On the Cohomologically Trivial Automorphisms of Elliptic Surfaces I:
$$\chi(S) = 0$$

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Dedicated to Professor Yurii (Gennadievich) Prokhorov on the occasion of his 60th
birthday

Abstract. In this first part we describe the group $\mathrm{Aut}_{\mathbb{Z}}(S)$ of cohomologically trivial automorphisms of a properly elliptic surface (a minimal surface S with Kodaira dimension $\kappa(S) = 1$), in the initial case $\chi(\mathcal{O}_S) = 0$.

In particular, in the case where $\mathrm{Aut}_{\mathbb{Z}}(S)$ is finite, we give the upper bound 4 for its cardinality, showing more precisely that if $\mathrm{Aut}_{\mathbb{Z}}(S)$ is nontrivial, it is one of the following groups: $\mathbb{Z}/2$, $\mathbb{Z}/3$, $(\mathbb{Z}/2)^2$. We also show with easy examples that the groups $\mathbb{Z}/2$, $\mathbb{Z}/3$ do effectively occur.

Respectively, in the case where $\mathrm{Aut}_{\mathbb{Z}}(S)$ is infinite, we give the sharp upper bound 2 for the number of its connected components.

1. Introduction

Let X be a compact connected complex manifold. Bochner and Montgomery [2,3] showed that the automorphism group $\mathrm{Aut}(X)$ (the group of biholomorphic maps $g\colon X \to X$, i.e., the group of diffeomorphisms $g \in \mathrm{Diff}(X)$ which preserve the complex structure of X) is a finite dimensional complex Lie group, possibly with infinitely many connected components, whose Lie algebra is the space $H^0(X, \Theta_X)$ of holomorphic vector fields on X.

The set $\mathrm{Aut}^0(X)$ denotes the connected component of the identity in $\mathrm{Aut}(X)$, and an important object of interest is the group of components $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$, which is in general at most countable.

For applications to Hodge theory and to period mappings quite important is the study of some larger subgroups, the group of numerically trivial automorphisms[1]

$$\mathrm{Aut}_{\mathbb{Q}}(X) := \{\sigma \in \mathrm{Aut}(X) \mid \sigma \text{ induces the trivial action on } H^*(X, \mathbb{Q})\},$$

and the group of cohomologically trivial automorphisms

$$\mathrm{Aut}_{\mathbb{Z}}(X) := \{\sigma \in \mathrm{Aut}(X) \mid \sigma \text{ induces the trivial action on } H^*(X, \mathbb{Z})\},$$

in which we are especially interested in this article.

It is an easy but important remark to observe that $\mathrm{Aut}_{\mathbb{Z}}(X) = \mathrm{Aut}_{\mathbb{Q}}(X)$ if $H^*(X, \mathbb{Z})$ has no torsion. In the case where $\dim(X) = 2$, this amounts by Poincaré duality (see Remark 2.7) to the condition that $H_1(X, \mathbb{Z})$ be a torsion free abelian group.

The case where X is a cKM (compact Kähler Manifold) endowed with a Kähler metric ω, was considered around 1978 by Lieberman [21] and Fujiki [17], in particular Lieberman [21] proved that $\mathrm{Aut}_{\mathbb{Q}}(X)/\mathrm{Aut}^0(X)$ is a finite group.

For surfaces S not of general type[2] the aim is to describe the group of numerically trivial automorphisms, this was begun in 1972 by Pjateckiĭ-Šapiro and Šafarevič [31] (see also Burns and Rapoport [5]) who proved that, for a K3 surface S, $\mathrm{Aut}_{\mathbb{Q}}(S)$ is a trivial group. Peters [29, 30] began the study of $\mathrm{Aut}_{\mathbb{Q}}(S)$ for compact Kähler surfaces. Automorphisms of surfaces were also investigated by Ueno [33] and Maruyama [25] in the 70's, then Barth and Peters [1], Mukai and Namikawa [28], Cai [8] and Mukai [27].

We are considering the case where S is minimal and the Kodaira dimension $\kappa(S) = 1$ (these are the so-called properly elliptic surfaces).

For these, we have a pluricanonical elliptic fibration $\Phi \colon S \to B$, and since $K_S^2 = 0$, by the Noether formula the basic invariant is $\chi(S) := p_g(S) - q(S) + 1$ such that $\chi(S) \geq 0$ (observe that the topological Euler number $e(S)$ is then determined by Noether's formula, asserting that $12\chi(S) = e(S)$).

A first easy but important tool is to observe that the group $\mathrm{Aut}(S)$ preserves this elliptic fibration. Our aim is to give upper bounds for the cardinality of the group $\mathrm{Aut}_{\mathbb{Z}}(S)$ (respectively $\mathrm{Aut}_{\mathbb{Z}}(S)/\mathrm{Aut}^0(S)$ if S admits vector fields) in terms of $\chi(S)$, hence in this first part we begin with the initial case $\chi(S) = 0$.

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[1] The reader should be aware that some authors called the numerically trivial automorphisms **automorphisms acting trivially on cohomology** without specifying that they meant cohomology with rational (and not integral) coefficients.

[2] For those of general type $\mathrm{Aut}(S)$ is a finite group and $|\mathrm{Aut}_{\mathbb{Q}}(S)|$ is universally bounded. In fact, $|\mathrm{Aut}_{\mathbb{Q}}(S)| \leq 4$ if $\chi(\mathcal{O}_S) \geq 189$ (see [7]). On the other hand, minimal surfaces of general type with $\chi(\mathcal{O}_S) < 189$ form a bounded family, and hence their full automorphism groups have a uniform bound. It is an open question whether there are surfaces of general type with $|\mathrm{Aut}_{\mathbb{Q}}(S)| > 25$.

If $\chi(S) = 0$ the surfaces, if minimal, are isogenous to a higher elliptic product, in the sense of [9] and [6, Chapter I], where the following fact[3] is proven.

Lemma 1.1. *Let S be a minimal surface with Kodaira dimension $\kappa(S) = 1$ and with $\chi(\mathcal{O}_S) = 0$. Then S is isogenous to a higher elliptic product, meaning that*

(1) $S = (C \times E)/\Delta_G$, *where C and E are smooth curves with $\mathrm{genus}(C) \geq 2$ and $\mathrm{genus}(E) = 1$, and G is a finite group acting faithfully on C and E such that the diagonal $\Delta_G \subset G \times G$ acts freely on $C \times E$.*

(2) *Moreover*

(I) *either G acts on E by translations, equivalently*

$$p_g(S) = \mathrm{genus}(C/G) = q(S) - 1 \iff \mathrm{genus}(E/G) = 1$$
$$\iff \mathrm{Aut}^0(S) \neq \{\mathrm{Id}_S\}$$

(II) *or G does not act on E only by translations, equivalently*

$$p_g(S) = q(S) - 1 = \mathrm{genus}(C/G) - 1 \iff \mathrm{genus}(E/G) = 0$$
$$\iff \mathrm{Aut}^0(S) = \{\mathrm{Id}_S\}.$$

(3) *The fibres of $f \colon S = (C \times E)/\Delta_G \to B = C/G$ are smooth elliptic or multiples of a smooth elliptic curves, that is, f is an elliptic quasi-bundle.*

In Case (I) of Lemma 1.1 one says that S is a **pseudo-elliptic surface**. If $S = (C \times E)/\Delta_G$, and the action is free, one can assume without loss of generality that G acts faithfully on C and on E, going to a minimal realization (see Remark 3.1).

The key problem now is whether $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathrm{Aut}^0(S)$. Our main results are summarized and reformulated as follows. Theorems 3.6 and 5.1 correct Cases (ii) and (iii) from [30] which correspond to Cases (II) and (I) of the previous lemma.[4]

Theorem 1.2. *Assume that S is a properly elliptic surface (S is minimal of Kodaira dimension 1) with $\chi(S) = 0$. Then $S = (C \times E)/G$ is isogenous to a higher elliptic product and*

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[3]One says instead, see [9], that S is isogenous to a higher product if S is isogenous to the product $C_1 \times C_2$ of two curves of respective genera at least 2.

[4]In Case (I) the flaw is an unproven assertion in the proof of Proposition 5.3, when treating Case (d) on page 265, lines 7–10. In Case (II) the analysis is more delicate, as shown in this paper. But, roughly speaking, the major flaw is to assert that if a branch point (downstairs) is fixed, then also a ramification point (upstairs) is fixed because $E \to E/G$ is the composition of an unramified morphism with a fully ramified morphism.

(I) *In the pseudo-elliptic case where* $\mathrm{Aut}^0(S)$ *is infinite, equivalently* G *acts by translations on* E, *the subgroup* $\mathrm{Aut}_{\mathbb{Z}}(S)$ *coincides with* $\mathrm{Aut}^0(S)$, *except for the pseudo-elliptic surfaces such that*

   (I-a) $G = \mathbb{Z}/(2m)$, *where* m *is an odd integer,*

   (I-b) $C/G = \mathbb{P}^1$ *and* $C \to \mathbb{P}^1$ *is branched in four points with local monodromies* $\{m, m, 2, -2\}$: *for these we have* $|\mathrm{Aut}_{\mathbb{Z}}(S)/\mathrm{Aut}^0(S)| = 2$.

(II) *If* $\mathrm{Aut}^0(S) = \{\mathrm{Id}_S\}$ *but* $\mathrm{Aut}_{\mathbb{Z}}(S) \neq \{\mathrm{Id}_S\}$, *then*

   (II-a) $B := C/G$ *has genus* $h \geq 1$.

   (II-b) $\mathrm{Aut}_{\mathbb{Z}}(S)$ *is isomorphic to one of the following groups:*

$$\mathbb{Z}/2, \quad \mathbb{Z}/3, \quad (\mathbb{Z}/2)^2.$$

   (II-c) *The first two cases:* $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathbb{Z}/2$, *and* $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathbb{Z}/3$, *do actually occur.*

Theorem 3.6 is devoted to Case (I), the pseudo-elliptic case where G acts on E by translations: there we describe $\mathrm{Aut}_{\mathbb{Q}}(S)$ explicitly, and show that the single notable exception in which $\mathrm{Aut}^0(S)$ has index two in $\mathrm{Aut}_{\mathbb{Z}}(S)$ does indeed exist: we just take the Galois covering described in (I-b) of the above theorem (which exists by Riemann's existence theorem) and show that the involution of $\mathbb{P}^1$ exchanging the first two branch points of $C \to B = \mathbb{P}^1$, and leaving the other two fixed, lifts to an automorphism of S.

The hard part of the proof is to show that this automorphism is cohomologically trivial: for showing this we have to perform calculations on the universal cover of S, which is the universal cover $\mathbb{H} \times \mathbb{C}$ of $C \times E$.

Observe that Theorem 3.1 of [13] uses surfaces in this class to produce examples, for all $n \in \mathbb{N}$, of pseudo-elliptic surfaces such that the index of $\mathrm{Aut}_{\mathbb{Z}}(S)$ in $\mathrm{Aut}_{\mathbb{Q}}(S)$ is at least n.

Moreover, Theorem 4.1 of [13] shows that if we drop the assumption of minimality, then there are blow ups of pseudo-elliptic surfaces such that $\mathrm{Aut}^0(S)$ becomes trivial, but $\mathrm{Aut}_{\mathbb{Z}}(S)$ does not: and we get examples, for all $n \in \mathbb{N}$, where $|\mathrm{Aut}_{\mathbb{Z}}(S)| = n$.

Case (II) where S is minimal with $\chi(S) = 0$, and where G does not act on E via translations is dealt with in Theorem 5.1. We have several concrete examples of automorphisms in $\mathrm{Aut}_{\mathbb{Z}}(S)$ as stated in (II-c): the simplest is one of order 3 and where the cohomology of S is torsion free.

We illustrate this main example in order to give a flavour of the arguments used. Firstly, a finite group G acting faithfully on an elliptic curve E is a semi-direct product $G = T \rtimes \mu_r$, where μ_r is the group of r-th roots of unity, $r \in \{2, 3, 4, 6\}$, and T is a finite group of translations on E, hence 2-generated.

To give instead the action of G on a curve C, it is equivalent to construct a Galois G-covering $(C \to B)$ of a curve B, determined (in view of the Riemann existence theorem) by a homomorphism onto G of the orbifold fundamental group, a quotient of the fundamental group $\pi_1(B \setminus \mathcal{B})$ of the complement in the curve B of the set $\mathcal{B}$ of branch points.

In this simple example there are no branch points, B has genus 2, and the group $G \cong \mathbb{Z}/3 \times \mu_3$. What is left is to give the monodromy, a surjection $\pi_1(B) \to G$, and we choose that the first two generators α_1, β_1 map to the two generators of G, while α_2, β_2 map to the identity.

In this way $S \to B$ is a fibre bundle with fibre E and it turns out that S is without torsion, indeed $H_1(S, \mathbb{Z}) \cong H_1(B, \mathbb{Z})$. It suffices then to take the group $H \cong \mathbb{Z}/3$ of automorphisms of $C \times E$ acting trivially on C and via translations by elements of T on E: these automorphisms induce automorphisms of S and clearly act trivially on $H^*(C \times E, \mathbb{Q})$: hence a fortiori they act trivially on the subspace of G-invariants, $H^*(C \times E, \mathbb{Q})^G = H^*(S, \mathbb{Q}) = H^*(S, \mathbb{Z})$.

We have more complicated examples (where $H_1(S, \mathbb{Z})$ has a nonzero torsion subgroup), and in order to prove Theorem 5.1 we need to run a long list of possible groups G and monodromy homomorphisms.

Finally, for applications to the Teichmüller space (stack) of X (see [11,26]) the primary object of interest is the subgroup of C^∞-isotopically trivial automorphisms

$$\mathrm{Aut}_*(X) := \{\sigma \in \mathrm{Aut}(X) \mid \sigma \in \mathrm{Diff}^0(X)\},$$

($\mathrm{Diff}^0(X)$ denotes the connected component of the identity in the group of diffeomorphisms), clearly contained in the group of homotopically trivial automorphisms

$$\mathrm{Aut}_\sharp(X) = \{\sigma \in \mathrm{Aut}(X) \mid \sigma \text{ is homotopic to } \mathrm{id}_X\}.$$

It is an intriguing question to find examples where these two subgroups differ.

The chain of subgroups

$$\mathrm{Aut}^0(X) \lhd \mathrm{Aut}_*(X) \lhd \mathrm{Aut}_\sharp(X) \lhd \mathrm{Aut}_\mathbb{Z}(X) \lhd \mathrm{Aut}_\mathbb{Q}(X) \lhd \mathrm{Aut}(X),$$

was discussed in our previous paper [13], especially dedicated to the case of complex dimension $n = 2$. In the present paper we prove in Proposition 5.6 that, for surfaces with Kodaira dimension 1 and $\chi(S) = 0$, $\mathrm{Aut}_\sharp(S) = \mathrm{Aut}^0(S)$.

The sequel to this paper, Part II [14], deals with cohomologically (and numerically) trivial automorphisms of properly elliptic surfaces S with $\chi(S) > 0$, showing in particular that $|\mathrm{Aut}_\mathbb{Q}(S)|$ is unbounded: this is a surprising result since for over 40 years it was believed that these groups are trivial for $p_g > 0$.[5]

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[5]Theorem 4.5 of [30] is confirmed by Corollary 0.3 of [8].

2. Preliminaries

2.1. Cohomology classes of multiple fibres

Lemma 2.1. *Let $f\colon S \to B$ be a fibred surface with $q_f := q(S) - g(B) = 0$. Then for two distinct multiple fibres m_1F_1 and m_2F_2, we have $[F_1] \neq [F_2] \in H^2(S, \mathbb{Z})$.*

Proof. Suppose on the contrary that $[F_1] = [F_2] \in H^2(S, \mathbb{Z})$. Then by the long exact sequence associated to the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 0$, we have $\mathcal{O}_S(F_1 - F_2) \in \mathrm{Pic}^0(S)$. Now $q_f = 0$ implies that the map $f_*\colon \mathrm{Alb}(S) \to \mathrm{Alb}(B)$ is an isomorphism. Therefore, its dual $f^*\colon \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(S)$ is also an isomorphism. Thus there is some $L \in \mathrm{Pic}^0(B)$ such that $f^*L = \mathcal{O}_S(F_1 - F_2)$. On the other hand, since $\mathcal{O}_S(F_1 - F_2)|_{F_1} \not\cong \mathcal{O}_{F_1}$, it cannot be of the form f^*L, which is a contradiction. $\qquad\square$

Corollary 2.2. *Let $f\colon S \to B$ be a fibred surface with $q(S) = 0$. Then for two distinct multiple fibres m_1F_1 and m_2F_2, we have $[F_1] \neq [F_2] \in H^2(S, \mathbb{Z})$.*

Remark 2.3. If $q_f > 0$, it is possible that $[F_1] = [F_2] \in H^2(S, \mathbb{Z})$, see Theorem 3.6. However, F_1 and F_2 cannot be linearly equivalent.

The following lemma is Principle 3, page 185 of [13].

Lemma 2.4. *Let $f\colon S \to B$ be a fibred surface, let $F'' = mF'$ be a multiple fibre with F' irreducible, and let σ be a cohomologically trivial automorphism of S. Then $\sigma(F') = F'$ unless possibly if the genus of B is at most 1, $m = 2$, there are only two multiple fibres with multiplicity 2, they are isomorphic to each other, and all the other multiple fibres have odd multiplicity.*

2.2. Integral first homology of an elliptic surface which is not a quasi-bundle

Here we say, as customary, that an elliptic surface is a quasi-bundle (as in [32]) if F_{red} is smooth for all fibres F.

In the contrary case, we have the following lemma which is essentially known (see [15, 32], but we state it and sketch its proof for the sake of clarity).

Lemma 2.5. *Let $f\colon S \to B$ be an elliptic surface with a singular fibre which is not a smooth multiple fibre (this is equivalent to the assumption that $e(S) > 0$, and in particular this holds if f admits a section and has a singular fibre). Then $H_1(S)$ is the abelianization of the orbifold fundamental group of the fibration,*

$$H_1(S) = H_1^{\mathrm{orb}}(f) := H_1(B, \mathbb{Z}) \oplus \left(\bigoplus_i (\mathbb{Z}/m_i)\gamma_i \right) \Big/ \mathbb{Z}\left(\sum_i \gamma_i \right).$$

Proof. Let g be the genus of B. Then we have the orbifold fundamental group exact sequence for f, see for instance [10]:

$$(2.1) \qquad \pi_1(F) \to \pi_1(S) \to \pi_1^{\mathrm{orb}}(f) \to 1,$$

where $m_1, \ldots, m_r$ are the multiplicities of the multiple fibres, and $\gamma_1, \ldots, \gamma_r$ are simple geometric loops around the branch points,

$$\pi_1^{\mathrm{orb}}(f) := \left\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_r \;\middle|\; \prod_i [\alpha_i, \beta_i]\gamma_1 \cdots \gamma_r = 1,\ \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = 1 \right\rangle.$$

Remark 2.6. In general, to an exact sequence of groups

$$A \to B \to C \to 1$$

corresponds another exact sequence

$$A^{ab} := A/[A, A] \to B^{ab} := B/[B, B] \to C^{ab} := C/[C, C] \to 0.$$

From this remark and (2.1) we deduce an exact sequence

$$H_1(F, \mathbb{Z}) = \pi_1(F) \to H_1(S, \mathbb{Z}) \to (\pi_1^{\mathrm{orb}}(f))^{ab} = H_1^{\mathrm{orb}}(f) \to 0.$$

The assumption made on the existence of a singular fibre which is not multiple of a smooth curve, and Kodaira's analysis of local monodromies implies by [15, Lemma 1.39] that the image of $H_1(F, \mathbb{Z})$ is zero in $H_1(S, \mathbb{Z})$, hence it follows then that $H_1(S, \mathbb{Z}) \cong (\pi_1^{\mathrm{orb}}(f))^{ab} = H_1^{\mathrm{orb}}(f)$ as claimed. $\qquad\square$

Remark 2.7. Lemma 2.5 determines in particular the torsion subgroup of $H_1(S, \mathbb{Z})$, which, by Poincaré's duality, is also the torsion subgroup of $H^3(S, \mathbb{Z})$. Moreover there is a canonical isomorphism of the respective torsion subgroups of $H^2(S, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$ and of $H_1(S, \mathbb{Z})$ (see [18, Theorem 23.18]).

Since this isomorphism is canonical, and is associated to the complexes of groups of singular chains and cochains, on which any homeomorphism acts compatibly, an automorphism which acts as the identity on the torsion subgroup of $H_1(S, \mathbb{Z})$ will also act as the identity on the torsion subgroup of $H_2(S, \mathbb{Z})$. Finally, to verify that an automorphism is cohomologically trivial, it suffices to verify that it acts as the identity on $H_1(S, \mathbb{Z})$ and on $H^2(S, \mathbb{Z})$.

The next section is devoted to the case where F_{red} is smooth for all fibres F.

2.3. Further notations

Given a variety X, we shall often denote $\pi_1(X)$ simply by π_X. The fundamental group $\pi_1(S)$ of our surfaces S shall be denoted by Γ, especially when we are seeing S as a quotient of its universal covering $\widetilde{S} = \mathbb{H} \times \mathbb{C}$. We shall also consider more generally groups Γ acting on $\mathbb{H} \times \mathbb{C}$.

3. Surfaces isogenous to a higher elliptic product and their automorphisms

We consider in this section a surface S isogenous to a higher elliptic product, namely, a quotient

$$S = (C \times E)/\Delta_G,$$

where C and E are smooth curves with respective genera genus(C) =: $g \geq 2$ and genus$(E) = 1$, and G is a finite group acting faithfully on C and on E such that the diagonal $\Delta_G \subset G \times G$ acts freely on $C \times E$.

Remark 3.1. If $S = (C \times E)/\Delta_G$ and the action is free, we can assume without loss of generality that G acts faithfully on C and on E, going to a minimal realization. In fact, if the action is not faithful on E, the subgroup $G' := \mathrm{Ker}(G \to \mathrm{Aut}(E))$ is normal and we can replace C by C/G' since G' acts freely on C, also C/G' has genus at least 2.

If the action is not faithful on C, the subgroup $G'' := \mathrm{Ker}(G \to \mathrm{Aut}(C))$ acts freely on E, hence by translations, and we can replace E by the elliptic curve E/G''.

Our aim is to describe explicitly the group $\mathrm{Aut}(S)$, in order to make precise calculations of the action of automorphisms on cohomology groups.

3.1. Lifts of automorphisms to the universal covering

As a first step we recall in this particular case some general theory saying that, if $\widetilde{S}$ is the universal covering of S and $S = \widetilde{S}/\Gamma$, then the group $\mathrm{Aut}(S)$ is the quotient group of the normalizer of Γ inside $\mathrm{Aut}(\widetilde{S})$ by the subgroup $\Gamma \cong \pi_1(S)$; see (3.2).

Such a surface S admits two fibrations

$$(3.1) \qquad\qquad f \colon S \to C/G, \quad p \colon S \to E/G.$$

We have an exact sequence of fundamental groups

$$1 \to \pi_C \times \pi_E \to \pi_1(S) =: \Gamma \to G \to 1,$$

where $\pi_C := \pi_1(C)$, $\pi_E := \pi_1(E)$.

Elementary covering spaces theory shows that each self-map $\psi \colon S \to S$ lifts to the universal covering, which is here

$$q \colon \mathbb{H} \times \mathbb{C} \to S = (\mathbb{H} \times \mathbb{C})/\Gamma.$$

As in [12, p. 316], we consider the cyclic semigroup H generated by ψ, or any other group or semigroup of self-maps of S, and we consider the semigroup of lifts of elements of H, namely

$$H' := \{\psi' \colon \mathbb{H} \times \mathbb{C} \to \mathbb{H} \times \mathbb{C} \mid \exists \psi \in H \text{ such that } \psi' \circ q = q \circ \psi\}.$$

There is a short exact sequence of semigroups

$$1 \to \Gamma \to H' \to H \to 1.$$

Assume now that $\psi \colon S \to S$ is a homeomorphism. Then we let H be the cyclic group generated by ψ and there is a homomorphism

$$H \to \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$$

induced by conjugation by a lift ψ' of ψ. The fact that this action is defined only up to inner conjugation amounts to the fact that changing the base point x_0 to y_0 we get an isomorphism of fundamental groups $\pi_1(S, x_0) \cong \pi_1(S, y_0)$ which is only defined up to inner conjugation. In particular, the group $\mathrm{Aut}(S)$ is the quotient

(3.2) $$\mathrm{Aut}(S) = \widetilde{\mathcal{N}}_\Gamma/\Gamma,$$

where $\widetilde{\mathcal{N}}_\Gamma$ is the normalizer of Γ inside $\mathrm{Aut}(\mathbb{H} \times \mathbb{C})$.

Indeed any lift ψ' of an automorphism $\psi \in \mathrm{Aut}(S)$ yields a commutative diagram

$$
\begin{array}{ccc}
\mathbb{H} \times \mathbb{C} & \xrightarrow{\psi'} & \mathbb{H} \times \mathbb{C} \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\
S & \xrightarrow{\psi} & S
\end{array}
$$

where q is the quotient map. It follows that $\psi'\Gamma = \Gamma\psi'$, that is, $\psi' \in \widetilde{\mathcal{N}}_\Gamma$. Conversely, any $\psi' \in \widetilde{\mathcal{N}}_\Gamma$ descends to S.

Consider then the exact sequence

$$1 \to \Gamma \to \widetilde{\mathcal{N}}_\Gamma \to \mathrm{Aut}(S) \to 1,$$

where any lift ψ' of ψ defines, via conjugation, an automorphism of Γ which is well defined up to inner conjugation.

Take x_0 to be a base point on S and consider the fundamental group $\Gamma = \pi_1(S, x_0)$. Setting $y_0 := \psi(x_0)$, we have $\psi_* \colon \pi_1(S, x_0) \to \pi_1(S, y_0)$, and choosing a path δ from x_0 to y_0 we get a fixed isomorphism

$$\pi_1(S, y_0) \cong \pi_1(S, x_0),$$

obtained via conjugation by δ (we take δ to be the trivial path if $y_0 = x_0$) and, composing ψ_* with this isomorphism, we get an automorphism of Γ.

By changing the lift we can make the automorphism of Γ induced by conjugation equal to

$$\psi_* \colon \pi_1(S, x_0) \to \pi_1(S, y_0)$$

(see especially [12, p. 316], where H' is called the orbifold fundamental group associated to a properly discontinuous subgroup H of $\mathrm{Aut}(S)$. If the action of H is free, H' is the fundamental group of S/H, otherwise, if x_0 is a fixed point of H, it is the semidirect product $\Gamma \rtimes H$, where conjugation is given by the action of H on $\Gamma = \pi_1(S, x_0)$, where $\psi \mapsto \psi_*$).

This property defines a lift $\widetilde{\psi}$ of ψ, such that, considering the isomorphism ψ_* indicated above (depending on the choice of δ), and setting, for $\gamma \in \Gamma$, $\gamma' := \psi_*(\gamma)$,

$$\widetilde{\psi} \colon \mathbb{H} \times \mathbb{C} \to \mathbb{H} \times \mathbb{C}$$

enjoys the following important property

$$(3.3) \qquad\qquad \gamma' \circ \widetilde{\psi} = \widetilde{\psi} \circ \gamma$$

(observe that both left-hand side and right-hand side are lifts of ψ which take the same value on the same base point x_0' lying above x_0).

Write now

$$\widetilde{\psi}(t, z) = (\widetilde{\psi}_1(t, z), \widetilde{\psi}_2(t, z)) = (\widetilde{\psi}_1(t), \widetilde{\psi}_2(t, z)),$$

where the last equality follows from Liouville's theorem, which also implies that for $\gamma \in \Gamma$ (which is a lift of the identity of S)

$$\gamma(t, z) = (\gamma_1(t), \gamma_2(t, z)).$$

3.2. Useful formulae and lifts to $C \times E$

Here we want to give another description of $\mathrm{Aut}(S)$ as the quotient of the normalizer of the diagonal subgroup Δ_G (inside the automorphism group of $C \times E$) divided by Δ_G, leading to (3.6).

Observe that the condition (3.3) spells out as

$$(3.4) \qquad\qquad \widetilde{\psi}_1(\gamma_1(t)) = \gamma_1'(\widetilde{\psi}_1(t)),$$
$$(3.5) \qquad\qquad \widetilde{\psi}_2(\gamma_1(t), \gamma_2(t, z)) = \gamma_2'(\widetilde{\psi}_1(t), \widetilde{\psi}_2(t, z)).$$

Conversely, any such map $\widetilde{\psi}$ satisfying the above two equations descends to S.

Lemma 3.2. *The two subgroups π_E, $\pi_C \times \pi_E$ are invariant under the action of $\mathrm{Aut}(S)$.*

Proof. $f \colon S \to C/G$ is the pluricanonical map of S (Iitaka fibration of S), hence f is preserved by automorphisms.

f is a quasi-bundle, that is, all smooth fibres are isomorphic to E, and the product $C \times E$ is the normalization of the fibre product $C \times_{C/G} S$, where the covering $C \to C/G$

corresponds to the kernel of the monodromy map of $\pi_1(C^*) \to \mathrm{Aut}(E)$, where C^* is the complement of the critical values of f.

Hence each automorphism of S also acts on C and on $C \times E$; and since every holomorphic map of E into C is constant, it also acts on E. Therefore the subgroups π_E, $\pi_C \times \pi_E$ are invariant under the action of $\mathrm{Aut}(S)$. □

Looking at elements γ in the subgroup $\pi_E < \Gamma$, acting by translations on $\mathbb{C}$, we obtain from (3.3), since we have $\gamma(t, z) = (t, z + \gamma_2)$,

$$\widetilde{\psi}_2(t, z + \gamma_2) = \widetilde{\psi}_2(t, z) + \gamma_2'.$$

Hence $\partial \widetilde{\psi}_2(t, z)/\partial z$ is π_E-periodic, thus constant as a function of z, and as usual we can write

$$\widetilde{\psi}_2(t, z) = \lambda z + \phi(t),$$

and, for t fixed, $\widetilde{\psi}_2$ descends, as already claimed, to an automorphism of $E = \mathbb{C}/\pi_E$.

A similar calculation shows that $\gamma_2(t, z)$ is an affine function of z, $\gamma_2(t, z) = \lambda_\gamma z + c_\gamma(t)$ (here λ_γ is a constant).

Looking at elements γ in the subgroup $\pi_C < \Gamma$, we see that (3.4) says that $\widetilde{\psi}_1$ descends to an automorphism of $C = \mathbb{H}/\pi_C$, which we shall denote by ψ_1.

Hence by Lemma 3.2 $\widetilde{\psi} \in \widetilde{\mathcal{N}}_\Gamma$ descends to an automorphism Ψ of $C \times E$ and we conclude that

$$(3.6) \qquad \qquad \mathrm{Aut}(S) = \mathcal{N}_G/G,$$

where $\mathcal{N}_G$ is the normalizer of G inside $\mathrm{Aut}(C \times E)$. Thus Ψ is a lift of ψ to $C \times E$, and we record that

$$\Psi(x, z) = (\psi_1(x), \lambda z + \Phi(x)),$$

because $\phi(\gamma_1(t)) = \phi(t)$ for $\gamma_1 \in \pi_C$ implies that ϕ descends to a holomorphic map $\Phi \colon C \to E$.

Indeed, putting $g' := \Psi_*(g)$ for $g \in G$, we have that the conditions (3.4) and (3.5) read out as

$$g'\psi_1(x) = \psi_1(gx), \quad g'(\lambda z + \Phi(x)) = \lambda(gz) + \Phi(gx).$$

Observing then that, for $g \in G$, $g \mapsto g'$ is the effect of conjugating by Ψ, we obtain, setting $g(z) = \epsilon z + b$, that

$$(3.7) \qquad \qquad g'(z) = \epsilon z + \lambda b - \epsilon \Phi(\psi_1^{-1}(x)) + \Phi g(\psi_1^{-1}(x)),$$

hence condition (3.5) is equivalent, setting

$$(3.8) \qquad \qquad U(x) := \Phi(gx) - \epsilon \Phi(x),$$

to

$$U(x) = U(\psi_1(x)).$$

This formula is particularly interesting in the following first case (where $\epsilon = 1$, $\forall\, g$).

3.3. The case where G acts on the elliptic curve E by translations, and S is called a pseudo-elliptic surface

In this case, we fix a point of E, called 0, and thus E is a group which acts on S by translations, namely to $w \in E$ corresponds

$$\tau_w \colon C \times E \to C \times E, \quad \tau_w(x, z) := (x, z + w)$$

which descends to an automorphism of S, which we denote again by τ_w, and indeed we see easily that $\mathrm{Aut}^0(S) \cong E$. Moreover, since Γ centralizes π_E under this assumption, it follows that

$$\gamma_2(t, z) = z + c_\gamma(t)$$

for all elements $\gamma \in \Gamma$.

Proposition 3.3. *Assume that $S = (C \times E)/\Delta_G$ is a pseudo-elliptic surface, that is, G acts on the elliptic curve E via translations. Then $\mathrm{Aut}^0(S) \cong E$ and $\mathrm{Aut}(S) \cong \mathcal{N}_G/G$, where $\mathcal{N}_G$ is the normalizer of G inside $\mathrm{Aut}(C \times E)$. Moreover, letting $Z(\Delta_G)$ be the centre of $\Delta_G \cong G$, and letting $\mathcal{C}(G)$ be the centralizer of G inside $\mathrm{Aut}(C)$, then*

$$\mathrm{Aut}_{\mathbb{Q}}(S) \cong (E \times \mathcal{C}(G))/Z(\Delta_G) \quad \Longrightarrow \quad \mathrm{Aut}_{\mathbb{Q}}(S)/\mathrm{Aut}^0(S) \cong \mathcal{C}(G)/Z(G).$$

Proof. The description of $\mathrm{Aut}(S)$ was already given before.

Assume first that ψ is numerically trivial. Then the fibration $S \to E/G$ is preserved, hence $\Phi(x)$ must be constant. Up to using the subgroup $E \cong \mathrm{Aut}^0(S)$ (the elements of E satisfy $\lambda = 1$, $\Phi = \mathrm{const}$) we may assume that $\Phi = 0$. We are then left with automorphisms ψ such that $\Psi(x, z) = (\psi_1(x), \lambda z)$.

Now, the fibration $S \to E/G$ has all the fibres smooth, hence there is an exact sequence

$$1 \to \pi_C \to \Gamma \to \pi_1(E/G) =: \pi'_E \to 1.$$

Since π_C and π_E commute, the homomorphism of $\pi'_E \to \mathrm{Out}(\pi_C)$ factors through the surjection $\pi'_E \to G$ and the action of G on π_C. Hence, abelianizing, for the first homology of S we have an exact sequence (see [32, Proposition 2.2])

$$0 \to \mathbb{Z}/m \to H_1(C, \mathbb{Z})_G \to H_1(S, \mathbb{Z}) \to \pi'_E \to 0,$$

where $H_1(C, \mathbb{Z})_G$ is the group of coinvariants (the quotient of $H_1(C, \mathbb{Z})$ by the subgroup generated by $\{gv - v\}$), and m is the largest integer dividing the class in $H^2(S, \mathbb{Z})/H^2(S, \mathbb{Z})_{\text{tor}}$ of a fibre of $p\colon S \to E/G$.

Since ψ is numerically trivial, it acts on the fibration p hence on this exact sequence and we conclude that it acts trivially on π'_E, since it acts trivially on $\pi'_E \otimes \mathbb{Q}$ (this amounts to saying that $\lambda = 1$; alternatively this can be deduced using the Künneth formula for $H^1(S, \mathbb{Q}) = H^1(C, \mathbb{Q})^G \oplus H^1(E, \mathbb{Q})^G = H^1(C, \mathbb{Q})^G \oplus H^1(E, \mathbb{Q}))$.

We can thus continue assuming that $\lambda = 1$, $\Phi = 0$. Then $\psi_1(gx) = g\psi_1(x)$, which precisely means that $\psi_1 \in \operatorname{Aut}(C)$ centralizes G. Hence the group of numerically trivial automorphisms is a quotient group of the direct product $E \times \mathcal{C}(G)$. The kernel of the surjection is the intersection with Δ_G: but $(\psi_1(x), z + u)$ is an element of Δ_G if and only if $u = g$ and $\psi_1 = g$, hence our assertions concerning numerically trivial automorphisms follow. $\qquad\square$

Remark 3.4. Proposition 3.3 allows to give examples of arbitrarily large quotient groups $\operatorname{Aut}_{\mathbb{Q}}(S)/\operatorname{Aut}^0(S)$, as already done in [13].

Remark 3.5. In the proof we saw more concretely that

$$\operatorname{Aut}(S) = \operatorname{Aut}^S(C \times E)/\Delta_G,$$

where

$$\operatorname{Aut}^S(C \times E) := \big\{ \Psi \mid \Psi(x, z) = (\psi_1(x), \lambda z + \Phi(x)), \lambda \in \mathbb{C}, \lambda \in \operatorname{Aut}(E, 0),$$
$$\lambda G = G \subset E, g'\psi_1(x) = \psi_1(gx), U(x) = U(\psi_1(x)) \big\},$$

g' is as in (3.7), and U is as in (3.8). And that $\mathcal{C}(G)$ is the subgroup $\mathcal{H}$ of the elements for which $\lambda = 1$, $\Phi = 0$.

Theorem 3.6. *Assume that S is a pseudo-elliptic surface, that is,*

$$S = (C \times E)/\Delta_G$$

is isogenous to a higher elliptic product, and G acts on the elliptic curve E via translations. Then $\operatorname{Aut}^0(S) \cong E$ and the subgroup $\operatorname{Aut}_{\mathbb{Z}}(S)$ coincides with $\operatorname{Aut}^0(S)$, except exactly when we have pseudo-elliptic surfaces such that

(i) *$G = \mathbb{Z}/(2m)$, where m is an odd integer, and*

(ii) *$C/G = \mathbb{P}^1$ with $C \to \mathbb{P}^1$ branched in four points with local monodromies $\{m, m, 2, -2\}$. For these surfaces we have*

$$|\operatorname{Aut}_{\mathbb{Z}}(S)/\operatorname{Aut}^0(S)| = 2$$

and $\operatorname{Aut}_{\mathbb{Z}}(S)$ contains an involution permuting the first two multiple fibres.

Proof. Using the notation of Proposition 3.3 and its proof we start with a preliminary observation: if we can show that ψ_1 acts as the identity on C/G, then ψ_1 is a lift of the identity, therefore ψ_1 is an element $g_0 \in G$. But then the product of Ψ and τ_{g_0} belongs to Δ_G, hence we conclude that $\psi \in \mathrm{Aut}^0(S)$.

Let us then see when the condition that ψ is cohomologically trivial implies that ψ_1 acts as the identity on C/G.

Since ψ acts trivially on the cohomology groups of the quotient C/G, it acts trivially on C/G if C/G has genus ≥ 2, and by translations on C/G if C/G has genus 1. Hence we are reduced to considering the two cases where C/G has genus 1, respectively where $C/G \cong \mathbb{P}^1$.

In both cases we can use Lemma 2.4 (= Principle 3 of [13]), saying that a permutation of two branch points is only possible if the local (G-valued)-monodromies are the same, have order 2, and moreover all other monodromies have odd order.

We begin now with the hard part of the proof, which actually requires doing calculations on the universal covering: because, if we have a translation t on an elliptic curve $E = \mathbb{C}/\Lambda$, and pick a specified lift of t to $\mathbb{C}$, a translation $z \mapsto z + x$, in order to infer that $x = 0$ it is not sufficient to show that the translation t is trivial.

3.4. Step (genus 1)

Assume now that C/G has genus 1. Since the action of ψ_1 on C/G is via a translation, it follows that if two branch points are permuted, then they have the same monodromies of order 2, they differ by translation by a 2-torsion element, and there are no more branch points than these two.

In order to proceed with our analysis, let us describe the group $H_1(S, \mathbb{Z})$ in general. We have an exact sequence $1 \to \pi_C \to \pi_1^{\mathrm{orb}}(C/G) \to G \to 1$ with

$$(3.9) \quad \pi_1^{\mathrm{orb}}(C/G) = \left\langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \gamma_1, \ldots, \gamma_r \;\middle|\; \prod_i [\alpha_i, \beta_i]\gamma_1 \cdots \gamma_r = 1, \; \gamma_j^{m_j} = 1, \forall j \right\rangle$$

where h denotes the genus of C/G, $\{\alpha_1, \beta_1, \ldots, \alpha_h, \beta_h\}$ is a symplectic basis of $H_1(C/G, \mathbb{Z})$, γ_j is the image of a geometric loop around the branch point p_j, and m_j is the order of the local monodromy at p_j. Set also for convenience $\pi^{\mathrm{orb}} := \pi_1^{\mathrm{orb}}(C/G)$. Then Γ is a subgroup of $\pi^{\mathrm{orb}} \times \pi'_E$, which has a surjection onto $G \times G$, and $\Gamma = \pi_1(S)$ is the subgroup inverse image of $\Delta_G < G \times G$.

Since G is abelian, $\Delta_G = \mathrm{Ker}(G \times G \to G)$, via the surjection $(g_1, g_2) \mapsto g_1 - g_2$. Hence also $\Gamma = \mathrm{Ker}(\pi^{\mathrm{orb}} \times \pi'_E \to G)$. The surjection $\Gamma \to \pi^{\mathrm{orb}}$ has kernel $\{1\} \times \Lambda$, where $\Lambda := \pi_E$. In particular $\Gamma \subset \pi^{\mathrm{orb}} \times \pi'_E$ is generated by $\{1\} \times \Lambda$, and by lifts to Γ of generators of π^{orb}, namely of

$$\gamma_1, \ldots, \gamma_r, \; \gamma_{r+1} := \alpha_1, \; \gamma_{r+2} := \beta_1, \ldots, \gamma_{r+2h} := \beta_h,$$

acting on $C \times E$ and whose further lifts to $\mathbb{H} \times \mathbb{C}$ have the form

$$\gamma_j'(t, z) = (\gamma_{j,1}'(t), z + x_j),$$

where the translation $z \mapsto z + x_j$ is in π_E', and m_j is the smallest multiple such that $m_j x_j \in \pi_E$.

Observation 3.7. The first homology group $H_1(S, \mathbb{Z})$, the abelianization of Γ, is generated by the image of $\pi_E = \Lambda$ and by the images of $\gamma_1', \ldots, \gamma_{r+2h}'$.

Moreover, setting $H^{\mathrm{orb}} := (\pi^{\mathrm{orb}})^{ab}$, we get

$$H^{\mathrm{orb}} = H_1^{\mathrm{orb}}(C, \mathbb{Z}) = \left(\bigoplus_{j=1}^{h} \mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{h} \mathbb{Z}\beta_j \oplus \bigoplus_{i=1}^{r} (\mathbb{Z}/m_i)\gamma_i \right) \Big/ \left\langle \sum_j \gamma_j \right\rangle,$$

and we obtain a homomorphism

$$H_1(S, \mathbb{Z}) = \Gamma^{ab} \to H^{\mathrm{orb}} \times \pi_E'.$$

The exact sequence

$$1 \to \Lambda \to \Gamma \to \pi^{\mathrm{orb}} \to 1$$

yields an exact sequence

$$0 \to \Lambda \to H_1(S, \mathbb{Z}) \to H^{\mathrm{orb}} \to 0,$$

because every commutator in Γ is a commutator in π^{orb}.

By the same argument we have an inclusion

$$H_1(S, \mathbb{Z}) \subset H^{\mathrm{orb}} \times \pi_E' = \left(H_1(B, \mathbb{Z}) \oplus \left(\left(\bigoplus_{i=1}^{r} (\mathbb{Z}/m_i)\gamma_i \right) \Big/ \left\langle \sum_j \gamma_j \right\rangle \right) \right) \times \pi_E',$$

because the commutators of $\pi^{\mathrm{orb}} \times \pi_E'$ consist of commutators in π^{orb}.

Finally, the generators γ_j' map to (γ_j, x_j). Hence we infer that if $\gamma_i' = \gamma_j'$ then $x_i = x_j$. The above description of $H_1(S, \mathbb{Z})$ is equivalent to the following presentation, given by Friedman and Morgan in [16, p. 200, Lemma 7.6 and Theorem 7.7] (but they obtained it just in the special case where the surface is obtained from a product $D \times E$ via logarithmic transformations):

$$H_1(S, \mathbb{Z}) \cong \left(H_1(B, \mathbb{Z}) \oplus \Lambda \oplus \left(\bigoplus_{i=1}^{r} \mathbb{Z}\gamma_i \right) \right) \Big/ \left\langle \sum_j \gamma_j, m_i\gamma_i - m_i x_i, 1 \leq i \leq r \right\rangle,$$

where $\sum_i x_i = 0$. This condition will lead in most cases to a contradiction.[6]

In the case where C/G has genus 1, this is indeed a contradiction since then $x_1 = x_2$ and $x_1 + x_2 = 0$ implies that $x_1 = x_2 = 0$, a contradiction.

3.5. Step (genus 0)

In the case where $C/G \cong \mathbb{P}^1$, the action of ψ on C/G is the identity (and we are done) if there are at least 5 branch points, because by Lemma 2.4 only two branch points can be moved by ψ.

Since the number of branch points is at least 4, if ψ_1 does not act trivially on C/G, then exactly two local multiplicities are equal to 2, and it remains to consider the case where we have exactly four branch points and, since the sum of the local monodromies into the abelian group G is zero, we must have monodromies $g_1 = g_2$ of order 2, and $g_3 = -g_4$ of odd order m.

In the above notation, again we must have that if $\gamma_1' = \gamma_2'$ then $x_1 = x_2$ and conversely, if $x_1 = x_2$ then $\gamma_1' = (g_1, x_1) = (g_2, x_2) = \gamma_2'$. Hence, eliminating γ_2', we remain with the relations

$$2\gamma_1' + \gamma_3' + \gamma_4' = 0, \quad 2\gamma_1' = (0, 2x_1), \quad m\gamma_3' = (0, mx_3).$$

With the first we eliminate γ_4', and remain with the two relations $2\gamma_1' = (0, 2x_1)$, $m\gamma_3' = (0, mx_3)$.

Since however $2x_1 \in \Lambda$ is not divisible by 2 in Λ, and $mx_3 \in \Lambda$ is not divisible by any nontrivial divisor of m, it follows that $H_1(S, \mathbb{Z})$ is torsion free. Moreover, we have that $G = \mathbb{Z}/2 \oplus \mathbb{Z}/m \cong \mathbb{Z}/(2m)$.

3.6. Existence part

To show that this case occurs we construct a $G = \mathbb{Z}/(2m)$-covering C of $\mathbb{P}^1$ branched over four points with local monodromies $(m, m, 2, -2)$, and we consider an embedding of G as a subgroup of an elliptic curve E (thus acting by translations).

Clearly the involution ι which exchanges the first two branch points lifts to an involution σ of C. Then the corresponding $S := (C \times E)/\Delta_G$ has $q(S) = 1$, $\chi(S) = 0$, hence $p_g(S) = 0$.

We take the automorphism ψ of S induced by $\Psi := \sigma \times \mathrm{Id}_E$. Then its action on $H^1(S, \mathbb{Z}) \cong H^1(E/G, \mathbb{Z})$ is trivial. Moreover, since the Euler number of S is zero, $H^2(S, \mathbb{Z})$

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[6]Serrano (cf. [32, Proof of Theorem 4.1]) established a weaker result, the exact sequence of abelian groups, which we shall use later on

$$0 \to H_1(E, \mathbb{Z}) \to H_1(S, \mathbb{Z}) \to H_1^{\mathrm{orb}}(C, \mathbb{Z}) \to 0.$$

has rank 2, and, being torsion free, it is isomorphic to $\mathbb{Z}^2$. Since ψ preserves the two fibrations, it follows that it acts trivially on the cohomology of S.

Since any other automorphism ψ_1 that will not permute the two branch points with local monodromy of order 2 must act as the identity on C/G, and is therefore the identity, as we have argued before, our assertion that $|\operatorname{Aut}_{\mathbb{Z}}(S)/\operatorname{Aut}^0(S)| = 2$ is proven. This completes the proof of Theorem 3.6. $\qquad\square$

4. Surfaces isogenous to a higher elliptic product, where G does not act on the elliptic curve E by translations

We collect here some general observations and formulae concerning the numerically trivial and cohomologically trivial automorphisms for this class of surfaces.

A key point is that any automorphism ψ of S preserves the canonical elliptic fibration, and the first question is to determine when it acts trivially on the rational cohomology of S. This is easier, since the rational cohomology of S is the G-invariant part of the cohomology of $C \times E$. Whereas, there is no simple commutative algebra recipe for determining the integral cohomology of the quotient: because, commutative algebra and homological algebra provide spectral sequences which boil down to several exact sequences of abelian groups, such as $0 \to A_1 \to A_2 \to A_3 \to 0$.

But if an automorphism ψ acts trivially on A_1, A_3, it does not necessarily act trivially on the extension A_2. This is the reason why in most cases we need to calculate the fundamental group $\Gamma = \pi_1(S)$ and, for doing so, we need the orbifold fundamental groups of the coverings $C \to B = C/G$ and $E \to E/G$, and their monodromy homomorphisms onto G.

Let us consider first the group $\operatorname{Aut}_{\mathbb{Q}}(S)$ of numerically trivial automorphisms. Assume that ψ is numerically trivial, hence it preserves the two fibrations of S (this is shown more generally in Lemma 3.2 to be true for each automorphism); hence, again, the function Φ is a constant c, and we may write

$$\Psi(x, z) = (\psi_1(x), \lambda z + c).$$

Recalling the formulae for $g(z) = \epsilon z + b$ we get

$$\Psi_*(g)(z) =: g'(z) = (\epsilon z + \lambda b) \implies g'\psi_1(x) = \psi_1(gx), \ \Phi(gx) = \epsilon\Phi(x),$$

and we conclude from the last equality that $c = \epsilon c$.

We first look at the condition of a trivial action of ψ on the first cohomology group $H^1(S, \mathbb{Q}) = H^1(C, \mathbb{Q})^G$. This means again that ψ_1 acts trivially on the cohomology group of the quotient C/G, hence by Lefschetz' fixpoint formula trivially on C/G if the genus of C/G, which equals $q(S)$, is ≥ 2, and by translations on C/G if C/G has genus 1.

Observe that G is a semidirect product $T \rtimes \mu_r$, where T is a subgroup of translations of E, and $\mu_r \cong \mathbb{Z}/r$ is the group of r-th roots of unity ($r = 2, 3, 4, 6$), and that the surjection $G \to \mu_r$ yields the character ρ of the representation of G on $H^0(E, \Omega_E^1)$.

The further condition that ψ is cohomologically trivial, that is, it lies in $\text{Aut}_{\mathbb{Z}}(S)$, requires in particular that its action on $\text{Tors}(H_1(S, \mathbb{Z}))$ is trivial.

We shall see in the next section that in this latter case we may assume that ψ_1 is the identity, and that the automorphism $\psi_2(z) = \lambda z + c$ is an element of the centre of G.

Assume from now on that $\psi = (\psi_1, \psi_2)$, that ψ_1 is the identity and ψ_2 is in the centre of G. Then the action of ψ on the fundamental group Γ of S is such that, in view of the exact sequence

$$(4.1) \qquad 1 \to \pi_C \times \pi_E \to \pi_1(S) =: \Gamma \to G \to 1,$$

the induced action on the subgroup π_C and on the quotient G is trivial, while the action on π_E is via the quotient $G \to \mu_r$.

In particular, if ψ_2 is a translation then ψ acts trivially on kernel and cokernel of (4.1). Passing to the first homology group $H_1(S, \mathbb{Z})$ we have an induced exact sequence (see for instance [12, Section 6.7])

$$(4.2) \qquad H_1(C, \mathbb{Z})_G \oplus H_1(E, \mathbb{Z})_G \to H_1(S, \mathbb{Z}) \to G^{ab} \to 0.$$

Facts 4.1. (1) The left-hand homomorphism in (4.2) is injective when the group G is cyclic (which is not necessarily the case here).

(2) In general the kernel of the left-hand homomorphism in (4.2) is generated by the commutators of the lifts to Γ of a system of generators of G^{ab}.

Since ψ_1 is assumed to be the identity, the action on $H_1(C, \mathbb{Z})_G$ is trivial; moreover, since $\psi_2 \in G$, the action of ψ_2 on $H_1(E, \mathbb{Z})_G$ is the identity.

Remark 4.2. Knowing that the action is trivial on the subgroup $H' < H_1(S, \mathbb{Z})$ which is the image of $H_1(C, \mathbb{Z})_G \oplus H_1(E, \mathbb{Z})_G$ and on the cokernel G^{ab} is unfortunately not sufficient to conclude that the action is trivial on $H_1(S, \mathbb{Z})$.

To show triviality of the action of ψ on $H_1(S, \mathbb{Z})$ it is necessary and sufficient to show that ψ fixes the lifts to $H_1(S, \mathbb{Z})$ of a system of generators of G^{ab}. Hence, in order to see whether the action is trivial on $H_1(S, \mathbb{Z})$ we proceed as in the pseudo-elliptic case searching for a presentation of $H_1(S, \mathbb{Z})$ (see Section 5.4 for an explicit example).

Definition 4.3. We introduce now a new bit of notation. Write $G = T \rtimes \mu_r$, where T is a finite group of translations of the elliptic curve $E = \mathbb{C}/\Lambda$. Then there is an overlattice $\Lambda_T \supset \Lambda$ such that

$$T = \Lambda_T/\Lambda.$$

Moreover, the group G lifts to a group of affine transformations $\mathcal{G}$ of $\mathbb{C}$ such that $\mathcal{G}/\Lambda \cong G$, hence

$$\mathcal{G} = \Lambda_T \rtimes \mu_r.$$

We have indeed that $\mathcal{G} := \pi_1^{\mathrm{orb}}(E/G)$.

Recall the formula (3.9) and set

$$\pi^{\mathrm{orb}} := \pi_1^{\mathrm{orb}}(C/G)$$
$$= \left\langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \gamma_1, \ldots, \gamma_r \;\middle|\; \prod_i [\alpha_i, \beta_i]\gamma_1 \cdots \gamma_r = 1, \; \gamma_j^{m_j} = 1, \forall j \right\rangle$$

and set again

$$\Lambda := \pi_E \quad \text{and} \quad H^{\mathrm{orb}} := (\pi_1^{\mathrm{orb}}(C/G))^{ab}.$$

We have an inclusion

$$\pi_1(S) = \Gamma \subset \pi_1^{\mathrm{orb}}(C/G) \times \pi_1^{\mathrm{orb}}(E/G) = \pi^{\mathrm{orb}} \times \mathcal{G},$$

and the latter direct product group has a surjection onto $G \times G$, such that Γ is the subgroup inverse image of $\Delta_G < G \times G$. The above inclusion of Γ then leads to a homomorphism

$$H_1(S, \mathbb{Z}) \to H^{\mathrm{orb}} \times \mathcal{G}^{ab},$$

and since we have already described the first summand, we turn to $\mathcal{G}^{ab}$.

We have the exact sequences

$$1 \to \Lambda \to \mathcal{G} \to G \to 1 \quad \Longrightarrow \quad \Lambda_G \to \mathcal{G}^{ab} \to G^{ab} \to 0,$$

and moreover

$$\mathcal{G}^{ab} \cong (\Lambda_T)_{\mu_r} \oplus \mu_r.$$

Moreover, we have that μ_r acts on Λ, Λ_T as an automorphism of period exactly r. This means that they are both $\mathbb{Z}[x]/P_r(x)$-modules, where $P_r(x)$ is the r-th cyclotomic polynomial (e.g., $P_3(x) = x^2 + x + 1$, $P_4(x) = x^2 + 1$, $P_6(x) = x^2 - x + 1$); and since $\mathbb{Z}[x]/P_r(x)$ is a PID, they are free modules of rank 1 for $r \geq 3$. Whence easy calculations (essentially the same as the ones performed in Lemma 4.4) show that

$$\Lambda_G \cong \begin{cases} (\Lambda_T)_{\mu_2} \cong (\mathbb{Z}/2)^2 & \text{if } r = 2, \\ (\Lambda_T)_{\mu_3} \cong \mathbb{Z}/3 & \text{if } r = 3, \\ (\Lambda_T)_{\mu_r} = \mathbb{Z}/2 & \text{if } r = 4, \\ (\Lambda_T)_{\mu_r} = 0 & \text{if } r = 6. \end{cases} \quad \text{Hence} \quad \mathcal{G}^{ab} \cong \begin{cases} (\mathbb{Z}/2)^2 \times \mu_2 & \text{for } r = 2, \\ (\mathbb{Z}/3) \times \mu_3 & \text{for } r = 3, \\ (\mathbb{Z}/2) \times \mu_4 & \text{for } r = 4, \\ \mu_6 & \text{for } r = 6. \end{cases}$$

For the upcoming classification, we explicitly associate the group G with its centre $Z(G)$.

Lemma 4.4. *Let*

$$G = T \rtimes \mu_r,$$

be a group of (affine) automorphisms of an elliptic curve $E = \mathbb{C}/\Lambda$, where the 2-generated group $T = \Lambda_T/\Lambda$ is a group of translations and $r \in \{2, 3, 4, 6\}$. Let ϵ be a generator of μ_r, and write

$$T \cong \mathbb{Z}/m_1 \oplus \mathbb{Z}/m_2$$

with $m_2 \mid m_1$. More precisely, let $\{\eta_1, \eta_2\}$ be a basis of Λ_T, such that $\{m_1 \eta_1, m_2 \eta_2\}$ is a basis of Λ, where m_1, m_2 can be chosen arbitrarily for $r = 2$. For $r \geq 3$, writing $\Lambda_T = \mathbb{Z}[x]/P_r(x)$, then there is an element η such that $\Lambda = m_2 \eta \Lambda_T$.

Write $\eta = a' + b'x$ with $a', b' \in \mathbb{Z}$. Then $m := m_1/m_2$ is unbounded, since it equals, in the respective cases $r = 3, 4, 6$,

$$(a')^2 + (b')^2 - a'b', \quad (a')^2 + (b')^2, \quad (a')^2 + (b')^2 + a'b'.$$

Setting $A = \mu_r$, we have the following cases for $Z(G)$:

(1) *If $A = \mu_2$ then $Z(G)$ is isomorphic to a subgroup of $E[2] \times \mu_2$, where $E[2]$ denotes the subgroup of 2-torsion elements of E.*

(2) *If $A = \mu_3$ then $Z(G)$ is isomorphic to a subgroup of*

$$(\mathbb{Z}/3)\frac{1 - \omega}{3} \times \mu_3.$$

(3) *If $A = \mu_4$ then $Z(G)$ is a subgroup of*

$$(\mathbb{Z}/2)\frac{1}{2}(1 + i) \times \mu_4.$$

(4) *If $A = \mu_6$ then $Z(G)$ is a subgroup of $A = \mu_6$.*

Finally, either

(i) *$Z(G)$ is contained in T, the nontrivial possibilities being $(\mathbb{Z}/2)^2$, $\mathbb{Z}/3$, $\mathbb{Z}/2$, or*

(ii) *$G = Z(G)$, hence G is abelian and a subgroup of the above groups, or*

(iii) *we have the following group that we shall call **sporadic**:*

$$(4.3) \qquad G = E[2] \rtimes \mu_4, \quad \text{where } Z(G) = (\mathbb{Z}/2)\frac{1}{2}(1 + i) \times \mu_2.$$

With a view towards Theorem 1.2, we point out that $|Z(G)| > 4$ only occurs, in the maximal cases from (1)–(4), when G is abelian.

Proof of Lemma 4.4. Let us now view Λ as a μ_r-invariant $\mathbb{Z}$-submodule of Λ_T.

For $r = 2$, μ_r acts as multiplication by -1, hence we can choose Λ as an arbitrary submodule of Λ_T, and the assertion $T \cong \mathbb{Z}/m_1 \oplus \mathbb{Z}/m_2$ with $m_2 \mid m_1$ a special case of the theorem regarding the Frobenius normal form for finite abelian groups[7] (see Theorem 3.9 of [20]).

For $r \geq 3$, then $\Lambda \subset \Lambda_T = \mathbb{Z}[x]/P_r(x)$ is a principal ideal domain, associated to an element $a + bx$. Let m_2 be the divisibility index of $a + bx$ and write

$$(a + bx) := m_2 \eta = m_2(a' + b'x)$$

so that $\Lambda = m_2 \Lambda'$, where Λ' is the principal ideal domain generated by $\eta = (a' + b'x)$ (we then set $T' = \Lambda_T/\Lambda'$ so that $T' = T/(\mathbb{Z}/m_2)^2$). Clearly, a second $\mathbb{Z}$-generator of Λ is $x(m_2\eta)$.

According to $r = 3, 4, 6$, we have

$$x(m_2\eta) = -b + (a - b)x, \quad -b + ax, \quad -b + (a + b)x.$$

Clearly $m_2 = \gcd(a, b)$, while then the matrix determinant equals $m_1 m_2$, hence our assertion concerning $m = m_1/m_2$. More precisely, we have that $(b' + a'x)\eta = mx$, and then Λ' is generated by

$$\eta, \quad x\eta, \quad mx = (b' + a'x)\eta,$$

hence, if r, s are integers such that $ra' - sb' = 1$, also by $(r + sx)\eta$, mx (since $(b' + a'x)$, $(r + sx)$ are a basis for Λ_T). Hence, we may explicitly set

$$\eta_1 := x, \quad \eta_2 := (r + sx)\eta = 1 + x(sa' + b'r + sb'\delta), \quad \delta = -1, 0, 1.$$

Then

$$\eta_1 = x, \quad \eta_2 =: 1 + nx$$

are a basis of Λ_T, and $m_1\eta_1 = mm_2\eta_1$, $m_2\eta_2$ are a basis of Λ.

We now determine the centre $Z(G)$ of $G = T \rtimes \mu_r$. It consists of the elements $g = (y, c)$ with $y \in \mathrm{Ker}(1 - \epsilon)$, $T \subset \mathrm{Ker}(c - 1)$.

For $r = 2$, this means that $2y = 0$, and $c = 1$ unless $m_1 = 2$. This verifies Case (1). More precisely, it leads to the abelian case

$$Z(G) = G = T \times \mu_2 \quad \text{if } T \subset E[2]$$

and to $Z(G) = T \cap E[2] \subseteq (\mathbb{Z}/2)^2 \subsetneq G$ in case $T \not\subset E[2]$ (where of course G is not abelian).

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[7]also known as the invariant factor decomposition.

For $r \geq 3$, assume first that there is an element $(y, c) \in Z(G)$ such that c is a generator of μ_r. Then $(1 - x)T = 0 \in T$, that is,

$$(1 - x)\Lambda_T \subset m_2 \eta_2 \Lambda_T.$$

This immediately implies first of all that $m_2 = 1$, and moreover, since the ideal I generated by $1 - x$ has respective index 3, 2, 1 inside Λ_T, it follows that either $T = 0$, or η_2 and $1 - x$ generate the same ideal, hence we may in particular assume that $\eta_2 = 1 - x$. It follows in particular that G is abelian.

The only alternative, where c is never a generator of μ_r and $Z(G) \not\subset T$, is that $r = 4$ while c is equal to ± 1. This implies that $2T = 0$, and then the condition $y \in \mathrm{Ker}(1 - i)$ says that y is a multiple of $\frac{1}{2}(1 + i)$. Hence in this case $G \subset E[2] \rtimes \mu_4$ which is non-abelian if and only if the inclusion is an equality, and then $Z(G) = (\mathbb{Z}/2)\frac{1}{2}(1 + i) \times \mu_2$ as stated. To conclude, if $Z(G) \not\subset T$, then we are in Cases (2)–(4) with $G = Z(G)$, or in the sporadic case (iii) (see (4.3)).

Finally, in order to deal with the case where $Z(G)$ is contained in the group of translations[8], the condition that $y = [v] \in \mathrm{Ker}(1 - \epsilon)$ says that

$$Z(G) \cap T = \{v \mid (1 - x)v \in \Lambda\}/\Lambda.$$

For simplicity of notation, let us use the isomorphism $\Lambda \cong \mathbb{Z}[x]/P_r(x)$ whose inverse is multiplication by η. Then

$$v \in (1 - x)^{-1}\Lambda \in \Lambda \otimes \mathbb{Q},$$

and since

$$(1 - x)^{-1} = \frac{1}{3}(2 + x), \quad \frac{1}{2}(1 + x), \quad x,$$

respectively, we see in all three cases that $Z(G) \cap T$ is a respective subgroup of

$$(\mathbb{Z}/3)\frac{1}{3}(2 + x), \quad (\mathbb{Z}/2)\frac{1}{2}(1 + x), \quad 0.$$

Together with the analysis for $r = 2$, this verifies the final statement for the case where $Z(G) \subset T$. $\qquad \square$

Corollary 4.5. *Assume that G is not abelian, $r \geq 3$. Then the centre $Z(G)$ consists of a nontrivial subgroup of T if and only if*

- *$r = 3$ and $3 \mid (a + b) = m_2(a' + b')$ (then $Z(G) = \mathbb{Z}/3$),*

- *$r = 4$ and $2 \mid m_1$ (then $Z(G) = \mathbb{Z}/2$).*

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[8]For instance, $r = 2$ and $T = (\mathbb{Z}/n)^2$, $n \geq 3$.

Proof. We have seen in Lemma 4.4 that it must be $r = 3, 4$. Moreover, for $r = 3$, the centre $Z(G)$ consists of multiples of $\frac{1}{3}(2+x)\eta$. Hence it is nontrivial if and only if $\frac{1}{3}(2+x)\eta \in \Lambda_T$, equivalently if and only if

$$\frac{1}{3}m_2(2+x)(a'+b'x) = \frac{1}{3}m_2(2a'-b') + x(a'+b')$$

has integral coefficients. This however amounts to $3 \mid m_2$, or $3 \mid (a'+b')$, the latter condition meaning that $\eta \equiv \pm 1(1-x) \mod 3\Lambda_T$.

For $r = 4$, the condition of nontriviality of $Z(G)$ amounts instead to

$$\frac{1}{2}(1+x)\eta \in \Lambda_T \quad \Longleftrightarrow \quad \frac{1}{2}m_2(a'-b') + (a'+b')x.$$

This holds true if either $2 \mid m_2$, or $a' \equiv b' \pmod 2$, which is equivalent to $2 \mid m$. The conclusion is that this holds if and only if $2 \mid m_1$. □

4.1. Normal forms for the monodromies of $C \to B$ in the case G abelian or sporadic

Lemma 4.4 classifies the possible groups $Z(G)$ appearing, hence gives a non sharp upper bound $|\operatorname{Aut}_{\mathbb{Z}}(S)| \leq 9$ for the group of cohomologically trivial automorphisms.

In order to obtain the better bound $|\operatorname{Aut}_{\mathbb{Z}}(S)| \leq 4$ in Theorem 5.1, we need to exclude several groups G and to consider all the corresponding surfaces S, which are determined by the G-Galois covering $C \to C/G = B$.

These in turn, by the Riemann existence theorem, are determined by their branch sets and by the isomorphism classes of their monodromy homomorphisms. For this reason we have to describe the normal forms for these monodromies.

These are a finite number in the case where we want to exclude the possibility $|\operatorname{Aut}_{\mathbb{Z}}(S)| \geq 5$, but especially in the case where the genus of B is at least 2, the classes of the possible monodromies are an infinite set.

Yet we found it worthwhile to collect a significant set of low genus (for C) cases, in order to establish evidence for the difficulty of finding an example with $|\operatorname{Aut}_{\mathbb{Z}}(S)| = 4$.

We assume first that the group G is as in Lemma 4.4 and abelian (i.e., with $G = Z(G)$, as detailed in (1)–(4) of Lemma 4.4). We moreover assume that the base curve $B = C/G$ has genus $h \geq 1$, and we want to find a normal form for the monodromy, that is, for the surjection $\operatorname{Mon}: \pi^{\mathrm{orb}} \twoheadrightarrow G$, where

$$\pi^{\mathrm{orb}} := \left\langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \gamma_1, \ldots, \gamma_k \,\middle|\, \prod_i [\alpha_i, \beta_i]\gamma_1 \cdots \gamma_k = 1, \ \gamma_j^{m_j} = 1, \ \forall j \right\rangle,$$

and we assume that $t_j := \operatorname{Mon}(\gamma_j)$ has order precisely m_j.

Lemma 4.6. *The case $G = \mu_r$ ($r = 2, 3, 4, 6$) cannot occur for $h = 1$, and for $h \geq 2$ it leads to a trivial group $\operatorname{Aut}_{\mathbb{Z}}(S)$. For G abelian, since C has genus ≥ 2, the number of branch points satisfies $k \geq 2$ for $h = 1$.*

Proof. The first and third assertions are clear, because if $T = 0$, then the local monodromies must be trivial, hence there are no branch points for $C \to C/G = B$. But since we require that C has genus ≥ 2 and G is abelian, if B has genus $h = 1$ the number of branch points for $C \to C/G = B$ is at least 2.

For $h \geq 2$, we shall see in the proof of Theorem 5.1 that $\mathrm{Aut}_{\mathbb{Z}}(S) \subset T$, hence the second assertion follows. $\square$

In order to find a normal form for the possible monodromies, we can use in general the following transformations:

(a) replace a generator α_i by $\alpha_i' := \alpha_i \gamma_j^{\pm 1}$ (or similarly for β_i),

(b) replace α_i by $\alpha_i'' := \alpha_i \beta_i^{\pm 1}$ (or analogously $\beta_i'' := \beta_i \alpha_i^{\pm 1}$),

(c) change the indices $i = 1, \ldots, h$ via any permutation.

These operations (the second is the effect of a Dehn twist along a path β_i, α_i or its inverse) provide us with a different symplectic basis, and the complement of the corresponding loops is again a disk, whence we can complete to a new basis of the orbifold fundamental group by adding $\gamma_1', \ldots, \gamma_k'$ where each γ_j' is a conjugate of γ_j. In particular, if G is abelian, the local monodromies do not change.

Finally, there is a fourth operation (see [34, 2.3 on p. 250], where however the roles of a_i, b_i are reversed) which, in the case where G is abelian, exchanges the monodromies $a_i := \mathrm{Mon}(\alpha_i)$, $b_i := \mathrm{Mon}(\beta_i)$ via

(d) $a_i \mapsto a_i$, $b_i \mapsto b_i b_{i+1} a_{i+1}$, $b_{i+1} \mapsto b_{i+1} a_i^{-1}$, $a_{i+1} \mapsto a_{i+1} a_i^{-1}$.

Note that applying (b) successively allows to replace α_i by β_i^{-1} (and β_i by $\beta_i \alpha_i \beta_i^{-1}$, for instance).

Lemma 4.7. *Suppose that we are in the case where $G = T \times \mu_r$ is abelian, $r < 6$, $T \neq 0$, and all the local monodromies $\mathrm{Mon}(\gamma_j) =: t_j$ are translations of order $m_j \geq 2$. Then, after a change of system of geometric generators for π^{orb}, we may assume that $\mathrm{Mon}(\alpha_1)$ is a generator of μ_r and that $\mathrm{Mon}(\beta_j), \mathrm{Mon}(\alpha_i) \in T$, $\forall j$ and $\forall i \geq 2$. If the local monodromies t_j generate T, we moreover assume that $\mathrm{Mon}(\beta_j)$ and $\mathrm{Mon}(\alpha_i)$ are trivial for all j and for all $i \geq 2$.*

Every monodromy is then equivalent to one of the following normal forms for $h = 1, 2$.

(I) *if $h = 1$ or $h = 2$ and $\mathrm{Mon}(\alpha_2)$ and $\mathrm{Mon}(\beta_2)$ are trivial, then we have the subcases:*

(I-1) $\mathrm{Mon}(\beta_1)$ *is trivial, $k \geq 2$ and t_1, t_2 generate T, or*

(I-2) $k = 0$, $h = 2$, and $\mathrm{Mon}(\beta_1)$ generates T, or

(I-3) $r = 2$, $T = (\mathbb{Z}/2)e_1 \oplus (\mathbb{Z}/2)e_2$, $k \geq 2$ and we have

$$(4.4) \qquad \mathrm{Mon}(\alpha_1) = \epsilon = -1 \in \mu_2, \quad \mathrm{Mon}(\beta_1) = e_1, \quad t_j = e_2, \ \ \forall\, j.$$

(II-III) If $h = 2$, and we are not in Case (I), then either $k = 0$, or $k \geq 2$, $r = 2$, $T = (\mathbb{Z}/2)e_1 \oplus (\mathbb{Z}/2)e_2$. We have $\mathrm{Mon}(\alpha_1) = \epsilon \in \mu_2$ and either

(II) $k = 0$, $r = 2$, $T = (\mathbb{Z}/2)e_1 \oplus (\mathbb{Z}/2)e_2$,

$$(4.5) \qquad \mathrm{Mon}(\beta_1) = e_1, \quad \mathrm{Mon}(\alpha_2) = e_2, \quad \mathrm{Mon}(\beta_2) \in \{0, e_1\}; \ \ or$$

(II-1) $k = 0$, $r = 2$, $T = (\mathbb{Z}/2)e_1 \oplus (\mathbb{Z}/2)e_2$, $\mathrm{Mon}(\beta_1)$ is trivial, $\mathrm{Mon}(\alpha_2) = e_1$, $\mathrm{Mon}(\beta_2) = e_2$; or

(II-2) $k = 0$, $r \geq 2$, T is cyclic, $\mathrm{Mon}(\beta_1)$ is trivial, $\mathrm{Mon}(\alpha_2) = e_1$, $\mathrm{Mon}(\beta_2) = 0$; or

(II-3) $k = 0$, $r \geq 2$, T is cyclic, $\mathrm{Mon}(\beta_1) = e_1$, $\mathrm{Mon}(\alpha_2) \in T \setminus \{0\}$, $\mathrm{Mon}(\beta_2) = 0$; or

(III) $k \geq 2$ even, $r = 2$, $T = (\mathbb{Z}/2)e_1 \oplus (\mathbb{Z}/2)e_2$. Here $t_j = e_2$ for all j, $\mathrm{Mon}(\beta_1) \in \{0, e_1\}$, $\mathrm{Mon}(\alpha_2) = e_1$, $\mathrm{Mon}(\beta_2) = 0$.

Proof. The composition of Mon with the surjection $G \twoheadrightarrow \mu_r$ is surjective, hence there is a global monodromy mapping onto a generator of μ_r, since $r < 6$ is a prime power. Without loss of generality, we may assume that this global monodromy is $\mathrm{Mon}(\alpha_1)$ and it equals ϵ.

With a transformation of type (b), we can then achieve that $\mathrm{Mon}(\beta_1) \in T$. For $j \geq 2$, after a permutation of indices exchanging j with $i+1$, we apply a transformation of type (d) with $i = 1$ to achieve that $\mathrm{Mon}(\alpha_j) \in T$; then a transformation of type (b) ensures that both $\mathrm{Mon}(\alpha_j)$, $\mathrm{Mon}(\beta_j)$ have the same image in μ_r whence another transformation of type (d) with $i = 1$ yields that $\mathrm{Mon}(\alpha_j), \mathrm{Mon}(\beta_j) \in T$, proving the first statement of the lemma (after another transformation of type (b) adjusting $\mathrm{Mon}(\beta_1)$ to be in T again). This kind of presentation will thus be fixed throughout the remainder of the proof.

If $k > 0$, we can add to each global monodromy any multiple of the local monodromies (transformation of type (a)). Hence, if the local monodromies generate T, we can obtain that all the global monodromies except $\mathrm{Mon}(\alpha_1)$ are trivial. This proves the second statement of the lemma.

Observe further that, if there are local monodromies, then $k \geq 2$ since $\sum_j t_j = 0$. In particular, $k \geq 2$ if $h = 1$. Since we can change the ordering of the branch points, we may always assume that the subgroup of T generated by the local monodromies is generated

by t_1, t_2. Moreover, if $k > 0$, the local monodromies generate T (of prime order) unless $r = 2$, $T = (\mathbb{Z}/2)e_1 \oplus (\mathbb{Z}/2)e_2$.

From now on, we assume that $h \leq 2$. Assume first that $h = 1$ or $h = 2$ and $\mathrm{Mon}(\alpha_2)$ and $\mathrm{Mon}(\beta_2)$ are trivial.

If $k = 0$, then necessarily $h \geq 2$ by the same argument as in the proof of Lemma 4.6, hence we are in Case (I-2).

In the case $k \geq 2$ and $r \geq 3$, since there are local monodromies, these generate T (cyclic of prime order), hence we reach Case (I-1).

For $r = 2$, there is the possibility that the local monodromies do not generate T, in which case we reach the normal form (4.4) of Case (I-3).

Assume now that $h = 2$, but $\mathrm{Mon}(\alpha_2)$ or $\mathrm{Mon}(\beta_2)$ are nontrivial elements of T. We may assume, possibly replacing α_2 by β_2^{-1}, that $\mathrm{Mon}(\alpha_2) \in T$ is nontrivial. In particular, if $r \geq 3$, we may assume that $\mathrm{Mon}(\alpha_2)$ generates T, while $\mathrm{Mon}(\beta_2) = 0$.

We also in general split into two cases according to (i) $\mathrm{Mon}(\beta_1)$ is trivial, or (ii) $\mathrm{Mon}(\beta_1)$ is nontrivial. Hence, for $r \geq 3$, and more generally for T cyclic (T is then of prime order), we get Cases (II-2), (II-3), since $k = 0$ is automatic (else the local monodromies would generate T).

Look now at the non-cyclic case $r = 2$, $T = (\mathbb{Z}/2)e_1 \oplus (\mathbb{Z}/2)e_2$ and assume first that $k = 0$; in the alternative (i) we have that $\mathrm{Mon}(\alpha_2), \mathrm{Mon}(\beta_2) \in T$ generate T, hence we get Case (II-1).

In the alternative (ii), we may assume $\mathrm{Mon}(\beta_1) = e_1$, and, using that the elements $\mathrm{Mon}(\alpha_2), \mathrm{Mon}(\beta_2) \in T$, we get the normal form (4.5) of Case (II), after some transformation of type (b).

It remains to study the case $k \geq 2$. Since the local monodromies do not generate T, we may assume that $t_j = e_2$ always, and $\mathrm{Mon}(\alpha_2), \mathrm{Mon}(\beta_j) \in \{0, e_1\}$. Since $\mathrm{Mon}(\alpha_2)$ is nontrivial by assumption, this leads to the normal form (III). $\square$

Definition 4.8. Assume that we have a monodromy $\mathrm{Mon}\colon \pi^{\mathrm{orb}} \twoheadrightarrow G$ in normal form as in Lemma 4.7 with G abelian (so $h = 1, 2$). Mon is said to be **minimal** if either

- $h = 2$ and we are in Case (I-2), or (II), or (II-i) for $i = 1, 2, 3$ (all with $k = 0$), or in Case (III) with $k = 2$; or

- $h = 1$ and $k = 2$, $r \in \{2, 3, 4\}$, and we are in Case (I-1), or $r = 2$ and Case (I-3); or

- $h = 1$, $r = 2$, and we are in Case (I-1) with $k = 3$.

Mon is said to be **minimal and big** if it is minimal, $h = 1$ and $|G| \geq 8$, which is equivalent to

- $h = 1$ and $k = 2$, $r \in \{3, 4\}$, and we are in Case (I-1);

- $h = 1$ and $k = 2$, $r = 2$ and we are in Case (I-3);

- $h = 1$, $r = 2$, and we are in Case (I-1) with $k = 3$.

Then a new homomorphism $\mathrm{Mon}' \colon (\pi^{\mathrm{orb}})' \to G$ is called a **simplication** of Mon if either

(i) $h' = h - 1 = 1$, $a_2 = b_2 = 0$ and $k = k' \geq 2$, or

(ii) $h' = h$, $k' = k - 1 \geq 2$ and $t'_{k-1} = t_{k-1} + t_k \neq 0$, or

(iii) $h' = h$, $k' = k - 2$, $k' \geq 2$ if $h = 1$, and $t_{k-1} + t_k = 0$,

and $\mathrm{Mon}' = \mathrm{Mon}$ on all the remaining generators, except for γ_{k-1} in Case (ii).

Observe that the simplified homomorphism

$$\mathrm{Mon}' \colon (\pi^{\mathrm{orb}})' \twoheadrightarrow G,$$

is a monodromy (that is, it is surjective) and in normal form, except when $r = 2$, we are in Case (I-1) with T non-cyclic and $k = 3$, or in Case (III) with $\mathrm{Mon}(\beta_1) = 0$, $\mathrm{Mon}(\alpha_2) = e_1$.

Remark 4.9. Clearly every monodromy as in Lemma 4.7 can be brought to a minimal one through a sequence of simplifications.

Lemma 4.10. *Assume that we are in the sporadic case where*

$$G = ((\mathbb{Z}/2)e_1 \oplus (\mathbb{Z}/2)e_2) \rtimes \mu_4 := T \rtimes \mu_4,$$

and all the local monodromies are translations. Then, after a change of system of geometric generators for π^{orb}, we may assume that $\mathrm{Mon}(\alpha_1)$ is the generator $\epsilon = i$ of μ_4 and that the other global monodromies $\mathrm{Mon}(\alpha_i), \mathrm{Mon}(\beta_j) \in T$ for all j and for $i \geq 2$.

We have the following normal forms with $\mathrm{Mon}(\alpha_i) = \mathrm{Mon}(\beta_j) = 0$ for $i \geq 2$ and for all j, where k is the number of branch points:

(III-k) $\beta_1 \mapsto 0$, $\gamma_1 \mapsto e_1$, $\gamma_2 \mapsto e_2$, $\gamma_j \mapsto d_j \in T$, $\sum_3^k d_j = e_1 + e_2$;

(IV-$2k'$) $\beta_1 \mapsto 0$, $\gamma_1, \ldots, \gamma_{2k'} \mapsto e_2$;

(V-$(1 + 2k')$) $\beta_1 \mapsto e_1$, $\gamma_j \mapsto e_1 + e_2$.

The above monodromies simplify to the minimal cases (IV-2) and (V-1). Indeed, for $k \geq 1$, we can achieve that $\mathrm{Mon}(\alpha_i) = 0$ for $i \geq 2$, and we have the following other normal forms.[9]

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[9]which we do not claim to form a complete list.

(V*(2k′)) $\beta_1 \mapsto 0$, $\beta_2 \mapsto e_1$, $\gamma_j \mapsto e_2$, $\forall\, j$, $\beta_j \mapsto b_j = 0$, $\forall\, j \geq 3$, *which simplifies to* (V*(2))*;*

(V**(2k′)) $\beta_1 \mapsto 0$, $\beta_2 \mapsto e_1$, $\gamma_j \mapsto e_1 + e_2$, $\forall\, j$, $\beta_j \mapsto b_j = 0$, $\forall\, j \geq 3$, *which simplifies to* (V**(2)).

For $k = 0$ we also have the monodromies:

(VI) $\alpha_1 \mapsto \epsilon$, $\alpha_2 \mapsto e_1$, $\beta_2 \mapsto e_2$ *and all other monodromies trivial, or*

(VI*) $\alpha_1 \mapsto \epsilon$, $\beta_1 \mapsto e_1 + e_2$, $\alpha_2 \mapsto e_1$ *and all other monodromies trivial, or*

(VII) $\alpha_1 \mapsto \epsilon$, $\alpha_2 \mapsto e_1 + e_2$, $\alpha_3 \mapsto e_2$ *and all other monodromies trivial,*

(VII*) $\alpha_1 \mapsto \epsilon$, $\alpha_2 \mapsto e_1$, $\alpha_3 \mapsto e_2$ *and all other monodromies trivial,*

(VIII) $\alpha_1 \mapsto \epsilon$, $\alpha_2 \mapsto e_1$ *and all other monodromies trivial.*

Case (VI) simplifies to (VI) with $h = 2$, Case (VI) simplifies to (VI*) with $h = 2$, Case (VII) simplifies to (VII) with $h = 3$, Case (VII*) simplifies to (VII*) with $h = 3$, Case (VIII) simplifies to (VIII) with $h = 2$.*

Proof. Since the local monodromies are in T, we may assume that α_1 maps onto a generator of μ_4 via the surjection $G \to \mu_4$; and, changing the origin in the genus one curve E, we may assume that $\mathrm{Mon}(\alpha_1) = \epsilon$.

Then, by transformations as in Lemma 4.7 we may assume that all other elements $\mathrm{Mon}(\alpha_i)$, $\mathrm{Mon}(\beta_j)$ belong to T: for the transformation (d), even if G is not abelian, yet we may apply it to the quotient group μ_4 of G.

In the case where the local monodromies generate T, we may multiply the global generators by local generators, and obtain that $\mathrm{Mon}(\alpha_i)$, $\mathrm{Mon}(\beta_j)$ for $i \geq 2$, $j \geq 1$ are trivial. Then we are exactly in Case (III-k).

If instead there are local monodromies and they generate a proper subgroup T' of T, we may assume that $\gamma_j \mapsto e_2$ for all j, or $\gamma_j \mapsto e_1 + e_2$ for all j, depending whether T' is not ϵ-invariant or conversely.

Then we may multiply the global generators by local generators and obtain that the other global monodromies are either trivial, or equal to e_1. If they are all trivial, then we are in Case (IV-k), for which we observe that ϵ, e_2 generate G, while ϵ and $e_1 + e_2$ do not generate G.

If instead they cannot be all made trivial, we may achieve by moves of type (b) that all other $\mathrm{Mon}(\alpha_j)$ are trivial for $j \geq 2$.

If $\beta_1 \mapsto e_1$, and if $\mathrm{Mon}(\beta_j)$ are trivial, for $j \geq 2$, we are in Case (V-$(1 + 2k')$). Since $[\epsilon, e_1] = e_1 + e_2$ the sum of the local monodromies must be equal to $e_1 + e_2$, hence $\gamma_j \mapsto e_1 + e_2$ for all j, since we assume that T is not generated by the local monodromies.

If $\beta_1 \mapsto 0$, we may assume $\beta_2 \mapsto e_1$, and if $\mathrm{Mon}(\beta_j)$ are trivial for $j \geq 3$, we are in Case (V*($2k'$)) or (V**($2k'$)).

Now, Case (III-k) simplifies to Case (III-3), which in turn simplifies to Case (IV-2), as well as Case (IV-$2k'$) does simplify to Case (IV-2); while Case (V-k) simplifies to (V-1).

In the case $k = 0$ we observe that the commutators $[\alpha_j, \beta_j]$ for $j \geq 2$ map to the identity, hence also $[\alpha_1, \beta_1]$ maps to the identity and it follows that the monodromy of β_1 is either trivial or equal to $e_1 + e_2$.

In this latter case, we may assume that $\alpha_2 \mapsto e_1$, and if all the other monodromies are trivial, we get (VI*).

In the former case, i.e., β_1 is trivial, assume that the monodromy restricted to the subgroup generated by the α_i, β_j with $i, j \geq 2$ corresponds to a surjection to T, hence to a surjection of the homology of a curve of genus $h - 1$ to T, and we have several normal forms, (depending on the nontriviality or conversely of the cup product of the two corresponding first cohomology elements, and on the action of ϵ) leading to Cases (VI), (VII), (VII*), (VIII).

We can simplify these monodromies by reducing the genus, until we find nontrivial monodromies, that is, until $h = 3$ in Cases (VII), (VII*), or until we reach $h = 2$ for Cases (VI), (VIII). $\qquad\square$

Arguing as in Lemmas 4.7 and 4.10, one can classify the normal forms for the monodromies in the sporadic case, but the list gets longer and longer as h increases. We report here the complete list for $h = 1, 2$.

Remark 4.11. Assume $G = ((\mathbb{Z}/2)e_1 \oplus (\mathbb{Z}/2)e_2) \rtimes \mu_4$, and that all local monodromies are translations, then every monodromy is then equivalent to one of the following normal forms for $h = 1, 2$.

- If $h = 1$, $k \geq 1$ and we are in Case (III-k), or Case (IV-$2k'$), or Case (V-$(1 + 2k')$).

- If $h = 2$ and $k \geq 1$, we are in Case (III-k), or Case (IV-$2k'$), or Case (V-$(1 + 2k')$), or Case (V*-$(2k')$), or Case (V**-$(2k')$), or

  (V***-$(1 + 2k')$) $\alpha_1 \mapsto \epsilon$, $\beta_1 \mapsto e_1$, $\alpha_2 \mapsto 0$, $\beta_2 \mapsto e_1$, $\gamma_j \mapsto e_1 + e_2$,

  which simplifies to Case (V***-1).

- If $h = 2$ and $k = 0$, we are in Case (VI), or Case (VI*), or Case (VIII), or

  (VI**) $(\alpha_1, \beta_1, \alpha_2, \beta_2) \mapsto (\epsilon, e_1 + e_2, e_1, e_2)$.

Lemma 4.12. *The effect of simplification replaces the curve C (of genus $h \geq 2$) by a curve C^0 such that $H_1(C^0, \mathbb{Z})$ is a direct summand in $H_1(C, \mathbb{Z})$. In particular, if we have a homology class $0 \neq c \in H_1(C^0, \mathbb{Z})$ then c does not map to zero in $H_1(C, \mathbb{Z})$.*

Proof. In Case (i), C is obtained from C^0 by attaching $|G|$ handles, hence the assertion follows right away.

In Cases (ii) and (iii), we have a family C_t tending to C_0, letting the last two branch points coalesce to a single one, possibly with trivial monodromy. In general, C^0 will be the normalization of C_0. By the second van Kampen's theorem,

$$\pi_1(C_0) = \pi_1(C^0) * \mathbb{Z} * \cdots * \mathbb{Z} \quad \Longrightarrow \quad H_1(C_0, \mathbb{Z}) = H_1(C^0, \mathbb{Z}) \oplus \mathbb{Z}^m.$$

Whereas C is homeomorphic to C_t for $t \neq 0$, and we have a surjection

$$H_1(C_t, \mathbb{Z}) \to H_1(C_0, \mathbb{Z})$$

whose kernel is generated by the vanishing cycles, which generate a free abelian group V; here V is a direct summand of $H_1(C_t, \mathbb{Z})$ since the quotient is torsion free. Hence

$$H_1(C, \mathbb{Z}) \cong V \oplus H_1(C_0, \mathbb{Z}) = V \oplus \mathbb{Z}^m \oplus H_1(C^0, \mathbb{Z}). \qquad \square$$

4.2. Some further useful observations

We collect here some observations and a criterion which shall be useful in order to achieve the full classification of the possible actions of groups of cohomologically trivial transformations. Γ is generated by the subgroup $\Lambda = \pi_E$ and by elements

$$\gamma'_1, \ldots, \gamma'_{r+2h}$$

acting on $\mathbb{H} \times \mathbb{C}$ and having the form

$$\gamma'_j(t, z) = (\gamma'_{j,1}(t), \epsilon_j z + x_j),$$

where the affine map $z \mapsto \epsilon_j z + x_j$ is in $\mathcal{G}$, and we let m_j be its smallest power which lies in π_E; this means for $\epsilon_j \neq 1$ that m_j is the multiplicative order of ϵ_j, while, if $\epsilon_j = 1$, that m_j is the smallest multiple such that $m_j x_j \in \pi_E$.

Since the action of Δ_G is free on $C \times E$, all local monodromies of $C \to B := C/G$ must act as translations on E. This implies that all the reductions of the singular fibres are smooth elliptic and means for $i = 1, \ldots, r$ that we have $\epsilon_{2h+i} = 1$, hence we shall see that the commutators of the last r generators lie in the commutator subgroup of π^{orb}. In particular, if $G = \mu_r$, there are no branch points for $C \to B = C/G$.

The first homology group $H_1(S, \mathbb{Z})$, the abelianization of Γ, is generated by the image of $\Lambda = \pi_E$ and by the images of $\gamma'_1, \ldots, \gamma'_{r+2h}$, which we continue to denote by the same symbol.

A simple calculation shows that the commutators of the generators are just the commutators of the elements γ'_j, which are of the form

$$(4.6) \qquad \qquad \left([\gamma'_{j,1}, \gamma'_{k,1}], z + (\epsilon_j - 1)x_k - (\epsilon_k - 1)x_j \right).$$

Hence we see that in the case where $\epsilon_j = \epsilon_k = 1$ the commutator is in π^{orb}.

Writing as usual $G = T \rtimes \mu_r$, where $T < E$ is a finite group of translations, then the second component of (4.6) lies in the lattice Λ_T, such that $\Lambda_T/\Lambda = T$. Writing the elements $f \in G$ through their action on E, the centralizer $\mathcal{C}(G)$ of G consists of the elements $f(z) = \lambda z + c$, such that

$$(\lambda - 1)T = 0, \quad c \in \mathrm{Ker}(\epsilon - 1).$$

In particular, since ψ_2 is in the centre of G, this applies to $f = \psi_2$, and we want to see whether the automorphism ψ with first component equal to the identity and with second component ψ_2 leaves the generators γ_i' fixed.

For the first component $\gamma_{i,1}'$ this is obvious since ψ_1 is the identity, while for $\epsilon_i z + x_i$ we see that conjugation by $f = \psi_2$ yields

$$(\epsilon_i z + x_i) \mapsto (\epsilon_i z + \lambda^{-1}(x_i + \omega_i)), \quad (\epsilon_i - 1)c =: \omega_i \in \Lambda,$$

and the question is whether $(\mathrm{Id}, \lambda^{-1}(x_i + \omega_i) - x_i)$ lies in the commutator subgroup of Γ.

For $\lambda = 1$, this amounts to $\omega_i \in [\Gamma, \Gamma]$, for $c = 0$, it amounts to $(\lambda^{-1} - 1)x_i \in [\Gamma, \Gamma]$. In practice, in view of the exact sequences (4.1) and (4.2), and since ψ_1 acts trivially on $\pi_1(C)$, it suffices by Remark 4.2 to check this for the lifts to Γ of a system of generators of G^{ab}.

From now on, we assume that the elements γ_i' are just elements of Γ which map onto a system of generators for G. From the previous discussion we immediately derive the following criterion. This criterion will be checked in what follows by hand or with machine assistance.

Criterion 4.13. (i) In the case where T is cyclic, then G is 2-generated and the kernel of the map $H_1(C, \mathbb{Z})_G \oplus H_1(E, \mathbb{Z})_G \to H_1(S, \mathbb{Z})$ (with cokernel G^{ab}) is generated by the commutator of two lifts to Γ of the two generators of G.

(ii) In the general case with G abelian, then G is 3-generated, and the kernel is generated by the three commutators of the respective lifts to π^{orb} of the three generators of G.

(iii) Assume now that there is a j such that $\gamma_j' = (\gamma_{j,1}', \psi_2)$ and pick other elements γ_i' mapping to a system of generators of G. Then

$$[\gamma_j', \gamma_i'] = \big([\gamma_{j,1}', \gamma_{i,1}'], [\psi_2, \gamma_{i,2}']\big),$$

hence we see that the condition that ψ acts trivially on $H_1(S, \mathbb{Z})$, which is equivalent to the fact that $(\mathrm{Id}, [\psi_2, \gamma_{i,2}'])$ lies in $[\Gamma, \Gamma]$ for all i, boils down to the following property:

the commutators $[\gamma_{j,1}', \gamma_{i,1}']$, of the respective lifts to π^{orb} of the

chosen system of generators of G, are zero in $H_1(C, \mathbb{Z})_G$.

(iv) Finally, if ψ_2 is a translation, since ψ_1 is the identity, it suffices to check that ψ leaves fixed the generator $\epsilon \in \mu_r \subset G$. Hence it suffices to verify that the commutator of respective lifts to π^{orb} of ψ_2 and ϵ is zero in $H_1(C, \mathbb{Z})_G$.

5. Properly elliptic surfaces with $\chi(\mathcal{O}_S) = 0$

As observed in Lemma 1.1 minimal surfaces with Kodaira dimension 1 and $\chi(S) = 0$ are isogenous to a higher elliptic product, and Case (I) where G acts by translations on the elliptic curve E was treated in Theorem 3.6; while case (II)—to be treated now—was prepared in Section 4.

Theorem 5.1. *Let S be a minimal surface with $\kappa(S) = 1$ and $\chi(\mathcal{O}_S) = 0$. Write $S = (C \times E)/\Delta_G$ as in Lemma 1.1 and assume that we are in Case (II), where $\mathrm{genus}(E/G) = 0$. Then the following statements hold.*

(0) *$B := C/G$ has genus $h \geq 1$.*

(1) *$\mathrm{Aut}_{\mathbb{Z}}(S)$ induces a trivial action on C/G and on E/G.*

(2) *$\mathrm{Aut}_{\mathbb{Z}}(S)$ is a priori isomorphic to a subgroup of one of the following groups:*

$$(\mathbb{Z}/2)^2 \times \mu_2, \quad \mathbb{Z}/2 \times \mu_4, \quad \mathbb{Z}/3 \times \mu_3,$$

   and it is trivial for $G = T \rtimes \mu_6$.

(3) *Indeed, $\mathrm{Aut}_{\mathbb{Z}}(S)$ is isomorphic to a subgroup of the following groups:*

$$(\mathbb{Z}/2)^2, \quad \mathbb{Z}/3.$$

(4) *If $h \geq 2$ or if G is not abelian, then $\mathrm{Aut}_{\mathbb{Z}}(S)$ is a subgroup of one of the following groups:*

$$(\mathbb{Z}/2)^2, \quad \mathbb{Z}/3.$$

   The cases $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathbb{Z}/2, \mathbb{Z}/3$ do in fact occur. More precisely, if $h \geq 2$ or if G is not abelian and not sporadic (in the sense of Lemma 4.4), then $\mathrm{Aut}_{\mathbb{Z}}(S)$ acts on E as a group of translations.

(5) *When $h = 1$, $\mathrm{Aut}_{\mathbb{Z}}(S)$ is trivial in the cases*

$$G \cong (\mathbb{Z}/2)^2 \times \mu_2, \quad \mathbb{Z}/3 \times \mu_3, \quad \mathbb{Z}/2 \times \mu_2.$$

   If instead $h = 1$, $G \cong \mathbb{Z}/2 \times \mu_4$, $\mathrm{Aut}_{\mathbb{Z}}(S)$ is either trivial or $\mathbb{Z}/2$, and the latter case does effectively occur.

(6) *We have* $\max |\operatorname{Aut}_{\mathbb{Z}}(S)| \in \{3, 4\}$.

Proof. (0) As indicated before, this follows from Lemma 1.1(2).

(1) Note that $\operatorname{Aut}_{\mathbb{Z}}(S)$ preserves any fibration structure. Let $f\colon S \to B = C/G$ be the fibration induced by the first projection of $C \times E$. Since genus$(E/G) = 0$, we have $q_f = 0$, and hence the cohomology classes of (the reduced divisors associated to) multiple fibres are different from one another by Lemma 2.1. Therefore, $\operatorname{Aut}_{\mathbb{Z}}(S)$ preserves each multiple fibre of f. Since the multiple fibres are exactly f^*b for the branch points b of the quotient map $\pi\colon C \to C/G$, the induced action of $\operatorname{Aut}_{\mathbb{Z}}(S)$ on C/G fixes each branch point of π.

We claim that $\operatorname{Aut}_{\mathbb{Z}}(S)$ acts trivially on C/G. In fact, $\operatorname{Aut}_{\mathbb{Z}}(S)$ acts trivially on $H^1(C/G, \mathbb{C}) \hookrightarrow H^1(S, \mathbb{C})$, hence this is clear for $h \geq 2$ by the Lefschetz formula. If $h = 1$ then $\operatorname{Aut}_{\mathbb{Z}}(S)$ acts by translations on C/G, and since moreover $\operatorname{Aut}_{\mathbb{Z}}(S)$ fixes pointwise the nonempty set of branch points of $C \to C/G$, it must act trivially on C/G.

Next, we show that $\operatorname{Aut}_{\mathbb{Z}}(S)$ acts trivially on E/G. Indeed, by the Riemann–Hurwitz formula, the branch multiplicities of $E \to E/G \cong \mathbb{P}^1$ cannot be of the form $(2, 2, m_1, \dots, m_r)$, where the m_i are odd numbers (since $\sum_j (1 - \frac{1}{m_j}) = 1$). It follows from Lemma 2.4 that $\operatorname{Aut}_{\mathbb{Z}}(S)$ preserves each multiple fibre of the induced fibration

$$p\colon S \to E/G,$$

and hence the induced action of $\operatorname{Aut}_{\mathbb{Z}}(S)$ on E/G fixes each branch point of $E \to E/G$. By the Riemann–Hurwitz formula again, there are at least three branch points of $E \to E/G$, which are then fixed by the induced action of $\operatorname{Aut}_{\mathbb{Z}}(S)$. It follows that $\operatorname{Aut}_{\mathbb{Z}}(S)$ acts trivially on E/G.

(2) Since $\operatorname{Aut}_{\mathbb{Z}}(S)$ acts trivially on C/G and on E/G, there is an inclusion

$$\operatorname{Aut}_{\mathbb{Z}}(S) \subset N_{G \times G}(\Delta_G)/\Delta_G \cong Z(G),$$

where $N_{G \times G}(\Delta_G)$ denotes the normalizer of Δ_G in $G \times G$, $Z(G)$ denotes the centre of G, and the isomorphism $N_{G \times G}(\Delta_G)/\Delta_G \cong Z(G)$ is given by $(\sigma_1, \sigma_2) \mapsto \sigma_1 \sigma_2^{-1}$. Since by assumption G acts faithfully on E and $g(E/G) = 0$, we have

$$G = T \rtimes A,$$

where T consists of translations and A is a nontrivial group $A \cong \mu_r \cong \mathbb{Z}/r$ with $r \in \{2, 3, 4, 6\}$ as studied in Lemma 4.4.

Since $\operatorname{Aut}_{\mathbb{Z}}(S)|_{C/G}$ is trivial, we also have an induced action of $\operatorname{Aut}_{\mathbb{Z}}(S)$ on the fibre E of $f\colon S \to B$. Since $\operatorname{Aut}_{\mathbb{Z}}(S)$ can be identified with a subgroup of $Z(G)$, it is isomorphic to a subgroup of one of the following groups (here the μ_r appearing means that $G = T \rtimes \mu_r$ as in Lemma 4.4):

$$(\mathbb{Z}/2)^2 \times \mu_2, \quad \mathbb{Z}/3 \times \mu_3, \quad \mathbb{Z}/2 \times \mu_4, \quad \mu_6.$$

However, if $\mathrm{Aut}_{\mathbb{Z}}(S) \subset \mu_6$, and it is not trivial, then by Lemma 4.4 $G = Z(G)$, hence $G = \mu_6$, which is excluded by Lemma 4.6 if $h = 1$, while if $h \geq 2$ it will be excluded in point (4). This completes the proof of the second assertion of the proposition.

(3) follows clearly from (4) and (5).

(4) Since ψ_1 induces the identity on C/G, and as observed we may assume that ψ_1 is the identity, we claim that the condition that ψ is numerically trivial is equivalent to the condition that the representations of G corresponding to the inclusion character $A = \mu_r \to \mathbb{C}^*$ and to its complex conjugate do not occur in the representation of G on $H^1(C, \mathbb{C})$.

In fact, the cohomology of the quotient is the G-invariant cohomology of the product $C \times E$, which, by the Künneth formula, is the tensor product of the respective cohomologies of C, E.

For the first cohomology, we have $H^1(S, \mathbb{C}) = H^1(C, \mathbb{C})^G = H^1(C/G, \mathbb{C})$, because $H^1(E, \mathbb{C})^G = 0$, and since we know that ψ_1 acts as the identity on C/G, the action on this first cohomology group is trivial.

For the second cohomology

$$H^2(S, \mathbb{C}) = \left(H^2(C, \mathbb{C}) \otimes H^0(E, \mathbb{C})\right)^G \oplus \left(H^1(C, \mathbb{C}) \otimes H^1(E, \mathbb{C})\right)^G$$
$$\oplus \left(H^0(C, \mathbb{C}) \otimes H^2(E, \mathbb{C})\right)^G,$$

and the action is trivial on the first and third summand. Write here $H^1(C, \mathbb{C}) = \bigoplus_j V_j$ as a sum of irreducible representations. We have $H^1(E, \mathbb{C}) = W \oplus \overline{W}$, where the 1-dimensional representation $W = H^0(E, \Omega_E^1)$ corresponds to the inclusion character $A = \mu_r \to \mathbb{C}^*$.

Set $W_1 := W$ and $W_2 := \overline{W}$. By Schur's lemma, if V_j, W_i are irreducible representations, then $(V_j \otimes W_i)^G = 0$ unless $V_j \cong W_i^\vee \cong \overline{W_i}$. Since ψ_1 is the identity on C, it acts as the identity on $H^1(C, \mathbb{C})$, hence on each summand V_j. Hence, if there is a j such that $V_j \cong \overline{W_i}$, and ψ_2 is not a translation, then the action on second cohomology is nontrivial. Hence, either

(i) ψ_2 is a translation for all $\psi \in \mathrm{Aut}_{\mathbb{Z}}(S)$, or

(ii) there is a $\psi \in \mathrm{Aut}_{\mathbb{Z}}(S)$ such that ψ_2 is not a translation.

In Case (i), Lemma 4.4 leads exactly to the possible groups $Z(G) \cap T$ stated in (4). It is important to observe that, again by Lemma 4.4, if G is not abelian and not sporadic, then $Z(G) \subset T$, hence we are in Case (i).

In Case (ii), we moreover have that no V_j can be isomorphic to W_1, nor W_2, which is equivalent to requiring that

$$\left(H^1(C, \mathbb{C}) \otimes H^1(E, \mathbb{C})\right)^G = 0,$$

and in turn this is equivalent to requiring $b_2(S) = 2$.

In turn, the condition $b_2(S) = 2$, since $q(S)$ equals the genus of $B = C/G$, implies that $e(S) = 4 - 4h$. Since we know that $e(S) = 0 = e(C \times E)$, the conclusion is that in Case (ii) it must be $h = 1$, that is, B is an elliptic curve. In Sections 5.2 and 5.3 we exhibit examples having $h = 2$ and realizing $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathbb{Z}/3$, $\mathbb{Z}/2$. This completes the proof of (4).

(5) By what we have seen so far, it suffices to concentrate on Case (ii): here B has genus 1, the local monodromies of $C \to C/G = B$ are just translations (observe that the branch locus of $C \to C/G = B$ is nonempty because we want that C has genus ≥ 2), and our automorphism is induced by (ψ_1, ψ_2), where ψ_1 is the identity, $G = Z(G)$ and $\psi_2 \in Z(G)$ is not a translation. Moreover, $q(S) = 1$, $p_g(S) = 0$, and $b_2(S) = 2$. Hence the two fibre images E, C of the horizontal and vertical curves in $C \times E$ span a lattice $\mathcal{L}$ of rank two, on which the group of automorphisms of S acts trivially.

The next lemma shows that, in case where T is cyclic, we may just focus on showing that the action of ψ is the identity on $H_1(S, \mathbb{Z})$.

Lemma 5.2. *Assume that $G = T \rtimes \mu_r$, where T is cyclic, that B has genus 1, that the local monodromies of $C \to C/G = B$ are translations, and there is one local monodromy generating T. Then, if an automorphism ψ, induced by (ψ_1, ψ_2), where ψ_1 is the identity and $\psi_2 \in Z(G)$, acts as the identity on $H_1(S, \mathbb{Z})$, it also acts as the identity on $H^2(S, \mathbb{Z})$.*

Proof. Assuming that ψ_1 is the identity implies that the submultiple fibres (the reduced multiple fibres) of the fibration $f \colon S \to B$ are left fixed by ψ. Then ψ acts as the identity on the subgroup of $H^2(S, \mathbb{Z})$ generated by these submultiple fibres of f and by the smooth fibres E, C of the two fibrations f, p (see (3.1)). We can moreover use Lemma 2.4 to infer that the submultiple fibres of the fibration $S \to E/G$ are left fixed, since the branching indices are, by Hurwitz' formula, never of the form $(2, 2, m_3, \ldots)$ where m_j is odd for $j \geq 3$.

It is not true in general (see Section 5.4) that these submultiple fibres generate the torsion subgroup, but by our assumption the action on the torsion subgroup of $H^2(S, \mathbb{Z})$ is the identity.

It suffices to exhibit two divisor classes with self intersection zero which are left fixed by $Z(G)$, and such that their intersection product equals 1: because then their image in the quotient group $H^2(S, \mathbb{Z})/H^2(S, \mathbb{Z})_{\mathrm{tor}}$ yields a basis, hence the two divisor classes and the torsion classes yield a system of generators for $H^2(S, \mathbb{Z})$ which are left fixed by $Z(G)$.

These two divisor classes are found as follows. By assumption, there is a fibre of f which is multiple of order $|T|$, which we write as $|T|M_1$. Now, the fibration $p \colon S \to E/G$ is induced by $C \times E \to E \to E/G$, and we observe that there is a fixed point y in E for μ_r, hence there is a multiple fibre of p occurring with multiplicity r, that we write as rM_2 (with M_2 isomorphic to C/μ_r).

Now, denoting by $\pi\colon C \times E \to S$ the projection, we have

$$M_1 \cdot M_2 = \frac{1}{|T|r} E \cdot C = \frac{1}{|G|} E \cdot C$$

and

$$|G| E \cdot C = \pi^*(E) \cdot \pi^*(C) = |G|^2 \quad \Longrightarrow \quad M_1 \cdot M_2 = 1.$$

Hence we have proven that ψ acts as the identity on a system of generators of $H^2(S, \mathbb{Z})$, so it acts as the identity on the full cohomology group. $\square$

We conclude now the proof of step (5) using the fact that the monodromies of the covering $C \to B$ in these cases simplify to the four minimal big cases, or to the case of Proposition 5.4. We shall use this proposition and the computer assisted result Theorem A.1 which is shown in the appendix, implying that in the four minimal big cases with $G = Z(G)$ of order ≥ 8, and in the case of $h = 1$, and $G = \mathbb{Z}/2 \times \mu_2$, the group $\mathrm{Aut}_{\mathbb{Z}}(S)$ is trivial except for Case (I-1) of $G = \mathbb{Z}/2 \times \mu_4$, where $\mathrm{Aut}_{\mathbb{Z}}(S) \cong \mathbb{Z}/2$.

The condition that $\mathrm{Aut}_{\mathbb{Z}}(S)$ is trivial is equivalent, in view of Criterion 4.13, to the fact that, for any element $g \in Z(G)$, the commutators of the lifts to π^{orb} of a minimal system of generators of G and of g are all nontrivial in $H_1(C, \mathbb{Z})_G$.

Now, for the four minimal big cases, with respective groups

$$\mathbb{Z}/3 \times \mu_3, \quad \mathbb{Z}/2 \times \mu_4, \quad (\mathbb{Z}/2)^2 \times \mu_2, \quad (\mathbb{Z}/2)^2 \times \mu_2,$$

the monodromy is in normal form and then the lifts of the generators are respectively

$$(\alpha_1, \gamma_1), \quad (\alpha_1, \gamma_1), \quad (\alpha_1, \beta_1, \gamma_1), \quad (\alpha_1, \gamma_1, \gamma_2).$$

An immediate extension of Lemma 4.12 shows that under the simplification process $H_1(C, \mathbb{Z}) = H_1(C^0, \mathbb{Z}) \oplus W$ is a G-equivariant splitting, thus we have equality $H_1(C, \mathbb{Z})_G = H_1(C^0, \mathbb{Z})_G \oplus W_G$. Therefore, if a commutator is not zero in $H_1(C^0, \mathbb{Z})_G$, then it is also nonzero inside $H_1(C, \mathbb{Z})_G$.

Because by simplification every monodromy can be carried to a minimal one, statement (5) follows now from Theorem A.1 and Proposition 5.4.

(6) The final statement follows by combining (3), (4) and (5). $\square$

Remark 5.3. Concerning the case where the genus h of B is ≥ 2, or more generally where we have a subgroup H of $Z(G)$ consisting only of translations:

  (i) If $h \geq 2$ we can prove that the subgroups in the list (3) of Theorem 5.1 are exactly the groups of numerically trivial automorphisms.

 (ii) If we have a subgroup H of $Z(G)$ consisting only of translations, then H acts trivially on $\pi^{\mathrm{orb}} \times \Lambda_T$, and we must see what happens for μ_r, using Criterion 4.13.

5.1. Proof of Theorem 1.2

Theorem 1.2 follows immediately from Theorems 3.6 and 5.1.

5.2. The main example

We show here that the case where $G = \mathbb{Z}_3 \times \mu_3$, $h = 2$, $k = 0$, and the monodromies are of type (I-2) leads to $\mathrm{Aut}_{\mathbb{Z}}(S) \cong \mathbb{Z}/3$. Here, if t is a generator of T, the monodromies are

$$\alpha_1 \mapsto \epsilon, \quad \beta_1 \mapsto t, \quad \mathrm{Mon}(\alpha_2) = \mathrm{Mon}(\beta_2) \text{ is trivial.}$$

In this case we have that the orbifold fundamental group π' on the side of the elliptic curve E is such that

$$\pi' \supset \pi_1(E) = \Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}\omega,$$

π' is generated by $e_1 := 1 \in \mathbb{C}$, by $\epsilon \in \mu_3$ and by $\tau := \frac{1-\omega}{3}$. Here t is the class of τ modulo Λ, $\omega = e_1 - 3\tau$, and conjugation by ϵ sends

$$e_1 \mapsto e_1^\epsilon = \omega = e_1 - 3\tau, \quad \tau \mapsto \tau^\epsilon = \frac{\omega - \omega^2}{3} = \frac{2\omega + e_1}{3} = e_1 - 2\tau.$$

Since $S \to B$ is a fibre bundle with fibre E, we have an exact sequence

$$1 \to \Lambda \to \Gamma \to \pi_1(B) \to 1,$$

hence $\Gamma \subset \pi_1(B) \times \pi'$ is generated by $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}\omega$ and by lifts of the generators of $\pi_1(B)$, namely

$$\widetilde{\alpha_1} := (\alpha_1, \epsilon), \quad \widetilde{\beta_1} := (\beta_1, \tau), \quad \alpha_2 = (\alpha_2, 0), \quad \beta_2 = (\beta_2, 0).$$

Clearly α_2, β_2, $\widetilde{\beta_1}$ centralize Λ, while

$$e_1^{\widetilde{\alpha_1}} = e_1^\epsilon = \omega, \quad \omega^{\widetilde{\alpha_1}} = \omega^2 = -e_1 - \omega.$$

Finally,

$$[\widetilde{\alpha_1}, \widetilde{\beta_1}][\alpha_2, \beta_2] = (1, [\epsilon, \tau]) = (1, \epsilon\tau\epsilon^{-1}\tau^{-1}) = (1, e_1 - 3\tau) = (1, \omega).$$

Because of the relations $\omega \sim 0$, $e_1 \sim \omega$, $\omega \sim -e_1 - \omega$, the image of Λ inside $H_1(S, \mathbb{Z})$ is zero, hence $H_1(S, \mathbb{Z}) \cong H_1(B, \mathbb{Z}) = \mathbb{Z}^4$ is torsion free and we conclude that $\mathrm{Aut}_{\mathbb{Q}}(S) = \mathrm{Aut}_{\mathbb{Z}}(S)$. Then since $h = 2$, $\mathrm{Aut}_{\mathbb{Q}}(S) = T$, hence $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathbb{Z}/3$.

5.3. A relative of the main example, with $\operatorname{Aut}_{\mathbb{Z}}(S) = \mathbb{Z}/2$

We assume now that $h = 2$, $k = 0$, $T \cong (\mathbb{Z}/2)t$, $r = 4$ and we have a monodromy of type (I-2), i.e.,

$$\alpha_1 \mapsto \epsilon, \quad \beta_1 \mapsto t, \quad \operatorname{Mon}(\alpha_2) = \operatorname{Mon}(\beta_2) \text{ is trivial.}$$

Here π' is generated by $e_1 := 1 \in \mathbb{C}$, by $\epsilon \in \mu_4$ and by $\tau := \frac{e_1 + e_2}{2}$, where $e_2 = i$ since $r = 4$. Again we have a fibre bundle and if we show that the image of Λ is zero in $H_1(S, \mathbb{Z})$, then again there is no torsion and $\operatorname{Aut}_{\mathbb{Z}}(S) = \mathbb{Z}/2$. Again

$$[\widetilde{\alpha_1}, \widetilde{\beta_1}][\alpha_2, \beta_2] = (1, [\epsilon, \tau]) = (1, -e_1).$$

Hence we get $e_1 \sim e_2$, $e_1 = 0$ and we are done.

5.4. Examples using the Reidemeister Schreier method

In this section we analyse whether, in the case $h = 1$ and $G = \mathbb{Z}/2 \times \mu_2$, the automorphisms in $Z(G)$ are cohomologically trivial.

In order to do so, we need to use the Reidemeister–Schreier method, which, given a group presentation (here for π^{orb}) with generators and relations, allows to find a presentation for a subgroup (here $\pi_1(C)$), see [24, pp. 86–95] and [23, pp. 102–104].

Proposition 5.4. *Assume that $G \cong \mathbb{Z}/2 \times \mu_2$. Then $\operatorname{Aut}_{\mathbb{Z}}(S)$ is trivial*

- *for the G-covering C of an elliptic curve B with minimal monodromy of type* (I-1)*, and*

- *more generally if $h = 1$.*

Proof. Here we have two branch points and the orbifold fundamental group is generated by α, β, γ_1 subject to the relations

$$\gamma_1^2 = ([\alpha, \beta]\gamma_1)^2 = 1.$$

To study the action on $H_1(S, \mathbb{Z})$, we use a presentation of Γ obtained using the Reidemeister–Schreier method. Let

$$\xi := (\alpha, \epsilon), \quad \eta := (\gamma_1, \eta_1) \quad \Longrightarrow \quad \xi^2 = \alpha^2 \in \pi_1(C), \quad \eta^2 = 2\eta_1 \in \Lambda.$$

Recall the exact sequence

$$H_1(C, \mathbb{Z})_G \times \Lambda_G \to H_1(S, \mathbb{Z}) \to G = (\mathbb{Z}/2)^2 \to 0$$

where the kernel of the first homomorphism to the left is generated by the commutator

$$\tau := [(\alpha, \epsilon), (\gamma_1, \eta_1)] = (\alpha\gamma_1\alpha^{-1}\gamma_1, -2\eta_1).$$

As we already calculated, $\Lambda_G = \Lambda/2\Lambda = (\mathbb{Z}/2)2\eta_1 \oplus (\mathbb{Z}/2)\eta_2$. C has genus 3, and the Reidemeister–Schreier method shows that $\pi_1(C)$ is generated by

$$\beta, \quad \alpha\beta\alpha^{-1}, \quad \gamma_1\beta\gamma_1, \quad \alpha\gamma_1\beta\gamma_1\alpha^{-1}, \quad \alpha^2, \quad \gamma_1\alpha^2\gamma_1, \quad \alpha\gamma_1\alpha\gamma_1.$$

Denote the above sequence of generators by b_1, b_2, b_3, b_4, x_5, x_6, y_7. These are too many, but it is immediate to verify that

$$b_2 b_1^{-1} x_6 y_7^{-1} b_4 y_7 x_6^{-1} b_3^{-1} = (\alpha\beta\alpha^{-1}\beta^{-1}\gamma_1)^2 = 1.$$

Hence any one of the b_i's can be eliminated from the system of generators and we are left with exactly six generators, whose image in $H_1(C, \mathbb{Z})$ will be a free $\mathbb{Z}$-basis.

Denoting their image in $H_1(C, \mathbb{Z})$ by the same symbol and writing $a \sim b$ if two elements have the same image in $H_1(C, \mathbb{Z})_G$, we see that, conjugating by ξ, η the first six generators, we get exactly the relations

$$\beta \sim \alpha\beta\alpha^{-1} \sim \gamma_1\beta\gamma_1 \sim \alpha\gamma_1\beta\gamma_1\alpha^{-1}, \quad \alpha^2 \sim \gamma_1\alpha^2\gamma_1$$

and we denote by b the image of the first four generators, by x the image of the fifth and sixth, and by y the image of $\alpha\gamma_1\alpha\gamma_1$.

Moreover, conjugating the last generator by $^{-1}\eta$ we get exactly the relations

$$\alpha\gamma_1\alpha\gamma_1 \sim \gamma_1\alpha\gamma_1\alpha = \gamma_1\alpha^2\gamma_1(\alpha\gamma_1\alpha\gamma_1)^{-1}\alpha^2,$$

hence $y = 2x - y$, so that $2(x - y) = 0$. We also observe that $\alpha\gamma_1\alpha^{-1}\gamma_1 = \alpha\gamma_1\alpha\gamma_1\gamma_1\alpha^{-2}\gamma_1$, hence the image of τ equals $(y - x, -2\eta_1) = (y - x, 0) - 2\eta$.

Putting everything together, we get that $H_1(S, \mathbb{Z})$ is generated by

$$\xi, \quad b, \quad \eta, \quad \eta_2, \quad y,$$

with relations (since $x = 2\xi$):

$$4\xi = 2y, \quad (2\xi - y) = -2\eta, \quad 4\eta = 0, \quad 2\eta_2 = 0,$$

where however the first relation is implied by the others. Hence, eliminating y, we get

$$H_1(S, \mathbb{Z}) = \mathbb{Z}b \oplus \mathbb{Z}\xi \oplus (\mathbb{Z}/2)\eta_2 \oplus (\mathbb{Z}/4)\eta.$$

This group is larger than H^{orb}, and the action of ψ, if $\psi_2 = -1$, equals the identity on ξ, b, η_2, but not on η, which is sent to $-\eta \neq \eta$.

Similarly if $\psi_2(z) = z + \eta_1$, see the calculations of Criterion 4.13, the action is not the identity since $\omega = 2\eta_1 = 2\eta \neq 0$ inside $H_1(S, \mathbb{Z})$, hence the translation by η_1 sends ξ to $\xi + 2\eta$, and leaves η fixed. Hence, not only these two elements do not belong to $\mathrm{Aut}_{\mathbb{Z}}(S)$, but also their product g does not, since it does not keep fixed ξ and η.

For the second assertion, it suffices to observe that for $h = 1$ the monodromy simplifies to a minimal monodromy of type (I-1). $\qquad\square$

Remark 5.5. One can ask whether similarly this is not the case also for $G \cong \mathbb{Z}/3 \times \mu_3$, and for the G-covering C of an elliptic curve B with minimal monodromy of type (I-1).

We have here $r = 3$, $G = T \times \mu_3$, where T is generated by $\eta_1 = \frac{1-\epsilon}{3}$. For T being cyclic, we take only two branch points, and we take the following two elements of Γ mapping onto a set of generators for Δ_G:

$$(\alpha, \epsilon), \quad (\gamma_1, \eta_1).$$

We now use the criterion given in Criterion 4.13. We have that $\psi(x, z) = (x, \lambda z + c)$, where $\lambda^3 = 1$ and c is a multiple of η_1. Hence the generators are left invariant by ψ if and only if

$$\lambda^{-1}(\epsilon - 1)c, \quad (1 - \lambda^{-1})\eta_1$$

are in the commutator subgroup of Γ. For this a sufficient condition would be that they map to zero in Λ_G. This holds for all λ, c if and only if $(\epsilon - 1)\eta_1 = 0 \in [\Gamma, \Gamma]$.

Since $(\epsilon - 1)\eta_1 = -\frac{1}{3}(\epsilon - 1)^2 = \epsilon$, which is the second period e_2 of Λ, we conclude that we do not have zero in Λ_G because

$$\Lambda_G = (\Lambda)_{\mu_3} = \langle 1, \epsilon \rangle / \langle \epsilon - 1, \epsilon - \epsilon^2 \rangle \cong (\mathbb{Z}/3)\epsilon.$$

However, in calculating the image of Λ_G inside Γ^{ab}, we must divide by the subgroup generated by the commutator

$$[(\alpha, \epsilon), (\gamma_1, \eta_1)] = (\alpha\gamma_1\alpha^{-1}\gamma_1^{-1}, (\epsilon - 1)\eta_1) = (\alpha\gamma_1\alpha^{-1}\gamma_1^{-1}, \epsilon),$$

hence $(0, \epsilon)$ is in $[\Gamma, \Gamma]$ if and only if $\alpha\gamma_1\alpha^{-1}\gamma_1^{-1}$ is trivial in $H_1(C, \mathbb{Z})_G$, as predicted by Criterion 4.13.

The answer is again negative, as shown in the appendix, and is found again via the Reidemeister Schreier method. Since the calculations here are quite complicated, it was necessary to use MAGMA for computer calculations (see Theorem A.1).

5.5. The group of homotopically trivial automorphisms of properly elliptic surfaces with $\chi(\mathcal{O}_S) = 0$ is just $\mathrm{Aut}^0(S)$

Proposition 5.6. *Let S be a smooth projective surface with $\kappa(S) = 1$ and $\chi(\mathcal{O}_S) = 0$. Then $\mathrm{Aut}_\#(S) = \mathrm{Aut}^0(S)$.*

Proof. If S is not minimal then $e(S) > 0$ and hence any $\sigma \in \mathrm{Aut}_\#(S)$ has fixed points. Using the natural Kähler metric with nonpositive sectional curvature on the minimal model $S_{\min}$ of S, one sees that $\sigma = \mathrm{id}_S$ as in the proof of [22, Corollary 0.2].

Now we may assume that S is minimal. Again, using the natural Kähler metric with nonpositive sectional curvature on S, σ is smoothly homotopic to id_S through harmonic maps $\sigma_s \colon S \to S$ with $\sigma_0 = \mathrm{id}_S$ and $\sigma_1 = \sigma$ by [19]. Moreover, for any $p \in S$, the path

$s \mapsto \sigma_s(p)$ is a geodesic segment (parametrized proportionally to arc length) with length independent of $p \in S$. The vector field corresponding to the homotopy σ_s lifts to $C \times E$ since $\pi \colon C \times E \to S$ is étale. It follows that σ_s lifts to $\widetilde{\sigma}_s \colon C \times E \to C \times E$ for $0 \le s \le 1$. For any $p \in S$, since σ preserves each fibre of $f \colon S \to B$, the geodesic $\sigma_s(p)$ from p to $\sigma(p)$ lies in the fibre of f containing p. Now one sees that

$$\widetilde{\sigma}_s(\widetilde{b}, x) = (\widetilde{b}, x + t_s(\widetilde{b}))$$

for some morphism $t_s \colon C \to E$. Thus $\widetilde{\sigma}_s$ and hence σ_s is holomorphic for each $0 \le s \le 1$. It follows that $\sigma \in \mathrm{Aut}^0(S)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

A. Appendix

A.1. G abelian

Let us consider first the case where $G = T \times \mu_r$ is an abelian group.

We consider the following List I, encompassing 20 cases, corresponding to all minimal monodromies for G abelian (see Lemma 4.7 and Definition 4.8). Here, ϵ is a generator of μ_r and the type of the monodromy $(\alpha_1, \beta_1, \ldots, \alpha_h, \beta_h; \gamma_1, \ldots, \gamma_k)$ is defined as the sequence $(h; m_1, \ldots, m_k)$, where the m_j's are the orders of the local monodromies $t_j := \mathrm{Mon}(\gamma_j)$.

(1) $T = (\mathbb{Z}/3)t$, $r = 3$, and the image of the monodromies are

    (I-1) $(\epsilon, 0; t, 2t)$, type: (1;3,3) BIG,

    (I-2) $(\epsilon, t, 0, 0)$, type: (2;-),

    (II-2) $(\epsilon, 0, t, 0)$, type: (2;-),

    (II-3) $(\epsilon, t, t, 0)$ or $(\epsilon, t, 2t, 0)$, type: (2;-).

(2) $T = (\mathbb{Z}/2)t$, $r = 4$, and the image of the monodromies are

    (I-1) $(\epsilon, 0; t, t)$, type: (1;2,2) BIG,

    (I-2) $(\epsilon, t, 0, 0)$, type: (2;-),

    (II-2) $(\epsilon, 0, t, 0)$, type: (2;-),

    (II-3) $(\epsilon, t, t, 0)$, type: (2;-).

(3) $T = (\mathbb{Z}/2)t \oplus (\mathbb{Z}/2)s$, $r = 2$, and the image of the monodromies are

    (I-1) $(\epsilon, 0; t, s, t + s)$, type: (1;2,2,2) BIG,

    (I-3) $(\epsilon, t; s, s)$, type: (1;2,2) BIG,

    (II) $(\epsilon, t, s, 0)$ or (ϵ, t, s, t), type: (2;-),

   (II-1) $(\epsilon, 0, t, s)$, type: (2;-),

    (III) $(\epsilon, t, t, 0; s, s)$ or $(\epsilon, 0, t, 0; s, s)$, type: (2;2,2).

(4) $T = (\mathbb{Z}/2)t$, $r = 2$, and image of the monodromies are

   (I-1) $(\epsilon, 0; t, t)$, type: (1;2,2),

   (I-2) $(\epsilon, t, 0, 0)$, type: (2;-),

   (II-2) $(\epsilon, 0, t, 0)$, type: (2;-),

   (II-3) $(\epsilon, t, t, 0)$, type: (2;-).

Theorem A.1. *Let $G = T \times \mu_r$ be an abelian group, and assume that the monodromy is one of List I. Then the following statements hold.*

(1) *If $T = (\mathbb{Z}/3)t$, $r = 3$ and the monodromy is*

   (i) *$(\epsilon, t, 0, 0)$ (Case (I-2)), or*

   (ii) *$(\epsilon, t, a_2, 0)$ ($a_2 \in T \setminus \{0\}$, Case (II-3)),*

   then $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathbb{Z}/3$.

(2) *If $T = (\mathbb{Z}/2)t$, $r = 4$ and the monodromy is*

   (i) *$(\epsilon, 0; t, t)$ (Case (I-1), big), or*

   (ii) *$(\epsilon, t, 0, 0)$ (Case (I-2)), or*

   (iii) *$(\epsilon, t, t, 0)$ (Case (II-3)),*

   then $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathbb{Z}/2$.

(3) *If $T = (\mathbb{Z}/2)t$, $r = 2$ and the monodromy is*

   (i) *$(\epsilon, t, 0, 0)$ (Case (I-2)), or*

   (ii) *$(\epsilon, t, t, 0)$ (Case (II-3)),*

   then $\mathrm{Aut}_{\mathbb{Z}}(S) \subseteq \mathbb{Z}/2$.

(4) *In all other cases $\mathrm{Aut}_{\mathbb{Z}}(S)$ is trivial.*

Proof. The MAGMA [4] script below shows that there are nontrivial elements in $\mathrm{Aut}_{\mathbb{Q}}(S)$ acting trivially on $H_1(S, \mathbb{Z})$ (cf. Criterion 4.13) only for the cases listed in (1), (2), (3) and for $G = \mathbb{Z}/2 \times \mu_4$, Case (II-2).

In Case (1-i) indeed, as explained in Subsection 5.2, the cohomology of S is torsion free, hence $\mathrm{Aut}_{\mathbb{Q}}(S) = \mathrm{Aut}_{\mathbb{Z}}(S)$. We recall that if $h \geq 2$ (see Theorem 5.1(4)) then only the translations are elements of $\mathrm{Aut}_{\mathbb{Q}}(S)$. The same occurs also for (1-ii), (2-ii) and (2-iii).

Instead, for (3-i) and (3-ii), every element of G induces an automorphism acting trivially on $H_1(S, \mathbb{Z})$ (which has torsion) but the non-translations act nontrivially on $H^2(S, \mathbb{Z})$, so $\mathrm{Aut}_{\mathbb{Z}}(S) \subseteq T = \mathbb{Z}/2$.

Finally, if $T = (\mathbb{Z}/2)t$, $r = 4$ and the monodromy is (I-1) or (II-2) the scripts show that only the element $\epsilon^2 \colon z \mapsto -z$ induces an automorphism acting trivially on $H_1(S, \mathbb{Z})$. In the latter case, we have a non-translation and $h = 2$, so this automorphism acts nontrivially on $H^2(S, \mathbb{Z})$. In the former case, by Lemma 5.2 the automorphism acts trivially on $H^2(S, \mathbb{Z})$ as well, so $\mathrm{Aut}_{\mathbb{Z}}(S)$ has order 2. $\qquad\square$

A.2. Sporadic case $G = (\mathbb{Z}/2)^2 \rtimes \mu_4$

Here we consider the case where $G = (\mathbb{Z}/2)^2 \rtimes \mu_4$, and assume that the monodromy images of the geometric basis elements $(\alpha_1, \beta_1, \ldots, \alpha_h, \beta_h; \gamma_1, \ldots, \gamma_k)$ are as follows, where ϵ is a generator of μ_4 and $T = (\mathbb{Z}/2)t \oplus (\mathbb{Z}/2)s$.

We treat a List II consisting of 11 cases, including all the minimal monodromies for $h = 1, 2$ (see Lemma 4.10 and Remark 4.11).

(IV-2) $(\epsilon, 0; s, s)$, type (1;2,2),

(V-1) $(\epsilon, t; t + s)$, type (1;2),

(V*-2) $(\epsilon, 0, 0, t; s, s)$, type (2;2,2),

(V**-2) $(\epsilon, 0, 0, t; t + s, t + s)$, type (2;2,2),

(V***-1) $(\epsilon, t, 0, t; t + s)$, type (2;2),

(VI) $(\epsilon, 0, t, s)$, type (2;-),

(VI*) $(\epsilon, t + s, t, 0)$, type (2;-),

(VI**) $(\epsilon, t + s, t, s)$, type (2;-),

(VIII) $(\epsilon, 0, t, 0)$, type (2;-),

(VII) $(\epsilon, 0, t + s, 0, s, 0)$, type (3;-),

(VII*) $(\epsilon, 0, t, 0, s, 0)$, type (3;-).

Theorem A.2. *Let $G = (\mathbb{Z}/2)^2 \rtimes \mu_4$, and assume that the monodromy is one of List II. Then $\mathrm{Aut}_{\mathbb{Z}}(S)$ is trivial.*

Proof. The MAGMA script below shows that in all cases there are no nontrivial elements in $\mathrm{Aut}_{\mathbb{Q}}(S)$ acting trivially on $H_1(S, \mathbb{Z})$, except in Cases (VI*) and (VI**).

In Cases (VI*) and (VI**), the element $z \mapsto -z + t + s$ (respectively $z \mapsto -z$) is the unique nontrivial element in $\mathrm{Aut}_{\mathbb{Q}}(S)$ acting trivially on $H_1(S, \mathbb{Z})$, but it is a non-translation and $h = 2$, so it acts nontrivially on $H^2(S, \mathbb{Z})$ by Theorem 5.1(4).   □

The above examples, although not providing a full classification, suggest that it may be difficult to find a case where $\mathrm{Aut}_{\mathbb{Z}}(S) \cong (\mathbb{Z}/2)^2$.

B. MAGMA script

The script can be run at `http://magma.maths.usyd.edu.au/calc/`.

```
 1  /* Input: i) the group: G; ii) the monodromy images of pi_orb(B)-> G: mon;
 2  iii) the genus of the quotient curve B: h;
 3
 4  Output: the number of elements of Z(G)=Aut_Q(S), acting trivially on H_1(S,Z)*/
 5
 6  // First of all, given the monodromy images,
 7  // we construct the group pi^orb and the monodromy pi^orb-->>G
 8
 9  Orbi:=function(gr,mon, h)
10  F:=FreeGroup(#mon); Rel:={}; G:=Id(F);
11  for i in {1..h} do G:=G*(F.(2*i-1)^-1,F.(2*i)^-1); end for;
12  for i in {2*h+1..#mon} do G:=G*F.(i); Include(~Rel,F.(i)^(Order(mon[i]))); end for;
13  Include(~Rel,G); P:=quo<F|Rel>;
14  return P, hom<P->gr|mon>;
15  end function;
16
17  // MapProd computes given two maps f,g:A->B the map product
18  // induced by the product on B
19
20  MapProd:=function(map1,map2)
21  seq:=[];
22  A:=Domain(map1);
23  B:=Codomain(map1);
24  if Category(A) eq GrpPC then
25      n:=NPCgens(A);
26      else n:=NumberOfGenerators(A);
27  end if;
28  for i in [1..n] do Append(~seq, map1(A.i)*map2(A.i)); end for;
29  return hom<A->B|seq>;
30  end function;
31
32  TrivialActionH1:=function(G,mon, h)
33  // First of all we construct the group pi^orb and the monodromy pi^orb-->>G
34
35  PiOrb,f:=Orbi(G,mon,h);
36
37  // We compute a set of generators U for ker(f) = pi_1(C),
38   //using the Reidemeister--Schreier method
39
40  R:=[PiOrb!1];
```

```
41  for g in {g: g in G| g ne G!1} do Append(~R,g@@f); end for;
42  // R is a set of representative of the cosets of ker(f) = pi_1(C)<PiOrb
43  U:={};
44  for r in R do
45  for t in Generators(PiOrb) do
46      h:=r*t;
47      if exists(k){s:s in R | f(s) eq f(h)} then
48          Include(~U,h*k^-1);
49  end if; end for; end for;
50
51  Pi1:=sub<PiOrb| U>; // Pi_1 of C
52  H1,t:=AbelianQuotient(Pi1); //H_1(C,Z)
53
54  /* The group of coinvariants is H_1(C,Z) modulo the subgroup generated by
55  {g u g^-1u^-1},
56  where u is a generator of H_1(C,Z) and g are lifts of generators of G.
57  We do it in 2 steps, so the presentation given by MAGMA is simpler */
58
59  U_new:={}; //the relations for the coinvariants
60  for u in U do
61  for g in Generators(G) do
62      Include(~U_new,t((g@@f)*u*(g@@f)^-1*u^-1));
63  end for; end for;
64
65  H1G, s:=quo<H1|U_new>;//H_1(C,Z)_{G}
66  triv:={G!1};
67  H:={g: g in Center(G)| g ne G!1};
68
69  for g in H do
70      l1:=g@@f; act:={};
71      for gen in G do
72          l2:=gen@@f; Include(~act,s(t((l1,l2))));
73      end for;
74      // check if g in Z(G)=Aut_Q(S), acts trivially on H_1(S,Z)
75      if #act eq 1 then Include(~triv,g); end if;
76  end for;
77
78  return #sub<G|triv>;
79  end function;
80
81  /* Input: i) the group: G; ii) the monodromy images of pi_orb(B)-> G: mon;
82  iii) the genus of the quotient curve B: h;
83  iv) the monodromy images of pi_orb(E)-> G: monE.
84  Output: the torsion subgroup of H_1(S,Z) */
85
86  Tors_H1S:=function(G,mon1, h1, mon2, h2)
87  T1,f1:=Orbi(G,mon1, h1);
88  T2,f2:=Orbi(G,mon2, h2);
89  T1xT2,inT,proT:=DirectProduct([T1,T2]);
90  GxG,inG:=DirectProduct(G,G);
91  Diag:=MapProd(inG[1],inG[2])(G);
92  f:=MapProd(proT[1]*f1*inG[1],proT[2]*f2*inG[2]);
93  HH:=Rewrite(T1xT2,Diag@@f); // This is the fundamental group of S=(CxE)/G
```

```
94 return TorsionSubgroup(AbelianQuotient(HH)); //Tors(H_1(S,Z))
95 end function;
```

We use the previous script, to check, for the monodromies of Lists I and II, which elements act nontrivially on $H_1(S, \mathbb{Z})$. If there are nontrivial elements acting trivially on $H_1(S, \mathbb{Z})$, we compute also the torsion subgroup of $H_1(S, \mathbb{Z})$.

- $G = \mathbb{Z}/3 \times \mu_3$

```
1  G:=SmallGroup(9,2);
2  // Z/3xZ/3 t=G.1=(1,0), e=G.2=(0,1)
3  monE:=[G.2,G.1*G.2,G.1^2*G.2]; //[e, (t,e), (2t,e)]
4
5  mon:=[G.2,G!1,G.1,G.1^2]; h:=1; // I-1, BIG
6  TrivialActionH1(G,mon,h);
7  > 1
8
9  mon:=[G.2,G.1,G!1,G!1]; h:=2; // I-2
10 TrivialActionH1(G,mon,h);
11 > 9 // whole group acts trivially on H_1(S,Z)
12 Tors_H1S(G,mon,h, monE,0);
13 > Abelian Group of order 1
14
15 mon:=[G.2,G!1,G.1,G!1]; h:=2; // II-2
16 TrivialActionH1(G,mon,h);
17 > 1
18
19 mon:=[G.2,G.1,G.1,G!1]; h:=2; // II-3 v1
20 TrivialActionH1(G,mon,h);
21 > 9 // whole group acts trivially on H_1(S,Z)
22 Tors_H1S(G,mon,h, monE,0);
23 > Abelian Group of order 1
24
25 mon:=[G.2,G.1,G.1^2,G!1]; h:=2; // II-3 v2
26 TrivialActionH1(G,mon,h);
27 > 9 // whole group acts trivially on H_1(S,Z)
28 Tors_H1S(G,mon,h, monE,0);
29 > Abelian Group of order 1
```

- $G = \mathbb{Z}/2 \times \mu_4$

```
1  G:=SmallGroup(8,2);
2   // Z/2xZ/4 t=G.2=(1,0), e=G.1=(0,1)
3   monE:=[G.1,G.1*G.2, G.1^2*G.2]; //[e, (t,e), (t,e^2)]
4
5  mon:=[G.1,G!1,G.2,G.2]; h:=1; // I-1, BIG
6  TrivialActionH1(G,mon,h);
7  > 2 // G.3=G.1^2=e^2
8  Tors_H1S(G,mon,h, monE,0);
9  > Z/2 + Z/2
10
11 mon:=[G.1,G.2,G!1,G!1]; h:=2; // I-2
```

```
12 TrivialActionH1(G,mon,h);
13 > 8 // whole group acts trivially on H_1(S,Z)
14 Tors_H1S(G,mon,h, monE,0);
15 > Abelian Group of order 1
16
17 mon:=[G.1,G!1,G.2,G!1]; h:=2; // II-2
18 TrivialActionH1(G,mon,h);
19 > 2 // G.3=G.1^2=e^2
20 Tors_H1S(G,mon,h, monE,0);
21 > Z/2
22
23 mon:=[G.1,G.2,G.2,G!1]; h:=2; // II-3
24 TrivialActionH1(G,mon,h);
25 > 8 // whole group acts trivially on H_1(S,Z)
26 Tors_H1S(G,mon,h, monE,0);
27 > Abelian Group of order 1
```

- $G = (\mathbb{Z}/2)^2 \times \mu_2$

```
1  G:=SmallGroup(8,5);
2  // (Z/2xZ/2)xZ/2 t=G.1=(1,0,0), s=G.2=(0,1,0), e=G.3=(0,0,1)
3  monE:=[G.3,G.1*G.3,G.2*G.3,G.1*G.2*G.3];//[e, (t,e), (s,e), (t+s,e)]
4
5  mon:=[G.3,G!1,G.1,G.2,G.1*G.2]; h:=1; // I-1 BIG
6  TrivialActionH1(G,mon,h);
7  > 1
8
9  mon:=[G.3,G.1,G.2,G.2]; h:=1; // I-3 BIG
10 TrivialActionH1(G,mon,h);
11 > 1
12
13 mon:=[G.3,G.1,G.2,G!1]; h:=2; // II v1
14 TrivialActionH1(G,mon,h);
15 > 1
16
17 mon:=[G.3,G.1,G.2,G.1]; h:=2; // II v2
18 TrivialActionH1(G,mon,h);
19 > 1
20
21 mon:=[G.3,G!1,G.1,G.2]; h:=2; // II-1
22 TrivialActionH1(G,mon,h);
23 > 1
24
25 mon:=[G.3,G.1,G.1,G!1,G.2,G.2]; h:=2; // III v1
26 TrivialActionH1(G,mon,h);
27 > 1
28
29 mon:=[G.3,G!1,G.1,G!1,G.2,G.2]; h:=2;// III v2
30 TrivialActionH1(G,mon,h);
31 > 1
```

- $G = \mathbb{Z}/2 \times \mu_2$

```
1  G:=SmallGroup(4,2);
2  // Z/2xZ/2 t=G.1=(1,0), e=G.2=(0,1)
3  monE:=[G.2,G.2,G.1*G.2,G.1*G.2]; // [e,e,(t,e),(t,e)]
4
5  mon:=[G.2,G!1,G.1,G.1]; h:=1; // I-1
6  TrivialActionH1(G,mon,h);
7  > 1
8
9  mon:=[G.2,G.1,G!1,G!1]; h:=2; // I-2
10 TrivialActionH1(G,mon,h);
11 > 4 // whole group acts trivially on H_1(S,Z)
12 Tors_H1S(G,mon,h, monE,0);
13 > Z/2
14
15 mon:=[G.2,G!1,G.1,G!1]; h:=2; // II-2
16 TrivialActionH1(G,mon,h);
17 > 1
18
19 mon:=[G.2,G.1,G.1,G!1]; h:=2; // II-3
20 TrivialActionH1(G,mon,h);
21 > 4 // whole group acts trivially on H_1(S,Z)
22 Tors_H1S(G,mon,h, monE,0);
23 > Z/2
```

- $G = (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mu_4$

```
1  G:=SmallGroup(16,3);
2  // epsilon= G.1, t=G.2, s=G.2*G.3, t+s=G.3, epsilon^2=G.4
3  monE:=[G.2*G.4,G.2*G.1,G.1]; //[(t,e^2),(t,e),e]
4
5  mon:=[G.1,G!1,G.2*G.3,G.2*G.3]; h:=1; // IV-2
6  TrivialActionH1(G,mon,h);
7  > 1
8
9  mon:=[G.1,G.2,G.3]; h:=1; // V-1
10 TrivialActionH1(G,mon,h);
11 > 1
12
13 mon:=[G.1,G!1,G!1,G.2,G.2*G.3,G.2*G.3]; h:=2; // V*-2
14 TrivialActionH1(G,mon,h);
15 > 1
16
17 mon:=[G.1,G!1,G!1,G.2,G.3,G.3]; h:=2; // V**-2
18 TrivialActionH1(G,mon,h);
19 > 1
20
21 mon:=[G.1,G.2,G!1,G.2,G.3]; h:=2; // V***-1
22 TrivialActionH1(G,mon,h);
23 > 1
24
25 mon:=[G.1,G!1,G.2,G.2*G.3]; h:=2; // VI
26 TrivialActionH1(G,mon,h);
```

```
27 > 1
28
29 mon:=[G.1,G.3,G.2,G!1]; h:=2; // VI*
30 TrivialActionH1(G,mon,h);
31 > 2 // G.3*G.4 = z-> -z+t+s acts trivially on H1(S,Z)
32 Tors_H1S(G,mon,h, monE,0);
33 > Abelian Group of order 1
34
35 mon:=[G.1,G.3,G.2,G.2*G.3]; h:=2; // VI**
36 TrivialActionH1(G,mon,h);
37 >2 // G.4 = z-> -z acts trivially on H1(S,Z)
38 Tors_H1S(G,mon,h, monE,0);
39 > Abelian Group of order 1
40
41 mon:=[G.1,G!1,G.2,G!1]; h:=2; // VIII
42 TrivialActionH1(G,mon,h);
43 > 1
44
45 mon:=[G.1,G!1,G.3,G!1, G.2*G.3,G!1]; h:=3; // VII
46 TrivialActionH1(G,mon,h);
47 > 1
48
49 mon:=[G.1,G!1,G.2,G!1, G.2*G.3,G!1]; h:=3; // VII*
50 TrivialActionH1(G,mon,h);
51 > 1
```

Acknowledgments

References

[1] Wolf Barth and Chris Peters, *Automorphisms of Enriques surfaces*, Invent. Math. **73** (1983), no. 3, 383–411.

[2] Salomon Bochner and Deane Montgomery, *Locally compact groups of differentiable transformations*, Ann. of Math. (2) **47** (1946), 639–653.

[3] _____, *Groups on analytic manifolds*, Ann. of Math. (2) **48** (1947), 659–669.

[4] Wieb Bosma, John Cannon and Catherine Playoust, *The Magma algebra system I: The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265.

[5] Dan Burns Jr. and Michael Rapoport, *On the Torelli problem for kählerian $K-3$ surfaces*, Ann. Sci. École Norm. Sup. (4) **8** (1975), no. 2, 235–273.

[6] Laurent Busé, Fabrizio Catanese and Elisa Postinghel, *Algebraic Curves and Surfaces—A history of shapes*, SISSA Springer Series **4**, Springer, Cham, 2023.

[7] Jin-Xing Cai, *Automorphisms of a surface of general type acting trivially in cohomology*, Tohoku Math. J. (2) **56** (2004), no. 3, 341–355.

[8] _____, *Automorphisms of elliptic surfaces, inducing the identity in cohomology*, J. Algebra **322** (2009), no. 12, 4228–4246.

[9] Fabrizio Catanese, *Fibred surfaces, varieties isogenous to a product and related moduli spaces*, Amer. J. Math. **122** (2000), no. 1, 1–44.

[10] _____, *Fibred Kähler and quasi-projective groups*, Special issue dedicated to Adriano Barlotti, Adv. Geom. **2003**, suppl., S13–S27.

[11] _____, *A superficial working guide to deformations and moduli*, in: *Handbook of Moduli I*, 161–215, Adv. Lect. Math. (ALM) **24**, Int. Press, Somerville, MA, 2013.

[12] _____, *Topological methods in moduli theory*, Bull. Math. Sci. **5** (2015), no. 3, 287–449.

[13] Fabrizio Catanese and Wenfei Liu, *On topologically trivial automorphisms of compact Kähler manifolds and algebraic surfaces*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **32** (2021), no. 2, 181–211.

[14] Fabrizio Catanese, Wenfei Liu and Matthias Schütt, *On the numerically and cohomologically trivial automorphisms of elliptic surfaces with $\chi(\mathcal{O}_S) > 0$*, preliminary version, November 2024.

[15] David A. Cox and Steven Zucker, *Intersection numbers of sections of elliptic surfaces*, Invent. Math. **53** (1979), no. 1, 1–44.

[16] Robert Friedman and John W. Morgan, *Smooth Four-manifolds and Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **27**, Springer-Verlag, Berlin, 1994.

[17] Akira Fujiki, *On automorphism groups of compact Kähler manifolds*, Invent. Math. **44** (1978), no. 3, 225–258.

[18] Marvin Jay Greenberg, *Lectures on Algebraic Topology*, W. A. Benjamin, New York, 1967.

[19] Philip Hartman, *On homotopic harmonic maps*, Canadian J. Math. **19** (1967), 673–687.

[20] Nathan Jacobson, *Basic Algebra I*, W. H. Freeman and Co., San Francisco, CA, 1974.

[21] David I. Lieberman, *Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds*, in: *Fonctions de Plusieurs Variables Complexes III (Sém. François Norguet, 1975–1977)*, 140–186, Lecture Notes in Math. **670**, Springer, Berlin, 1978.

[22] Wenfei Liu, *Surfaces of general type with $q = 2$ are rigidified*, Commun. Contemp. Math. **20** (2018), no. 7, 1750084, 12 pp.

[23] Roger C. Lyndon and Paul E. Schupp, *Combinatorial Group Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89, Springer-Verlag, Berlin, 1977.

[24] Wilhelm Magnus, Abraham Karrass and Donald Solitar, *Combinatorial Group Theory: Presentations of groups in terms of generators and relations*, Interscience Publishers, New York, 1966.

[25] Masaki Maruyama, *On automorphism groups of ruled surfaces*, J. Math. Kyoto Univ. **11** (1971), 89–112.

[26] Laurent Meersseman, *The Teichmüller and Riemann moduli stacks*, J. Éc. polytech. Math. **6** (2019), 879–945.

[27] Shigeru Mukai, *Numerically trivial involutions of Kummer type of an Enriques surface*, Kyoto J. Math. **50** (2010), no. 4, 889–902.

[28] Shigeru Mukai and Yukihiko Namikawa, *Automorphisms of Enriques surfaces which act trivially on the cohomology groups*, Invent. Math. **77** (1984), no. 3, 383–397.

[29] Chris A. M. Peters, *Holomorphic automorphisms of compact Kähler surfaces and their induced actions in cohomology*, Invent. Math. **52** (1979), no. 2, 143–148.

[30] ———, *On automorphisms of compact Kähler surfaces*, in: *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, 249–267, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[31] Ilia I. Pjateckiĭ-Šapiro and Igor Rotislav Šafarevič, *Torelli's theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 530–572.

[32] Fernando Serrano, *The Picard group of a quasi-bundle*, Manuscripta Math. **73** (1991), no. 1, 63–82.

[33] Kenji Ueno, *A remark on automorphisms of Enriques surfaces*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **23** (1976), no. 1, 149–165.

[34] Bruno Zimmermann, *Surfaces and the second homology of a group*, Monatsh. Math. **104** (1987), no. 3, 247–253.

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