element $g \in G$, there exists $n \in N$ such that $Q^g = Q^n$, so $C = \bigcup_{g\in G} C_N(Q)^g$. If $C = N$, then $C_N(Q) = N$. Since $[Q, H] \leq [G, QN] = N$, we conclude that $Q$ acts trivially on $H/N$. Therefore, $(hN)^w = hN$ for every $h \in H$ and $w \in Q$, so there exists $n \in N$ such that $h = h^w n$. Let $|w| = t$. Then $h = hn^t$. It follows that $n = 1$ since $(|w|, |n|) = 1$, and thus $Q$ acts trivially on $H$. Now we have that $Q \leq Z(G)$, which is a contradiction. So $C < N$. Let $y$ be a primary element in $N - C$ such that $q$ divides $\left|y^G\right|$. Then the hypothesis implies that $p$ does not divide $\left|y^G\right|$. Choose $P^v$ to be a Sylow $p$-subgroup of $G$ which is contained in $C_G(y)$. Since $P^v$ is not abelian, there exists $z \in P^v$ such that $p$ divides $\left|z^G\right|$. Therefore, $pq$ divides $\left|(yz)^G\right|$, which is a contradiction. □

## 4. Proof of Theorem 1.2

In [1], R. Baer took interest in a group whose conjugacy class size of every primary element is a prime power, and we call such a group a Baer-group. In the same paper, he gave the structure of such a group. In 1998, A. R. Camina and R. D. Camina [4] investigated $q$-Baer groups, the conjugacy class sizes of whose $q$-elements, for just one prime $q$, are prime powers. Later, A. Beltrán and M. J. Felipe [2] investigated the structure of a group $G$ if the conjugacy class sizes of its $\pi$-elements are prime powers, where $\pi$ is a set of primes.

In the rest of this paper, we are just interested in the conjugacy class sizes of primary or biprimary $\pi$-elements for some primes set $\pi$, and we obtain Theorem 1.2.

*Proof of Theorem* 1.2. If $|\pi| = 1$, then the theorem is true by [4, Theorem A]. Therefore, we can suppose that $|\pi| > 1$. Furthermore, we can assume that $G$ does not have central Hall $\pi$-subgroup. Since $G$ is not simple, the hypothesis holds for every proper normal subgroup and every factor group of $G$, and thus every proper normal subgroup and every factor group of $G$ is $\pi$-separable by induction. It follows that $G$ is $\pi$-separable.

First suppose that there exists a prime $p \in \pi$ and a $p$-element $x \in G$ such that $\left|x^G\right|$ is a power of $q$ with $q$ a prime not in $\pi$. Then $C_G(x)$ contains a Sylow $r$-subgroup $R$ of $G$ for every prime $r \in \pi - p$. If $y \in R$, then $C_G(xy) = C_G(x) \cap C_G(y)$. Since $\left|(xy)^G\right|$ is a prime power and $\left|x^G\right|$ is a power of $q$, we see that $\left|y^G\right|$ is also a power of $q$. On the other hand, the conjugacy class size of every $p$-element is a power of $q$ by [4, Theorem 2]. Therefore, $G$ has abelian Hall $\pi$-subgroups by [11, Theorem 3.3].

Now, suppose that the conjugacy class size of every primary or biprimary $\pi$-element in $G$ is a $\pi$-number. Then by [11, Theorem 3.1], $G = H \times K$ with $H$ a Hall $\pi$-subgroup and $K$ a $\pi$-complement of $G$, respectively. Therefore, the conjugacy class size of every primary element of $H$ is a prime power by Lemma 2.4. It follows that $H$ is a Baer-group, and thus $H = H_1 \times \cdots \times H_t$ such that $(|H_i|, |H_j|) = 1$ for $i \neq j$ and that if $H_i$ is not of prime power order, then the order of $H_i$ is divisible by exactly two different primes and its Sylow subgroups are abelian by [1, Theorem]. If two direct factors, say $H_1$ and $H_2$, are not contained in $Z(G)$, then we can choose $x \in H_1$ and $y \in H_2$. It follows that $C_G(xy) = C_G(x) \cap C_G(y)$, and thus both $\left|x^G\right|$ and $\left|y^G\right|$ divide $\left|(xy)^G\right|$, which contradicts the fact that $\left|(xy)^G\right|$ is a prime power. Therefore, there is at most one direct factor of $H$ which is not contained in $Z(G)$. Now we conclude that the conjugacy class size of every $\pi$-element of $H$ is a prime power, and thus $H$ has the described structure by [5, Theorem 2]. □

**Corollary 4.1.** *Let $G$ be a group. Then the conjugacy class size of every primary or biprimary element of $G$ is a prime power if and only if the conjugacy class size of every element of $G$ is a prime power.*

*Proof.* Let $\pi = \pi(G)$ in Theorem 1.2. Then $G$ is abelian or $G = H \times K$, where $(|H|, |K|) = 1$, $K \leq Z(G)$ and $H$ is a Sylow subgroup of $G$ or $|H|$ is divisible by exactly two different primes and its Sylow subgroups are abelian. In particular, the conjugacy class size of every element in $G$ is a prime power. □

*Remark* 4.2. Let $n > 1$ be a natural number and $n = \prod_{i=1}^{k} p_i^{a_i}$, where $p_i$ are distinct primes and $a_i > 0$ for all $i = 1, 2, \ldots, k$. We define $\sigma(n) = k$. Furthermore, for a group $G$, we set $\sigma'(G) = \max_{g \in G^*} \sigma(|g^G|)$, where $G^* = \{g \in G \mid g \text{ is a primary or biprimary element}\}$. We see that $G$ is solvable if $\sigma'(G) = 1$. However, we cannot derive that $G$ is solvable, or we cannot even have that $G$ is non-simple if $\sigma'(G) = 2$ since $G = A_5$ is an example.

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Ruifang Chen and Xianhe Zhao
School of Mathematics and Information Science, Henan Normal University, Henan 453007, China
*E-mail address*: fang119128@126.com, zhaoxianhe989@163.com