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Normalized Solutions for the Fractional Choquard Equations with Lower Critical Exponent and Nonlocal Perturbation

Zilin Chen and Yang Yang*

Abstract. We study the existence and nonexistence of normalized solutions for the following fractional Choquard equations with Hardy–Littlewood–Sobolev lower critical exponent and nonlocal perturbation:

$$\begin{cases} (-\Delta)^s u + \lambda u = \gamma(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + \mu(I_\alpha * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x = c^2, \end{cases}$$

where $N \geq 3$, $s \in (0,1)$, $\alpha \in (0,N)$, $\gamma, \mu, c > 0$ and $2_\alpha := \frac{N+\alpha}{N} < q \leq 2^*_{\alpha,s} := \frac{N+\alpha}{N-2s}$. I_α is the Riesz potential and $\lambda \in \mathbb{R}$ appears as an unknown Lagrange multiplier. By precisely restricting parameters γ, μ and c, using constrained variational method and introducing new relevant arguments, we establish several existence and nonexistence results. In particular, we consider the case $q = 2^*_{\alpha,s}$ which corresponds to equations involving double critical exponents, and the Hardy–Littlewood–Sobolev subcritical approximation method is used to solve the case.

1. Introduction and main results

In this paper, we are concerned with the following lower critical fractional Choquard equation with a nonlocal perturbation:

$$(-\Delta)^s u + \lambda u = \gamma(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + \mu(I_\alpha * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

having prescribed L^2-norm

$$\int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x = c^2, \tag{1.2}$$

where $N \geq 3$, $s \in (0,1)$, $\alpha \in (0,N)$, $\gamma, \mu, c > 0$, $2_\alpha < q \leq 2^*_{\alpha,s}$, and $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier and is part of the unknowns. In particular, $2_\alpha := \frac{N+\alpha}{N}$ is

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*Corresponding author.

the Hardy–Littlewood–Sobolev lower critical exponent and $2^*_{\alpha,s} := \frac{N+\alpha}{N-2s}$ is the fractional Hardy–Littlewood–Sobolev upper critical exponent. Here, $I_\alpha\colon \mathbb{R}^N \setminus \{0\} \mapsto \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) := \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{with } A_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^\alpha \pi^{\frac{N}{2}} \Gamma\left(\frac{\alpha}{2}\right)},$$

and the fractional Laplacian $(-\Delta)^s$ is defined for $u \in S(\mathbb{R}^N)$ by

$$(-\Delta)^s u(x) = C_{N,s}\,\mathrm{P.\,V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $S(\mathbb{R}^N)$ denotes the Schwartz space of rapidly decreasing smooth functions, P. V. stands for the principle value of the integral and $C_{N,s}$ is the positive normalization constant. The fractional Laplacian appears in many diverse domains, including optimization, phase transitions, conservation laws, anomalous diffusion, stratified materials, ultra-relativistic limits of quantum mechanics, crystal dislocation, water waves and so on. We refer to [4, 11] for more applied backgrounds.

Recently, the following Choquard equation

$$-\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

has been investigated by many scholars. Physically speaking, Choquard equation appears as several models in quantum mechanics. For $N = 3$, $\alpha = 2$, $p = 2$ and $V = 1$ in (1.3), Pekar [39] introduced it to describe the quantum theory of a polaron at rest, Choquard [29] adopted it as a certain approximation to Hartree–Fock theory of one component plasma to model electron trapped in its own hole, and Penrose [40] put forward its application in investigating the self-gravitational collapse of a quantum wave function. We shall pay more attention to the mathematical aspects. When $V = 1$ in (1.3), Moroz and Schaftingen [35] established the existence, regularity and radially symmetry of ground state solutions to (1.3) with an optimal range exponent p satisfying $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$. For more related topics, we advise readers to read a survey paper [36] and its references.

Meanwhile, when it comes to the fractional Choquard equations, d'Avenia et al. [10] studied the following fractional Choquard equation:

$$(-\Delta)^s u + \omega u = (|x|^{\alpha-N} * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $\omega > 0$, $N \geq 3$ and $p > 1$, and they obtained regularity, existence, nonexistence, symmetry and decays properties of solutions. Shen et al. [42] investigated the existence of ground state solutions for a fractional Choquard equation involving a nonlinearity which is subcritical and satisfies the general Berestycki–Lions type conditions. Chen and Liu [6] considered an autonomous fractional Choquard equation via Nehari manifold and concentration-compactness arguments. By using variational methods, Mukherjee and Sreenadh [37] obtained some existence, nonexistence and regularity results for weak solution of the Brezis–Nirenberg type problem of nonlinear fractional Choquard equation. He and Rădulescu [18] studied the critical fractional Choquard equations with small linear perturbation, and they established a nonlocal global compactness property in the framework of fractional Choquard equations and obtained the existence of at least one positive solution. For more results about the fractional Choquard equations, we refer to [2, 3, 15, 33, 34] and the references therein.

For the last few years, normalized solutions to nonlinear elliptic problems have been widely concerned by scholars. Jeanjean [20] introduced a stretched functional, constructing the mountain pass structure for the functional on a natural constraint related to the Pohozaev identity to obtain the existence of at least one normalized solution for a nonlinear elliptic problem. Luo [32] approached a Hartree equation by multiple constrained minimization methods which differs from the methods in [20]. In [43,44], Soave considered the existence of normalized solutions to the following nonlinear Schrödinger equation with local perturbation:

$$\begin{cases} -\Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x = a^2. \end{cases}$$

When $N \geq 1$, $\mu \in \mathbb{R}$ and $2 < q \leq 2 + \frac{4}{N} \leq p \leq 2^* := \frac{2N}{N-2}$, $q \neq p$, Soave [43] proved that if the perturbation term is small, there exists at least one normalized radial ground state. In particular, if $2 + \frac{4}{N} < p \leq 2^*$, $2 < q < 2^*$ and $\mu > 0$ small, there are two radial positive solutions. When $N \geq 3$, $\mu > 0$, $\lambda \in \mathbb{R}$ and $p = 2^*$, in the context of L^2-subcritical, L^2-critical and L^2-supercritical perturbation $\mu|u|^{q-2}u$, Soave [44] obtained several existence/non-existence and stability/instability results. Later, when $N \geq 3$, $\mu > 0$, $p = 2^*$ and $2 < q < 2 + \frac{4}{N}$, Jeanjean et al. [21] established the existence of ground state and demonstrated that the set of ground states is orbitally stable. By the Sobolev subcritical approximation approach to mass constrained problem, Li [25] got the existence of normalized ground states with $p = 2^*$ for L^2-critical and L^2-supercritical perturbation.

Yao et al. [47] considered normalized solutions of the following Choquard equation:

$$\begin{cases} -\Delta u + \lambda u = \gamma (I_\alpha * |u|^p)|u|^{p-2}u + \mu |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x = c^2, \end{cases}$$

where $p = \frac{N+\alpha}{N}$ is the Hardy–Littlewood–Sobolev lower critical exponent. Under different assumptions on q, p, γ and μ, they established several existence and nonexistence results. When $\gamma = 1$ and $p = 2^*_\alpha := \frac{N+\alpha}{N-2}$, Li [26] considered the case $2 < q < 2 + \frac{4}{N}$, then the existence and orbital stability of the ground states were obtained. Moreover, the case $\frac{N+2+\alpha}{N} < p < 2_\alpha^*$ and $2+\frac{4}{N} < q < 2^*$ was investigated in [27].

Moreover, Ding et al. [12] explored the following equation:

$$\begin{cases} -\Delta u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u + \mu(I_\alpha * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x = a^2, \end{cases}$$

where $N \geq 3$ and $\frac{N+\alpha}{N} < q < p \leq \frac{N+\alpha}{N-2}$. They got the existence and nonexistence of normalized solutions to this problem. In particular, when $p = 2_\alpha^*$, Ye et al. [48] and Shang et al. [41], using the method in [44], studied the critical Choquard equation with nonlocal perturbation. Then, for L^2-subcritical, L^2-critical and L^2-supercritical perturbation $\mu(I_\alpha * |u|^q)|u|^{q-2}u$, the normalized ground states and mountain-pass type solutions were obtained.

When it comes to investigating normalized solutions for the fractional Choquard equations, Li and Luo [23] studied the following equation:

$$(-\Delta)^s u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$ and $\max\left\{2, \frac{N+2s+\alpha}{N}\right\} < p < 2_{\alpha,s}^*$. By using the constrained minimization method, they got the existence of normalized ground state. For the following equation:

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{q-2}u + \mu(I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x = a^2, \end{cases}$$

when $N \geq 2$, $2+\frac{4s}{N} < q < \frac{2N}{N-2s}$ and $\frac{N+2s+\alpha}{N} < p < \frac{N+\alpha}{N-2s}$, Yang [46] used a refined version of the min-max principle, and under suitable assumptions on the related parameters they obtained the existence and asymptotic properties of normalized solutions. When $2 \leq p < \frac{N+2s+\alpha}{N}$, Li et al. [24] obtained the existence and asymptotic properties of normalized solutions. Relied on a Lagrange formulation and new deformation arguments, Cingolani et al. [7] obtained the existence of a symmetric ground state solution with a general nonlinearity.

Moreover, He et al. [19] considered the following equation:

$$\begin{cases} (-\Delta)^s u = \lambda u + \mu|u|^{q-2}u + (I_\alpha * |u|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^*-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x = a^2. \end{cases}$$

For $2 < q < \frac{2N}{N-2s}$, they obtained the existence and asymptotic properties of normalized solutions. When $\mu > 0$ is large enough, Feng et al. [16] got the existence and multiplicity of normalized solutions by the concentration-compactness principle and truncation technique. Besides, Yu et al. [49] researched the following fractional lower critical Choquard equation:

$$\begin{cases}(-\Delta)^s u = \lambda u + \gamma(I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha - 2}u + \mu|u|^{q-2}u \quad \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x = a^2,\end{cases}$$

where $2 < q \leq \frac{2N}{N-2s}$, they got the existence and symmetry of normalized ground states. Lan et al. [22] studied the following fractional critical Choquard equation with a nonlocal perturbation:

$$\begin{cases}(-\Delta)^s u = \lambda u + a(I_\alpha * |u|^q)|u|^{q-2}u + (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u \quad \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x = c^2.\end{cases}$$

Under L^2-subcritical, L^2-critical and L^2-supercritical perturbation $a(I_\alpha * |u|^q)|u|^{q-2}u$, they obtained the existence of normalized ground states and mountain-pass-type solutions.

However, there seems to be no result on normalized solutions to the fractional Choquard equations involving Hardy–Littlewood–Sobolev lower critical exponent and a nonlocal perturbation in $\mathbb{R}^N$. For the fractional Choquard equation, the Hardy–Littlewood–Sobolev lower critical exponent 2_α is a new feature which is associated with the phenomenon of "bubbling at infinity", and this makes it differ from the case of the Hardy–Littlewood–Sobolev upper critical exponent.

Motivated by the aforementioned works, especially by [41,47,49], we shall focus on the problem (1.1) and (1.2) for four different scenarios: (i) L^2-subcritical case: $2_\alpha < q < \overline{q} := \frac{N+2s+\alpha}{N}$; (ii) L^2-critical case: $q = \overline{q}$; (iii) L^2-supercritical case: $\overline{q} < q < 2^*_{\alpha,s}$; (iv) doubly critical case: $q = 2^*_{\alpha,s}$, respectively.

Before we state our main results, we first introduce some notations. Throughout this paper, $L^r(\mathbb{R}^N)$ denotes the Lebesgue space with the norm $\|u\|_r = \big(\int_{\mathbb{R}^N} |u|^r\,\mathrm{d}x\big)^{1/r}$ for any $1 \leq r < \infty$. $H^s(\mathbb{R}^N)$ is the fractional Hilbert space defined as

$$H^s(\mathbb{R}^N) := \big\{u \in L^2(\mathbb{R}^N) : (-\Delta)^{\frac{s}{2}}u \in L^2(\mathbb{R}^N)\big\},$$

which is endowed with the standard inner product and norm given respectively by

$$\langle u, v\rangle := \int_{\mathbb{R}^N} \big((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + uv\big)\,\mathrm{d}x, \quad \|u\|^2_{H^s} = \langle u, u\rangle = \big\|(-\Delta)^{\frac{s}{2}}\big\|_2^2 + \|u\|_2^2,$$

where

$$\big\|(-\Delta)^{\frac{s}{2}}u\big\|_2^2 = \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y.$$

Define $E_q(u)\colon H^s(\mathbb{R}^N) \to \mathbb{R}$ by

$$E_q(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big|(-\Delta)^{\frac{s}{2}}u\big|^2\,\mathrm{d}x - \frac{\gamma}{22_\alpha}\int_{\mathbb{R}^N} (I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha}\,\mathrm{d}x - \frac{\mu}{2q}\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q\,\mathrm{d}x. \tag{1.4}$$

It is standard to check that the energy functional is of class C^1, and a critical point of $E_q(u)$ restricted on the constraint

$$S(c) = \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x = c^2 \right\}$$

corresponds to a solution of the problem (1.1) and (1.2).

Now, we recall the following definitions.

Definition 1.1. We say that $u \in H^s(\mathbb{R}^N)$ is a weak solution to (1.1) if

$$\begin{aligned} &\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v \, \mathrm{d}x + \lambda \int_{\mathbb{R}^N} uv \, \mathrm{d}x \\ &= \gamma \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_\alpha}) |u|^{2_\alpha - 2} uv \, \mathrm{d}x + \mu \int_{\mathbb{R}^N} (I_\alpha * |u|^q) |u|^{q-2} uv \, \mathrm{d}x \end{aligned}$$

for any $v \in H^s(\mathbb{R}^N)$. Moreover, $(u_c, \lambda_c) \in H^s(\mathbb{R}^N) \times \mathbb{R}$ is a couple of weak solution to (1.1) if u_c is a weak solution to (1.1) with $\lambda = \lambda_c$, where λ_c arises as an unknown Lagrange multiplier which depends on the solution u_c.

Definition 1.2. We say that u_c is a normalized ground state solution of the problem (1.1) if $(u_c, \lambda_c) \in H^s(\mathbb{R}^N) \times \mathbb{R}$ is the solution to the problem (1.1), and u_c has the minimal energy among all the solutions which belong to $S(c)$, that is,

$$E_q'|_{S(c)}(u_c) = 0 \quad \text{and} \quad E_q(u_c) = \inf\{E_q(v) \mid E_q'|_{S(c)}(v) = 0 \text{ and } v \in S(c)\}.$$

Here, λ_c depending on the solution u_c is an unknown Lagrange multiplier.

Let we set

$$c^* := \left(\frac{N + \alpha}{N\gamma} \right)^{\frac{N}{2\alpha}} S_\alpha^{\frac{N+\alpha}{2\alpha}}$$

and

$$\begin{aligned} c_* := &\left[\frac{2qs}{\mu \widetilde{C} (Nq - N - \alpha)} \right]^{\frac{s}{(2_\alpha - 1)(Nq - N - \alpha) + 2qs - 2s \cdot 2_\alpha}} \\ &\times \left[\frac{\gamma (Nq - N - \alpha)}{2_\alpha S_\alpha^{2_\alpha} (Nq - N - \alpha - 2s)} \right]^{\frac{N + \alpha + 2s - Nq}{2(2_\alpha - 1)(Nq - N - \alpha) + 4qs - 4s \cdot 2_\alpha}}, \end{aligned}$$

where $\widetilde{C}$ and S_α are defined in (2.3) and (2.6) in Section 2, respectively. In addition, the definitions of S and C_α are given in (2.1) and (2.4) respectively in Section 2.

The main results read as follows.

Theorem 1.3. *Let $\gamma, \mu > 0$ and $2_\alpha < q < \overline{q}$. It results that the infimum*

$$\sigma(c) := \inf_{u \in S(c)} E_q(u) < -\frac{\gamma}{22_\alpha} S_\alpha^{-2_\alpha} c^{22_\alpha}$$

is achieved by $u^* \in S(c)$ *which is a normalized ground state solution of the problem* (1.1) *and* (1.2) *with the corresponding Lagrange multiplier* $\lambda^* > \frac{\gamma}{2_\alpha} S_\alpha^{-2_\alpha} c^{\frac{2\alpha}{N}}$*. Besides,* u^* *is a real-valued positive function in* $\mathbb{R}^N$*, which is radially symmetric and non-increasing.*

Theorem 1.4. *Let* $\gamma > 0$ *and* $q = \overline{q}$*. For* $0 < \mu < \frac{N+2s+\alpha}{N\widetilde{C}} c^{-\frac{4s+2\alpha}{N}}$*, problem* (1.1) *and* (1.2) *has no solution for any* $\lambda \in \mathbb{R}$*.*

Theorem 1.5. *Let* $\gamma > 0$*,* $\overline{q} < q < 2^*_{\alpha,s}$ *and* $0 < c < c_*$*. Then there exists* $\widehat{\mu} > 0$ *such that for every* $\mu > \widehat{\mu}$*, problem* (1.1) *and* (1.2) *possesses a normalized ground state solution* $\widehat{u} \in H^s(\mathbb{R}^N)$ *with the corresponding Lagrange multiplier* $\widehat{\lambda} > 0$*, which is positive and radially symmetric.*

Theorem 1.6. *Let* $q = 2^*_{\alpha,s}$*,* $0 < c < \min\{c_*, c^*\}$*, and*

$$\gamma > \left(\frac{\alpha}{\alpha+2s}\right)^{\frac{\alpha}{N}} S_\alpha^{\frac{N+\alpha}{N}} \left(\frac{C_\alpha^{N-2s}}{S^{N+\alpha}}\right)^{\frac{\alpha}{N(2s+\alpha)}} \mu^{\frac{\alpha(N-2s)}{N(2s+\alpha)}}.$$

Then, there exists $\widetilde{\mu} \geq \widehat{\mu}$ *such that for every* $\mu > \widetilde{\mu}$*, problem* (1.1) *and* (1.2) *possesses a normalized ground state solution* $\widetilde{u} \in H^s(\mathbb{R}^N)$ *which is positive and radially symmetric, and the corresponding Lagrange multiplier* $\widetilde{\lambda}$ *satisfies* $0 < \widetilde{\lambda} \leq \gamma S_\alpha^{-2_\alpha} c^{\frac{2\alpha}{N}}$*.*

To better understand the context of this paper, let us briefly state the strategies and methodologies which are used to prove the above theorems. Firstly, when $2_\alpha < q < \overline{q}$, the functional E_q is bounded from below on $S(c)$. To prove that the infimum $\sigma(c)$ is achieved, we utilize the fractional concentration-compactness principle. Specifically, we rule out the vanishing and dichotomy of the minimizing sequence to obtain the compactness of the minimizing sequence. The dichotomy can be excluded by applying the strict inequality $\sigma(c) < \sigma(c_1) + \sigma(c_2)$. Moreover, we take advantage of the extremal function to make an estimate of $\sigma(c)$, which allows us to exclude the vanishing. Secondly, when $q = \overline{q}$, we make restriction on μ and then obtain the nonexistence result. Next, when it comes to $q \in (\overline{q}, 2^*_{\alpha,s}]$, we find that E_q is no longer bounded from below on $S(c)$. Hence, we make use of the Pohozaev manifold $\mathcal{P}_q(c)$ as a natural constraint of E_q that contains all the critical points of E_q restricted to $S(c)$ to make sure that the functional E_q restricted to $\mathcal{P}_q(c)$ is bounded from below. Then, for $\overline{q} < q < 2^*_{\alpha,s}$, inspired by [9], we introduce the homotopy-stable family to establish the existence of Palais–Smale sequence, and take advantage of a similar idea in [47, 49] to get the compactness result and then illustrate the existence of normalized ground states. Finally, when it comes to the case $q = 2^*_{\alpha,s}$ which involves the interaction of the double critical terms, due to the assistance of some processes and the solutions obtained in proving Theorem 1.5, we make full use of the Hardy–Littlewood–Sobolev subcritical approximation method combined with the new estimate trick to achieve our results. Due to the dual influence of fractional Laplacian and nonlocal perturbation, the problem becomes more complex and challenging. This requires us to precisely restrict parameters. We further estimate accurately, mainly use constrained variational method, and then under different assumptions on parameters we obtain new and interesting results about the existence and nonexistence of normalized solutions for the fractional Choquard equation.

The paper is organized as follows. In Section 2, we give some preliminary results which will be used later. In Section 3, we deal with the case $2_\alpha < q < \overline{q}$ and prove Theorem 1.3. In Section 4, we consider the case $q = \overline{q}$ and prove Theorem 1.4. In Section 5, we treat the case $\overline{q} < q < 2^*_{\alpha,s}$ and the case $q = 2^*_{\alpha,s}$ respectively, and give the proofs of Theorems 1.5 and 1.6.

2. Preliminaries

In this section, we present various preliminary results that will be used later. Firstly, let us recall the following fractional Sobolev embedding, see [11, Theorem 6.5].

Lemma 2.1. *Let $0 < s < 1$ and $N > 2s$. Then there exists a constant $S = S(N, s) > 0$ such that*

$$S = \inf_{u \in H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^2}{\|u\|_{2_s^*}^2}, \tag{2.1}$$

where $2_s^* = \frac{2N}{N-2s}$. Moreover, $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for any $q \in [2, 2_s^*]$ and compactly embedded into $L^q_{\text{loc}}(\mathbb{R}^N)$ for every $q \in [2, 2_s^*)$.

Next, we introduce the fractional Gagliardo–Nirenberg inequality of Hartree type established in [15].

Lemma 2.2. *Let $0 < s < 1$, $N > 2s$ and $q \in (2_\alpha, 2^*_{\alpha,s})$. Then, for all $u \in H^s(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \, \mathrm{d}x \le \widetilde{C} \left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^{2q\gamma_{q,s}} \|u\|_2^{2q(1-\gamma_{q,s})}, \tag{2.2}$$

where $\gamma_{q,s} = \frac{Nq-N-\alpha}{2qs}$ and the optimal constant $\widetilde{C}$ is given by

$$\widetilde{C} = \frac{2sq}{2sq - Nq + N + \alpha} \left(\frac{2sq - Nq + N + \alpha}{Nq - N - \alpha} \right)^{\frac{Nq-N-\alpha}{2s}} \|w\|_2^{2-2q}, \tag{2.3}$$

where W is the ground state solution of $(-\Delta)^s W + W - (I_\alpha * |W|^q)|W|^{q-2}W = 0$.

The following Hardy–Littlewood–Sobolev inequality is of importance, see [31].

Lemma 2.3. *Let $\alpha \in (0, N)$ and $r, t > 1$ with $\frac{1}{r} + \frac{1}{t} = 1 + \frac{\alpha}{N}$. Let $f \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$, then there exists a constant $C(r, t, \alpha, N)$ such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{N-\alpha}} \, \mathrm{d}x\mathrm{d}y \right| \leq C(r, t, \alpha, N) \|f\|_r \|h\|_t.$$

In particular, if $r = t = \frac{2N}{N+\alpha}$, then

$$C(r, t, \alpha, N) = C_\alpha := \pi^{\frac{N-\alpha}{2}} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{N+\alpha}{2}\right)} \left\{ \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)} \right\}^{-\frac{\alpha}{N}}. \tag{2.4}$$

Remark 2.4. Let $r = t = \frac{2N}{N+\alpha}$ and $f = h = |u|^p \in L^r(\mathbb{R}^N)$, then according to Lemma 2.3 the integral

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha}} \, \mathrm{d}x\mathrm{d}y$$

is well defined. Thus, for $u \in H^s(\mathbb{R}^N)$, by Lemma 2.1, we have

$$\frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{N-2s}.$$

In particular, when $p = \frac{N+\alpha}{N}$, for any $u \in H^s(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{N+\alpha}{N}}\right) |u|^{\frac{N+\alpha}{N}} \, \mathrm{d}x \leq S_\alpha^{-\frac{N+\alpha}{N}} \left(\int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x \right)^{\frac{N+\alpha}{N}}, \tag{2.5}$$

where S_α is relevant to the following minimization problem:

$$S_\alpha = \inf \left\{ \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x : u \in L^2(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{N+\alpha}{N}}\right) |u|^{\frac{N+\alpha}{N}} \, \mathrm{d}x = 1 \right\} > 0. \tag{2.6}$$

Furthermore, by [31, Theorem 4.3] and [30, Theorem 3.1], S_α is achieved by the function

$$V_\epsilon(x) = C \left(\frac{\epsilon}{\epsilon^2 + |x-z|^2} \right)^{\frac{N}{2}} \tag{2.7}$$

for some $C \in \mathbb{R}$, $\epsilon \in \mathbb{R}^+$, and $z \in \mathbb{R}^N$.

In what follows, we show the weak compactness result for the nonlocal nonlinearities, see [49, Lemma 2.7].

Lemma 2.5. *Suppose that $q \in \left[\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2s}\right]$. If $\{u_n\}$ is a sequence satisfying $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^N)$, then, for any $\varphi \in H^s(\mathbb{R}^N)$, we have*

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^q) |u_n|^{q-2} u_n \varphi \, \mathrm{d}x \to \int_{\mathbb{R}^N} (I_\alpha * |u|^q) |u|^{q-2} u \varphi \, \mathrm{d}x$$

as $n \to \infty$.

In view of the Brezis–Lieb lemma for the Riesz potential (see [10, 18]), we obtain the following lemma.

Lemma 2.6. *Let* $q \in \left[\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2s}\right]$ *and* $\{u_n\}$ *be a bounded sequence in* $H^s(\mathbb{R}^N)$*. If* $u_n \to u$ *a.e. on* $\mathbb{R}^N$ *as* $n \to \infty$*, then*

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^q)|u_n - u|^q \, \mathrm{d}x = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \, \mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \, \mathrm{d}x + o_n(1).$$

Now, we introduce the following Pohozaev identity, which can be derived from [8, 28].

Proposition 2.7. *Let* $u \in H^s(\mathbb{R}^N)$ *be a weak solution of* (1.1)*, then* u *satisfies the Pohozaev identity:*

$$\begin{aligned} &\frac{N-2s}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 + \frac{\lambda N}{2}\|u\|_2^2 \\ &= \frac{\gamma(N+\alpha)}{22_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha}\,\mathrm{d}x + \frac{\mu(N+\alpha)}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,\mathrm{d}x. \end{aligned} \tag{2.8}$$

Next, we show the Pohozaev manifold.

Lemma 2.8. *Let* $u \in H^s(\mathbb{R}^N)$ *be a weak solution of* (1.1)*, the Pohozaev manifold is the following:*

$$\mathcal{P}_q(c) = \{u \in S(c) : P_q(u) = 0\},$$

where

$$P_q(u) = \left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 - \frac{\mu}{2s}\left(N - \frac{N+\alpha}{q}\right)\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,\mathrm{d}x.$$

Proof. In view of Proposition 2.7, we know that u satisfies the Pohozaev identity as follows:

$$\begin{aligned} &\frac{N-2s}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 + \frac{\lambda N}{2}\|u\|_2^2 \\ &= \frac{\gamma N}{2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha}\,\mathrm{d}x + \frac{\mu(N+\alpha)}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,\mathrm{d}x. \end{aligned}$$

Moreover, since u is the weak solution of (1.1), we have

$$\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 + \lambda\|u\|_2^2 = \gamma\int_{\mathbb{R}^N}(I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha}\,\mathrm{d}x + \mu\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,\mathrm{d}x.$$

Therefore, we have

$$\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 - \frac{\mu}{2s}\left(N - \frac{N+\alpha}{q}\right)\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,\mathrm{d}x = 0,$$

which finishes the proof. □

In light of Lemma 2.8, it is clear that any critical point of $E_q|_{S(c)}$ belongs to $\mathcal{P}_q(c)$. Thus, the properties of $\mathcal{P}_q(c)$ are related to the minimax structure of $E_q|_{S(c)}$. Actually, for each $u \in S(c)$ and $t \in \mathbb{R}$, we define

$$u_t(x) := t^{\frac{N}{2}} u(tx) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

It results that $u_t \in S(c)$. Then we define the fibering map $t \in (0, \infty) \mapsto \Phi_u(t) := E_q(u_t)$ given by

$$\begin{aligned} \Phi_u(t) &= \frac{1}{2} t^{2s} \big\|(-\Delta)^{\frac{s}{2}} u\big\|_2^2 - \frac{\gamma}{22_\alpha} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha} \,\mathrm{d}x \\ &\quad - \frac{\mu}{2q} t^{Nq-N-\alpha} \int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \,\mathrm{d}x. \end{aligned} \tag{2.9}$$

Lemma 2.9. *Let $u \in S(c)$, then $t \in \mathbb{R}$ is the critical point of $\Phi_u(t)$ if and only if $u_t \in \mathcal{P}_q(c)$.*

Proof. For $u \in S(c)$ and $t \in \mathbb{R}$, it is easy to check that

$$\begin{aligned} \Phi_u'(t) &= s t^{2s-1} \big\|(-\Delta)^{\frac{s}{2}} u\big\|_2^2 - \frac{\mu}{2} \left(N - \frac{N+\alpha}{q}\right) t^{Nq-N-\alpha-1} \int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \,\mathrm{d}x \\ &= \frac{s}{t} \big\|(-\Delta)^{\frac{s}{2}} u_t\big\|_2^2 - \frac{\mu}{2t} \left(N - \frac{N+\alpha}{q}\right) \int_{\mathbb{R}^N} (I_\alpha * |u_t|^q)|u_t|^q \,\mathrm{d}x \\ &= \frac{s P_q(u_t)}{t}. \end{aligned} \tag{2.10}$$

Then, by Lemma 2.8, we can easily draw this conclusion. □

It is natural to consider the decomposition of $\mathcal{P}_q(c)$ into the disjoint union $\mathcal{P}_q(c) = \mathcal{P}_q^+(c) \cup \mathcal{P}_q^0(c) \cup \mathcal{P}_q^-(c)$, where

$$\begin{aligned} \mathcal{P}_q^+(c) &:= \{u \in S(c) \mid \Phi_u'(1) = 0, \Phi_u''(1) > 0\}, \\ \mathcal{P}_q^-(c) &:= \{u \in S(c) \mid \Phi_u'(1) = 0, \Phi_u''(1) < 0\}, \\ \mathcal{P}_q^0(c) &:= \{u \in S(c) \mid \Phi_u'(1) = 0, \Phi_u''(1) = 0\}. \end{aligned}$$

Moreover, for $u \in \mathcal{P}_q(c)$, we have

$$\begin{aligned} \Phi_u''(1) &= s(2s-1) \big\|(-\Delta)^{\frac{s}{2}} u\big\|_2^2 \\ &\quad - \frac{\mu}{2} \left(N - \frac{N+\alpha}{q}\right) (Nq - N - \alpha - 1) \int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \,\mathrm{d}x \\ &= s(2s - Nq + N + \alpha) \big\|(-\Delta)^{\frac{s}{2}} u\big\|_2^2. \end{aligned} \tag{2.11}$$

We now analyze the structure of the Pohozaev manifold $\mathcal{P}_q(c)$. Similar to the arguments in [22, Lemma 3.1, Proposition 3.1] and [43], the following proposition holds.

Proposition 2.10. *Assume that $\mathcal{P}_q^0(c) = \emptyset$. Then $\mathcal{P}_q(c)$ is a smooth manifold of codimension 2 in $H^s(\mathbb{R}^N)$ and a smooth manifold of codimension 1 in $S(c)$. Moreover, if $u \in \mathcal{P}_q(c)$ is a critical point for $E_q|_{\mathcal{P}_q(c)}$, then u is a critical point for $E_q|_{S(c)}$.*

3. The case $2_\alpha < q < \overline{q}$

Lemma 3.1. *Let $\gamma, \mu > 0$ and $q \in (2_\alpha, \overline{q})$. Then the following statements are true.*

(i) *The functional E_q is bounded below and coercive on $S(c)$;*

(ii) $\sigma(c) := \inf_{u \in S(c)} E_q(u) < -\frac{\gamma}{22_\alpha} S_\alpha^{-2_\alpha} c^{22_\alpha} < 0$;

(iii) *Let $c_1, c_2 > 0$ be such that $c_1^2 + c_2^2 = c^2$. Then $\sigma(c) < \sigma(c_1) + \sigma(c_2)$.*

Proof. (i) By (1.4), (2.2) and (2.5), for every $u \in S(c)$, we have

$$E_q(u) = \frac{1}{2}\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^2 - \frac{\gamma}{22_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha}\,\mathrm{d}x - \frac{\mu}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,\mathrm{d}x$$
$$\geq \frac{1}{2}\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^2 - \frac{\gamma}{22_\alpha} S_\alpha^{-2_\alpha} c^{22_\alpha} - \frac{\mu\widetilde{C}}{2q} c^{2q - \frac{Nq-N-\alpha}{s}} \left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^{\frac{Nq-N-\alpha}{s}}.$$

Since $2_\alpha < q < \overline{q}$, we can deduce that the functional E_q is bounded below and coercive on $S(c)$.

(ii) By (2.5) and (2.7), we have

$$\int_{\mathbb{R}^N}(I_\alpha * |V_\epsilon|^{2_\alpha})|V_\epsilon|^{2_\alpha}\,\mathrm{d}x = S_\alpha^{-2_\alpha}\left(\int_{\mathbb{R}^N}|V_\epsilon|^2\,\mathrm{d}x\right)^{2_\alpha}. \tag{3.1}$$

Then we define $\upsilon := c\frac{V_\epsilon}{\|V_\epsilon\|_2}$ and $\upsilon_t := t^{\frac{N}{2}}\upsilon(tx)$. It is clear that $\upsilon \in S(c)$ and $\upsilon_t \in S(c)$. By (2.9) and (3.1), we have

$$\begin{aligned}
&E_q(\upsilon_t)\\
&= \frac{1}{2}t^{2s}\left\|(-\Delta)^{\frac{s}{2}}\upsilon\right\|_2^2 - \frac{\gamma}{22_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\upsilon|^{2_\alpha})|\upsilon|^{2_\alpha}\,\mathrm{d}x - \frac{\mu}{2q}t^{Nq-N-\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\upsilon|^q)|\upsilon|^q\,\mathrm{d}x\\
&= \frac{1}{2}t^{2s}\left\|(-\Delta)^{\frac{s}{2}}\upsilon\right\|_2^2 - \frac{\mu}{2q}t^{Nq-N-\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\upsilon|^q)|\upsilon|^q\,\mathrm{d}x - \frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha}.
\end{aligned}$$

Thus, in view of $2_\alpha < q < \overline{q}$, there exists $t_0 \in (0,1)$ such that $\frac{1}{2}t^{2s}\left\|(-\Delta)^{\frac{s}{2}}\upsilon\right\|_2^2 - \frac{\mu}{2q}t^{Nq-N-\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\upsilon|^q)|\upsilon|^q\,\mathrm{d}x < 0$. Then, we have

$$E_q(\upsilon_t) < -\frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha} < 0,$$

namely, there exists some $\Upsilon = \upsilon_t \in S(c)$ such that $E_q(\Upsilon) < -\frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha} < 0$. Thus, we can deduce that $\sigma(c) < -\frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha} < 0$.

(iii) Let $\{u_n\} \subset S(c)$ be a bounded minimizing sequence for $\sigma(c)$. Then for any $\theta \in (1, \sqrt{2})$ and $u \in S(c)$, it holds that $\theta u \in S(\theta c)$ and

$$\begin{aligned}
E_q(\theta u_n) - \theta^2 E_q(u_n) &= \frac{\theta^2 - \theta^{22_\alpha}}{22_\alpha}\gamma\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2_\alpha})|u_n|^{2_\alpha}\,\mathrm{d}x\\
&\quad + \frac{\theta^2 - \theta^{2q}}{2q}\mu\int_{\mathbb{R}^N}(I_\alpha * |u_n|^q)|u_n|^q\,\mathrm{d}x\\
&< 0.
\end{aligned}$$

This implies that $\sigma(\theta c) \le \theta^2 \sigma(c)$, where the equality holds if and only if

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2_\alpha})|u_n|^{2_\alpha}\,\mathrm{d}x + \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

However, this is impossible, since otherwise we find that

$$0 > \sigma(c) = \lim_{n\to\infty} E_q(u_n) \ge \liminf_{n\to\infty} \frac{1}{2}\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 \ge 0.$$

Therefore, we have the strict inequality $\sigma(\theta c) < \theta^2\sigma(c)$. It follows that

$$\sigma(c) < \theta^2\sigma\left(\frac{c}{\theta}\right) = \sigma\left(\frac{c}{\theta}\right) + (\theta^2-1)\sigma\left(\frac{1}{\sqrt{\theta^2-1}}\cdot\frac{\sqrt{\theta^2-1}}{\theta}c\right) < \sigma\left(\frac{c}{\theta}\right) + \sigma\left(\frac{\sqrt{\theta^2-1}}{\theta}c\right).$$

Then, choosing $c_1 = \frac{c}{\theta}$, $c_2 = \frac{\sqrt{\theta^2-1}}{\theta}c$, we have $c_1^2 + c_2^2 = c^2$ and $\sigma(c) < \sigma(c_1) + \sigma(c_2)$. The proof is completed. □

Lemma 3.2. *Assume that $\gamma, \mu > 0$ and $2_\alpha < q < \overline{q}$. Let $\{u_n\} \subset H^s(\mathbb{R}^N)$ be a sequence such that $E_q(u_n) \to \sigma(c)$ and $\|u_n\|_2 = c_n \to c$. Then the sequence $\{u_n\}$ is relatively compact in $H^s(\mathbb{R}^N)$ up to translations, that is, there exist a subsequence, still denoted by $\{u_n\}$, a sequence of points $\{y_n\} \subset \mathbb{R}^N$, and a function $u_0 \in S(c)$ such that $u_n(\cdot + y_n) \to u_0$ strongly in $H^s(\mathbb{R}^N)$.*

Proof. It follows from Lemma 3.1(i) and $c_n \to c$ that the sequence $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Then according to the fractional concentration-compactness principle (see [14, Lemma 2.4]), we take $\varsigma_n := \frac{c}{c_n}u_n$, and there exists a subsequence, still denoted by $\{\varsigma_n\}$, for which one of the following properties holds:

(i) Compactness: there exists a sequence $\{y_n\}$ in $\mathbb{R}^N$ such that, for any $\varepsilon > 0$, there exists $0 < r < \infty$ with

$$\int_{|x-y_n|\le r} |\varsigma_n(x)|^2\,\mathrm{d}x \ge c^2 - \varepsilon.$$

(ii) Vanishing: for all $r < \infty$, it follows that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{|x-y_n|\le r} |\varsigma_n(x)|^2\,\mathrm{d}x = 0.$$

(iii) Dichotomy: there exist a constant $c_1 \in (0, c)$ and two bounded sequences $\{\nu_n\}$, $\{\omega_n\}$ such that

$$\operatorname{supp}\nu_n \cap \operatorname{supp}\omega_n = \emptyset; \quad |\nu_n| + |\omega_n| \le |\varsigma_n|;$$
$$\|\nu_n\|_2^2 \to c_1^2, \quad \|\omega_n\|_2^2 \to c_2^2 := c^2 - c_1^2 \quad \text{as } n \to \infty;$$
$$\|\varsigma_n - \nu_n - \omega_n\|_r \to 0 \quad \text{for } 2 \le r < \frac{2N}{N-2s};$$
$$\liminf_{n\to\infty}\left\{\left\|(-\Delta)^{\frac{s}{2}}\varsigma_n\right\|_2^2 - \left\|(-\Delta)^{\frac{s}{2}}\nu_n\right\|_2^2 - \left\|(-\Delta)^{\frac{s}{2}}\omega_n\right\|_2^2\right\} \ge 0.$$

Firstly, we verify that the vanishing cannot occur. Suppose the contrary. Then by [13, Lemma 2.2] we have that $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_s^*$. Noticing $q \in (2_\alpha, 2_{\alpha,s}^*)$ implying that $\frac{2Nq}{N+\alpha} \in (2, 2_s^*)$, by (1.4), (2.5) and Lemma 2.3, it follows that

$$\begin{aligned}
&\sigma(c) + o_n(1) \\
&= \frac{1}{2}\big\|(-\Delta)^{\frac{s}{2}} u_n\big\|_2^2 - \frac{\gamma}{22_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2_\alpha})|u_n|^{2_\alpha}\,\mathrm{d}x - \frac{\mu}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^q)|u_n|^q\,\mathrm{d}x \\
&\geq \frac{1}{2}\big\|(-\Delta)^{\frac{s}{2}} u_n\big\|_2^2 - \frac{\gamma}{22_\alpha} S_\alpha^{-2_\alpha} c^{22_\alpha} - \frac{\mu C_\alpha}{2q}\|u_n\|_{\frac{2Nq}{N+\alpha}}^{2q} \\
&\geq \frac{1}{2}\big\|(-\Delta)^{\frac{s}{2}} u_n\big\|_2^2 - \frac{\gamma}{22_\alpha} S_\alpha^{-2_\alpha} c^{22_\alpha} + o_n(1) \\
&\geq -\frac{\gamma}{22_\alpha} S_\alpha^{-2_\alpha} c^{22_\alpha},
\end{aligned}$$

which contradicts Lemma 3.1(ii).

Next we claim dichotomy cannot occur. Otherwise, according to [2, Lemma 2.14] and [5, Proposition 1.7.6 with Lemma 1.7.5(ii)] which state that

$$\int_{\mathbb{R}^N}(I_\alpha * |\varsigma_n|^q)|\varsigma_n|^q\,\mathrm{d}x = \int_{\mathbb{R}^N}(I_\alpha * |\nu_n|^q)|\nu_n|^q\,\mathrm{d}x + \int_{\mathbb{R}^N}(I_\alpha * |\omega_n|^q)|\omega_n|^q\,\mathrm{d}x + o_n(1)$$

for $q \in [2_\alpha, \overline{q})$. Then we have

$$\sigma(c) = \lim_{n\to\infty} E_q(u_n) = \lim_{n\to\infty} E_q(\varsigma_n) \geq \limsup_{n\to\infty}(E_q(\nu_n) + E_q(\omega_n)) \geq \sigma(c_1) + \sigma(c_2),$$

which contradicts Lemma 3.1(iii). Thus, the compactness holds, then there exist $\{y_n\} \subset \mathbb{R}^N$ and $u_0 \in S(c)$ such that $\varsigma_n(\cdot + y_n) \to u_0$ in $L^2(\mathbb{R}^N)$. Since $c_n \to c$ and $\{u_n\}$ is bounded, then we have that $\overline{u}_n := u_n(\cdot + y_n) \to u_0$ in $L^2(\mathbb{R}^N)$. Meanwhile, by interpolation inequality and fractional Sobolev embedding theorem, we have

$$\|\overline{u}_n - u_0\|_r^r \leq \|\overline{u}_n - u_0\|_2^{\theta r}\|\overline{u}_n - u_0\|_{2_s^*}^{(1-\theta)r} \leq C\|\overline{u}_n - u_0\|_2^{\theta r} \to 0 \quad \text{as } n \to \infty,$$

where $r \in (2, 2_s^*)$ and $\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{2_s^*}$, then we have that $\overline{u}_n := u_n(\cdot + y_n) \to u_0$ in $L^r(\mathbb{R}^N)$ for $2 \leq r < 2_s^*$. Then, by Lemmas 2.6 and 2.3, we can imply that

$$\begin{aligned}
\int_{\mathbb{R}^N}(I_\alpha * |\overline{u}_n|^{2_\alpha})|\overline{u}_n|^{2_\alpha}\,\mathrm{d}x &= \int_{\mathbb{R}^N}(I_\alpha * |u_0|^{2_\alpha})|u_0|^{2_\alpha}\,\mathrm{d}x + o_n(1), \\
\int_{\mathbb{R}^N}(I_\alpha * |\overline{u}_n|^{q})|\overline{u}_n|^{q}\,\mathrm{d}x &= \int_{\mathbb{R}^N}(I_\alpha * |u_0|^{q})|u_0|^{q}\,\mathrm{d}x + o_n(1).
\end{aligned}$$

Hence, we can deduce that

$$\sigma(c) \leq E_q(u_0) \leq \liminf_{n\to\infty} E_q(\overline{u}_n) = \liminf_{n\to\infty} E_q(u_n) = \sigma(c),$$

which implies that $E_q(u_0) = \sigma(c)$ and $\big\|(-\Delta)^{\frac{s}{2}}\overline{u}_n\big\|_2^2 \to \big\|(-\Delta)^{\frac{s}{2}}u_0\big\|_2^2$. Consequently, through the above analysis, we have that $\|\overline{u}_n\|_{H^s(\mathbb{R}^N)} \to \|u_0\|_{H^s(\mathbb{R}^N)}$, namely, $u_n(\cdot + y_n) \to u_0$ strongly in $H^s(\mathbb{R}^N)$, and u_0 is a minimizer for $\sigma(c)$. We complete the proof. □

Proof of Theorem 1.3. It follows from Lemma 3.2 that there exists a minimizer u_0 for E_q on $S(c)$. Hence, the infimum $\sigma(c) < -\frac{\gamma}{22_\alpha} S_\alpha^{-2_\alpha} c^{22_\alpha}$ is achieved by $u_0 \in S(c)$ which is a ground state of problem (1.1) and (1.2). Then, let u^* denote the Schwartz rearrangement of $|u_0|$ and we have

$$\|u^*\|_2^2 = \|u_0\|_2^2 = c^2. \tag{3.2}$$

By the Riesz's rearrangement inequality [31, Theorem 3.7], we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u^*|^q)|u^*|^q \,\mathrm{d}x \geq \int_{\mathbb{R}^N} (I_\alpha * |u_0|^q)|u_0|^q \,\mathrm{d}x, \tag{3.3}$$

$$\int_{\mathbb{R}^N} (I_\alpha * |u^*|^{2_\alpha})|u^*|^{2_\alpha} \,\mathrm{d}x \geq \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{2_\alpha})|u_0|^{2_\alpha} \,\mathrm{d}x. \tag{3.4}$$

And the fractional Polya–Szegö inequality [38] shows that

$$\left\|(-\Delta)^{\frac{s}{2}} u^*\right\|_2^2 \leq \left\|(-\Delta)^{\frac{s}{2}} u_0\right\|_2^2. \tag{3.5}$$

Thus, $u^* \in S(c)$ and $E_q(u^*) \leq E_q(u_0) = \sigma(c)$. According to the definition of $\sigma(c)$, we have $E_q(u^*) = \sigma(c)$, which implies that $\sigma(c)$ is achieved by the real-valued positive and radially symmetric nonincreasing function. Moreover, corresponding to u^*, there exists a Lagrange multiplier $\lambda^* \in R$ such that

$$\begin{aligned}
\lambda^* c^2 &= -\left\|(-\Delta)^{\frac{s}{2}} u^*\right\|_2^2 + \gamma \int_{\mathbb{R}^N} (I_\alpha * |u^*|^{2_\alpha})|u^*|^{2_\alpha} \,\mathrm{d}x + \mu \int_{\mathbb{R}^N} (I_\alpha * |u^*|^q)|u^*|^q \,\mathrm{d}x \\
&= -2\sigma(c) + \frac{\gamma(2_\alpha - 1)}{2_\alpha} \int_{\mathbb{R}^N} (I_\alpha * |u^*|^{2_\alpha})|u^*|^{2_\alpha} \,\mathrm{d}x + \frac{\mu(q-1)}{q} \int_{\mathbb{R}^N} (I_\alpha * |u^*|^q)|u^*|^q \,\mathrm{d}x \\
&> -2\sigma(c),
\end{aligned}$$

together with Lemma 3.1(ii), we have

$$\lambda^* > \frac{\gamma}{2_\alpha} S_\alpha^{-2_\alpha} c^{\frac{2\alpha}{N}}.$$

We complete the proof. □

4. The case $q = \overline{q} = \frac{N+2s+\alpha}{N}$

Proof of Theorem 1.4. Let $u \in S(c)$ and $t > 0$. By (2.2) and (2.10), we have

$$\begin{aligned}
\Phi_u'(t) &= st^{2s-1}\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^2 - \frac{\mu s N}{N+2s+\alpha} t^{2s-1} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\overline{q}})|u|^{\overline{q}} \,\mathrm{d}x \\
&\geq st^{2s-1}\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^2 - \frac{\mu s N \widetilde{C}}{N+2s+\alpha} c^{\frac{4s+2\alpha}{N}} t^{2s-1}\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^2 \\
&= \left(s - \frac{\mu s N \widetilde{C}}{N+2s+\alpha} c^{\frac{4s+2\alpha}{N}}\right) t^{2s-1}\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^2 \\
&> 0 \quad \text{if } \mu < \frac{N+2s+\alpha}{N\widetilde{C}} c^{-\frac{4s+2\alpha}{N}} := \mu^*.
\end{aligned}$$

This implies that the fiber map $\Phi_u(t)$ is strictly increasing, and then the functional E_q has no critical point on $S(c)$ for $0<\mu<\mu^*$. Hence, when $0<\mu<\mu^*$, problem (1.1) and (1.2) has no solution for any $\lambda\in\mathbb{R}$. We complete the proof. □

5. The case $\overline{q}<q\leq 2^*_{\alpha,s}$

In this case, we notice that the functional E_q is unbounded from below on $S(c)$. Thus, we shall restrict it to a natural constraint manifold $\mathcal{P}_q(c)$ on which E_q is bounded below and then we may find critical points of E_q.

Lemma 5.1. *Assume that $\gamma,\mu>0$ and $\overline{q}<q\leq 2^*_{\alpha,s}$. Then the following statements are true.*

(i) *There exists a unique $\widetilde{t}_u:=t(u)>0$ such that $u_{\widetilde{t}_u}\in\mathcal{P}_q(c)=\mathcal{P}_q^-(c)$ and*

$$E_q(u_{\widetilde{t}_u})=\max_{t>0}E_q(u_t);$$

(ii) *The map $u\in S(c)\mapsto\widetilde{t}_u\in\mathbb{R}$ is of class C^1.*

Proof. (i) In view of $\overline{q}<q\leq 2^*_{\alpha,s}$ and (2.11), we have $\mathcal{P}_q(c)=\mathcal{P}_q^-(c)$. Fix $u\in S(c)$ and we define

$$F(t):=\frac{2qs}{\mu(Nq-N-\alpha)}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 t^{N+\alpha+2s-Nq}\quad\text{for } t>0.$$

According to Lemma 2.9, we have $u_t\in\mathcal{P}_q(c)$ if and only if $F(t)=\int_{\mathbb{R}^N}(I_\alpha*|u|^q)|u|^q\,\mathrm{d}x$. We notice that $F(t)$ is decreasing on $(0,+\infty)$, $\lim_{t\to0^+}F(t)=+\infty$, $\lim_{t\to+\infty}F(t)=0$. This implies that there exists a unique $\widetilde{t}_u>0$ such that $u_{\widetilde{t}_u}\in\mathcal{P}_q(c)$. Furthermore, we also obtain that $\Phi_u'(t)>0$ on $(0,\widetilde{t}_u)$ and $\Phi_u'(t)<0$ on $(\widetilde{t}_u,+\infty)$, namely, $E_q(u_{\widetilde{t}_u})=\max_{t>0}E_q(u_t)$.

(ii) By a direct application of the implicit function theorem on the C^1 function $G\colon\mathbb{R}\times S(c)\to\mathbb{R}$ defined by $G(t,u)=\Phi_u'(t)$, we easily reach the conclusion. □

Lemma 5.2. *Assume that $\gamma,\mu>0$ and $q\in(\overline{q},2^*_{\alpha,s}]$. Then the functional E_q is bounded below and coercive on $\mathcal{P}_q(c)$ for $0<c<c_*$, where c_* is defined as*

$$\begin{aligned}c_*:=&\left[\frac{2qs}{\mu\widetilde{C}(Nq-N-\alpha)}\right]^{\frac{s}{(2\alpha-1)(Nq-N-\alpha)+2qs-2s\cdot2_\alpha}}\\&\times\left[\frac{\gamma(Nq-N-\alpha)}{2_\alpha S_\alpha^{2_\alpha}(Nq-N-\alpha-2s)}\right]^{\frac{N+\alpha+2s-Nq}{2(2\alpha-1)(Nq-N-\alpha)+4qs-4s\cdot2_\alpha}}.\end{aligned}$$

Proof. Let $u\in\mathcal{P}_q(c)$, then we have

$$\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2=\frac{\mu}{2s}\left(N-\frac{N+\alpha}{q}\right)\int_{\mathbb{R}^N}(I_\alpha*|u|^q)|u|^q\,\mathrm{d}x.$$

Combined with Lemma 2.2, then

$$\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 \geq \left(\frac{2qs}{\mu\widetilde{C}(Nq-N-\alpha)}\cdot c^{-\frac{2qs+N+\alpha-Nq}{s}}\right)^{\frac{2s}{Nq-N-\alpha-2s}}.$$

It follows that

$$\begin{aligned}
&E_q(u)\\
&=\frac{1}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2-\frac{\gamma}{22_\alpha}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2\alpha})|u|^{2\alpha}\,\mathrm{d}x-\frac{\mu}{2q}\int_{\mathbb{R}^N}(I_\alpha*|u|^q)|u|^q\,\mathrm{d}x\\
&=\left(\frac{1}{2}-\frac{s}{Nq-N-\alpha}\right)\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2-\frac{\gamma}{22_\alpha}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2\alpha})|u|^{2\alpha}\,\mathrm{d}x\\
&\geq\left(\frac{1}{2}-\frac{s}{Nq-N-\alpha}\right)\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2-\frac{\gamma}{22_\alpha}S_\alpha^{-2\alpha}c^{22_\alpha}\\
&\geq\left(\frac{1}{2}-\frac{s}{Nq-N-\alpha}\right)\left(\frac{2qs}{\mu\widetilde{C}(Nq-N-\alpha)}\cdot c^{-\frac{2qs+N+\alpha-Nq}{s}}\right)^{\frac{2s}{Nq-N-\alpha-2s}}-\frac{\gamma}{22_\alpha}S_\alpha^{-2\alpha}c^{22_\alpha}\\
&>0,
\end{aligned}$$

provided that $c<c_*$. We complete the proof. □

Through the above discussion, we can define

$$m_q(c):=\inf_{u\in\mathcal{P}_q(c)}E_q(u).$$

We now work in the subspace of functions in $H^s(\mathbb{R}^N)$ which are radially symmetric with respect to 0, denoted by $H^s_r(\mathbb{R}^N)$, and we define

$$S_r(c):=S(c)\cap H^s_r(\mathbb{R}^N)\quad\text{and}\quad\mathcal{P}^r_q(c):=\mathcal{P}_q\cap H^s_r(\mathbb{R}^N).$$

Lemma 5.3. *Suppose that $\gamma,\mu>0$ and $q\in(\overline{q},2^*_{\alpha,s}]$. Then for $0<c<c_*$, the following statements are true.*

(i) $m_q(c)=\inf_{u\in\mathcal{P}_q(c)}E_q(u)=\inf_{u\in\mathcal{P}^r_q(c)}E_q(u)>0$;

(ii) *If $m_q(c)$ is achieved, then it is achieved by a Schwarz symmetric function.*

Proof. (i) Since $\mathcal{P}^r_q(c)\subset\mathcal{P}_q(c)$, then we obtain that

$$\inf_{u\in\mathcal{P}_q(c)}E_q(u)\leq\inf_{u\in\mathcal{P}^r_q(c)}E_q(u).\tag{5.1}$$

Next, we claim that $\inf_{u\in\mathcal{P}_q(c)}E_q(u)\geq\inf_{u\in\mathcal{P}^r_q(c)}E_q(u)$. Indeed, by Lemma 5.1(i), it is clear to see that

$$m_q(c)=\inf_{u\in\mathcal{P}_q(c)}E_q(u)=\inf_{u\in S(c)}\max_{t>0}E_q(u_t)\tag{5.2}$$

and $E_q(u_{\widetilde{t}_u}) = \max_{t>0} E_q(u_t)$. Let $u \in S(c)$ and $\vartheta \in S(c)$ be the Schwartz rearrangement of $|u|$. Then by (3.2)–(3.5), we obtain that

$$\begin{aligned}\Phi_\vartheta(t) &= \frac{1}{2}t^{2s}\big\|(-\Delta)^{\frac{s}{2}}\vartheta\big\|_2^2 - \frac{\gamma}{22_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\vartheta|^{2_\alpha})|\vartheta|^{2_\alpha}\,\mathrm{d}x \\ &\quad - \frac{\mu}{2q}t^{Nq-N-\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\vartheta|^q)|\vartheta|^q\,\mathrm{d}x \\ &\le \frac{1}{2}t^{2s}\big\|(-\Delta)^{\frac{s}{2}}u\big\|_2^2 - \frac{\gamma}{22_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha}\,\mathrm{d}x \\ &\quad - \frac{\mu}{2q}t^{Nq-N-\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,\mathrm{d}x \\ &= \Phi_u(t).\end{aligned} \tag{5.3}$$

Moreover, we also have $\Phi'_\vartheta(t) \le \Phi'_u(t)$ for $t \in (0,+\infty)$, which implies $\widetilde{t}_\vartheta \le \widetilde{t}_u$. Then it follows from (5.3) that

$$\inf_{u\in\mathcal{P}_q^r(c)} E_q(u) \le \max_{t>0} E_q(\vartheta_t) = \Phi_\vartheta(\widetilde{t}_\vartheta) \le \Phi_u(\widetilde{t}_\vartheta) = E_q(u_{\widetilde{t}_\vartheta}) \le \max_{t>0} E_q(u_t). \tag{5.4}$$

Then,by (5.2) and (5.4), we have

$$\inf_{u\in\mathcal{P}_q(c)} E_q(u) \ge \inf_{u\in\mathcal{P}_q^r(c)} E_q(u). \tag{5.5}$$

Thus, it follows from (5.1), (5.5) and Lemma 5.2 that

$$\inf_{u\in\mathcal{P}_q(c)} E_q(u) = \inf_{u\in\mathcal{P}_q^r(c)} E_q(u) > 0.$$

(ii) Let $v \in \mathcal{P}_q(c)$ satisfy $E_q(v) = m_q(c)$, and let κ be the Schwartz rearrangement of $|v|$. Indeed, according to (3.2)–(3.5), if $\big\|(-\Delta)^{\frac{s}{2}}\kappa\big\|_2^2 < \big\|(-\Delta)^{\frac{s}{2}}v\big\|_2^2$, or $\int_{\mathbb{R}^N}(I_\alpha * |\kappa|^q)|\kappa|^q\,\mathrm{d}x > \int_{\mathbb{R}^N}(I_\alpha * |v|^q)|v|^q\,\mathrm{d}x$, or $\int_{\mathbb{R}^N}(I_\alpha * |\kappa|^{2_\alpha})|\kappa|^{2_\alpha}\,\mathrm{d}x > \int_{\mathbb{R}^N}(I_\alpha * |v|^{2_\alpha})|v|^{2_\alpha}\,\mathrm{d}x$ hold, then, based on (5.2) and (5.4), we can deduce that

$$m_q(c) = \inf_{\kappa\in S(c)}\max_{t>0} E_q(\kappa_t) \le \max_{t>0} E_q(\kappa_t) = \Phi_\kappa(\widetilde{t}_\kappa) < \Phi_v(\widetilde{t}_\kappa) = E_q(v_{\widetilde{t}_\kappa}) \le E_q(v) = m_q(c),$$

which is a contradiction. As a result, we have

$$\big\|(-\Delta)^{\frac{s}{2}}\kappa\big\|_2^2 = \big\|(-\Delta)^{\frac{s}{2}}v\big\|_2^2, \quad \int_{\mathbb{R}^N}(I_\alpha * |\kappa|^q)|\kappa|^q\,\mathrm{d}x = \int_{\mathbb{R}^N}(I_\alpha * |v|^q)|v|^q\,\mathrm{d}x,$$
$$\int_{\mathbb{R}^N}(I_\alpha * |\kappa|^{2_\alpha})|\kappa|^{2_\alpha}\,\mathrm{d}x = \int_{\mathbb{R}^N}(I_\alpha * |v|^{2_\alpha})|v|^{2_\alpha}\,\mathrm{d}x.$$

Then combined with (3.2) we can imply that $\kappa \in \mathcal{P}_q^r(c)$ and $E_q(\kappa) = E_q(v) = m_q(c)$. We complete the proof. $\square$

Next, we shall take advantage of the homotopy-stable family and some related known results in order to establish the existence of Palais–Smale sequence.

Definition 5.4. [45, Definition 3.1] Let $\mathcal{D}$ be a closed subset of a metric space X. We say that a class $\mathcal{F}$ of compact subsets of X is a homotopy-stable family with boundary $\mathcal{D}$ provided that

(i) every set in $\mathcal{F}$ contains $\mathcal{D}$;

(ii) for any set $\Xi \in \mathcal{F}$ and any $\mathcal{E} \in C([0,1] \times X, X)$ satisfying $\mathcal{E}(t,x) = x$ for all $(t,x) \in (\{0\} \times X) \cup ([0,1] \times \mathcal{D})$, we have that $\mathcal{E}(\{1\} \times \Xi) \in \mathcal{F}$.

In particular, the above definition is still valid if the boundary $\mathcal{D}$ is empty.

Lemma 5.5. *For $u \in S_r(c)$, $t \in \mathbb{R}$, the map*

$$T_u S_r(c) \to T_{u_{\tilde{t}_u}} S_r(c), \quad \psi \mapsto \psi_{\tilde{t}_u}$$

is an linear isomorphism, where $T_u S_r(c)$ denotes the tangent space to $S_r(c)$ in u.

Proof. The proof is standard, see [47, Lemma 5.5] or [1, Lemma 3.6], so we omit the details. □

Now we define the functional $\mathcal{J} \colon S_r(c) \mapsto \mathbb{R}$ by

$$\mathcal{J}(u) = E_q(u_{\tilde{t}_u}).$$

According to Lemma 5.1(ii), we obtain that the functional $\mathcal{J}$ is of class C^1.

Lemma 5.6. *It holds that*

$$\mathcal{J}'(u)[\psi] = E_q'(u_{\tilde{t}_u})[\psi_{\tilde{t}_u}]$$

for any $u \in S_r(c)$ and $\psi \in T_u S_r(c)$.

Proof. The proof is similar to [9, Lemma 3.15]; we omit it here. □

Then, the result for the existence of Palais–Smale sequences to a general homotopy-stable family is as follows.

Lemma 5.7. *Let $\mathcal{F}$ be a homotopy-stable family of compact subsets of $S_r(c)$ with closed boundary $\mathcal{D}$, and let*

$$e_{\mathcal{F}} := \inf_{\Xi \in \mathcal{F}} \max_{u \in \Xi} \mathcal{J}(u).$$

Assume that $\mathcal{D}$ is contained in a connected component of $\mathcal{P}_q^r(c)$ and

$$\max\{\sup \mathcal{J}(\mathcal{D}), 0\} < e_{\mathcal{F}} < \infty.$$

Then there exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{P}_q^r(c)$ for E_q restricted to $S_r(c)$ at level $e_{\mathcal{F}}$.

Proof. By Lemmas 5.5 and 5.6, similar to the arguments of [9, Lemma 3.16], we can easily obtain the conclusion. □

Lemma 5.8. *Suppose that $\gamma, \mu > 0$ and $q \in (\overline{q}, 2^*_{\alpha,s}]$. Then for $0 < c < c_*$, there exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{P}^r_q(c)$ for E_q restricted to $S_r(c)$ at level $m_q(c) > 0$.*

Proof. Let $\mathcal{F}_1$ be a family of all singletons belonging to $S_r(c)$. Clearly, the boundary $\mathcal{D}$ is empty. Then, by Definition 5.4 it is clearly a homotopy-stable family of compact subsets of $S_r(c)$ (without boundary). Besides, by Lemma 5.3 one has

$$e_{\mathcal{F}_1} = \inf_{\Xi\in\mathcal{F}_1} \max_{u\in\Xi} \mathcal{J}(u) = \inf_{u\in S_r(c)} \mathcal{J}(u) = \inf_{u\in\mathcal{P}^r_q(c)} E_q(u) = \inf_{u\in\mathcal{P}_q(c)} E_q(u) = m_q(c).$$

Consequently, choosing $\mathcal{F} = \mathcal{F}_1$, the lemma follows directly from Lemma 5.7. We complete the proof. □

5.1. The subcritical perturbation

Lemma 5.9. *Let $\gamma, \mu > 0$, $q \in (\overline{q}, 2^*_{\alpha,s})$ and $\{u_n\} \subset \mathcal{P}^r_q(c)$ be a bounded Palais–Smale sequence for E_q restricted to $S_r(c)$ at level $m_q(c) > 0$. Then there exists a constant $\widehat{\mu} > 0$ such that for every $\mu > \widehat{\mu}$, up to a subsequence, $u_n \to \widehat{u}$ strongly in $H^s_r(\mathbb{R}^N)$ for $0 < c < c_*$.*

Proof. Since $\{u_n\} \subset \mathcal{P}^r_q(c)$ is a bounded Palais–Smale sequence, then there exists $\widehat{u} \in H^s_r(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup \widehat{u} \text{ in } H^s_r(\mathbb{R}^N), \quad u_n \to \widehat{u} \text{ in } L^r(\mathbb{R}^N) \quad \text{for } 2 < r < 2^*_s,\ u_n \to \widehat{u} \text{ a.e. on } \mathbb{R}^N. \tag{5.6}$$

Based on the Lagrange multipliers rule, there exists $\lambda_n \in \mathbb{R}$ such that for every $\varphi \in H^s_r(\mathbb{R}^N)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi \,\mathrm{d}x + \lambda_n \int_{\mathbb{R}^N} u_n\varphi \,\mathrm{d}x - \gamma \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2\alpha})|u_n|^{2\alpha-2} u_n\varphi \,\mathrm{d}x \\ &- \mu \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^{q-2} u_n\varphi \,\mathrm{d}x \\ = {}& o_n(1)\|\varphi\|. \end{aligned} \tag{5.7}$$

It follows that

$$\begin{aligned} \lambda_n c^2 = \lambda_n \|u_n\|_2^2 = \gamma \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2\alpha})|u_n|^{2\alpha} \,\mathrm{d}x \\ + \mu \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \,\mathrm{d}x - \left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 + o_n(1), \end{aligned} \tag{5.8}$$

which implies that $\{\lambda_n\}$ is bounded. Then, up to a subsequence, there exists $\widehat{\lambda} \in \mathbb{R}$ such that $\lambda_n \to \widehat{\lambda}$ as $n \to \infty$. In view of $q \in (\overline{q}, 2^*_{\alpha,s})$, (5.8) and the fact of $P_q(u_n) = o_n(1)$, we have

$$
\begin{aligned}
\widehat{\lambda}c^2 &= \lim_{n\to\infty} \lambda_n c^2 \\
&= \lim_{n\to\infty} \left[\gamma \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2_\alpha})|u_n|^{2_\alpha} \,\mathrm{d}x \right. \\
&\qquad \left. + \mu \left(1 - \frac{Nq - N - \alpha}{2qs}\right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \,\mathrm{d}x\right] \\
&\geq 0,
\end{aligned} \tag{5.9}
$$

which implies $\widehat{\lambda} \geq 0$. Then, we claim that $\widehat{\lambda} \neq 0$. Otherwise, by (5.9), we have that $\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2_\alpha})|u_n|^{2_\alpha} \,\mathrm{d}x = o_n(1)$ and $\int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \,\mathrm{d}x = o_n(1)$, and then combined with $P_q(u_n) = o_n(1)$ we deduced that $\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 = o_n(1)$. Thus, we have $E_q(u_n) = m_q(c) + o_n(1) = o_n(1)$, which is contradictory to $m_q(c) > 0$. Consequently, $\widehat{\lambda} > 0$.

Next, we claim that $\widehat{u} \neq 0$. Suppose by contradiction that $\widehat{u} = 0$, obviously we have $\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 = o_n(1)$. Moreover, by Lemma 2.3 and (5.6), we have $\int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \,\mathrm{d}x = o_n(1)$. Then, in view of (5.8) and $\widehat{\lambda} > 0$, we have

$$
0 < \widehat{\lambda}c^2 = \lim_{n\to\infty} \gamma \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2_\alpha})|u_n|^{2_\alpha} \,\mathrm{d}x,
$$

which contradicts

$$
0 < m_q(c) = \lim_{n\to\infty} E_q(u_n) = -\frac{\gamma}{22_\alpha} \lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2_\alpha})|u_n|^{2_\alpha} \,\mathrm{d}x.
$$

As a consequence, $\widehat{u} \neq 0$.

Now, according to $u_n \rightharpoonup \widehat{u}$ in $H_r^s(\mathbb{R}^N)$, (5.7) and Lemma 2.5, it follows that $\widehat{u}$ is a weak solution such that

$$
(-\Delta)^s \widehat{u} + \widehat{\lambda}\widehat{u} = \gamma (I_\alpha * |\widehat{u}|^{2_\alpha})|\widehat{u}|^{2_\alpha - 2}\widehat{u} + \mu (I_\alpha * |\widehat{u}|^q)|\widehat{u}|^{q-2}\widehat{u} \quad \text{in } \mathbb{R}^N. \tag{5.10}
$$

Then, it follows from Lemma 2.8 that $P_q(\widehat{u}) = 0$.

Let $v_n := u_n - \widehat{u}$, then by (5.6) we have $v_n \rightharpoonup 0$ weakly in $H_r^s(\mathbb{R}^N)$. Thus, one has

$$
\left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 = \left\|(-\Delta)^{\frac{s}{2}} \widehat{u}\right\|_2^2 + \left\|(-\Delta)^{\frac{s}{2}} v_n\right\|_2^2 + o_n(1). \tag{5.11}
$$

Moreover, it follows from Lemma 2.6 and (5.6) that

$$
\begin{aligned}
\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2_\alpha})|u_n|^{2_\alpha} \,\mathrm{d}x &= \int_{\mathbb{R}^N} (I_\alpha * |\widehat{u}|^{2_\alpha})|\widehat{u}|^{2_\alpha} \,\mathrm{d}x + \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2_\alpha})|v_n|^{2_\alpha} \,\mathrm{d}x + o_n(1), \\
\int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \,\mathrm{d}x &= \int_{\mathbb{R}^N} (I_\alpha * |\widehat{u}|^q)|\widehat{u}|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} (I_\alpha * |v_n|^q)|v_n|^q \,\mathrm{d}x + o_n(1),
\end{aligned}
$$

and then by Lemma 2.3 we know that

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^q)|v_n|^q \,\mathrm{d}x = 0,
$$

which implies that

$$\int_{\mathbb{R}^N}(I_\alpha * |u_n|^q)|u_n|^q\,\mathrm{d}x = \int_{\mathbb{R}^N}(I_\alpha * |\widehat{u}|^q)|\widehat{u}|^q\,\mathrm{d}x + o_n(1). \tag{5.12}$$

By (5.11), (5.12) and the fact of $P_q(u_n) = o_n(1)$, we have

$$\left\|(-\Delta)^{\frac{s}{2}}\widehat{u}\right\|_2^2 + \left\|(-\Delta)^{\frac{s}{2}}v_n\right\|_2^2 = \frac{\mu}{2s}\left(N - \frac{N+\alpha}{q}\right)\int_{\mathbb{R}^N}(I_\alpha * |\widehat{u}|^q)|\widehat{u}|^q\,\mathrm{d}x + o_n(1),$$

which shows that $\left\|(-\Delta)^{\frac{s}{2}}v_n\right\|_2^2 = o_n(1)$ by the fact of $P_q(\widehat{u}) = 0$.

Moreover, by (5.10), we have that

$$E_q'(\widehat{u})\varphi + \widehat{\lambda}\int_{\mathbb{R}^N}\widehat{u}\varphi\,\mathrm{d}x = 0 \tag{5.13}$$

for every $\varphi \in H_r^s(\mathbb{R}^N)$. Taking $\varphi = v_n$ in (5.7) and (5.13), and then subtracting, we have

$$\left\|(-\Delta)^{\frac{s}{2}}v_n\right\|_2^2 + \widehat{\lambda}\|v_n\|_2^2 = \mu\int_{\mathbb{R}^N}(I_\alpha * |v_n|^q)|v_n|^q\,\mathrm{d}x + \gamma\int_{\mathbb{R}^N}(I_\alpha * |v_n|^{2_\alpha})|v_n|^{2_\alpha}\,\mathrm{d}x + o_n(1),$$

which implies that

$$\mathcal{L} := \lim_{n\to\infty}\widehat{\lambda}\|v_n\|_2^2 = \lim_{n\to\infty}\gamma\int_{\mathbb{R}^N}(I_\alpha * |v_n|^{2_\alpha})|v_n|^{2_\alpha}\,\mathrm{d}x, \tag{5.14}$$

where we also have used the fact that $\left\|(-\Delta)^{\frac{s}{2}}v_n\right\|_2^2 = o_n(1)$ and $\int_{\mathbb{R}^N}(I_\alpha * |v_n|^q)|v_n|^q\,\mathrm{d}x = o_n(1)$. It follows from (5.14) and (2.5) that either $\mathcal{L} = 0$ or $\mathcal{L} \geq \gamma^{-\frac{N}{\alpha}}(\widehat{\lambda}S_\alpha)^{\frac{N+\alpha}{\alpha}}$.

If $\mathcal{L} = 0$, by (5.14) and $\left\|(-\Delta)^{\frac{s}{2}}v_n\right\|_2^2 = o_n(1)$, then we obtain $u_n \to \widehat{u}$ in $H_r^s(\mathbb{R}^N)$.

If $\mathcal{L} \geq \gamma^{-\frac{N}{\alpha}}(\widehat{\lambda}S_\alpha)^{\frac{N+\alpha}{\alpha}}$, firstly, similar to the argument of Lemma 5.2, according to $P_q(\widehat{u}) = 0$ and $\|\widehat{u}\|_2 \leq c$, we can easily deduce $E_q(\widehat{u}) > 0$ for $0 < c < c_*$. Then, for one thing, we have

$$\begin{aligned}
m_q(c) + \frac{\widehat{\lambda}}{2}c^2 &= m_q(c) + \frac{1}{2}\lim_{n\to\infty}\lambda_n\|u_n\|_2^2 \geq m_q(c) + \frac{1}{2}\lim_{n\to\infty}\lambda_n\|v_n\|_2^2 \\
&= E_q(\widehat{u}) + \lim_{n\to\infty}\left(E_q(v_n) + \frac{\lambda_n}{2}\|v_n\|_2^2\right) \\
&= E_q(\widehat{u}) + \lim_{n\to\infty}\left(\frac{\lambda_n}{2}\|v_n\|_2^2 - \frac{\gamma}{22_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |v_n|^{2_\alpha})|v_n|^{2_\alpha}\,\mathrm{d}x\right) \\
&= E_q(\widehat{u}) + \lim_{n\to\infty}\frac{\gamma\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |v_n|^{2_\alpha})|v_n|^{2_\alpha}\,\mathrm{d}x \\
&\geq E_q(\widehat{u}) + \frac{\alpha}{2(N+\alpha)}\gamma^{-\frac{N}{\alpha}}(\widehat{\lambda}S_\alpha)^{\frac{N+\alpha}{\alpha}} > \frac{\alpha}{2(N+\alpha)}\gamma^{-\frac{N}{\alpha}}(\widehat{\lambda}S_\alpha)^{\frac{N+\alpha}{\alpha}}.
\end{aligned} \tag{5.15}$$

For another, we make use of the extremal function V_ε in (2.7) to estimate $m_q(c) + \frac{\widehat{\lambda}}{2}c^2$. We define $\tau := c\frac{V_\epsilon}{\|V_\epsilon\|_2}$ and $\tau_t := t^{\frac{N}{2}}\tau(tx)$. Clearly, $\tau \in S(c)$ and $\tau_t \in S(c)$. It follows from Lemma 5.1 that there exists a unique $\widetilde{t}_\tau > 0$ such that $\tau_{\widetilde{t}_\tau} \in \mathcal{P}_q(c)$ and $E_q(\tau_{\widetilde{t}_\tau}) = \max_{t>0} E_q(\tau_t)$, then we have

$$m_q(c) \leq \max_{t>0} E_q(\tau_t) = E_q(\tau_{\widetilde{t}_\tau}).$$

Moreover, by (2.9) and (3.1) we have

$$\begin{aligned} m_q(c) &\leq E_q(\tau_{\widetilde{t}_\tau}) \\ &= \frac{1}{2}(\widetilde{t}_\tau)^{2s}\big\|(-\Delta)^{\frac{s}{2}}\tau\big\|_2^2 - \frac{\gamma}{22_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\tau|^{2_\alpha})|\tau|^{2_\alpha}\,\mathrm{d}x \\ &\quad - \frac{\mu}{2q}(\widetilde{t}_\tau)^{Nq-N-\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\tau|^q)|\tau|^q\,\mathrm{d}x \\ &= \frac{1}{2}(\widetilde{t}_\tau)^{2s}\big\|(-\Delta)^{\frac{s}{2}}\tau\big\|_2^2 - \frac{\mu}{2q}(\widetilde{t}_\tau)^{Nq-N-\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\tau|^q)|\tau|^q\,\mathrm{d}x - \frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha}. \end{aligned}$$

Then, we choose

$$\widehat{\mu} := \frac{q\big\|(-\Delta)^{\frac{s}{2}}\tau\big\|_2^2}{\int_{\mathbb{R}^N}(I_\alpha * |\tau|^q)|\tau|^q\,\mathrm{d}x}(\widetilde{t}_\tau)^{2s+N+\alpha-Nq}.$$

Thus, for every $\mu > \widehat{\mu}$, there holds that

$$m_q(c) < -\frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha},$$

which implies that

$$m_q(c) + \frac{\widehat{\lambda}}{2}c^2 < -\frac{N\gamma}{2(N+\alpha)}S_\alpha^{-\frac{N+\alpha}{N}}c^{\frac{2(N+\alpha)}{N}} + \frac{\widehat{\lambda}}{2}c^2. \tag{5.16}$$

Let us consider the function $\mathcal{K}\colon \mathbb{R}^+ \to \mathbb{R}$ given by

$$\mathcal{K}(c) := -\frac{N\gamma}{2(N+\alpha)}S_\alpha^{-\frac{N+\alpha}{N}}c^{\frac{2(N+\alpha)}{N}} + \frac{\widehat{\lambda}}{2}c^2. \tag{5.17}$$

By a direct calculation, we have that $\mathcal{K}$ has a unique critical point $c_0 = \big(\frac{\widehat{\lambda}}{\gamma}\big)^{\frac{N}{2\alpha}}S_\alpha^{\frac{N+\alpha}{2\alpha}}$, which is also a global maximum point. Then, the maximum of $\mathcal{K}$ is

$$\mathcal{K}(c_0) = \frac{\alpha}{2(N+\alpha)}\gamma^{-\frac{N}{\alpha}}(\widehat{\lambda}S_\alpha)^{\frac{N+\alpha}{\alpha}}. \tag{5.18}$$

It follows from (5.16), (5.17) and (5.18) that

$$m_q(c) + \frac{\widehat{\lambda}}{2}c^2 < \frac{\alpha}{2(N+\alpha)}\gamma^{-\frac{N}{\alpha}}(\widehat{\lambda}S_\alpha)^{\frac{N+\alpha}{\alpha}}, \tag{5.19}$$

which contradicts (5.15), that is, $\mathcal{L} \geq \gamma^{-\frac{N}{\alpha}}(\widehat{\lambda}S_\alpha)^{\frac{N+\alpha}{\alpha}}$ is not true. Consequently, we have $u_n \to \widehat{u}$ in $H_r^s(\mathbb{R}^N)$. The proof is completed. $\square$

Proof of Theorem 1.5. By Lemmas 5.2 and 5.8, for $0 < c < c_*$, there exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{P}_q^r(c)$ for E_q restricted to $S_r(c)$ at level $m_q(c) > 0$. Then in view of Lemma 5.9, we have that there exists a constant $\widehat{\mu} > 0$ such that for every $\mu > \widehat{\mu}$, up to a subsequence, $u_n \to \widehat{u}$ strongly in $H_r^s(\mathbb{R}^N)$ for $0 < c < c_*$. Combined with Lemma 5.3(i), we can deduce that $\widehat{u}$ is a ground state normalized solution of (1.1) for some $\widehat{\lambda} > 0$. Moreover, by Lemma 5.3(ii) and the maximum principle (see [17, Theorem 8.19]), $\widehat{u}$ is a positive radial ground state normalized solution of (1.1) for some $\widehat{\lambda} > 0$. □

5.2. The critical perturbation

In this part, we consider the case of $q = 2^*_{\alpha,s} = \frac{N+\alpha}{N-2s}$, which corresponds to the fractional doubly critical Choquard equation with prescribed L^2-norm. Here, we mainly apply the Hardy–Littlewood–Sobolev approximation method to study the problem. We first present a relevant property of $m_q(c)$.

Lemma 5.10. *Assume that $\gamma, \mu > 0$ and $\overline{q} < q < 2^*_{\alpha,s}$. Then*

$$\limsup_{q \to 2^{*-}_{\alpha,s}} m_q(c) \le m_{2^*_{\alpha,s}}(c).$$

Proof. By the definition of $m_{2^*_{\alpha,s}}(c)$, for any $\varepsilon \in (0,1)$, there exists $u \in \mathcal{P}_{2^*_{\alpha,s}}(c)$ such that

$$E_{2^*_{\alpha,s}}(u) < m_{2^*_{\alpha,s}}(c) + \varepsilon. \tag{5.20}$$

It follows from (2.9) that there exists $t_1 > 0$ large enough such that $E_{2^*_{\alpha,s}}(u_{t_1}) \le -2$. Moreover, by the Young inequality, we have

$$|u|^q \le \frac{2^*_{\alpha,s} - q}{2^*_{\alpha,s} - p}|u|^p + \frac{q-p}{2^*_{\alpha,s} - p}|u|^{2^*_{\alpha,s}} \quad \text{for } \overline{q} < p < q < 2^*_{\alpha,s}. \tag{5.21}$$

By the Lebesgue dominated convergence theorem, we have that

$$\frac{\mu}{2q} t^{Nq-N-\alpha} \int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \,\mathrm{d}x$$

is continuous on $q \in [p, 2^*_{\alpha,s}]$ uniformly with respect to $t \in (0, t_1]$. Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|E_q(u_t) - E_{2^*_{\alpha,s}}(u_t)| < \varepsilon \tag{5.22}$$

for $q \in [2^*_{\alpha,s} - \delta, 2^*_{\alpha,s}]$ and $t \in (0, t_1]$, which implies that $E_q(u_{t_1}) \le -1$ for any $q \in [2^*_{\alpha,s} - \delta, 2^*_{\alpha,s}]$. Then it follows from Lemma 5.1(i) that there exists a unique critical point $t_* \in (0, t_1)$ of $E_q(u_t)$ such that $P_q(u_{t_*}) = 0$. Noticing that $u \in \mathcal{P}_{2^*_{\alpha,s}}(c)$, we deduce that

$$E_{2^*_{\alpha,s}}(u) = \max_{t>0} E_{2^*_{\alpha,s}}(u_t). \tag{5.23}$$

As a consequence, by (5.20), (5.22) and (5.23), we have

$$m_q(c) \le E_q(u_{t_*}) \le E_{2^*_{\alpha,s}}(u_{t_*}) + \varepsilon \le E_{2^*_{\alpha,s}}(u) + \varepsilon < m_{2^*_{\alpha,s}}(c) + 2\varepsilon$$

for any $q \in [2^*_{\alpha,s} - \delta, 2^*_{\alpha,s}]$. Hence, $\limsup_{q\to 2^{*-}_{\alpha,s}} m_q(c) \le m_{2^*_{\alpha,s}}(c)$. □

Lemma 5.11. *Let* $q = 2^*_{\alpha,s}$, $c^* = \left(\frac{N+\alpha}{N\gamma}\right)^{\frac{N}{2\alpha}} S_\alpha^{\frac{N+\alpha}{2\alpha}}$, $0 < c < \min\{c_*, c^*\}$, *and*

$$\gamma > \left(\frac{\alpha}{\alpha+2s}\right)^{\frac{\alpha}{N}} S_\alpha^{\frac{N+\alpha}{N}} \left(\frac{C_\alpha^{N-2s}}{S^{N+\alpha}}\right)^{\frac{\alpha}{N(2s+\alpha)}} \mu^{\frac{\alpha(N-2s)}{N(2s+\alpha)}}.$$

Then, there exists a constant $\widetilde{\mu} \ge \widehat{\mu}$ *such that for every* $\mu > \widetilde{\mu}$, *the infimum* $m_{2^*_{\alpha,s}}(c)$ *is achieved by* $\widetilde{u}$, *where* $\widehat{\mu}$ *is as in Lemma* 5.9.

Proof. Let $q_n \to 2^{*-}_{\alpha,s}$ as $n \to \infty$. According to Theorem 1.5 and Lemma 5.10, for every $\mu > \widehat{\mu}$, there exists a sequence of positive and radially functions $\{u_n := u_{q_n}\} \subset \mathcal{P}^r q_n(c)$ such that

$$E_{q_n}(u_n) = m_{q_n}(c) \le m_{2^*_{\alpha,s}}(c) + 1,$$

and then combining Lemma 5.2 we have that $\{u_n\}$ is bounded in $H^s_r(\mathbb{R}^N)$. Thus, there exists $\overline{u} \in H^s_r(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup \overline{u}$ in $H^s_r(\mathbb{R}^N)$, $u_n \to \overline{u}$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2^*_s$, and $u_n \to \overline{u}$ a.e. on $\mathbb{R}^N$. Then by the Lagrange multipliers rule, there exists $\lambda_n \in \mathbb{R}$ such that for every $\varphi \in H^s_r(\mathbb{R}^N)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi \,\mathrm{d}x + \lambda_n \int_{\mathbb{R}^N} u_n\varphi \,\mathrm{d}x - \gamma \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2_\alpha})|u_n|^{2_\alpha-2} u_n\varphi \,\mathrm{d}x \\ &- \mu \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_n})|u_n|^{q_n-2} u_n \varphi \,\mathrm{d}x \\ = {}& o_n(1)\|\varphi\|. \end{aligned} \tag{5.24}$$

There holds that

$$\lambda_n c^2 = \gamma \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2_\alpha})|u_n|^{2_\alpha} \,\mathrm{d}x + \mu \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_n})|u_n|^{q_n} \,\mathrm{d}x - \left\|(-\Delta)^{\frac{s}{2}} u_n\right\|_2^2 + o_n(1),$$

which shows that $\{\lambda_n\}$ is bounded. Similar to the argument in Lemma 5.9, we know that there exists $\overline{\lambda} \in \mathbb{R}$ such that $\lambda_n \to \overline{\lambda}$ as $n \to \infty$ and $\overline{\lambda} > 0$.

Now, we claim that $\overline{u} \ne 0$. Assume by contradiction that $\overline{u} = 0$. Similar to the inequality (5.21) we have

$$|u_n|^{q_n} \le \frac{2^*_{\alpha,s} - q_n}{2^*_{\alpha,s} - p}|u_n|^p + \frac{q_n - p}{2^*_{\alpha,s} - p}|u_n|^{2^*_{\alpha,s}} \quad \text{for } \overline{q} < p < q_n. \tag{5.25}$$

Moreover, since $\{u_n\}$ is bounded and $u_n \to \overline{u}$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2^*_s$, using the Hardy–Littlewood–Sobolev inequality in Lemma 2.3, we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^{2^*_{\alpha,s}} \,\mathrm{d}x = o_n(1), \quad \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p \,\mathrm{d}x = o_n(1) \tag{5.26}$$

for $\overline{q} < p < q_n$. Hence, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{q_n})|u_n|^{q_n}\,\mathrm{d}x &\le \left(\frac{q_n-p}{2^*_{\alpha,s}-p}\right)^2\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}}\,\mathrm{d}x + o_n(1)\\ &\le \int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}}\,\mathrm{d}x + o_n(1),\end{aligned} \tag{5.27}$$

where we have used (5.25), (5.26) and $0 < \frac{q_n-p}{2^*_{\alpha,s}-p} < 1$. Besides, it follows from Lemmas 2.1 and 2.3 that

$$\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}}\,\mathrm{d}x \le C_\alpha\|u_n\|_{2^*_s}^{22^*_{\alpha,s}} \le C_\alpha S^{-2^*_{\alpha,s}}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^{22^*_{\alpha,s}}. \tag{5.28}$$

Thus, by $P_{q_n}(u_n)=0$, (5.27), (5.28) and $q_n \to {2^*_{\alpha,s}}^-$, we can infer that

$$\begin{aligned}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 &= \frac{\mu}{2s}\left(N-\frac{N+\alpha}{q_n}\right)\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{q_n})|u_n|^{q_n}\,\mathrm{d}x\\ &\le \frac{\mu}{2s}\left(N-\frac{N+\alpha}{q_n}\right)C_\alpha S^{-2^*_{\alpha,s}}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^{22^*_{\alpha,s}} + o_n(1)\\ &\le \mu C_\alpha S^{-2^*_{\alpha,s}}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^{22^*_{\alpha,s}} + o_n(1).\end{aligned} \tag{5.29}$$

Since $\liminf_{n\to\infty}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 > 0$, then it follows from (5.29) that

$$\limsup_{n\to\infty}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 \ge (\mu C_\alpha)^{-\frac{N-2s}{\alpha+2s}}S^{\frac{N+\alpha}{\alpha+2s}}. \tag{5.30}$$

Then, by Lemma 5.10, (5.30), (2.5) and $P_{q_n}(u_n)=0$, we have

$$\begin{aligned}m_{2^*_{\alpha,s}}(c) &\ge \limsup_{n\to\infty} m_{q_n}(c)\\ &= \limsup_{n\to\infty}\left[\frac{1}{2}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 - \frac{\gamma}{22_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2_\alpha})|u_n|^{2_\alpha}\,\mathrm{d}x\right.\\ &\qquad\left. - \frac{s}{Nq_n-N-\alpha}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2\right]\\ &\ge \limsup_{n\to\infty}\left[\left(\frac{1}{2}-\frac{s}{Nq_n-N-\alpha}\right)\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 - \frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha}\right]\\ &= \limsup_{n\to\infty}\left[\left(\frac{1}{2}-\frac{s}{N\cdot 2^*_{\alpha,s}-N-\alpha}\right)\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 - \frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha}\right]\\ &= \limsup_{n\to\infty}\left[\frac{\alpha+2s}{2(N+\alpha)}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 - \frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha}\right]\\ &\ge \frac{\alpha+2s}{2(N+\alpha)}(\mu C_\alpha)^{-\frac{N-2s}{\alpha+2s}}S^{\frac{N+\alpha}{\alpha+2s}} - \frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha}.\end{aligned} \tag{5.31}$$

Moreover, similar to the process of (5.19) in Lemma 5.9 with $q = 2^*_{\alpha,s}$ and $\widehat{\lambda}=1$, there holds that

$$m_{2^*_{\alpha,s}}(c) < \frac{\alpha}{2(N+\alpha)}\gamma^{-\frac{N}{\alpha}}S_\alpha^{\frac{N+\alpha}{\alpha}} - \frac{1}{2}c^2. \tag{5.32}$$

However, in view of

$$\gamma > \left(\frac{\alpha}{\alpha+2s}\right)^{\frac{\alpha}{N}} S_\alpha^{\frac{N+\alpha}{N}} \left(\frac{C_\alpha^{N-2s}}{S^{N+\alpha}}\right)^{\frac{\alpha}{N(2s+\alpha)}} \mu^{\frac{\alpha(N-2s)}{N(2s+\alpha)}} \text{ and } 0 < c < c^* := \left(\frac{N+\alpha}{N\gamma}\right)^{\frac{N}{2\alpha}} S_\alpha^{\frac{N+\alpha}{2\alpha}},$$

we get

$$\frac{\alpha+2s}{2(N+\alpha)}(\mu C_\alpha)^{-\frac{N-2s}{\alpha+2s}} S^{\frac{N+\alpha}{\alpha+2s}} - \frac{\gamma}{22_\alpha} S_\alpha^{-2\alpha} c^{22\alpha} > \frac{\alpha}{2(N+\alpha)}\gamma^{-\frac{N}{\alpha}} S_\alpha^{\frac{N+\alpha}{\alpha}} - \frac{1}{2}c^2,$$

which is contradictory to (5.31) and (5.32). Thus, $\overline{u} \neq 0$.

Next, we claim that $\overline{u}$ is a weak solution of (1.1) with $q = 2^*_{\alpha,s}$. Note $\widetilde{r} := \frac{2N}{N+\alpha}$. Since $\left\{\left||u_n|^{q_n-2}u_n - |\overline{u}|^{2^*_{\alpha,s}-2}\overline{u}\right|^{\widetilde{r}}\right\}$ is bounded in $L^{\frac{2^*_{\alpha,s}}{2^*_{\alpha,s}-1}}(\mathbb{R}^N)$ and $|u_n|^{q_n-2}u_n - |\overline{u}|^{2^*_{\alpha,s}-2}\overline{u} \to 0$ a.e. on $\mathbb{R}^N$, we can infer that $\left||u_n|^{q_n-2}u_n - |\overline{u}|^{2^*_{\alpha,s}-2}\overline{u}\right|^{\widetilde{r}} \rightharpoonup 0$ in $L^{\frac{2^*_{\alpha,s}}{2^*_{\alpha,s}-1}}(\mathbb{R}^N)$, and then $\left\|(|u_n|^{q_n-2}u_n - |\overline{u}|^{2^*_{\alpha,s}-2}\overline{u})\varphi\right\|_{\widetilde{r}} \to 0$ for any $\varphi \in H^s(\mathbb{R}^N)$. Since $\{|u_n|^{q_n}\}$ is bounded in $L^{\widetilde{r}}(\mathbb{R}^N)$, by the Hardy–Littlewood–Sobolev inequality, we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_n})\big(|u_n|^{q_n-2}u_n - |\overline{u}|^{2^*_{\alpha,s}-2}\overline{u}\big)\varphi \,\mathrm{d}x \to 0. \tag{5.33}$$

Since $|\overline{u}|^{2^*_{\alpha,s}-2}\overline{u}\varphi \in L^{\widetilde{r}}(\mathbb{R}^N)$, we have $I_\alpha * (|\overline{u}|^{2^*_{\alpha,s}-2}\overline{u}\varphi) \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$. Besides, $|u_n|^{q_n} \rightharpoonup |\overline{u}|^{2^*_{\alpha,s}}$ in $L^{\widetilde{r}}(\mathbb{R}^N)$. Therefore, we have

$$\int_{\mathbb{R}^N} \big(I_\alpha * (|\overline{u}|^{2^*_{\alpha,s}-2}\overline{u}\varphi)\big)(|u_n|^{q_n} - |\overline{u}|^{2^*_{\alpha,s}}) \,\mathrm{d}x \to 0. \tag{5.34}$$

It follows from (5.33) and (5.34) that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_n})|u_n|^{q_n-2}u_n\varphi \,\mathrm{d}x \to \int_{\mathbb{R}^N} (I_\alpha * |\overline{u}|^{2^*_{\alpha,s}})|\overline{u}|^{2^*_{\alpha,s}-2}\overline{u}\varphi \,\mathrm{d}x. \tag{5.35}$$

Then, by Lemma 2.5, (5.24) and (5.35), we have

$$E'_{q_n}(u_n)\varphi + \lambda_n \int_{\mathbb{R}^N} u_n\varphi \,\mathrm{d}x \to E'_{2^*_{\alpha,s}}(\overline{u})\varphi + \overline{\lambda}\int_{\mathbb{R}^N} \overline{u}\varphi \,\mathrm{d}x = 0 \quad \text{as } n \to \infty, \tag{5.36}$$

which implies that $\overline{u}$ is a weak solution of (1.1) with $q = 2^*_{\alpha,s}$. Then, it follows from Lemma 2.8 that $P_{2^*_{\alpha,s}}(\overline{u}) = 0$.

Finally, we shall prove that $m_{2^*_{\alpha,s}}(c)$ is achieved. Set $\int_{\mathbb{R}^N} |\overline{u}|^2 \,\mathrm{d}x = \beta^2 \leq c^2$ and define

$$\widetilde{u}(x) = \left(\frac{\beta}{c}\right)^{\frac{N}{2s}-1} \overline{u}\bigg(\left(\frac{\beta}{c}\right)^{\frac{1}{s}} x\bigg).$$

By a direct calculation, we have

$$\int_{\mathbb{R}^N} |\widetilde{u}|^2 \,\mathrm{d}x = c^2, \quad \big\|(-\Delta)^{\frac{s}{2}}\widetilde{u}\big\|_2^2 = \big\|(-\Delta)^{\frac{s}{2}}\overline{u}\big\|_2^2, \tag{5.37}$$

$$\int_{\mathbb{R}^N} (I_\alpha * |\widetilde{u}|^{2^*_{\alpha,s}})|\widetilde{u}|^{2^*_{\alpha,s}} \,\mathrm{d}x = \int_{\mathbb{R}^N} (I_\alpha * |\overline{u}|^{2^*_{\alpha,s}})|\overline{u}|^{2^*_{\alpha,s}} \,\mathrm{d}x, \tag{5.38}$$

and

$$\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^{2\alpha})|\widetilde{u}|^{2\alpha}\,\mathrm{d}x=\left(\frac{\beta}{c}\right)^{-\frac{2(N+\alpha)}{N}}\int_{\mathbb{R}^N}(I_\alpha*|\overline{u}|^{2\alpha})|\overline{u}|^{2\alpha}\,\mathrm{d}x \geq\int_{\mathbb{R}^N}(I_\alpha*|\overline{u}|^{2\alpha})|\overline{u}|^{2\alpha}\,\mathrm{d}x. \tag{5.39}$$

Thus, $\widetilde{u}\in S_r(c)$. Moreover, by $P_{2^*_{\alpha,s}}(\overline{u})=0$, (5.37) and (5.38), we have

$$\begin{aligned}\big\|(-\Delta)^{\frac{s}{2}}\widetilde{u}\big\|_2^2=\big\|(-\Delta)^{\frac{s}{2}}\overline{u}\big\|_2^2&=\frac{\mu}{2s}\left(N-\frac{N+\alpha}{q}\right)\int_{\mathbb{R}^N}(I_\alpha*|\overline{u}|^q)|\overline{u}|^q\,\mathrm{d}x\\&=\frac{\mu}{2s}\left(N-\frac{N+\alpha}{q}\right)\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^q)|\widetilde{u}|^q\,\mathrm{d}x,\end{aligned}$$

which implies that $\widetilde{u}\in\mathcal{P}^r_{2^*_{\alpha,s}}(c)$. According to Proposition 2.7 and (5.36), we have

$$\begin{aligned}&\frac{N-2s}{2}\big\|(-\Delta)^{\frac{s}{2}}\overline{u}\big\|_2^2+\frac{\overline{\lambda}N}{2}\|\overline{u}\|_2^2\\&=\frac{\gamma N}{2}\int_{\mathbb{R}^N}(I_\alpha*|\overline{u}|^{2\alpha})|\overline{u}|^{2\alpha}\,\mathrm{d}x+\frac{\mu(N+\alpha)}{2q}\int_{\mathbb{R}^N}(I_\alpha*|\overline{u}|^q)|\overline{u}|^q\,\mathrm{d}x,\end{aligned}$$

which shows that

$$\begin{aligned}&\frac{N-2s}{2}\big\|(-\Delta)^{\frac{s}{2}}\widetilde{u}\big\|_2^2+\frac{\overline{\lambda}N}{2}\left(\frac{\beta}{c}\right)^2\|\widetilde{u}\|_2^2\\&=\frac{\gamma N}{2}\left(\frac{\beta}{c}\right)^{\frac{2(N+\alpha)}{N}}\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^{2\alpha})|\widetilde{u}|^{2\alpha}\,\mathrm{d}x+\frac{\mu(N+\alpha)}{2q}\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^q)|\widetilde{u}|^q\,\mathrm{d}x,\end{aligned}\tag{5.40}$$

where we have used (5.37), (5.38) and (5.39). Then, by (5.37)–(5.40) and the Fatou's lemma, we have

$$\begin{aligned}m_{2^*_{\alpha,s}}(c)&\leq E_{2^*_{\alpha,s}}(\widetilde{u})\\&=\frac{1}{2}\big\|(-\Delta)^{\frac{s}{2}}\widetilde{u}\big\|_2^2-\frac{\gamma N}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^{2\alpha})|\widetilde{u}|^{2\alpha}\,\mathrm{d}x-\frac{\mu}{2q}\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^q)|\widetilde{u}|^q\,\mathrm{d}x\\&=\frac{\gamma N}{2}\left[\frac{1}{N-2s}\left(\frac{\beta}{c}\right)^{\frac{2(N+\alpha)}{N}}-\frac{1}{N+\alpha}\right]\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^{2\alpha})|\widetilde{u}|^{2\alpha}\,\mathrm{d}x\\&\quad+\frac{\mu(2s+\alpha)}{2q(N-2s)}\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^q)|\widetilde{u}|^q\,\mathrm{d}x-\frac{\overline{\lambda}N}{2(N-2s)}\left(\frac{\beta}{c}\right)^2\|\widetilde{u}\|_2^2\\&=\frac{\gamma N}{2}\left[\frac{1}{N-2s}-\frac{1}{N+\alpha}\left(\frac{\beta}{c}\right)^{-\frac{2(N+\alpha)}{N}}\right]\int_{\mathbb{R}^N}(I_\alpha*|\overline{u}|^{2\alpha})|\overline{u}|^{2\alpha}\,\mathrm{d}x\\&\quad+\frac{\mu(2s+\alpha)}{2q(N-2s)}\int_{\mathbb{R}^N}(I_\alpha*|\overline{u}|^q)|\overline{u}|^q\,\mathrm{d}x-\frac{\overline{\lambda}N}{2(N-2s)}\beta^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{\gamma N}{2}\left(\frac{1}{N-2s}-\frac{1}{N+\alpha}\right)\int_{\mathbb{R}^N}(I_\alpha * |\overline{u}|^{2\alpha})|\overline{u}|^{2\alpha}\,\mathrm{d}x \\
&\quad + \frac{\mu(2s+\alpha)}{2q(N-2s)}\int_{\mathbb{R}^N}(I_\alpha * |\overline{u}|^{q})|\overline{u}|^{q}\,\mathrm{d}x - \frac{\overline{\lambda}N}{2(N-2s)}\beta^2 \\
&\quad + \frac{\gamma N}{2(N+\alpha)}\left[1-\left(\frac{\beta}{c}\right)^{-\frac{2(N+\alpha)}{N}}\right]\int_{\mathbb{R}^N}(I_\alpha * |\overline{u}|^{2\alpha})|\overline{u}|^{2\alpha}\,\mathrm{d}x \\
&\le \liminf_{n\to\infty}\left[\frac{\gamma N}{2}\left(\frac{1}{N-2s}-\frac{1}{N+\alpha}\right)\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2\alpha})|u_n|^{2\alpha}\,\mathrm{d}x\right. \\
&\qquad\quad \left. + \frac{\mu(2s+\alpha)}{2q_n(N-2s)}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{q_n})|u_n|^{q_n}\,\mathrm{d}x\right] - \frac{\overline{\lambda}N}{2(N-2s)}\beta^2 \\
&\quad + \frac{\gamma N}{2(N+\alpha)}\left[1-\left(\frac{\beta}{c}\right)^{-\frac{2(N+\alpha)}{N}}\right]\int_{\mathbb{R}^N}(I_\alpha * |\overline{u}|^{2\alpha})|\overline{u}|^{2\alpha}\,\mathrm{d}x.
\end{aligned}
$$

Moreover, by the Pohozaev identity in (2.8), we know that

$$
\begin{aligned}
&\frac{\gamma N}{2}\left(\frac{1}{N-2s}-\frac{1}{N+\alpha}\right)\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2\alpha})|u_n|^{2\alpha}\,\mathrm{d}x \\
&\quad + \frac{\mu(2s+\alpha)}{2q_n(N-2s)}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{q_n})|u_n|^{q_n}\,\mathrm{d}x \\
&= \frac{\gamma N}{2(N-2s)}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2\alpha})|u_n|^{2\alpha}\,\mathrm{d}x + \frac{\mu(N+\alpha)}{2q_n(N-2s)}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{q_n})|u_n|^{q_n}\,\mathrm{d}x \\
&\quad - \frac{\gamma N}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2\alpha})|u_n|^{2\alpha}\,\mathrm{d}x - \frac{\mu}{2q_n}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{q_n})|u_n|^{q_n}\,\mathrm{d}x \\
&= \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 + \frac{\lambda_n\cdot N}{2(N-2s)}\|u_n\|_2^2 - \frac{\gamma N}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2\alpha})|u_n|^{2\alpha}\,\mathrm{d}x \\
&\quad - \frac{\mu}{2q_n}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{q_n})|u_n|^{q_n}\,\mathrm{d}x \\
&= E_{q_n}(u_n) + \frac{\lambda_n\cdot N}{2(N-2s)}\|u_n\|_2^2.
\end{aligned}
$$

Therefore, from the above, we can conclude that

$$
\begin{aligned}
m_{2^*_{\alpha,s}}(c) &\le E_{2^*_{\alpha,s}}(\widetilde{u}) \\
&\le \liminf_{n\to\infty} E_{q_n}(u_n) + \frac{\overline{\lambda}N}{2(N-2s)}(c^2-\beta^2) \\
&\quad + \frac{\gamma N}{2(N+\alpha)}\left[1-\left(\frac{\beta}{c}\right)^{-\frac{2(N+\alpha)}{N}}\right]\int_{\mathbb{R}^N}(I_\alpha * |\overline{u}|^{2\alpha})|\overline{u}|^{2\alpha}\,\mathrm{d}x \\
&\le \limsup_{n\to\infty} m_{q_n}(c) \le m_{2^*_{\alpha,s}}(c),
\end{aligned}
$$

provided that

$$\frac{\overline{\lambda}N}{2(N-2s)}(c^2-\beta^2)+\frac{\gamma N}{2(N+\alpha)}\left[1-\left(\frac{\beta}{c}\right)^{-\frac{2(N+\alpha)}{N}}\right]\int_{\mathbb{R}^N}(I_\alpha*|\overline{u}|^{2\alpha})|\overline{u}|^{2\alpha}\,\mathrm{d}x\le 0.$$

If $\beta=c$, then we can clearly obtain that $m_{2^*_{\alpha,s}}(c)$ is achieved by $\widetilde{u}$ for all $\gamma>0$. If $\beta<c$, then $m_{2^*_{\alpha,s}}(c)$ is achieved by $\widetilde{u}$ when

$$\gamma\ge\frac{\overline{\lambda}(c^2-\beta^2)(N+\alpha)}{(N-2s)\big((\frac{\beta}{c})^{-\frac{2(N+\alpha)}{N}}-1\big)\int_{\mathbb{R}^N}(I_\alpha*|\overline{u}|^{2\alpha})|\overline{u}|^{2\alpha}\,\mathrm{d}x}. \tag{5.41}$$

Note that

$$\gamma>\left(\frac{\alpha}{\alpha+2s}\right)^{\frac{\alpha}{N}}S_\alpha^{\frac{N+\alpha}{N}}\left(\frac{C_\alpha^{N-2s}}{S^{N+\alpha}}\right)^{\frac{\alpha}{N(2s+\alpha)}}\mu^{\frac{\alpha(N-2s)}{N(2s+\alpha)}},$$

then there exists $\widetilde{\mu}\ge\widehat{\mu}$ such that (5.41) holds. Thus, for every $\mu>\widetilde{\mu}$, the infimum $m_{2^*_{\alpha,s}}(c)$ is achieved. □

Proof of Theorem 1.6. According to Lemma 5.11 we know that $m_{2^*_{\alpha,s}}(c)$ is achieved by $\widetilde{u}$, then there exist $\widetilde{\lambda},\eta\in\mathbb{R}$ such that

$$\langle E'_{2^*_{\alpha,s}}(\widetilde{u}),\varphi\rangle+\widetilde{\lambda}\int_{\mathbb{R}^N}\widetilde{u}\varphi\,\mathrm{d}x=\eta\langle P'_{2^*_{\alpha,s}}(\widetilde{u}),\varphi\rangle\quad\text{for every }\varphi\in H^s_r(\mathbb{R}^N).$$

It follows that $\widetilde{u}$ satisfies

$$\begin{aligned}&(-\Delta)^s\widetilde{u}+\widetilde{\lambda}\widetilde{u}-\gamma(I_\alpha*|\widetilde{u}|^{2\alpha})|\widetilde{u}|^{2\alpha-2}\widetilde{u}-\mu(I_\alpha*|\widetilde{u}|^{2^*_{\alpha,s}})|\widetilde{u}|^{2^*_{\alpha,s}-2}\widetilde{u}\\&=\eta\big[2(-\Delta)^s\widetilde{u}-2\mu\cdot 2^*_{\alpha,s}(I_\alpha*|\widetilde{u}|^{2^*_{\alpha,s}})|\widetilde{u}|^{2^*_{\alpha,s}-2}\widetilde{u}\big],\end{aligned} \tag{5.42}$$

that is,

$$(1-2\eta)(-\Delta)^s\widetilde{u}+\widetilde{\lambda}\widetilde{u}=\gamma(I_\alpha*|\widetilde{u}|^{2\alpha})|\widetilde{u}|^{2\alpha-2}\widetilde{u}+\mu(1-2\eta\cdot 2^*_{\alpha,s})(I_\alpha*|\widetilde{u}|^{2^*_{\alpha,s}})|\widetilde{u}|^{2^*_{\alpha,s}-2}\widetilde{u}.$$

Similar to Lemma 2.8, $\widetilde{u}$ satisfies the following identity

$$(1-2\eta)\big\|(-\Delta)^{\frac{s}{2}}\widetilde{u}\big\|_2^2=\mu(1-2\eta\cdot 2^*_{\alpha,s})\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^{2^*_{\alpha,s}})|\widetilde{u}|^{2^*_{\alpha,s}}\,\mathrm{d}x.$$

Together with $\widetilde{u}\in\mathcal{P}^r_{2^*_{\alpha,s}}(c)$, we can infer that

$$2\mu\eta(2^*_{\alpha,s}-1)=0,$$

which implies that $\eta=0$. Moreover, it follows from (5.42) and $P_{2^*_{\alpha,s}}(\widetilde{u})=0$ that

$$\begin{aligned}\widetilde{\lambda}&=\frac{1}{c^2}\left[-\big\|(-\Delta)^{\frac{s}{2}}\widetilde{u}\big\|_2^2+\gamma\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^{2\alpha})|\widetilde{u}|^{2\alpha}\,\mathrm{d}x+\mu\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^{2^*_{\alpha,s}})|\widetilde{u}|^{2^*_{\alpha,s}}\,\mathrm{d}x\right]\\&=\frac{\gamma}{c^2}\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}|^{2\alpha})|\widetilde{u}|^{2\alpha}\,\mathrm{d}x\le\gamma S_\alpha^{-2\alpha}c^{\frac{2\alpha}{N}}.\end{aligned}$$

Thus, $\widetilde{u}\in H^s(\mathbb{R}^N)$ is a normalized ground state solution to the problem (1.1) and (1.2), and the corresponding Lagrange multiplier $\widetilde{\lambda}$ satisfies $0<\widetilde{\lambda}\le\gamma S_\alpha^{-2\alpha}c^{\frac{2\alpha}{N}}$. Moreover, by Lemma 5.3(ii), $\widetilde{u}$ is positive and radially symmetric. We complete the proof. □

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References

[1] T. Bartsch and N. Soave, *Multiple normalized solutions for a competing system of Schrödinger equations*, Calc. Var. Partial Differential Equations **58** (2019), no. 1, Paper No. 22, 24 pp.

[2] S. Bhattarai, *On fractional Schrödinger systems of Choquard type*, J. Differential Equations **263** (2017), no. 6, 3197–3229.

[3] F. Binhua, R. Chen and J. Liu, *Blow-up criteria and instability of normalized standing waves for the fractional Schrödinger–Choquard equation*, Adv. Nonlinear Anal. **10** (2021), no. 1, 311–330.

[4] C. Bucur and E. Valdinoci, *Nonlocal Diffusion and Applications*, Lecture Notes of the Unione Matematica Italiana **20**, Springer, Unione Matematica Italiana, Bologna, 2016.

[5] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics **10**, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[6] Y.-H. Chen and C. Liu, *Ground state solutions for non-autonomous fractional Choquard equations*, Nonlinearity **29** (2016), no. 6, 1827–1842.

[7] S. Cingolani, M. Gallo and K. Tanaka, *Symmetric ground states for doubly nonlocal equations with mass constraint*, Symmetry **13** (2021), 17 pp.

[8] ———, *On fractional Schrödinger equations with Hartree type nonlinearities*, Math. Eng. **4** (2022), no. 6, Paper No. 056, 33 pp.

[9] S. Cingolani and L. Jeanjean, *Stationary waves with prescribed L^2-norm for the planar Schrödinger–Poisson system*, SIAM J. Math. Anal. **51** (2019), no. 4, 3533–3568.

[10] P. d'Avenia, G. Siciliano and M. Squassina, *On fractional Choquard equations*, Math. Models Methods Appl. Sci. **25** (2015), no. 8, 1447–1476.

[11] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[12] Y. Ding and H.-Y. Wang, *Normalized solutions to Schrödinger equations with critical exponent and mixed nonlocal nonlinearities*, J. Geom. Anal. **34** (2024), no. 7, Paper No. 215, 56 pp.

[13] P. Felmer, A. Quaas and J. Tan, *Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), no. 6, 1237–1262.

[14] B. Feng, *Ground states for the fractional Schrödinger equation*, Electron. J. Differential Equations **2013**, No. 127, 11 pp.

[15] B. Feng and H. Zhang, *Stability of standing waves for the fractional Schrödinger–Hartree equation*, J. Math. Anal. Appl. **460** (2018), no. 1, 352–364.

[16] Z. Feng, X. He and Y. Meng, *Normalized solutions of fractional Choquard equation with critical nonlinearity*, Differential Integral Equations **36** (2023), no. 7-8, 593–620.

[17] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[18] X. He and V. D. Rădulescu, *Small linear perturbations of fractional Choquard equations with critical exponent*, J. Differential Equations **282** (2021), 481–540.

[19] X. He, V. D. Rădulescu and W. Zou, *Normalized ground states for the critical fractional Choquard equation with a local perturbation*, J. Geom. Anal. **32** (2022), no. 10, Paper No. 252, 51 pp.

[20] L. Jeanjean, *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal. **28** (1997), no. 10, 1633–1659.

[21] L. Jeanjean, J. Jendrej, T. T. Le and N. Visciglia, *Orbital stability of ground states for a Sobolev critical Schrödinger equation*, J. Math. Pures Appl. (9) **164** (2022), 158–179.

[22] J. Lan, X. He and Y. Meng, *Normalized solutions for a critical fractional Choquard equation with a nonlocal perturbation*, Adv. Nonlinear Anal. **12** (2023), no. 1, Paper No. 20230112, 40 pp.

[23] G. Li and X. Luo, *Existence and multiplicity of normalized solutions for a class of fractional Choquard equations*, Sci. China Math. **63** (2020), no. 3, 539–558.

[24] G. Li, X. Luo and T. Yang, *Normalized solutions for the fractional Schrödinger equation with a focusing nonlocal perturbation*, Math. Methods Appl. Sci. **44** (2021), no. 13, 10331–10360.

[25] X. Li, *Existence of normalized ground states for the Sobolev critical Schrödinger equation with combined nonlinearities*, Calc. Var. Partial Differential Equations **60** (2021), no. 5, Paper No. 169, 14 pp.

[26] ———, *Standing waves to upper critical Choquard equation with a local perturbation: multiplicity, qualitative properties and stability*, Adv. Nonlinear Anal. **11** (2022), no. 1, 1134–1164.

[27] ———, *Nonexistence, existence and symmetry of normalized ground states to Choquard equations with a local perturbation*, Complex Var. Elliptic Equ. **68** (2023), no. 4, 578–602.

[28] X. Li and S. Ma, *Choquard equations with critical nonlinearities*, Commun. Contemp. Math. **22** (2020), no. 4, 1950023, 28 pp.

[29] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Appl. Math. **57** (1977), no. 2, 93–105.

[30] ———, *Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities*, Ann. of Math. (2) **118** (1983), no. 2, 349–374.

[31] E. H. Lieb and M. Loss, *Analysis*, Second edition, Graduate Studies in Mathematics **14**, American Mathematical Society, Providence, RI, 2001.

[32] X. Luo, *Normalized standing waves for the Hartree equations*, J. Differential Equations **267** (2019), no. 7, 4493–4524.

[33] P. Ma and J. Zhang, *Existence and multiplicity of solutions for fractional Choquard equations*, Nonlinear Anal. **164** (2017), 100–117.

[34] Y. Meng and X. He, *Normalized solutions for the fractional Choquard equations with Hardy–Littlewood–Sobolev upper critical exponent*, Qual. Theory Dyn. Syst. **23** (2024), no. 1, Paper No. 19, 21 pp.

[35] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), no. 2, 153–184.

[36] ———, *A guide to the Choquard equation*, J. Fixed Point Theory Appl. **19** (2017), no. 1, 773–813.

[37] T. Mukherjee and K. Sreenadh, *Fractional Choquard equation with critical nonlinearities*, NoDEA Nonlinear Differential Equations Appl. **24** (2017), no. 6, Paper No. 63, 34 pp.

[38] Y. J. Park, *Fractional Polya–Szegö inequality*, J. Chungcheong Math. Soc. **24** (2011), no. 2, 267–271.

[39] S. I. Pekar, *Untersuchungen über die Elektronentheorie der Kristalle*, De Gruyter, Berlin, 1954.

[40] R. Penrose, *On gravity's role in quantum state reduction*, Gen. Relativity Gravitation **28** (1996), no. 5, 581–600.

[41] X. Shang and P. Ma, *Normalized solutions to the nonlinear Choquard equations with Hardy–Littlewood–Sobolev upper critical exponent*, J. Math. Anal. Appl. **521** (2023), no. 2, Paper No. 126916, 29 pp.

[42] Z. Shen, F. Gao and M. Yang, *Ground states for nonlinear fractional Choquard equations with general nonlinearities*, Math. Methods Appl. Sci. **39** (2016), no. 14, 4082–4098.

[43] N. Soave, *Normalized ground states for the NLS equation with combined nonlinearities*, J. Differential Equations **269** (2020), no. 9, 6941–6987.

[44] ———, *Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case*, J. Funct. Anal. **279** (2020), no. 6, 108610, 43 pp.

[45] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372.

[46] T. Yang, *Normalized solutions for the fractional Schrödinger equation with a focusing nonlocal L^2-critical or L^2-supercritical perturbation*, J. Math. Phys. **61** (2020), no. 5, 051505, 26 pp.

[47] S. Yao, H. Chen, V. D. Rădulescu and J. Sun, *Normalized solutions for lower critical Choquard equations with critical Sobolev perturbation*, SIAM J. Math. Anal. **54** (2022), no. 3, 3696–3723.

[48] W. Ye, Z. Shen and M. Yang, *Normalized solutions for a critical Hartree equation with perturbation*, J. Geom. Anal. **32** (2022), no. 9, Paper No. 242, 44 pp.

[49] S. Yu, C. Tang and Z. Zhang, *Normalized ground states for the lower critical fractional Choquard equation with a focusing local perturbation*, Discrete Contin. Dyn. Syst. Ser. S **16** (2023), no. 11, 3369–3393.

Zilin Chen and Yang Yang
School of Science, Jiangnan University, Wuxi, Jiangsu, 214122, China
E-mail addresses: `6231204001@stu.jiangnan.edu.cn`, `yangyangli@jiangnan.edu.cn`