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On Various Space-time Properties of Solutions to the One-dimensional Heat Equation on Semi-infinite Rod

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Abstract. We discuss many interesting properties of the initial-boundary value problem for the heat equation $u_t(x,t) = u_{xx}(x,t)$ on $(x,t) \in (0,\infty) \times (0,\infty)$. In particular, we can prescribe the space, time, and space-time oscillation limits (limsup and liminf) of $u(x,t)$ by choosing suitable initial data $h(x)$ and boundary dada $g(t)$. We also consider singular initial-boundary value problem and oblique initial-boundary value problem for the heat equation.

1. Introduction

The purpose of this paper is to study the space-time asymptotic behavior of solution to the one-dimensional heat equation $u_t = u_{xx}$ on the half space $x \in (0,\infty)$, i.e., solution to the following **initial-boundary value problem (ibvp)**:

$$(1.1) \qquad \begin{cases} u_t(x,t) = u_{xx}(x,t), & (x,t) \in (0,\infty) \times (0,\infty), \\ u(x,0) = h(x), & x \in (0,\infty), \\ u(0,t) = g(t), & t \in (0,\infty), \end{cases}$$

where $h(x)$ and $g(t)$ are given **continuous** functions on $(0,\infty)$. The readers can view this paper as a survey article and at the same time it also contains many new interesting results not explored before.

For (1.1), since we are not interested in the existence of the limit $\lim_{(x,t)\to(0^+,0^+)} u(x,t)$ in general (except in Lemma 2.3 below), $h(x)$ and $g(t)$ **may not** be defined and continuous up to $x = 0$ and $t = 0$ respectively. In case both are defined at $x = 0$ and $t = 0$, they **may not** satisfy $h(0) = g(0)$.

As for the existence of a solution, by Theorem 4.3.1 of the book [2, p. 50], if there exist positive constants $C_1, C_2, \alpha \in [0,1)$, $\varepsilon > 0$ small, such that $h(x)$ and $g(t)$ satisfy

$$(1.2) \qquad |h(x)| \le C_1 e^{C_2|x|^{1+\alpha}}, \ \ \forall\, x \in (0,\infty); \quad |g(t)| \le \frac{C_1}{t^\alpha}, \ \ \forall\, t \in (0,\varepsilon),$$

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then the function on $(0,\infty)\times(0,\infty)$:

$$(1.3)\qquad u(x,t)=\begin{cases}\frac{1}{\sqrt{4\pi t}}\int_0^\infty \left(e^{-\frac{(x-\xi)^2}{4t}}-e^{-\frac{(x+\xi)^2}{4t}}\right)h(\xi)\,d\xi & (\textbf{space convolution})\\ +\frac{x}{\sqrt{4\pi}}\int_0^t \frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}g(s)\,ds & (\textbf{time convolution})\end{cases}$$

is a **smooth** solution of the ibvp (1.1) on the domain $(0,\infty)\times(0,\infty)$ and it satisfies the initial-boundary condition in the following sense:

$$(1.4)\qquad \lim_{(x,t)\to(x_0,0^+)} u(x,t)=h(x_0),\quad \forall\, x_0\in(0,\infty)$$

and

$$(1.5)\qquad \lim_{(x,t)\to(0^+,t_0)} u(x,t)=g(t_0),\quad \forall\, t_0\in(0,\infty).$$

That is to say, $u(x,t)$ lies in the space

$$(1.6)\qquad u(x,t)\in C^\infty((0,\infty)\times(0,\infty))\cap C^0(([0,\infty)\times[0,\infty))\setminus\{(0,0)\}).$$

From now on, we shall denote the two integrals for $u(x,t)$ in (1.3) as $u_h(x,t)$ and $u_g(x,t)$ respectively, i.e., $u(x,t)=u_h(x,t)+u_g(x,t)$, where $(x,t)\in(0,\infty)\times(0,\infty)$.

The ibvp (1.1) is an old problem. There have been many papers discussing it. We refer the readers to the references in the two classic books by Cannon [2] and Widder [15] for related papers. Historically, the people who first studied the ibvp (1.1) with deeper analytical tools are mostly Russian mathematicians, around the 1950–1970 period. For papers related to heat equation $u_t=u_{xx}$ on the half space $x\in(0,\infty)$, we can cite a few Russian papers in [1, 6, 10, 13].

Remark 1.1. The condition on $h(x)$ in (1.2) automatically implies that it is **bounded** near $x=0$. The assumption (1.2) is to guarantee that both improper integrals in (1.3) converge. **It may not be optimal**. For example, one can allow $h(x)$ to become singular as $x\to0^+$ (but still satisfies $|h(x)|\le C_1e^{C_2|x|^{1+\alpha}}$ as $x\to\infty$) with the order

$$(1.7)\qquad |h(x)|\le\frac{M}{x^p}\quad\text{on } x\in(0,\varepsilon)$$

for some constants $\varepsilon>0$, $M>0$, $p\in[0,2)$. Then the space convolution integral in (1.3) still converges. See Lemma 6.1 and Remark 6.2 below. In conclusion, as long as $h(x)$ satisfies (1.7) and $|h(x)|\le C_1e^{C_2|x|^{1+\alpha}}$ as $x\to\infty$ and $g(t)$ satisfies (1.2), then both convolution integrals in (1.3) will converge.

Remark 1.2. The second integral of (1.3) is equal to

$$-2\int_0^t\frac{\partial\Phi}{\partial x}(x,t-s)g(s)\,ds,\quad (x,t)\in(0,\infty)\times(0,\infty),$$

where $\Phi(x,t) = (\sqrt{4\pi t})^{-1}e^{-x^2/(4t)}$ is the fundamental solution of the heat equation. Also note that

$$0 < e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} < 1, \quad \forall\, x \in (0,\infty),\ t \in (0,\infty),\ \xi \in (0,\infty) \tag{1.8}$$

and for fixed $x \in (0,\infty)$ we have

$$0 < \frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}} \le \left(\frac{6}{e}\right)^{3/2}\frac{1}{x^3}, \quad \forall\, t \in (0,\infty),\ s \in (0,t).$$

Remark 1.3. In the formula (1.3), $u(x,t)$ depends on h over the whole space domain $(0,\infty)$, but depends on g only on the time domain $(0,t)$. This matches with the principle that $u(x,t)$ should not depend on future time $\widetilde{t} > t$.

Remark 1.4. If $h(x)$ and $g(t)$ are both continuous on $[0,\infty)$ with $h(0) = g(0) = \lambda$ and $h(x)$ satisfies the growth condition in (1.2), then the function $u(x,t)$ given by (1.3) will satisfy the two-dimensional limit

$$\lim_{(x,t)\to(0^+,0^+)} u(x,t) = \lambda.$$

In this case, $u(x,t)$ is continuous up to the point $(x,t) = (0,0)$ if we define $u(0,0) = \lambda$, i.e.,

$$u(x,t) \in C^\infty((0,\infty)\times(0,\infty)) \cap C^0([0,\infty)\times[0,\infty)). \tag{1.9}$$

1.1. Non-uniqueness of the ibvp (1.1)

For continuous initial-boundary data $h(x)$ and $g(t)$ on $(0,\infty)$ satisfying (1.2), the representation formula (1.3) only gives **one possible** solution. Similar to the heat equation on the entire space $x \in (-\infty,\infty)$ with given initial data, without further restriction on the growth behavior of $u(x,t)$ on the domain $(0,\infty)\times(0,\infty)$, the ibvp (1.1) **does not have unique solution** for given continuous initial-boundary data $h(x)$ and $g(t)$ on $(0,\infty)$. In fact, it has **infinitely many** solutions.

In fact, as demonstrated in [2, 15], there exist several **nonzero** solutions $u(x,t)$ to the ibvp (1.1) for the case $h(x) \equiv 0$ and $g(t) \equiv 0$. One quick example is the following: Let $\Phi(x,t) = (\sqrt{4\pi t})^{-1}e^{-x^2/(4t)}$ be the fundamental solution of the heat equation. The function $u(x,t) = \partial_x\Phi(x,t)$ satisfies the heat equation on the domain $(0,\infty)\times(0,\infty)$ with

$$\lim_{(x,t)\to(x_0,0^+)} u(x,t) = 0 \quad \text{and} \quad \lim_{(x,t)\to(0^+,t_0)} u(x,t) = 0$$

for fixed $x_0 \in (0,\infty)$, $t_0 \in (0,\infty)$, and it decays to zero as $t \to \infty$ (for fixed $x \in (0,\infty)$) and $x \to \infty$ (for fixed $t \in (0,\infty)$). However, on the parabola $x = \sqrt{4t}$, $t > 0$, we have

$u(\sqrt{4t}, t) \to -\infty$ as $t \to 0^+$. Therefore, $u(x,t)$ is not continuous at the corner point $(0,0)$ and is unbounded on any neighborhood of $(0,0)$.

On the other hand, one can also find a **nonzero** solution $u(x,t)$ of the ibvp (1.1) with $h(x) \equiv 0$ and $g(t) \equiv 0$, which is **smooth up to the boundary** of the domain $(0,\infty) \times (0,\infty)$, i.e., smooth on $[0,\infty) \times [0,\infty)$. Specifically, if we let $v(x,t) \in C^\infty(\mathbb{R}^2)$ be the **Tychonoff solution** (a power series solution) as constructed in pp. 211–213 of the book [7], given by

$$v(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}, \quad (x,t) \in (-\infty,\infty) \times (-\infty,\infty),$$

where $g(t)$ is chosen as

$$g(t) = \begin{cases} \exp\left(-\frac{1}{t^\alpha}\right), & t > 0,\ \alpha > 1 \text{ is a constant}, \\ 0, & t \le 0, \end{cases}$$

then $v(x,t) \in C^\infty(\mathbb{R}^2)$ will be a solution of the heat equation on the entire space $\mathbb{R}^2$ with $v(x,0) \equiv 0$ for all $x \in (-\infty,\infty)$. Since for each $t > 0$, $v(x,t)$ is an **even function** of $x \in (-\infty,\infty)$, we have $v_x(0,t) \equiv 0$ for all $t \in (-\infty,\infty)$. Therefore, the function $u(x,t) := \partial_x v(x,t) \in C^\infty(\mathbb{R}^2)$ will be a nonzero smooth solution of the ibvp (1.1) satisfying $u(x,0) \equiv 0$ for all $x \in [0,\infty)$ and $u(0,t) \equiv 0$ for all $t \in [0,\infty)$.

Clearly, for $h(x) \equiv g(t) \equiv 0$, the above two nonzero solutions $u(x,t) = \partial_x \Phi(x,t)$ and $u(x,t) = \partial_x v(x,t)$ of (1.1) are not obtained through the representation formula (1.3). Both solutions have defect even if the initial-boundary data are smooth on $[0,\infty)$. The first solution blows up near the origin $(0,0)$ and the second solution oscillates very rapidly as $|x| \to \infty$. Unlike the physically correct solution $u(x,t) \equiv 0$ given by the representation formula (1.3), they are "**non-physical**" solutions (see Evans PDE book [5, p. 59]). In the following we mention an uniqueness criteria, which can eliminate these non-physical solutions.

Lemma 1.5 (Uniqueness property)**.** *Assume $h(x)$ and $g(t)$ are continuous functions on $[0,\infty)$ with $h(0) = g(0)$ and $h(x)$ also satisfies the growth condition in* (1.2). *Let $u(x,t)$ be given by the formula* (1.3) *(and define $u(0,0) = h(0)$). Then it lies in the function space S, given by*

$$S = C^\infty((0,\infty) \times (0,\infty)) \cap C^0([0,\infty) \times [0,\infty)) \tag{1.10}$$

and satisfies the ibvp (1.1). *Moreover, for each fixed $T > 0$, it satisfies the growth estimate*

$$|u(x,t)| \le M e^{bx^2}, \quad \forall\, x \in [0,\infty),\ t \in [0,T] \tag{1.11}$$

for some positive constants M, b depending on C_1, C_2, α in (1.2) and h, g, T. Conversely, for the above given $h(x)$ and $g(t)$, if $v(x,t) \in S$ is a solution of the ibvp (1.1) and for each fixed $T > 0$ it satisfies (1.11) for some positive constants M, b depending on C_1, C_2, α, h, g, T, then $v(x,t)$ is given by the formula (1.3) on $(x,t) \in (0,\infty) \times (0,\infty)$.

Proof. For $u(x,t) = u_h(x,t) + u_g(x,t)$ given by (1.3), it lies in the space S due to Remark 1.4. By (2.1) below, we can write $u_h(x,t)$ as

$$\begin{aligned} u_h(x,t) &= \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} h(x+\sqrt{4t}z)\,dz - \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} h(-x+\sqrt{4t}z)\,dz \\ &:= u_h^{(1)}(x,t) - u_h^{(2)}(x,t), \end{aligned}$$

and by (1.2), we first obtain

$$\begin{aligned} |u_h^{(1)}(x,t)| &\le \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} |h(x+\sqrt{4t}z)|\,dz \\ &\le \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} C_1 e^{C_2|x+\sqrt{4t}z|^{1+\alpha}}\,dz, \quad \alpha \in [0,1). \end{aligned} \tag{1.12}$$

One can split the integral in (1.12) as $\int_{-x/\sqrt{4t}}^{0} (*)\,dz + \int_0^\infty (*)\,dz$ and for the first integral we have

$$\begin{aligned} &\frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{0} e^{-z^2} C_1 e^{C_2|x+\sqrt{4t}z|^{1+\alpha}}\,dz \\ &\le C_1 e^{C_2 x^{1+\alpha}} \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{0} e^{-z^2}\,dz \le \frac{1}{2} C_1 e^{C_2 x^{1+\alpha}}. \end{aligned} \tag{1.13}$$

As for the second integral, we can use the elementary inequality

$$a^p + b^p < (a+b)^p \le 2^{p-1}(a^p + b^p), \quad \forall\, a > 0,\ b > 0,\ 1 < p < \infty$$

to get, for $x \in [0,\infty)$, $t \in [0,T]$, the estimate

$$\begin{aligned} &\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-z^2} C_1 e^{C_2|x+\sqrt{4t}z|^{1+\alpha}}\,dz \\ &\le \frac{1}{\sqrt{\pi}} C_1 e^{C_2 2^\alpha x^{1+\alpha}} \int_0^\infty e^{-z^2} e^{C_2 2^\alpha |\sqrt{4t}z|^{1+\alpha}}\,dz \\ &\le \frac{1}{\sqrt{\pi}} C_1 e^{C_2 2^\alpha x^{1+\alpha}} \int_0^\infty e^{-z^2} e^{C_2 2^\alpha |\sqrt{4T}z|^{1+\alpha}}\,dz, \quad \alpha \in [0,1). \end{aligned} \tag{1.14}$$

By (1.13) and (1.14), $u_h^{(1)}(x,t)$ clearly satisfies the estimate (1.11). The proof for $u_h^{(2)}(x,t)$ is similar and we conclude the estimate (1.11) for $u_h(x,t)$.

As for $u_g(x,t)$, by (2.1) below again, we have

$$u_g(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} g\left(t - \left(\frac{x}{2z}\right)^2\right) dz, \quad (x,t) \in (0,\infty) \times (0,\infty)$$

and for $x \in [0,\infty)$, $t \in [0,T]$, it implies

$$\begin{aligned}|u_g(x,t)| &\le \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\left|g\left(t-\left(\frac{x}{2z}\right)^2\right)\right| dz\\ &\le \max_{t\in[0,T]}|g(t)|\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-z^2}\,dz = \max_{t\in[0,T]}|g(t)|.\end{aligned}$$

Since both $u_h(x,t)$ and $u_g(x,t)$ satisfy (1.11), $u(x,t)$ also satisfies (1.11).

Conversely, if $v(x,t) \in S$ is a solution of (1.1) and for each fixed $T > 0$ it satisfies (1.11), then we look at $w(x,t) = u(x,t) - v(x,t)$, where $u(x,t)$ is from formula (1.3). We have $w(x,t) \in S$ with

$$\begin{cases}\partial_t w(x,t) = w_{xx}(x,t) & \text{in } (0,\infty)\times(0,\infty),\\ w(x,0) = 0, & x \in (0,\infty),\\ w(0,t) = 0, & t \in (0,\infty),\end{cases}$$

and for each $T > 0$ it also satisfies

$$|w(x,t)| \le Me^{bx^2}, \quad \forall\, x \in [0,\infty),\ t \in [0,T]$$

for some positive constants M, b. By the well-known uniqueness theory for heat equation on the semi-infinite rod (see Theorem 6.2 in Widder's book [15, p. 29], which requires solution continuous up to the point $(x,t) = (0,0)$), we must have $w(x,t) \equiv 0$ for all $(x,t) \in [0,\infty)\times[0,T]$. Since $T > 0$ can be arbitrary, we must have $w(x,t) \equiv 0$ for all $(x,t) \in [0,\infty)\times[0,\infty)$. The proof is done. □

Motivated by Lemma 1.5, throughout this paper, we shall focus only on the solution $u(x,t)$ **given by the representation formula** (1.3)**, where the data** $h(x)$ **and** $g(t)$ **are assumed to be continuous on** $(0,\infty)$ **satisfying the basic assumption** (1.2) (except in Section 6).

The main results of the paper are explained as follows. In each section below, we explore certain interesting properties of the solution $u(x,t)$ given by (1.3). In most sections, we discuss the space-time asymptotic behavior of the solution $u(x,t)$ under some assumptions on $h(x)$ and $g(t)$ or under some special choices of $h(x)$ and $g(t)$ on $(0,\infty)$.

In Section 2, we do general discussion on the solution properties of $u(x,t)$ given by (1.3), including its space-time energy and space-time gradient estimate.

In Section 3, we discuss the asymptotic behavior of solution $u(x,t)$ as $x \to \infty$ or as $t \to \infty$ with polynomial, trigonometric, and logarithmic initial-boundary data respectively.

In Section 4.1, we prescribe the oscillation behavior of $u(x,t)$ as $x \to \infty$ or as $t \to \infty$ using certain slow-oscillation initial-boundary data.

In Sections 5 and 6, we discuss space and time periodic solutions and solutions with singular initial-boundary data.

Finally, we would like to mention some useful books on parabolic partial differential equations. For one-dimensional equation, one can see [2, 15] and for high-dimensional equation, one can see [4, 5, 7–9].

2. Solution properties of the ibvp (1.1): general discussion

2.1. Rewriting the formula (1.3)

Lemma 2.1. *Assume $h(x)$ and $g(t)$ are continuous on $(0,\infty)$ satisfying (1.2). Then the formula (1.3) can be rewritten as*

$$
\begin{aligned}
u(x,t) &= u_h(x,t) + u_g(x,t) \\
&= \left(\frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} h(x+\sqrt{4t}z)\,dz - \frac{1}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2} h(-x+\sqrt{4t}z)\,dz\right) \\
&\quad + \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2} g\left(t-\left(\frac{x}{2z}\right)^2\right) dz, \quad (x,t)\in(0,\infty)\times(0,\infty),
\end{aligned} \tag{2.1}
$$

which has certain advantage for analysis and computation.

Proof. The rewriting for the integral of h is trivial. For the integral of g in (1.3), we do the change of variables $\theta = t - s$ first and then let $z = x/\sqrt{4\theta}$. The proof is done. □

As a consequence of (2.1), we have the following corollary.

Corollary 2.2. *Assume the same assumption as in Lemma 2.1. Along the parabola $P(\lambda) : x/\sqrt{4t} = \lambda$, where $\lambda \in (0,\infty)$ is a parameter, we can write $u(x,t)$ in (2.1) as*

$$
\begin{aligned}
u(x,t) &= \left(\frac{1}{\sqrt{\pi}}\int_{-\lambda}^{\infty} e^{-z^2} h\left(x\left(1+\frac{z}{\lambda}\right)\right) dz - \frac{1}{\sqrt{\pi}}\int_{\lambda}^{\infty} e^{-z^2} h\left(x\left(-1+\frac{z}{\lambda}\right)\right) dz\right) \\
&\quad + \frac{2}{\sqrt{\pi}}\int_{\lambda}^{\infty} e^{-z^2} g\left(t\left(1-\left(\frac{\lambda}{z}\right)^2\right)\right) dz, \quad (x,t)\in P(\lambda).
\end{aligned} \tag{2.2}
$$

The next lemma is an immediate application of (2.2).

Lemma 2.3. *Assume $h(x)$ and $g(t)$ are continuous on $[0,\infty)$ satisfying (1.2). Then along each fixed parabola $P(\lambda) : x/\sqrt{4t} = \lambda$, $\lambda \in (0,\infty)$, the solution $u(x,t)$ given by (1.3) satisfies*

$$
\lim_{(x,t)\in P(\lambda)\to(0,0)} u(x,t) = \left(\frac{1}{\sqrt{\pi}}\int_{-\lambda}^{\lambda} e^{-z^2}\,dz\right) h(0) + \left(\frac{2}{\sqrt{\pi}}\int_{\lambda}^{\infty} e^{-z^2}\,dz\right) g(0), \tag{2.3}
$$

which, by (2.7) below, is an ***interpolation*** *between $h(0)$ and $g(0)$.*

Remark 2.4. In particular, if $h(0) \neq g(0)$, the limit on the right-hand side of (2.3) depends on λ and $u(x,t)$ cannot be continuous up to the point $(0,0)$. But if $h(0) = g(0)$, the limit is independent of λ and is equal to $h(0)$ (same as $g(0)$). In such a case, $u(x,t)$ is actually continuous up to $(0,0)$ (see Remark 1.4 also).

Proof of Lemma 2.3. For $(x,t) \in P(\lambda)$ we have $x = \sqrt{4t}\lambda$ in (2.2) and by the assumption on $h(x)$ and $g(t)$, the Lebesgue Dominated Convergence Theorem (denoted as "**LDCT**" from now on) in analysis can be applied here. We obtain

$$\begin{aligned}\lim_{(x,t)\in P(\lambda)\to(0,0)} u(x,t) &= \lim_{t\to 0^+} u(\sqrt{4t}\lambda, t) \\ &= \left(\frac{1}{\sqrt{\pi}}\int_{-\lambda}^{\lambda} e^{-z^2}\,dz\right) h(0) + \left(\frac{2}{\sqrt{\pi}}\int_{\lambda}^{\infty} e^{-z^2}\,dz\right) g(0).\end{aligned}$$

The proof is done. □

We will make use of (2.2) again in Section 3.1.1.

2.2. The case when $h(x)$ and $g(t)$ are bounded on $(0,\infty)$; maximum and minimum principle

In this section, we assume $h(x)$ and $g(t)$ are continuous and **bounded** on $(0,\infty)$ where $h(x)$ and $g(t)$ may not be defined and continuous up to $x = 0$ and $t = 0$ respectively. They clearly satisfy the basic assumption (1.2). We first observe the following simple estimate:

Lemma 2.5. *We have the following simple estimate*

$$0 < \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\,dz < e^{-\frac{x^2}{4t}}, \quad \forall\,(x,t) \in (0,\infty)\times(0,\infty). \tag{2.4}$$

Moreover, for fixed $t \in (0,\infty)$, *we have*

$$\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\,dz = O\left(\frac{\sqrt{t}}{x}e^{-\frac{x^2}{4t}}\right) \quad \textit{as } x \to \infty. \tag{2.5}$$

Proof. Let

$$F(x) = \frac{2}{\sqrt{\pi}}e^{x^2}\int_x^{\infty} e^{-z^2}\,dz, \quad x \in (0,\infty), \quad F(0) = 1.$$

It satisfies

$$\begin{aligned}F'(x) &= \frac{2}{\sqrt{\pi}}\left(e^{x^2}\int_x^{\infty} 2xe^{-z^2}\,dz - 1\right) \\ &< \frac{2}{\sqrt{\pi}}\left(e^{x^2}\int_x^{\infty} 2ze^{-z^2}\,dz - 1\right) = 0, \quad \forall\,x \in (0,\infty),\end{aligned}$$

which implies (2.4) if we replace x by $x/\sqrt{4t}$.

For (2.5), for fixed $t \in (0, \infty)$, we can use L'Hospital rule to get

$$\lim_{x\to\infty} \frac{\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\, dz}{\sqrt{t}x^{-1}e^{-x^2/(4t)}} = 1. \qquad \square$$

Lemma 2.6. *Assume $h(x)$ and $g(t)$ are continuous bounded functions with $|h(x)| \le M$ and $|g(t)| \le N$ for all $x \in (0, \infty)$, $t \in (0, \infty)$ and some positive constants M, N. Then the solution $u(x, t)$ given by (1.3) is also a bounded function satisfying*

(2.6)
$$|u(x,t)| \le \left(\frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{x/\sqrt{4t}} e^{-z^2}\, dz\right) M + \left(\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\, dz\right) N, \quad \forall\, (x,t) \in (0,\infty)\times(0,\infty),$$

where by the identity

$$\frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{x/\sqrt{4t}} e^{-z^2}\, dz + \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\, dz \equiv 1, \quad \forall\, (x,t) \in (0,\infty)\times(0,\infty), \tag{2.7}$$

we see that the bound on $|u(x, t)|$ in (2.6) is an ***interpolation*** *between M and N.*

Remark 2.7. The bound on $|u(x, t)|$ in (2.6) is **sharp** in the sense that when $h(x) \equiv g(t) \equiv M > 0$, we have $u(x, t) \equiv M$ and the inequality in (2.6) becomes equality for all $(x, t) \in (0, \infty) \times (0, \infty)$.

Remark 2.8. Note that, in (2.6), the coefficient function of M tends to 1 as $x \to \infty$ (for fixed t) and tends to 0 as $t \to \infty$ (for fixed x). Similarly, the coefficient function of N tends to 0 as $x \to \infty$ and tends to 1 as $t \to \infty$. For general $(x, t) \to (\infty, \infty)$, the bound on $|u(x, t)|$ in (2.6) does not tend to 0 in general. Along the parabola $P(\lambda) : x/\sqrt{4t} = \lambda$, $\lambda \in (0, \infty)$, we have

$$|u(x,t)| \le \left(\frac{1}{\sqrt{\pi}} \int_{-\lambda}^{\lambda} e^{-z^2}\, dz\right) M + \left(\frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} e^{-z^2}\, dz\right) N, \quad \forall\, (x,t) \in P(\lambda).$$

Proof of Lemma 2.6. By (1.3), (1.8), and (2.1), we have for $(x, t) \in (0, \infty) \times (0, \infty)$ the estimate

$$\begin{aligned}
|u(x,t)| &= \left| \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right) h(\xi)\, d\xi + \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} g\left(t - \left(\frac{x}{2z}\right)^2\right) dz \right| \\
&\le \frac{M}{\sqrt{4\pi t}} \int_0^{\infty} \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right) d\xi + \frac{2N}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\, dz \\
&= M \left(\frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{x/\sqrt{4t}} e^{-z^2}\, dz\right) + N \left(\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\, dz\right).
\end{aligned}$$

The proof is done. $\square$

Similar to the proof of Lemma 2.6, one can obtain the following **maximum principle** result. In order for $u(x,t)$ to be continuous up to the boundary, we assume $h(x)$ and $g(t)$ are continuous on $[0,\infty)$ with $h(0)=g(0)$.

Corollary 2.9 (Maximum principle)**.** *Assume $h(x)$ and $g(t)$ are **bounded continuous functions on** $[0,\infty)$ with $h(0)=g(0)$, $\sup_{[0,\infty)} h(x)=M$, $\sup_{[0,\infty)} g(t)=N$, where M, N are finite numbers. If $M>N$, then it is impossible for the solution $u(x,t)$ (given by* (1.3)*) to have $u(x_0,t_0)=M$ at some $(x_0,t_0)\in(0,\infty)\times(0,\infty)$. Similarly, if $N>M$, then it is impossible for $u(x,t)$ to have $u(x_0,t_0)=N$ at some $(x_0,t_0)\in(0,\infty)\times(0,\infty)$. Finally, if $M=N$ and $u(x_0,t_0)=M$ at some $(x_0,t_0)\in(0,\infty)\times(0,\infty)$, then we must have $h(x)\equiv g(t)\equiv M$ for all $x\in[0,\infty)$ and $t\in[0,t_0]$ and $u(x,t)\equiv M$ on* $[0,\infty)\times[0,t_0]$.

Remark 2.10. Similar results hold for the **minimum principle** with supremum replaced by infimum.

Proof of Corollary 2.9. By the assumption we know that $u(x,t)$ is continuous on $[0,\infty)\times[0,\infty)$ and lies in the space (1.9) and for $M>N$ we have

$$\begin{aligned}
&u(x,t)\\
&=\frac{1}{\sqrt{4\pi t}}\int_0^\infty\left(e^{-\frac{(x-\xi)^2}{4t}}-e^{-\frac{(x+\xi)^2}{4t}}\right)h(\xi)\,d\xi+\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^\infty e^{-z^2}g\left(t-\left(\frac{x}{2z}\right)^2\right)dz \\
&\le\left(\frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{x/\sqrt{4t}}e^{-z^2}\,dz\right)M+\left(\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^\infty e^{-z^2}\,dz\right)N<M
\end{aligned}\tag{2.8}$$

for all $(x,t)\in(0,\infty)\times(0,\infty)$. By (2.8), the first assertion follows. The proof of the second assertion is similar. For the case $M=N$, (2.8) implies $u(x,t)\le M$ for all $(x,t)\in(0,\infty)\times(0,\infty)$. If we have $u(x_0,t_0)=M$ at some $(x_0,t_0)\in(0,\infty)\times(0,\infty)$, the standard strong maximum principle for the heat equation on the domain $U_{t_0}=(0,b)\times(0,t_0]$, where $b>x_0$, implies that $u(x,t)\equiv M$ on $\overline{U}_{t_0}=[0,b]\times[0,t_0]$ (see Evans PDE book [5, p. 55]). Since $b>x_0$ is arbitrary, we must have $u(x,t)\equiv M$ on $[0,\infty)\times[0,t_0]$, which also implies $h(x)\equiv g(t)\equiv M$ on $x\in[0,\infty)$ and $t\in[0,t_0]$. The proof is done. □

2.3. The characterization of solutions $u(x,t)$ which are constant along each parabola $x/\sqrt{4t}=\lambda$, $\lambda\in(0,\infty)$

Lemma 2.11. *Assume $u(x,t)$ is a solution of the heat equation on $(0,\infty)\times(0,\infty)$ which is **constant** along each parabola $P(\lambda): x/\sqrt{4t}=\lambda$, $\lambda\in(0,\infty)$, then it must have the form*

$$u(x,t)=C_1\int_0^{x/\sqrt{4t}}e^{-z^2}\,dz+C_2,\quad (x,t)\in(0,\infty)\times(0,\infty)\tag{2.9}$$

for some constants C_1 and C_2. It satisfies the **constant** *initial-boundary conditions:*

$$\begin{cases} u(x,0) = C_1\frac{\sqrt{\pi}}{2} + C_2, & x \in (0,\infty), \\ u(0,t) = C_2, & t \in (0,\infty). \end{cases}$$

Remark 2.12. Conversely, if we take $h(x) = M$, $g(t) = N$ in (1.3), where M, N are constants, then by (2.1), we have

$$u(x,t) = \frac{2}{\sqrt{\pi}}(M-N)\int_0^{x/\sqrt{4t}} e^{-z^2}\,dz + N,$$

which is **constant** along each parabola $P(\lambda)$ and has the form (2.9).

Proof of Lemma 2.11. Assume $u(x,t)$ is a solution on $(0,\infty)\times(0,\infty)$ satisfying the assumption. Then it can be expressed as $u(x,t) = v(x/\sqrt{4t})$ for some single-variable function $v(z)$, $z \in (0,\infty)$. By

$$u_t(x,t) = v'\left(\frac{x}{\sqrt{4t}}\right)\frac{x}{\sqrt{4t}}\left(-\frac{1}{2t}\right), \quad u_{xx}(x,t) = v''\left(\frac{x}{\sqrt{4t}}\right)\frac{1}{4t},$$

$v(z)$ must satisfy the equation

$$v''(z) + 2zv'(z) = 0, \quad z = \frac{x}{\sqrt{4t}} \in (0,\infty),$$

which has its general solution given by

$$v(z) = C_1\int_0^z e^{-\theta^2}\,d\theta + C_2, \quad \forall\, z \in (0,\infty),\ z = \frac{x}{\sqrt{4t}}$$

for some integration constants C_1 and C_2. The proof is done. □

Remark 2.13. For a solution $u(x,t)$ of the heat equation on $(0,\infty)\times(0,\infty)$, it is impossible for it to be **constant** along each curve of the form $x/t^\alpha = \lambda$, where $\alpha > 0$ is some constant and $\lambda \in (0,\infty)$ is a parameter, unless $\alpha = 1/2$. For such a solution, it must have the form $u(x,t) = v(x/t^\alpha)$ for some single-variable function $v(z)$, $z \in (0,\infty)$ and the identity $u_t(x,t) = u_{xx}(x,t)$ implies

$$v''(z)\frac{1}{t^{2\alpha}} + zv'(z)\frac{\alpha}{t} = 0,$$

which does not give rise to a self-contained equation unless $\alpha = 1/2$.

2.4. Conditions on $h(x)$ and $g(t)$ which imply $u(x,t) \to 0$ as $x \to \infty$

In this section, we are interested in the convergence of $u(x,t)$ to 0 as $x \to \infty$ or as $t \to \infty$. We first need the following simple calculus fact: for fixed $x \in (0,\infty)$ we have

$$\max_{\theta\in(0,\infty)}\left(\frac{1}{\theta^{3/2}}e^{-\frac{x^2}{4\theta}}\right) = \left(\frac{6}{e}\right)^{3/2}\frac{1}{x^3} > 0, \tag{2.10}$$

where the maximum is attained at $\theta = x^2/6$.

The following lemma is about the convergence of $u(x,t)$ to 0 as $x \to \infty$.

Lemma 2.14 (Conditions implying $u(x,t) \to 0$ as $x \to \infty$)**.** *Assume $h(x)$ and $g(t)$ are continuous functions on $(0,\infty)$ satisfying (1.2). If $h(x)$ also satisfies*

$$\text{(2.11)} \qquad \textit{either} \quad \lim_{x\to\infty} h(x) = 0 \quad \textit{or} \quad \int_0^\infty |h(x)|\, dx < \infty,$$

then we have

$$\text{(2.12)} \qquad \lim_{x\to\infty} u(x,t) = 0 \quad \textit{for fixed } t \in (0,\infty).$$

Here $u(x,t)$ is the solution of (1.1) given by (1.3).

Proof. Denote the two integrals for $u(x,t)$ in (1.3) as $u_h(x,t) + u_g(x,t)$, where $(x,t) \in (0,\infty)\times(0,\infty)$. We can express $u_h(x,t)$ as an integral over the whole space $\xi \in (-\infty,\infty)$ as

$$\text{(2.13)} \qquad \begin{aligned} u_h(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} H(\xi)\, d\xi \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H(x+\sqrt{4t}z)\, dz, \quad (x,t) \in (0,\infty)\times(0,\infty), \end{aligned}$$

where $H(\xi)$ is the **odd extension** of $h(\xi)$ to $\xi \in (-\infty,\infty)$ (we can define $H(0)=0$). By (2.11), we have either $\lim_{|\xi|\to\infty} H(\xi) = 0$ or $\int_{-\infty}^{\infty} |H(\xi)|\, d\xi < \infty$. Note that $H(\xi)$ is continuous on $(-\infty,0)\cup(0,\infty)$, bounded near $x=0$.

For the first case in (2.11), $|H(\xi)|$ is a **bounded** function on $(-\infty,\infty)$ with $\lim_{|\xi|\to\infty} H(\xi) = 0$. Standard result for heat equation with such initial data implies $\lim_{x\to\infty} u_h(x,t) = 0$ for fixed $t \in (0,\infty)$. The discontinuity of $H(\xi)$ at $\xi = 0$ will not cause any problem. For the second case in (2.11), we have $|H(\xi)| \in L^1(-\infty,\infty)$ and, for fixed $t \in (0,\infty)$, the LDCT implies

$$\lim_{x\to\infty} u_h(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \left(\lim_{x\to\infty} e^{-\frac{(x-\xi)^2}{4t}} \right) H(\xi)\, d\xi = 0.$$

Next, for fixed $t \in (0,\infty)$, we look at

$$u_g(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)\, ds = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\, d\theta,$$

and here we only assume $g(s)$ is continuous on $(0,\infty)$ satisfying (1.2), which implies that $g(s)$ is integrable near $s=0$ and so $g \in L^1(0,t)$. By (2.10), we have for fixed $t \in (0,\infty)$

$$|u_g(x,t)| \le \frac{x}{\sqrt{4\pi}} \left(\frac{6}{e}\right)^{3/2} \frac{1}{x^3} \int_0^t |g(t-\theta)|\, d\theta \to 0 \quad \text{as } x \to \infty.$$

The proof is done. □

Remark 2.15. In case $h(x)$ does not satisfy (2.11), then the conclusion (2.12) fails in general. A simple example is to take $h(x) = \sin x$ and $g(t) \equiv 0$, then the solution $u(x,t) = e^{-t} \sin x$ does not satisfy (2.12).

Remark 2.16. In Lemma 2.14, if we replace the condition (2.11) as (without changing the condition on $g(t)$)

$$\text{either} \quad \lim_{x\to\infty} h(x) = M \quad \text{or} \quad \int_0^\infty |h(x) - M|\, dx < \infty,$$

then we will have

$$\lim_{x\to\infty} u(x,t) = M \quad \text{for fixed } t \in (0,\infty).$$

This is due to the superposition principle if we decompose (1.1) into two problems with initial-boundary data $h(x) - M$, $g(t)$ and M, 0 respectively.

It is interesting to know that (2.11) is **not a necessary condition** for the conclusion (2.12) to hold. To see this, let $h_1(x) \geq 0$, $h_2(x) \geq 0$, $g_1(t)$, $g_2(t)$ be four continuous functions on $(0,\infty)$ satisfying (1.2), with

$$\lim_{x\to\infty} h_1(x) = 0 \quad \text{and} \quad \int_0^\infty h_1(x)\, dx = \infty \tag{2.14}$$

and

$$\lim_{x\to\infty} h_2(x) \quad \text{does not exist and} \quad \int_0^\infty h_2(x)\, dx < \infty. \tag{2.15}$$

Let $u(x,t)$ be the solution of (1.1) given by (1.3), where now the initial-boundary data $h(x)$, $g(t)$ are given by $h(x) = h_1(x) + h_2(x)$, $g(t) = g_1(t) + g_2(t)$. Clearly, by (2.14) and (2.15), $h(x)$ does not satisfy (2.11). By the superposition principle, we have $u(x,t) = u_1(x,t) + u_2(x,t)$, where $u_i(x,t)$ is the solution of (1.1) with initial-boundary data $h_i(x)$, $g_i(t)$ respectively, $i = 1,2$. By Lemma 2.14, for fixed $t \in (0,\infty)$, we have

$$\lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} u_1(x,t) + \lim_{x\to\infty} u_2(x,t) = 0 + 0 = 0.$$

Therefore, $u(x,t)$ satisfies (2.12) even if $h(x)$ does not satisfy (2.11).

2.5. Conditions on $h(x)$ and $g(t)$ which imply $u(x,t) \to 0$ as $t \to \infty$

The following lemma is about the convergence of $u(x,t)$ to 0 as $t \to \infty$.

Lemma 2.17 (Conditions implying $u(x,t) \to 0$ as $t \to \infty$)**.** *Assume $h(x)$ and $g(t)$ are continuous functions on $(0,\infty)$ satisfying (1.2). If $h(x)$ also satisfies*

$$\text{either } h(x) \text{ is bounded on } (0,\infty) \text{ or } \int_0^\infty |h(x)|\, dx < \infty \tag{2.16}$$

and $g(t)$ also satisfies

$$\text{(2.17)} \qquad \textit{either} \quad \lim_{t\to\infty} g(t) = 0 \quad \textit{or} \quad \int_0^\infty |g(t)|\,dt < \infty,$$

then we have

$$\text{(2.18)} \qquad \lim_{t\to\infty} u(x,t) = 0 \quad \textit{for fixed } x \in (0,\infty).$$

Here $u(x,t)$ is the solution of (1.1) *given by* (1.3).

Proof. Same as in the proof of Lemma 2.14, we write $u(x,t)$ as $u(x,t) = u_h(x,t) + u_g(x,t)$, where $u_h(x,t)$ can be expressed as (2.13).

We first estimate $u_h(x,t)$. By the assumption (2.16), $H(\xi)$ in (2.13) is either bounded on $(-\infty,\infty)$ or $\int_{-\infty}^{\infty} |H(\xi)|\,d\xi < \infty$. For the first case, we have $u_h(0,t) = 0$ for all $t \in (0,\infty)$ and note that $u_h(x,t)$ satisfies the following gradient estimate

$$\text{(2.19)} \qquad \begin{aligned} \left|\frac{\partial u_h}{\partial x}(x,t)\right| &= \left|\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-\xi)^2}{4t}} \frac{\xi - x}{2t} H(\xi)\,d\xi\right| \\ &= \left|\frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-z^2} z \cdot H(x + \sqrt{4t}z)\,dz\right| \\ &\leq \frac{M}{\sqrt{\pi t}}, \quad \forall\,(x,t) \in (0,\infty)\times(0,\infty), \end{aligned}$$

where $M > 0$ is the bound of $|H(\xi)|$ on $(-\infty,\infty)$. By the mean value theorem, for fixed $x \in (0,\infty)$, we have

$$|u_h(x,t)| = |u_h(x,t) - u_h(0,t)| \leq \frac{Mx}{\sqrt{\pi t}} \to 0 \quad \text{as } t \to \infty.$$

For the second case, we have $|H(\xi)| \in L^1(-\infty,\infty)$; hence

$$|u_h(x,t)| \leq \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} |H(\xi)|\,d\xi \to 0 \quad \text{as } t \to \infty.$$

Next we estimate $u_g(x,t)$, where $g(t)$ satisfies (2.17). For the first case in (2.17), for any $\varepsilon > 0$ there exists a number $M > 0$ such that $|g(t)| < \varepsilon$ on $[M,\infty)$. We have for large $t > M > 0$ the following

$$\text{(2.20)} \qquad \begin{aligned} u_g(x,t) &= \frac{x}{\sqrt{4\pi}} \int_0^M \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)\,ds + \frac{x}{\sqrt{4\pi}} \int_M^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)\,ds \\ &:= I + II \end{aligned}$$

and since $g \in L^1(0,M)$, for fixed $x \in (0,\infty)$ we have

$$\text{(2.21)} \qquad |I| \leq \frac{x}{\sqrt{4\pi}} \frac{1}{(t-M)^{3/2}} \int_0^M |g(s)|\,ds \to 0 \quad \text{as } t \to \infty.$$

On the other hand, by the change of variables $z = x/\sqrt{4(t-s)}$, the second integral in (2.20) satisfies

$$|II| \le \varepsilon \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}}\, ds = \varepsilon \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\, dz \le \varepsilon. \tag{2.22}$$

As $\varepsilon > 0$ can be arbitrarily small, by (2.21) and (2.22), we have $u_g(x,t) \to 0$ as $t \to \infty$ for fixed $x \in (0,\infty)$.

For the second case in (2.17), we first look at $u_g(x,t)$ at $t = n \in \mathbb{N}$ and let $n \to \infty$ eventually. For fixed $x \in (0,\infty)$, set

$$f_n(s) = \begin{cases} \frac{1}{(n-s)^{3/2}} e^{-\frac{x^2}{4(n-s)}} g(s), & s \in (0,n), \\ 0, & s \in [n,\infty), \end{cases}$$

which gives

$$u_g(x,n) = \frac{x}{\sqrt{4\pi}} \int_0^\infty f_n(s)\, ds, \quad \text{where } \lim_{n\to\infty} f_n(s) = 0,\ \forall\, s \in (0,\infty)$$

and note that

$$|f_n(s)| \le \left(\frac{6}{e}\right)^{3/2} \frac{1}{x^3} |g(s)| \in L^1(0,\infty), \quad \forall\, n,\ \forall\, s \in (0,\infty).$$

The LDCT implies

$$\begin{aligned} \lim_{n\to\infty} u_g(x,n) &= \frac{x}{\sqrt{4\pi}} \lim_{n\to\infty} \int_0^\infty f_n(s)\, ds \\ &= \frac{x}{\sqrt{4\pi}} \int_0^\infty \lim_{n\to\infty} f_n(s)\, ds = 0 \quad \text{for fixed } x \in (0,\infty). \end{aligned}$$

By analogy, for any sequence $a_n \to \infty$, we also have $\lim_{n\to\infty} u_g(x,a_n) = 0$. This implies $u_g(x,t) \to 0$ as $t \to \infty$.

Combining all of the above estimates, we have (2.18). □

Remark 2.18. In case $h(x)$ does not satisfy (2.16) or $g(t)$ does not satisfy (2.17), then the conclusion (2.18) fails in general. For the first example, we take $h(x) \equiv x$ and $g(t) \equiv 0$, then the solution $u(x,t) = x$ for all $(x,t) \in (0,\infty) \times (0,\infty)$. For the second example, see Section 3.2 with $h(x) = \sin x$ and $g(t) = \sin t$.

Remark 2.19. In Lemma 2.17, if we replace the condition (2.17) as (without changing the condition on $h(x)$)

$$\text{either} \quad \lim_{t\to\infty} g(t) = N \quad \text{or} \quad \int_0^\infty |g(t) - N|\, dt < \infty,$$

then we will have

$$\lim_{t\to\infty} u(x,t) = N \quad \text{for fixed } x \in (0,\infty).$$

This is due to the superposition principle if we decompose (1.1) into two problems with initial-boundary data $h(x)$, $g(t) - N$ and 0, N respectively.

Again, we note that, (2.16) and (2.17) are, in general, not necessary conditions for the conclusion (2.18) to hold. To see this, let $h_1(x)$, $h_2(x)$, $g_1(t)$, $g_2(t)$ be four nonnegative continuous functions on $(0,\infty)$ satisfying (1.2), with

$$\begin{cases} h_1(x) \text{ is bounded on } (0,\infty) \text{ and } \int_0^\infty h_1(x)\,dx = \infty, \\ h_2(x) \text{ is unbounded on } (0,\infty) \text{ and } \int_0^\infty h_2(x)\,dx < \infty, \\ \lim_{t\to\infty} g_1(t) = 0 \text{ and } \int_0^\infty g_1(t)\,dt = \infty, \\ \lim_{t\to\infty} g_2(t) \text{ does not exist and } \int_0^\infty g_2(t)\,dt < \infty. \end{cases}$$

Let $u(x,t)$ be the solution of (1.1) given by (1.3), where now the initial-boundary data $h(x)$, $g(t)$ are given by $h(x) = h_1(x) + h_2(x)$, $g(t) = g_1(t) + g_2(t)$. Clearly $h(x)$ does not satisfy (2.16) and $g(t)$ does not satisfy (2.17) either. By the superposition principle, we have $u(x,t) = u_1(x,t) + u_2(x,t)$, where $u_i(x,t)$ is the solution of (1.1) with initial-boundary data $h_i(x)$, $g_i(t)$ respectively, $i = 1, 2$. By Lemma 2.17, for fixed $x \in (0,\infty)$, we have

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} u_1(x,t) + \lim_{t\to\infty} u_2(x,t) = 0 + 0 = 0.$$

Therefore, $u(x,t)$ satisfies (2.18) even if $h(x)$ does not satisfy (2.16) and $g(t)$ does not satisfy (2.17).

2.6. Total space and total time energy

In this section, we study the total space energy and total time energy of $u(x,t)$. To begin with, we collect some useful integral identities, which will be used in this section.

Lemma 2.20. *We have the following integral identities:*

$$\text{(2.23a)} \quad \frac{1}{\sqrt{4\pi t}}\int_0^\infty \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right)dx = \frac{1}{\sqrt{\pi}}\int_{-\xi/\sqrt{4t}}^{\xi/\sqrt{4t}} e^{-z^2}\,dz \in (0,1), \quad \forall\, t, \xi > 0,$$

$$\text{(2.23b)} \quad \int_0^\infty \frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right)dt = \min\{x,\xi\}, \quad \forall\, x, \xi > 0,$$

$$\text{(2.23c)} \quad \frac{x}{\sqrt{4\pi}}\int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}}\,d\theta = \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^\infty e^{-z^2}\,dz \in (0,1), \quad \forall\, x, t > 0,$$

$$\text{(2.23d)} \quad \frac{x}{\sqrt{4\pi}}\int_0^\infty \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}}\,d\theta = 1, \quad \forall\, x > 0,$$

$$\text{(2.23e)} \quad \int_0^\infty \frac{x}{\sqrt{4\pi}}\frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}}\,dx = \frac{1}{\sqrt{\pi\theta}}, \quad \forall\, \theta > 0.$$

Proof. The verifications of (2.23a), (2.23c) and (2.23d) are straightforward. For fixed $\theta > 0$, if we do the change of variables $x = \sqrt{4\theta}z$, $z \in (0,\infty)$, we can obtain (2.23e). Finally for (2.23b) we first let $t = 1/(4s^2)$ to get

$$\int_0^\infty \frac{1}{\sqrt{4\pi t}} \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} \right) dt = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{e^{-a^2s^2} - e^{-b^2s^2}}{s^2}\, ds,$$

where $a = x - \xi$ and $b = x + \xi$. By the integration by parts, we have

$$\begin{aligned} \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{e^{-a^2s^2} - e^{-b^2s^2}}{s^2}\, ds &= \frac{1}{\sqrt{4\pi}} \int_0^\infty \left(e^{-a^2s^2} - e^{-b^2s^2}\right) d\left(-\frac{1}{s}\right) \\ &= -\frac{2}{\sqrt{4\pi}} \int_0^\infty \left(e^{-a^2s^2}a^2 - e^{-b^2s^2}b^2\right) ds. \end{aligned}$$

Note that we have

$$\int_0^\infty \left(e^{-a^2s^2}a^2\right) ds = |a| \int_0^\infty e^{-z^2}\, dz = |a|\frac{\sqrt{\pi}}{2}$$

and the same for $\int_0^\infty \left(e^{-b^2s^2}b^2\right) ds$. Therefore, we conclude

$$-\frac{2}{\sqrt{4\pi}} \int_0^\infty \left(e^{-a^2s^2}a^2 - e^{-b^2s^2}b^2\right) ds = \frac{1}{2}(|x+\xi| - |x-\xi|) = \min\{x,\xi\}$$

for all $x, \xi > 0$. The proof is done. □

Now we discuss the total space and total time energy respectively. For the total space energy, we have

Lemma 2.21 (Total space energy)**.** *Assume $h(x)$ and $g(t)$ are continuous functions on $(0,\infty)$ satisfying* (1.2) *and also $\int_0^\infty |h(\xi)|\, d\xi < \infty$. Let $u(x,t)$ be the solution of* (1.1) *given by* (1.3). *Then, for fixed $t \in (0,\infty)$, its* ***total space energy****, defined as $E_{\text{space}}(t) = \int_0^\infty u(x,t)\, dx$, is finite and can be expressed as*

(2.24)
$$E_{\text{space}}(t) = \int_0^\infty \left(\frac{1}{\sqrt{\pi}} \int_{-\xi/\sqrt{4t}}^{\xi/\sqrt{4t}} e^{-z^2}\, dz \right) h(\xi)\, d\xi + \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{\theta}} g(t-\theta)\, d\theta, \quad t \in (0,\infty).$$

Proof. By (1.3), we have $E_{\text{space}}(t) = I(t) + II(t)$, where

(2.25)
$$\begin{aligned} I(t) &= \int_0^\infty u_h(x,t)\, dx = \int_0^\infty \left[\frac{1}{\sqrt{4\pi t}} \int_0^\infty \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} \right) h(\xi)\, d\xi \right] dx, \\ II(t) &= \int_0^\infty u_g(x,t)\, dx = \int_0^\infty \left(\frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\, d\theta \right) dx. \end{aligned}$$

We will use the classical **Tonelli's Theorem** (see the book [14]) to see that we can change the order of integration in both $I(t)$ and $II(t)$. The theorem says that for any

nonnegative measurable function $f(x,y)$ defined on an interval $J = J_1 \times J_2 \subset \mathbb{R}^{n+m}$, we always have the identity

$$\iint_J f(x,y)\,dxdy = \int_{J_1}\left(\int_{J_2} f(x,y)\,dy\right)dx = \int_{J_2}\left(\int_{J_1} f(x,y)\,dx\right)dy. \tag{2.26}$$

In particular, for $f \geq 0$, the finiteness of any one of the three integrals in (2.26) implies that of the other two. Hence for any measurable $f(x,y)$, the finiteness of any one of the three integrals for $|f(x,y)|$ implies that $f(x,y)$ is integrable on $J = J_1 \times J_2$ and that all three integrals in (2.26) are equal due to the **Fubini's Theorem**.

For $I(t)$ in (2.25), we have for $x > 0$ and $\xi > 0$ the following

$$\left|\frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right)h(\xi)\right| = \frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right)|h(\xi)|$$

and by (2.23a) we get

$$\int_0^\infty\left(\frac{1}{\sqrt{4\pi t}}\int_0^\infty\left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right)dx\right)|h(\xi)|\,d\xi \leq \int_0^\infty |h(\xi)|\,d\xi < \infty.$$

Hence, by Tonelli's and Fubini's Theorems, we have

$$\begin{aligned} I(t) &= \int_0^\infty\left(\frac{1}{\sqrt{4\pi t}}\int_0^\infty\left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right)dx\right)h(\xi)\,d\xi \\ &= \int_0^\infty\left(\frac{1}{\sqrt{\pi}}\int_{-\xi/\sqrt{4t}}^{\xi/\sqrt{4t}} e^{-z^2}\,dz\right)h(\xi)\,d\xi. \end{aligned} \tag{2.27}$$

For $II(t)$, by (2.23e) and the assumption on $g(t)$ in (1.2), we have

$$\int_0^t\left(\int_0^\infty \frac{x}{\sqrt{4\pi}}\frac{1}{\theta^{3/2}}e^{-\frac{x^2}{4\theta}}\,dx\right)|g(t-\theta)|\,d\theta = \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\theta}}|g(t-\theta)|\,d\theta < \infty.$$

Hence Tonelli's and Fubini's Theorems imply

$$II(t) = \int_0^\infty\left(\frac{x}{\sqrt{4\pi}}\int_0^t \frac{1}{\theta^{3/2}}e^{-\frac{x^2}{4\theta}}g(t-\theta)\,d\theta\right)dx = \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\theta}}g(t-\theta)\,d\theta. \tag{2.28}$$

The proof of (2.24) is done due to (2.27) and (2.28). □

Remark 2.22. For fixed $\xi > 0$, the positive quantity

$$\frac{1}{\sqrt{\pi}}\int_{-\xi/\sqrt{4t}}^{\xi/\sqrt{4t}} e^{-z^2}\,dz \in (0,1), \quad t \in (0,\infty)$$

is strictly decreasing from 1 to 0 with respect to $t \in (0,\infty)$. Hence, by the assumption $\int_0^\infty |h(x)|\,dx < \infty$ and the LDCT, we can conclude

$$\lim_{t\to 0^+}\int_0^\infty\left(\frac{1}{\sqrt{\pi}}\int_{-\xi/\sqrt{4t}}^{\xi/\sqrt{4t}} e^{-z^2}\,dz\right)h(\xi)\,d\xi = \int_0^\infty h(\xi)\,d\xi,$$

$$\lim_{t\to \infty}\int_0^\infty\left(\frac{1}{\sqrt{\pi}}\int_{-\xi/\sqrt{4t}}^{\xi/\sqrt{4t}} e^{-z^2}\,dz\right)h(\xi)\,d\xi = 0.$$

However, if we only assume $g(t)$ satisfies the condition in (1.2), i.e.,

$$|g(t)| \le \frac{C_1}{t^{\alpha}}, \quad \forall\, t \in (0,\varepsilon)$$

for some positive constants C_1, $\varepsilon > 0$ small, and $\alpha \in [0,1)$, then in general, we have no convergence of the term $\int_0^t \frac{1}{\sqrt{\theta}} g(t-\theta)\, d\theta$ as $t \to 0^+$ or as $t \to \infty$. For example, we can take $g(t) = 1/t^{\alpha}$, $t \in (0,\varepsilon)$, with $\alpha \in (1/2, 1)$, then

$$\int_0^t \frac{1}{\sqrt{\theta}} g(t-\theta)\, d\theta \ge \frac{1}{\sqrt{t}} \int_0^t \frac{1}{(t-\theta)^{\alpha}}\, d\theta = \frac{1}{1-\alpha} t^{\frac{1}{2}-\alpha} \to \infty \quad \text{as } t \to 0^+.$$

Therefore, we do not have $\lim_{t\to 0^+} E_{\text{space}}(t) = \int_0^{\infty} h(\xi)\, d\xi$ in general unless we put more assumption on $g(t)$ on $(0,\infty)$ (for example, assume $g(t)$ is bounded near $t = 0$).

Remark 2.23. If $g(t) = t^n$, $n \in \mathbb{N} \cup \{0\}$, its contribution to $E_{\text{space}}(t)$ is equal to

$$\frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{\theta}} (t-\theta)^n\, d\theta = \left(\frac{1}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{s}} (1-s)^n\, ds \right) t^{n+\frac{1}{2}}, \quad n \in \mathbb{N} \cup \{0\},\ t \in (0,\infty).$$

For the total time energy, we have

Lemma 2.24 (Total time energy)**.** *Assume $h(x)$ and $g(t)$ are continuous functions on $(0,\infty)$ satisfying (1.2). In addition, we also assume*

$$\int_0^{\infty} |h(\xi)|\, d\xi < \infty \quad \textit{and} \quad \int_0^{\infty} |g(t)|\, dt < \infty. \tag{2.29}$$

Let $u(x,t)$ be the solution of (1.1) given by (1.3). Then, for fixed $x \in (0,\infty)$, its ***total time energy****, defined as $E_{\text{time}}(x) = \int_0^{\infty} u(x,t)\, dt$, is finite and can be expressed as*

$$E_{\text{time}}(x) = \int_0^x \xi h(\xi)\, d\xi + \int_x^{\infty} x h(\xi)\, d\xi + \int_0^{\infty} g(t)\, dt, \quad x \in (0,\infty). \tag{2.30}$$

In particular, we have

$$\lim_{x\to 0^+} E_{\text{time}}(x) = \int_0^{\infty} g(t)\, dt.$$

Proof. By (1.3), we have $E_{\text{time}}(x) = I(x) + II(x)$, where

$$\begin{aligned} I(x) &= \int_0^{\infty} u_h(x,t)\, dt = \int_0^{\infty} \left[\frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} \right) h(\xi)\, d\xi \right] dt, \\ II(x) &= \int_0^{\infty} u_g(x,t)\, dt = \int_0^{\infty} \left[\frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\, d\theta \right] dt. \end{aligned} \tag{2.31}$$

Again, we use Tonelli's Theorem to imply the change of order of integration in both $I(x)$ and $II(x)$. For $I(x)$, we first look at the integral

$$\widetilde{I}(x) := \int_0^{\infty} \left[\left(\int_0^{\infty} \frac{1}{\sqrt{4\pi t}} \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} \right) dt \right) |h(\xi)| \right] d\xi \tag{2.32}$$

and by the assumption (2.29) and (2.23b), we have for any fixed $x \in (0, \infty)$ the following

$$\widetilde{I}(x) = \int_0^\infty \min\{x, \xi\} \cdot |h(\xi)|\, d\xi = \int_0^x \xi |h(\xi)|\, d\xi + \int_x^\infty x|h(\xi)|\, d\xi < \infty,$$

which implies the convergence of the improper integral for $\widetilde{I}(x)$ in (2.32). Hence the Fubini Theorem is applicable to the improper integrals in $I(x)$ in (2.31) and we have

$$\begin{aligned} I(x) &= \int_0^\infty \left[\int_0^\infty \frac{1}{\sqrt{4\pi t}} \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} \right) dt \right] h(\xi)\, d\xi \\ &= \int_0^x \xi h(\xi)\, d\xi + \int_x^\infty x h(\xi)\, d\xi. \end{aligned} \tag{2.33}$$

For $II(x)$ in (2.31), for fixed $x \in (0, \infty)$, we first look at the integral

$$\int_0^M \left(\frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} |g(t-\theta)|\, d\theta \right) dt,$$

where $M > 0$ is temporarily a fixed number. By the assumption $\int_0^\infty |g(t)|\, dt < \infty$ and the estimate (2.10), we have

$$\int_0^M \left(\frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} |g(t-\theta)|\, d\theta \right) dt \le \left(\frac{x}{\sqrt{4\pi}} \left(\frac{6}{e}\right)^{3/2} \frac{1}{x^3} \int_0^\infty |g(s)|\, ds \right) M,$$

which, by Tonelli's Theorem, implies that the nonnegative function

$$\frac{x}{\sqrt{4\pi}} \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} |g(t-\theta)|$$

is integrable on the domain $\{(\theta, t) \in \mathbb{R}^2 : 0 < \theta \le t, 0 < t \le M\}$ for any $M > 0$ and Fubini's Theorem implies

$$\begin{aligned} II(x) &= \lim_{M\to\infty} \int_0^M \left(\frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\, d\theta \right) dt \\ &= \lim_{M\to\infty} \int_0^M \left(\frac{x}{\sqrt{4\pi}} \int_\theta^M \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\, dt \right) d\theta \\ &= \lim_{M\to\infty} \int_0^M \left(\frac{x}{\sqrt{4\pi}} \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} \int_0^{M-\theta} g(s)\, ds \right) d\theta. \end{aligned}$$

Next, for fixed $x \in (0, \infty)$, let

$$f_M(\theta) = \begin{cases} \frac{x}{\sqrt{4\pi}} \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} \int_0^{M-\theta} g(s)\, ds, & \theta \in (0, M], \\ 0, & \theta \in (M, \infty). \end{cases}$$

It satisfies

$$\lim_{M\to\infty} f_M(\theta) = \frac{x}{\sqrt{4\pi}} \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} \int_0^\infty g(s)\, ds, \quad \forall\, \theta \in (0, \infty),$$

$$\int_0^\infty f_M(\theta)\,d\theta = \int_0^M \left(\frac{x}{\sqrt{4\pi}}\frac{1}{\theta^{3/2}}e^{-\frac{x^2}{4\theta}}\int_0^{M-\theta} g(s)\,ds\right)d\theta,$$
$$|f_M(\theta)| \le \frac{x}{\sqrt{4\pi}}\frac{1}{\theta^{3/2}}e^{-\frac{x^2}{4\theta}}\int_0^\infty |g(s)|\,ds, \quad \forall\,\theta, M \in (0,\infty),$$

where, by the identity (2.23d), the function $(x/\sqrt{4\pi})\theta^{-3/2}e^{-x^2/4\theta}\int_0^\infty |g(s)|\,ds$ is integrable with respect to $\theta \in (0,\infty)$ for all $x \in (0,\infty)$ with

$$\int_0^\infty \left(\frac{x}{\sqrt{4\pi}}\frac{1}{\theta^{3/2}}e^{-\frac{x^2}{4\theta}}\int_0^\infty |g(s)|\,ds\right)d\theta = \int_0^\infty |g(s)|\,ds, \quad \forall\,x \in (0,\infty).$$

The LDCT implies for any $x \in (0,\infty)$ that

$$(2.34)\qquad \begin{aligned} II(x) &= \lim_{M\to\infty}\int_0^\infty f_M(\theta)\,d\theta = \int_0^\infty \lim_{M\to\infty} f_M(\theta)\,d\theta \\ &= \int_0^\infty \left(\frac{x}{\sqrt{4\pi}}\frac{1}{\theta^{3/2}}e^{-\frac{x^2}{4\theta}}\int_0^\infty g(s)\,ds\right)d\theta = \int_0^\infty g(s)\,ds. \end{aligned}$$

The proof of the identity (2.30) is done due to (2.33) and (2.34). □

Remark 2.25. If we have $h(x) \equiv 0$ in (2.30), then

$$E_{\text{time}}(x) = \int_0^\infty g(t)\,dt, \quad \forall\,x \in (0,\infty),$$

which is a **constant** independent of $x \in (0,\infty)$.

2.7. Gradient estimate of $u(x,t)$

In this section, we do estimate on u_t and u_x under the assumption that both $h(x)$ and $g(t)$ are bounded continuous functions on $(0,\infty)$.

Lemma 2.26 (Space derivative estimate)**.** *Assume $h(x)$ and $g(t)$ are **bounded** continuous functions with $|h(x)| \le M$ and $|g(t)| \le N$ for all $x \in (0,\infty)$ and $t \in (0,\infty)$. Then the solution $u(x,t)$ given by (1.3) is a bounded solution on $(x,t) \in (0,\infty)\times(0,\infty)$ satisfying the estimate*

$$(2.35)\qquad |u_x(x,t)| \le \frac{M}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{t}} + \left(\frac{2N}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^\infty e^{-y^2}(1+2y^2)\,dy\right)\cdot\frac{1}{x}$$

for all $(x,t) \in (0,\infty)\times(0,\infty)$.

Remark 2.27. As one can see from Theorem 3.10 or from Remark 3.15 below, the assumption that $h(x)$ and $g(t)$ are bounded is necessary for (2.35) to be valid. Also note that, for fixed $t \in (0,\infty)$ and fixed $x \in (0,\infty)$ respectively, we have

$$(2.36)\quad \limsup_{x\to\infty}|u_x(x,t)| \le \frac{M}{\sqrt{\pi}}\frac{1}{\sqrt{t}}, \quad \limsup_{t\to\infty}|u_x(x,t)| \le \left(\frac{2N}{\sqrt{\pi}}\int_0^\infty e^{-y^2}(1+2y^2)\,dy\right)\frac{1}{x}.$$

For bounded trigonometric initial-boundary data, the solution $u(x,t)$ may oscillate along x-direction and also along t-direction. Therefore, both limsup limits in (2.36) are **not zero** in general. See Lemma 3.12 below.

Remark 2.28. Similar to estimate (2.4), one can check that for any number $\sigma \in (0,1)$ there is a constant $C(\sigma) > 0$ such that

$$\int_{x/\sqrt{4t}}^{\infty} e^{-y^2}(1+2y^2)\,dy \le C(\sigma)e^{-\sigma\frac{x^2}{4t}}, \quad \forall\,(x,t)\in(0,\infty)\times(0,\infty). \tag{2.37}$$

Proof of Lemma 2.26. Let $H(\xi)$ be the odd extension of $h(\xi)$ to $\xi \in (-\infty,\infty)$ (we can define $H(0)=0$). We have

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} H(\xi)\,d\xi + \frac{x}{\sqrt{4\pi}}\int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\,d\theta \\ &:= I(x,t) + II(x,t), \quad (x,t)\in(0,\infty)\times(0,\infty) \end{aligned} \tag{2.38}$$

and so $u_x(x,t) = I_x(x,t) + II_x(x,t)$. Since $|H(\xi)| \le M$ for all $\xi \in (-\infty,\infty)$, we have

$$|I_x(x,t)| \le \frac{M}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}}\frac{|x-\xi|}{2t}\,d\xi = \frac{M}{\sqrt{\pi}}\frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-z^2}|z|\,dz = \frac{M}{\sqrt{\pi}}\frac{1}{\sqrt{t}}$$

and

$$\begin{aligned} |II_x(x,t)| &= \left|\frac{1}{\sqrt{4\pi}}\int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\,d\theta + \frac{x}{\sqrt{4\pi}}\int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}}\left(-\frac{x}{2\theta}\right) g(t-\theta)\,d\theta\right| \\ &\le \frac{N}{\sqrt{4\pi}}\int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}}\,d\theta + \frac{x^2 N}{2\sqrt{4\pi}}\int_0^t \frac{1}{\theta^{5/2}} e^{-\frac{x^2}{4\theta}}\,d\theta \\ &= \frac{N}{\sqrt{4\pi}}\frac{4}{x}\int_{x/\sqrt{4t}}^{\infty} e^{-y^2}\,dy + \frac{x^2 N}{2\sqrt{4\pi}}\frac{16}{x^3}\int_{x/\sqrt{4t}}^{\infty} e^{-y^2}y^2\,dy \\ &= \frac{N}{\sqrt{\pi}}\frac{2}{x}\int_{x/\sqrt{4t}}^{\infty} e^{-y^2}(1+2y^2)\,dy, \end{aligned}$$

which implies the conclusion. □

Lemma 2.29 (Time derivative estimate)**.** *Assume $h(x)$ and $g(t)$ are **bounded** continuous functions with $|h(x)| \le M$ and $|g(t)| \le N$ for all $x \in (0,\infty)$ and $t \in (0,\infty)$. Then the solution $u(x,t)$ given by* (1.3) *is a bounded solution on $(x,t)\in(0,\infty)\times(0,\infty)$ satisfying the estimate*

$$|u_t(x,t)| \le M\cdot\frac{1}{t} + \left(\frac{12N}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-y^2}y^2\left(1+\frac{2}{3}y^2\right)dy\right)\cdot\frac{1}{x^2} \tag{2.39}$$

for all $(x,t)\in(0,\infty)\times(0,\infty)$.

Remark 2.30. Similar to (2.37), for any number $\sigma \in (0,1)$, we have the estimate

$$\int_{x/\sqrt{4t}}^{\infty} e^{-y^2} y^2 \left(1 + \frac{2}{3}y^2\right) dy \le C(\sigma) e^{-\sigma \frac{x^2}{4t}}, \quad \forall\, (x,t) \in (0,\infty)\times(0,\infty),$$

where $C(\sigma) > 0$ is a constant depending only on σ.

Remark 2.31. By Lemmas 2.26 and 2.29, we have

$$\lim_{(x,t)\to(\infty,\infty)} u_x(x,t) = \lim_{(x,t)\to(\infty,\infty)} u_t(x,t) = 0$$

for bounded continuous functions $h(x)$ and $g(t)$ on $(0,\infty)$.

Proof of Lemma 2.29. In this case, by (2.38), we need to look at $I_t(x,t)$ and $II_t(x,t)$. We first have

$$\begin{aligned} I_t(x,t) &= \frac{-1}{2\sqrt{4\pi}} \frac{1}{t^{3/2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} H(\xi)\, d\xi + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} \frac{(x-\xi)^2}{4t^2} H(\xi)\, d\xi \\ &= \frac{-1}{2\sqrt{\pi}} \frac{1}{t} \int_{-\infty}^{\infty} e^{-z^2} H(x+\sqrt{4t}z)\, dz + \frac{1}{\sqrt{\pi}} \frac{1}{t} \int_{-\infty}^{\infty} e^{-z^2} z^2 H(x+\sqrt{4t}z)\, dz, \end{aligned}$$

which gives

$$|I_t(x,t)| \le \frac{M}{2t} + \frac{M}{2t} = \frac{M}{t}.$$

Next, we look at

$$\begin{aligned} II_t(x,t) = II_{xx}(x,t) &= \frac{\partial^2}{\partial x^2} \left(\frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\, d\theta \right) \\ &= \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\, d\theta + \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} \left(-\frac{x}{2\theta}\right) g(t-\theta)\, d\theta \right) \\ &= -\frac{3x}{2\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{5/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\, d\theta + \frac{x^3}{4\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{7/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\, d\theta \end{aligned}$$

and conclude

$$\begin{aligned} |II_t(x,t)| &\le \frac{3xN}{2\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{5/2}} e^{-\frac{x^2}{4\theta}}\, d\theta + \frac{x^3 N}{4\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{7/2}} e^{-\frac{x^2}{4\theta}}\, d\theta \\ &= \frac{3xN}{2\sqrt{4\pi}} \frac{16}{x^3} \int_{x/\sqrt{4t}}^{\infty} e^{-y^2} y^2\, dy + \frac{x^3 N}{4\sqrt{4\pi}} \frac{64}{x^5} \int_{x/\sqrt{4t}}^{\infty} e^{-y^2} y^4\, dy \\ &= \frac{12N}{\sqrt{\pi}} \frac{1}{x^2} \int_{x/\sqrt{4t}}^{\infty} e^{-y^2} \left(y^2 + \frac{2}{3}y^4\right) dy. \end{aligned}$$

The proof is done. □

Motivated by (2.35) and (2.39), we have the following interesting result.

Corollary 2.32. *Assume $h(x)$ and $g(t)$ are **bounded** continuous functions with $|h(x)| \leq M$ and $|g(t)| \leq N$ for all $x \in (0,\infty)$ and $t \in (0,\infty)$ and let $u(x,t)$ be the bounded function given by (1.3). Then the function*

$$xu_x(x,t) + 2tu_t(x,t), \quad (x,t) \in (0,\infty) \times (0,\infty) \tag{2.40}$$

*is also a **solution** of the heat equation on $(0,\infty)\times(0,\infty)$. Moreover, along the **parabola** $P(\lambda) : x/\sqrt{4t} = \lambda$, where $\lambda \in (0,\infty)$ is a parameter, we have the estimate*

$$\begin{aligned} |xu_x(x,t) + 2tu_t(x,t)| \leq & \left[2\lambda \frac{M}{\sqrt{\pi}} + \left(\frac{2N}{\sqrt{\pi}} \int_\lambda^\infty e^{-y^2}(1+2y^2)\,dy\right)\right] \\ & + \left[2M + \frac{1}{2\lambda^2}\left(\frac{12N}{\sqrt{\pi}} \int_\lambda^\infty e^{-y^2} y^2 \left(1 + \frac{2}{3}y^2\right)\,dy\right)\right] \end{aligned} \tag{2.41}$$

for all $(x,t) \in P(\lambda)$. Note that the bound in (2.41) depends only on M, N and λ.

Proof. By parabolic scaling, for each constant $c > 0$, the function $U(x,t,c) := u(cx, c^2 t)$ is a solution of the heat equation on $(0,\infty)\times(0,\infty)$. By this, the function

$$\frac{\partial U}{\partial c}(x,t,c) = x \cdot u_x(cx, c^2 t) + 2ct \cdot u_t(cx, c^2 t) \tag{2.42}$$

will also be a solution of the heat equation on $(0,\infty)\times(0,\infty)$ for each $c \in (0,\infty)$. Taking $c = 1$ in (2.42) will give rise to the solution in (2.40). As for the estimate in (2.41), it is a consequence of (2.35) and (2.39). □

2.8. Derivative up to the boundary

In this section, we discuss the values of the derivative $u_x(x,t)$ and $u_t(x,t)$ on the boundary of the domain $(0,\infty)\times(0,\infty)$ (except at the corner point $(0,0)$). For our purpose of discussion, $h(x)$ and $g(t)$ are assumed to be at least continuously differentiable on $[0,\infty)$. Since we have $u(x,t) = u_h(x,t) + u_g(x,t)$, we look at $u_h(x,t)$ and $u_g(x,t)$ separately.

Lemma 2.33 (Derivative of $u_h(x,t)$ on the boundary $t = 0$)**.** *Assume $h \in C^1[0,\infty) \cap C^2(0,\infty)$ and $|h(x)|$, $|h'(x)|$, $|h''(x)|$ all satisfy the growth condition in (1.2). Then we have*

$$\frac{\partial u_h}{\partial x}(x,0) = h'(x), \quad \frac{\partial u_h}{\partial t}(x,0) = h''(x), \quad \forall\, x \in (0,\infty). \tag{2.43}$$

Proof. The first identity in (2.43) is obvious since $u_h(x,0) = h(x)$ for all $x \in (0,\infty)$. For the second identity, since both $|h'(x)|$ and $|h''(x)|$ satisfy the growth condition (1.2), we have

(2.44)
$$\begin{aligned}\frac{\partial u_h}{\partial t}(x,t) &= \frac{\partial^2 u_h}{\partial x^2}(x,t)\\ &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} h(x+\sqrt{4t}z)\,dz - \frac{1}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2} h(-x+\sqrt{4t}z)\,dz\right)\\ &= \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{\pi t}}e^{-\frac{x^2}{4t}}h(0) + \frac{1}{\sqrt{\pi}}\left(\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} h'(x+\sqrt{4t}z)\,dz + \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} h'(-x+\sqrt{4t}z)\,dz\right)\right]\\ &= \frac{1}{\sqrt{\pi t}}e^{-\frac{x^2}{4t}}\left(-\frac{x}{2t}\right)h(0) + \frac{1}{\sqrt{\pi}}\left[\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} h''(x+\sqrt{4t}z)\,dz - \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} h''(-x+\sqrt{4t}z)\,dz\right]\end{aligned}$$

and for fixed $x \in (0,\infty)$ we obtain $\lim_{t\to 0^+} \big(\partial u_h/\partial t\big)(x,t) = h''(x)$. By the mean value theorem, there exists $\theta(t) \in (0,t)$ such that

$$\frac{\partial u_h}{\partial t}(x,0) = \lim_{t\to 0^+}\frac{u_h(x,t)-u_h(x,0)}{t} = \lim_{t\to 0^+}\frac{\partial u_h}{\partial t}(x,\theta(t)) = h''(x), \quad x\in(0,\infty). \tag{2.45}$$

Hence $\big(\partial u_h/\partial t\big)(x,0)$ exists and the second identity is verified. □

Remark 2.34. Be careful that we cannot use the identity (2.13) to derive the conclusion since the extended function $H(\xi)$, $\xi \in (-\infty,\infty)$, is not even continuous at $\xi = 0$ if $h(0) \neq 0$ and one cannot move the differentiation into the integral sign. Also, one cannot prove the second identity using the following argument

$$\frac{\partial u_h}{\partial t}(x,0) = \frac{\partial^2 u_h}{\partial x^2}(x,0) = h''(x), \quad x \in (0,\infty),$$

since $u_h(x,t)$ satisfies the heat equation only on the domain $(0,\infty)\times(0,\infty)$.

Lemma 2.35 (Derivative of $u_h(x,t)$ on the boundary $x=0$)**.** *Assume $h \in C^1[0,\infty)$ and both $|h(x)|$ and $|h'(x)|$ satisfy the growth condition in* (1.2)*. Then we have*

$$\frac{\partial u_h}{\partial x}(0,t) = \frac{1}{\sqrt{\pi t}}h(0) + \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-z^2}h'(\sqrt{4t}z)\,dz, \quad \frac{\partial u_h}{\partial t}(0,t) = 0, \quad \forall\, t\in(0,\infty). \tag{2.46}$$

Remark 2.36. By integration by parts, we can also express $\big(\partial u_h/\partial x\big)(0,t)$ as

$$\frac{\partial u_h}{\partial x}(0,t) = \frac{2}{\sqrt{\pi t}}\int_0^{\infty}(ze^{-z^2})h(\sqrt{4t}z)\,dz.$$

Proof of Lemma 2.35. The second identity in (2.46) is obvious due to $u_h(0,t) = 0$ for all $t \in (0,\infty)$. As for the first identity, by (2.44), we have

$$\begin{aligned}\frac{\partial u_h}{\partial x}(x,t) &= \frac{1}{\sqrt{\pi t}}e^{-\frac{x^2}{4t}}h(0)\\ &\quad + \frac{1}{\sqrt{\pi}}\left(\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}h'(x+\sqrt{4t}z)\,dz + \int_{x/\sqrt{4t}}^{\infty} e^{-z^2}h'(-x+\sqrt{4t}z)\,dz\right)\end{aligned} \tag{2.47}$$

and letting $x \to 0^+$ we obtain

$$\lim_{x\to 0^+} \frac{\partial u_h}{\partial x}(x,t) = \frac{1}{\sqrt{\pi t}} h(0) + \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} h'(\sqrt{4t} z)\, dz. \tag{2.48}$$

The first identity in (2.46) will follow due to (2.48) and the mean value theorem argument similar to (2.45). □

In the following we use two examples to verify the correctness of the formulas in Lemmas 2.33 and 2.35.

Example 2.37. We choose $h(x) = x^m$, where $m = 2k+1$, $k \in \mathbb{N} \cup \{0\}$, is an odd natural number. By (3.12) below, we have

$$u_h(x,t) = x^m + m(m-1)x^{m-2}t + \cdots + \frac{m!}{3!(k-1)!} x^3 t^{k-1} + \frac{m!}{k!} x t^k,$$

which gives

$$\begin{aligned}
\frac{\partial u_h}{\partial x}(x,t) &= m x^{m-1} + m(m-1)(m-2)x^{m-3} t + \cdots + \frac{m!}{3!(k-1)!} 3x^2 t^{k-1} + \frac{m!}{k!} t^k, \\
\frac{\partial u_h}{\partial t}(x,t) &= m(m-1)x^{m-2} + \cdots + \frac{m!}{3!(k-1)!} x^3 (k-1) t^{k-2} + \frac{m!}{k!} x k t^{k-1},
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{\partial u_h}{\partial x}(x,0) = m x^{m-1}, \qquad &&\frac{\partial u_h}{\partial t}(x,0) = m(m-1)x^{m-2}, \\
&\frac{\partial u_h}{\partial x}(0,t) = \frac{m!}{k!} t^k, &&\frac{\partial u_h}{\partial t}(0,t) = 0.
\end{aligned} \tag{2.49}$$

To see that (2.49) is consistent with (2.43) and (2.46), it suffices to check that

$$\frac{\partial u_h}{\partial x}(0,t) = \frac{m!}{k!} t^k = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} m(\sqrt{4t} z)^{m-1}\, dz, \quad m = 2k+1,$$

which is equivalent to the identity

$$\int_0^\infty e^{-z^2} z^{2k}\, dz = \frac{1}{2^{2k+1}} \frac{(2k)!}{k!} \sqrt{\pi}, \quad k \in \mathbb{N} \cup \{0\}. \tag{2.50}$$

One can verify (2.50) using the identities $\int_0^\infty z^2 e^{-z^2}\, dz = \sqrt{\pi}/4$ and

$$\int_0^\infty z^{2k+2} e^{-z^2}\, dz = \frac{2k+1}{2} \int_0^\infty z^{2k} e^{-z^2}\, dz,$$

together with the mathematical induction.

Example 2.38. We choose $h(x) = \sin x$, $x \in (0,\infty)$, and obtain $u_h(x,t) = e^{-t}\sin x$. We see that

$$\frac{\partial u_h}{\partial x}(x,0) = \cos x, \quad \frac{\partial u_h}{\partial t}(x,0) = -\sin x, \quad \frac{\partial u_h}{\partial x}(0,t) = e^{-t}, \quad \frac{\partial u_h}{\partial t}(0,t) = 0.$$

Again, to check the correctness of (2.46) and (2.43), it suffices to check that

$$e^{-t} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} \cos(\sqrt{4t}z)\, dz, \quad \forall\, t \in (0,\infty). \tag{2.51}$$

However, we note that (2.51) is a consequence of the familiar identity

$$e^{-t}\cos x = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{4t}} \cos y\, dy, \quad (x,t) \in (0,\infty)\times(0,\infty).$$

Lemma 2.39 (Derivative of $u_g(x,t)$ on the boundary $t=0$)**.** *Assume $g \in C^1[0,\infty)$ (which will satisfy* (1.2) *automatically). Then we have*

$$\frac{\partial u_g}{\partial x}(x,0) = 0, \quad \frac{\partial u_g}{\partial t}(x,0) = 0, \quad \forall\, x \in (0,\infty). \tag{2.52}$$

Proof. The first identity in (2.52) is obvious due to $u_g(x,0) = 0$ for all $x \in (0,\infty)$. For the second identity, we compute

$$\frac{\partial u_g}{\partial t}(x,t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} g(0) \frac{x}{2t} + \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^\infty e^{-z^2} g'\left(t - \left(\frac{x}{2z}\right)^2\right) dz$$

and for fixed $x \in (0,\infty)$ we have $\lim_{t\to 0^+} \big(\partial u_g/\partial t\big)(x,t) = 0$. Hence we have $\big(\partial u_g/\partial t\big)(x,0) = 0$ due to the mean value theorem. □

Lemma 2.40 (Derivative of $u_g(x,t)$ on the boundary $x=0$)**.** *Assume $g \in C^1[0,\infty)$. Then we have*

$$\frac{\partial u_g}{\partial x}(0,t) = -\frac{1}{\sqrt{\pi t}} g(0) - \frac{1}{\sqrt{\pi}} \int_0^t \frac{g'(s)}{\sqrt{t-s}}\, ds, \quad \frac{\partial u_g}{\partial t}(0,t) = g'(t), \quad \forall\, t \in (0,\infty). \tag{2.53}$$

Remark 2.41. One can also express $\big(\partial u_g/\partial x\big)(0,t)$ as

$$\frac{\partial u_g}{\partial x}(0,t) = -\frac{1}{\sqrt{\pi t}} g(0) - \frac{2\sqrt{t}}{\sqrt{\pi}} \int_0^1 g'(t(1-\theta^2))\, d\theta. \tag{2.54}$$

Proof of Lemma 2.40. The second identity in (2.53) is obvious due to $u_g(0,t) = g(t)$ for all $t \in (0,\infty)$. For the first identity, we compute

$$\begin{aligned}
\frac{\partial u_g}{\partial x}(x,t) &= \frac{\partial}{\partial x}\left(\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^\infty e^{-z^2} g\left(t - \left(\frac{x}{2z}\right)^2\right) dz\right) \\
&= -\frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} g(0) + \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^\infty e^{-z^2} g'\left(t - \left(\frac{x}{2z}\right)^2\right)\left(-\frac{x}{2z^2}\right) dz.
\end{aligned}$$

For fixed $t \in (0,\infty)$, if we do the change of variables $s = t - \left(\frac{x}{2z}\right)^2$, we will get

$$\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2} g'\left(t-\left(\frac{x}{2z}\right)^2\right)\left(-\frac{x}{2z^2}\right)dz = -\frac{1}{\sqrt{\pi}}\int_0^t e^{-\frac{x^2}{4(t-s)}}\frac{g'(s)}{\sqrt{t-s}}\,ds$$

with

$$\lim_{x\to 0^+}\left(-\frac{1}{\sqrt{\pi}}\int_0^t e^{-\frac{x^2}{4(t-s)}}\frac{g'(s)}{\sqrt{t-s}}\,ds\right) = -\frac{1}{\sqrt{\pi}}\int_0^t \frac{g'(s)}{\sqrt{t-s}}\,ds. \tag{2.55}$$

By (2.55), we conclude

$$\lim_{x\to 0^+}\frac{\partial u_g}{\partial x}(x,t) = -\frac{1}{\sqrt{\pi t}}g(0) - \frac{1}{\sqrt{\pi}}\int_0^t \frac{g'(s)}{\sqrt{t-s}}\,ds,$$

which implies the first identity in (2.53) due to the mean value theorem. □

Combining Lemmas 2.33, 2.35, 2.39, 2.40, and (2.54), we can conclude the following theorem.

Theorem 2.42 (Derivative of $u(x,t)$ on the boundary $x=0$ and $t=0$)**.** *Assume $h \in C^1[0,\infty)\cap C^2(0,\infty)$, $g \in C^1[0,\infty)$, and $|h(x)|$, $|h'(x)|$, $|h''(x)|$ all satisfy the growth condition in* (1.2)*. Let $u(x,t) = u_h(x,t)+u_g(x,t)$, $(x,t)\in(0,\infty)\times(0,\infty)$, be the solution given by* (1.3)*. Then we have*

$$\frac{\partial u}{\partial x}(x,0) = h'(x), \quad \frac{\partial u}{\partial t}(x,0) = h''(x)$$

and

$$\begin{aligned}\frac{\partial u}{\partial x}(0,t) &= \frac{1}{\sqrt{\pi t}}\left(h(0) + 2\sqrt{t}\int_0^{\infty} e^{-z^2}h'(\sqrt{4t}z)\,dz\right)\\ &\qquad -\frac{1}{\sqrt{\pi t}}\left(g(0) + 2t\int_0^1 g'(t(1-\theta^2))\,d\theta\right),\\ \frac{\partial u}{\partial t}(0,t) &= g'(t)\end{aligned} \tag{2.56}$$

for all $x\in(0,\infty)$ and all $t\in(0,\infty)$.

Example 2.43. Again, we use a simple example to check the correctness of $\left(\partial u/\partial x\right)(0,t)$ in (2.56). We choose $h(x) = e^x$, $x\in(0,\infty)$, and $g(t)=e^t$, $t\in(0,\infty)$. The solution $u(x,t)$ of the ibvp (1.1), given by (1.3), is equal to $u(x,t) = e^{x+t}$. To verify the correctness of (2.56), we need to check the identity

$$e^t = \frac{1}{\sqrt{\pi t}}\left(2\sqrt{t}\int_0^{\infty} e^{-z^2}e^{\sqrt{4t}z}\,dz - 2t\int_0^1 e^{t(1-\theta^2)}\,d\theta\right), \quad \forall\, t\in(0,\infty). \tag{2.57}$$

We first note that

$$\int_0^\infty e^{-z^2} e^{\sqrt{4t}z}\, dz = e^t \int_0^\infty e^{-(z-\sqrt{t})^2}\, dz = e^t \int_{-\sqrt{t}}^\infty e^{-s^2}\, ds$$

and

$$\int_0^1 e^{t(1-\theta^2)}\, d\theta = e^t \int_0^1 e^{-t\theta^2}\, d\theta = \frac{e^t}{\sqrt{t}} \int_0^{\sqrt{t}} e^{-s^2}\, ds.$$

Hence the right-hand side of (2.57) is equal to

$$\frac{2e^t}{\sqrt{\pi}} \left(\int_{-\sqrt{t}}^\infty e^{-s^2}\, ds - \int_0^{\sqrt{t}} e^{-s^2}\, d\theta \right) = \frac{2e^t}{\sqrt{\pi}} \int_0^\infty e^{-s^2}\, ds = e^t, \quad \forall\, t \in (0,\infty),$$

as verified.

2.9. Monotonicity of $u_h(x,t)$ and $u_g(x,t)$

In this section, we discuss the monotonicity of $u_h(x,t)$ and $u_g(x,t)$ under suitable assumptions on $h(x)$ and $g(t)$. However, the monotonicity of $u(x,t)$ is, in general, very difficult to determine even if we know that of $u_h(x,t)$ and $u_g(x,t)$.

Lemma 2.44 (Monotonicity of $u_h(x,t)$ in space direction)**.** *Assume $h \in C^1[0,\infty)$ and both $|h(x)|$ and $|h'(x)|$ satisfy the growth condition in* (1.2)*. If $h(0) \geq 0$, $h'(x) \geq 0$ on $(0,\infty)$, and $h(x)$ is not identically zero, then, for fixed $t \in (0,\infty)$, $u_h(x,t)$ is positive for all $x \in (0,\infty)$ and is* ***strictly increasing*** *in $x \in (0,\infty)$.*

Remark 2.45. $h(x)$ is increasing (strictly increasing) on $(0,\infty)$ means that, for $0 < x_1 < x_2$, we have $h(x_1) \leq h(x_2)$ $(h(x_1) < h(x_2))$.

Proof of Lemma 2.44. For fixed $t \in (0,\infty)$, by (1.3) and (1.8), $u_h(x,t)$ is clearly positive for all $x \in (0,\infty)$. Also, by (2.47), we have $(\partial u_h/\partial x)(x,t) > 0$ for all $x \in (0,\infty)$ due to the assumption on $h(x)$. The proof is done. □

The monotonicity of $u_h(x,t)$ in the time direction will depend on the sign of $h''(x)$ on $(0,\infty)$. However, the sign of $h'(x)$ on $(0,\infty)$ is irrelevant as one can see from Example 2.47 below.

Lemma 2.46 (Monotonicity of $u_h(x,t)$ in time direction)**.** *Assume $h \in C^1[0,\infty) \cap C^2(0,\infty)$ and $|h(x)|$, $|h'(x)|$, $|h''(x)|$ all satisfy the growth condition in* (1.2)*. If $h(0) \leq 0$, $h''(x) \geq 0$ on $(0,\infty)$, then, for fixed $x \in (0,\infty)$, $u_h(x,t)$ is* ***increasing*** *in $t \in (0,\infty)$.*

Proof. Assume $h(0) \leq 0$ and $h''(x) \geq 0$ on $(0,\infty)$. By (2.44), we have, for fixed $x \in (0,\infty)$, the following

$$\frac{\partial u_h}{\partial t}(x,t) = \frac{\partial^2 u_h}{\partial x^2}(x,t)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left(-\frac{x}{2t}\right) h(0) \\
&\quad + \frac{1}{\sqrt{\pi}} \left(\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} h''(x+\sqrt{4t}z)\, dz - \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} h''(-x+\sqrt{4t}z)\, dz \right) \\
&= \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left(-\frac{x}{2t}\right) h(0) + \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} \right) h''(\xi)\, d\xi \\
&\geq 0, \quad t \in (0,\infty).
\end{aligned} \tag{2.58}
$$

The conclusion follows due to (1.8) and the assumption on $h(x)$. □

Example 2.47. Assume $h(x) = \sin x$, $x \in (0,\infty)$, $h(0) = 0$. The corresponding $u_h(x,t)$ is given by

$$u_h(x,t) = e^{-t} \sin x, \quad (x,t) \in (0,\infty) \times (0,\infty), \quad u_h(0,t) \equiv 0, \quad u_h(x,0) = \sin x.$$

For fixed $x \in (0,\infty)$ with $\sin x > 0$ (i.e., $h''(x) < 0$), $u_h(x,t)$ is decreasing in $t \in (0,\infty)$ and for fixed $x \in (0,\infty)$ with $\sin x < 0$ (i.e., $h''(x) > 0$), $u_h(x,t)$ is increasing in $t \in (0,\infty)$. The sign of $h'(x)$ does not come into play at all.

Corollary 2.48. *Assume $h \in C^1[0,\infty) \cap C^2(0,\infty)$ and $|h(x)|$, $|h'(x)|$, $|h''(x)|$ all satisfy the growth condition in* (1.2). *If $h(x)$ satisfies the assumption in Lemma* 2.44 *and also $h'(x) + h''(x) \geq 0$ on $(0,\infty)$, then on the region $\{(x,t) \in (0,\infty) \times (0,\infty) : x < 2t\}$, we have*

$$\left(\frac{\partial u_h}{\partial x} + \frac{\partial u_h}{\partial t} \right)(x,t) > 0. \tag{2.59}$$

Proof. By (2.47) and (2.58), we have

$$
\begin{aligned}
&\left(\frac{\partial u_h}{\partial x} + \frac{\partial u_h}{\partial t} \right)(x,t) \\
&= \left(1 - \frac{x}{2t}\right) \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} h(0) + \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left(e^{-\frac{(x-\xi)^2}{4t}} + e^{-\frac{(x+\xi)^2}{4t}} \right) h'(\xi)\, d\xi \\
&\quad + \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} \right) h''(\xi)\, d\xi, \quad (x,t) \in (0,\infty) \times (0,\infty).
\end{aligned}
$$

Since $h'(x) \geq 0$ on $(0,\infty)$, we have

$$
\begin{aligned}
\left(\frac{\partial u_h}{\partial x} + \frac{\partial u_h}{\partial t} \right)(x,t) &\geq \left(1 - \frac{x}{2t}\right) \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} h(0) \\
&\quad + \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} \right) (h'(\xi) + h''(\xi))\, d\xi
\end{aligned} \tag{2.60}
$$

and note that the equality sign in (2.60) occurs only when $h'(x) \equiv 0$ on $(0,\infty)$ and it will imply $h(0) > 0$. By this, under the assumption on $h(x)$, we must have (2.59) on the region $x < 2t$. The proof is done. □

Example 2.49. Take $h(x) \equiv 1$ in Corollary 2.48. We have

$$u_h(x,t) = \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{x/\sqrt{4t}} e^{-z^2}\, dz, \quad (x,t) \in (0,\infty) \times (0,\infty)$$

and obtain

$$\left(\frac{\partial u_h}{\partial x} + \frac{\partial u_h}{\partial t} \right)(x,t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left(1 - \frac{x}{2t}\right) > 0 \quad \text{if } x < 2t.$$

This example says that the condition $x < 2t$ in Corollary 2.48 is necessary.

Lemma 2.50 (Monotonicity of $u_g(x,t)$ in time direction)**.** *Assume $g \in C^0(0,\infty)$ and satisfies* (1.2)*. If $g(t)$ is positive and increasing on $(0,\infty)$, then, for fixed $x \in (0,\infty)$, $u_g(x,t)$ is **strictly increasing** in $t \in (0,\infty)$. Moreover, we have*

$$0 < u_g(x,t) < g(t), \quad \forall\, (x,t) \in (0,\infty) \times (0,\infty). \tag{2.61}$$

Proof. For fixed $x \in (0,\infty)$ and $0 < t_1 < t_2$, we have

$$\begin{aligned} u_g(x,t_2) &= \frac{x}{\sqrt{4\pi}} \int_0^{t_1} \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t_2 - \theta)\, d\theta + \frac{x}{\sqrt{4\pi}} \int_{t_1}^{t_2} \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t_2 - \theta)\, d\theta \\ &> \frac{x}{\sqrt{4\pi}} \int_0^{t_1} \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t_2 - \theta)\, d\theta \geq \frac{x}{\sqrt{4\pi}} \int_0^{t_1} \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t_1 - \theta)\, d\theta = u_g(x,t_1). \end{aligned}$$

Hence $u_g(x,t)$ is strictly increasing in $t \in (0,\infty)$. For (2.61), we note that

$$0 < u_g(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} g\left(t - \left(\frac{x}{2z}\right)^2\right) dz \leq \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} g(t)\, dz < g(t)$$

for all $(x,t) \in (0,\infty) \times (0,\infty)$. The proof is done. □

Lemma 2.51 (Monotonicity of $u_g(x,t)$ in space direction)**.** *We have the following two results:*

(1) *Assume $g \in C^0(0,\infty)$ and satisfies* (1.2)*. If $g(t)$ is positive on $(0,\infty)$, then, for fixed $t \in (0,\infty)$, $u_g(x,t)$ is **strictly decreasing** in $x \in (\sqrt{2t},\infty)$.*

(2) *Assume $g \in C^0[0,\infty) \cap C^1(0,\infty)$ and $g'(t)$ also satisfies* (1.2)*. If $g(t)$ is positive and increasing on $(0,\infty)$, then, for fixed $t \in (0,\infty)$, $u_g(x,t)$ is **strictly decreasing** in $x \in (0,\infty)$.*

Proof. For (1), to avoid the differentiation on g, we use the formula for $u_g(x,t)$ in (1.3) and get, for fixed $t \in (0,\infty)$ and for $x \in (\sqrt{2t},\infty)$, the following

$$\begin{aligned} \frac{\partial u_g}{\partial x}(x,t) &= \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta)\, d\theta \right) \\ &= \frac{1}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta) \left(1 - \frac{x^2}{2\theta}\right) d\theta \\ &< \frac{1}{\sqrt{4\pi}} \int_0^t \frac{1}{\theta^{3/2}} e^{-\frac{x^2}{4\theta}} g(t-\theta) \left(1 - \frac{x^2}{2t}\right) d\theta < 0. \end{aligned}$$

The proof is done.

For (2), since we assume $g \in C^0[0,\infty) \cap C^1(0,\infty)$, we can differentiate it. Now we use a different formula for $u_g(x,t)$ and get

$$\begin{aligned}\frac{\partial u_g}{\partial x}(x,t) &= \frac{\partial}{\partial x}\left(\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2} g\left(t-\left(\frac{x}{2z}\right)^2\right) dz\right) \\ &= -\frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} g(0) + \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2} g'\left(t-\left(\frac{x}{2z}\right)^2\right)\left(-\frac{x}{2z^2}\right) dz < 0,\end{aligned} \tag{2.62}$$

for all $x \in (0,\infty)$. Note that the integral in (2.62) converges since $g'(t)$ satisfies the condition in (1.2) near $t = 0$. The inequality in (2.62) is due to the assumption that $g(t)$ is positive and increasing on $(0,\infty)$. The proof is done. □

The following result is analogous to Corollary 2.48.

Corollary 2.52. *Assume $g \in C^0[0,\infty) \cap C^1(0,\infty)$ and $g'(t)$ also satisfies* (1.2). *If $g(t)$ is positive and increasing on $(0,\infty)$, then, on the region $\{(x,t) \in (0,\infty) \times (0,\infty) : x > 2t\}$, we have*

$$\left(\frac{\partial u_g}{\partial x} + \frac{\partial u_g}{\partial t}\right)(x,t) > 0.$$

Proof. Since we can differentiate $g(t)$ on $(0,\infty)$, we have

$$\begin{aligned}\frac{\partial u_g}{\partial t}(x,t) &= \frac{\partial}{\partial t}\left(\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2} g\left(t-\left(\frac{x}{2z}\right)^2\right) dz\right) \\ &= \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} g(0)\frac{x}{2t} + \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2} g'\left(t-\left(\frac{x}{2z}\right)^2\right) dz > 0,\end{aligned} \tag{2.63}$$

where the inequality in (2.63) is due to our assumption on $g(t)$. Combining (2.62) and (2.63), we get

$$\begin{aligned}\left(\frac{\partial u_g}{\partial x} + \frac{\partial u_g}{\partial t}\right)(x,t) &= \frac{1}{\sqrt{\pi t}}\left(\frac{x}{2t} - 1\right) e^{-\frac{x^2}{4t}} g(0) \\ &\quad + \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2} g'\left(t-\left(\frac{x}{2z}\right)^2\right)\left(1-\frac{x}{2z^2}\right) dz.\end{aligned} \tag{2.64}$$

On the region $x > 2t$, we have $2t/x \in (0,1)$ and

$$1 - \frac{x}{2z^2} \in \left(1 - \frac{2t}{x}, 1\right) \subset (0,1) \quad \text{for } z \in (x/\sqrt{4t}, \infty).$$

Therefore, by the assumption on $g(t)$, the quantity in (2.64) is positive on the indicated region. The proof is done. □

Example 2.53. Take $g(t) \equiv 1$ in Corollary 2.52. We have

$$u_g(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\,dz, \quad (x,t) \in (0,\infty) \times (0,\infty)$$

and obtain

$$\left(\frac{\partial u_g}{\partial x} + \frac{\partial u_g}{\partial t}\right)(x,t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left(\frac{x}{2t} - 1\right) > 0 \quad \text{if } x > 2t.$$

This example says that the condition $x > 2t$ in Corollary 2.52 is necessary.

3. Solution properties of the ibvp (1.1): special initial-boundary data

3.1. Polynomial initial-boundary data

In this section we study the asymptotic behavior of $u(x,t)$ with **polynomial** initial and boundary data. We divide the discussions into 3 cases.

Case 1: $h(x) \equiv 0$, $x \in (0,\infty)$, $g(t) = t^n$, $t \in (0,\infty)$, $n \in \mathbb{N}$. By Lemma 2.14, we first have $u(x,t) = u_g(x,t)$ and

$$\lim_{x\to\infty} u(x,t) = 0 \quad \text{for fixed } t \in (0,\infty).$$

Clearly $u(x,t)$ will not tend to zero as $t \to \infty$. Now, by (2.1), we can express $u(x,t)$ as

$$u(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \left(t - \left(\frac{x}{2z}\right)^2\right)^n dz, \quad \text{where } t - \left(\frac{x}{2z}\right)^2 \in (0,t)$$

and along each parabola $P(\lambda) : x/\sqrt{4t} = \lambda$, $\lambda \in (0,\infty)$, we have

$$u(x,t) = t^n \cdot \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} e^{-z^2} \left(1 - \left(\frac{\lambda}{z}\right)^2\right)^n dz = B_n(\lambda) t^n, \tag{3.1}$$

where

$$B_n(\lambda) = \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} e^{-z^2} \left(1 - \left(\frac{\lambda}{z}\right)^2\right)^n dz \in (0,1), \quad \lambda = \frac{x}{\sqrt{4t}} \in (0,\infty). \tag{3.2}$$

Note that $B_n(\lambda)$ is a decreasing function of $\lambda \in (0,\infty)$ with $\lim_{\lambda\to 0^+} B_n(\lambda) = 1$, $\lim_{\lambda\to\infty} B_n(\lambda) = 0$.

By (3.1), along the parabola $P(\lambda) : x/\sqrt{4t} = \lambda$, we can express $u(x,t) - t^n$ as

$$u(x,t) - t^n = (B_n(\lambda) - 1)t^n = \frac{B_n(\lambda) - 1}{2\lambda} \cdot x t^{n-1/2}, \quad (x,t) \in P(\lambda). \tag{3.3}$$

Note that for fixed $x \in (0,\infty)$, $t \to \infty$ is equivalent to $\lambda \to 0^+$. We can evaluate the following limit using the L'Hospital rule and LDCT:

$$\begin{aligned}
&\lim_{\lambda\to 0^+} \frac{B_n(\lambda)-1}{2\lambda} \\
(3.4)\quad &= \lim_{\lambda\to 0^+} \frac{B_n'(\lambda)}{2} = \lim_{\lambda\to 0^+}\left(\frac{1}{\sqrt{\pi}}\int_\lambda^\infty e^{-z^2} n\left(1-\left(\frac{\lambda}{z}\right)^2\right)^{n-1}\left(-\frac{2\lambda}{z^2}\right)dz\right) \\
&= \lim_{\lambda\to 0^+}\left(-\frac{2n}{\sqrt{\pi}}\int_1^\infty e^{-\lambda^2 s^2}\left(1-\frac{1}{s^2}\right)^{n-1}\frac{1}{s^2}\,ds\right) = -\frac{2n}{\sqrt{\pi}}\int_1^\infty \left(1-\frac{1}{s^2}\right)^{n-1}\frac{1}{s^2}\,ds,
\end{aligned}$$

where the improper integral in (3.4) does converge. By (3.3) and (3.4), we can conclude the following result.

Lemma 3.1. *The solution $u(x,t)$ of the ibvp (1.1) with $h(x) \equiv 0$, $x \in (0,\infty)$, $g(t) = t^n$, $t \in (0,\infty)$, $n \in \mathbb{N}$, is given by*

$$(3.5)\qquad u(x,t) = \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^\infty e^{-z^2}\left(t-\left(\frac{x}{2z}\right)^2\right)^n dz, \quad (x,t)\in(0,\infty)\times(0,\infty).$$

For fixed $t \in (0,\infty)$ it satisfies

$$(3.6)\qquad \lim_{x\to\infty} u(x,t) = 0$$

and for fixed $x \in (0,\infty)$ and $t \to \infty$, it satisfies the following asymptotic behavior

$$(3.7)\qquad u(x,t) = t^n + (-C + o(1))\cdot x t^{n-1/2}, \quad \lim_{t\to\infty} o(1) = 0,$$

where $C > 0$ is the value of the integral

$$(3.8)\qquad C = \frac{2n}{\sqrt{\pi}}\int_1^\infty \left(1-\frac{1}{s^2}\right)^{n-1}\frac{1}{s^2}\,ds > 0.$$

Moreover, along each parabola $P(\lambda) : x/\sqrt{4t} = \lambda$, $u(x,t)$ can be expressed as

$$u(x,t) = B_n(\lambda)t^n, \quad (x,t)\in P(\lambda),\ \lambda\in(0,\infty),$$

where $B_n(\lambda) \in (0,1)$, given by (3.2), is a decreasing function of $\lambda \in (0,\infty)$ with $\lim_{\lambda\to 0^+} B_n(\lambda) = 1$ and $\lim_{\lambda\to\infty} B_n(\lambda) = 0$.

Remark 3.2. In case $n = 0$, we have $h(x) \equiv 0$, $g(t) \equiv 1$, and (3.5) is still correct with $n = 0$, i.e.,

$$u(x,t) = \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^\infty e^{-z^2}\,dz \in (0,1), \quad (x,t)\in(0,\infty)\times(0,\infty).$$

Remark 3.3. If we pick n to be the rational number $n = 1/2$, i.e., $g(t) = \sqrt{t}$, then the computation in (3.4) is still correct and (3.7) becomes

$$u(x,t) = \sqrt{t} + (-C + o(1))x, \quad \lim_{t\to\infty} o(1) = 0,$$

where now

$$C = \frac{1}{\sqrt{\pi}} \int_1^\infty \frac{1}{s\sqrt{s^2-1}}\, ds = \frac{\sqrt{\pi}}{2}.$$

Hence we conclude

$$\lim_{t\to\infty} \left| u(x,t) - \left(\sqrt{t} - \frac{\sqrt{\pi}}{2} x\right) \right| = 0 \quad \text{for fixed } x \in (0,\infty). \tag{3.9}$$

We will need (3.9) in the proof of Lemma 4.15 below.

Proof of Lemma 3.1. It suffices to explain (3.7) a little bit. By (3.3), we have

$$u(x,t) - t^n = -Cxt^{n-1/2} + \left(\frac{B_n(\lambda) - 1}{2\lambda} + C\right) xt^{n-1/2},$$

where by (3.4) we have for fixed $x \in (0,\infty)$ the limit

$$\lim_{t\to\infty} \left(\frac{B_n(\lambda) - 1}{2\lambda} + C\right) = \lim_{\lambda\to 0^+} \left(\frac{B_n(\lambda) - 1}{2\lambda} + C\right) = 0.$$

The proof is done. □

Case 2: $h(x) = x^m$, $x \in (0,\infty)$, $g(t) \equiv 0$, $t \in (0,\infty)$, $m \in \mathbb{N}$. In this case, the solution behavior depends on whether m is odd or even. We discuss it separately.

Case 2A: $m \in \mathbb{N}$ is odd. In this case, $h(x)$ is an odd function on $(-\infty,\infty)$ and by (2.13) we have $u(x,t) = u_h(x,t)$ and

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^\infty e^{-\frac{(x-\xi)^2}{4t}} \xi^m \, d\xi.$$

For $h(x) = x$, $g(t) = 0$, we have

$$u(x,t) = x, \quad (x,t) \in (0,\infty) \times (0,\infty)$$

and for $h(x) = x^3$, $g(t) = 0$, we have

$$u(x,t) = x^3 + 6xt, \quad (x,t) \in (0,\infty) \times (0,\infty)$$

and for $h(x) = x^5$, $g(t) = 0$, we have

$$u(x,t) = x^5 + 20x^3t + 60xt^2, \quad (x,t) \in (0,\infty) \times (0,\infty).$$

For general odd m, we can try $u(x,t)$ to have the space-time polynomial form

$$u(x,t) = p_0(x) + p_1(x)t + p_2(x)t^2 + p_3(x)t^3 + p_4(x)t^4 + \cdots \quad \text{(finite terms only)}, \tag{3.10}$$

where we want $u(x,0) = p_0(x) = h(x) = x^m$ and each $p_i(x)$, $i \geq 1$, is a polynomial in x to be determined. We can plug (3.10) into the heat equation to obtain each $p_i(x)$. Compute

$$u_t(x,t) = p_1(x) + 2p_2(x)t + 3p_3(x)t^2 + 4p_4(x)t^3 + \cdots$$

and

$$u_{xx}(x,t) = p_0''(x) + p_1''(x)t + p_2''(x)t^2 + p_3''(x)t^3 + \cdots,$$

and by comparing the coefficient functions, we need to require

$$\begin{gathered} p_1(x) = p_0''(x), \quad p_2(x) = \frac{1}{2}p_1''(x) = \frac{1}{2!}p_0^{(4)}(x), \\ p_3(x) = \frac{1}{3}p_2''(x) = \frac{1}{3!}p_0^{(6)}(x), \quad \ldots, \quad p_k(x) = \frac{1}{k!}p_0^{(2k)}(x), \quad k \in \mathbb{N}, \ k \geq 4. \end{gathered} \tag{3.11}$$

The above process will cease somewhere since $p_0(x) = x^m$ has a finite degree. As a result, we get a space-time polynomial solution $u(x,t)$ of the heat equation on the **entire space** $(-\infty,\infty) \times (-\infty,\infty)$ and it satisfies the initial condition

$$u(x,0) = p_0(x) = h(x) = x^m, \quad \forall\, x \in (0,\infty).$$

However, it may not satisfy the boundary condition $u(0,t) = g(t) \equiv 0$ unless m is an **odd** natural number.

If $m = 2k+1$ for some $k \in \mathbb{N}$, then the last term $p_k(x)$ in (3.11) is given by

$$p_k(x) = \frac{1}{k!}p_0^{(2k)}(x) = \frac{1}{k!}\frac{d^{2k}}{dx^{2k}}x^{2k+1} = \frac{(2k+1)!}{k!}x$$

and so

$$u(x,t) = x^{2k+1} + (2k+1)(2k)x^{2k-1}t + \cdots + \frac{(2k+1)!}{k!}xt^k, \quad (x,t) \in (0,\infty) \times (0,\infty),$$

which is a solution of the ibvp (1.1) with

$$u(x,0) = h(x) = x^{2k+1}, \quad u(0,t) = 0, \quad \forall\, (x,t) \in (0,\infty) \times (0,\infty).$$

We can conclude the following result.

Lemma 3.4. *Let $m = 2k+1$, $k \in \mathbb{N} \cup \{0\}$, be an **odd** natural number. The solution $u(x,t)$ of the ibvp (1.1) with $h(x) = x^m$, $x \in (0,\infty)$, $g(t) \equiv 0$, $t \in (0,\infty)$, is given by*

$$u(x,t) = x^m + m(m-1)x^{m-2}t + \cdots + \frac{m!}{3!(k-1)!}x^3t^{k-1} + \frac{m!}{k!}xt^k, \tag{3.12}$$

where $(x,t) \in (0,\infty) \times (0,\infty)$.

Remark 3.5. For fixed $t \in (0,\infty)$ (fixed $x \in (0,\infty)$), one can determine the asymptotic behavior of $u(x,t)$ as $x \to \infty$ (as $t \to \infty$) easily from (3.12).

Remark 3.6. Note that the space-time polynomial in (3.12) is also a solution of the heat equation on the **entire space** $x \in (-\infty,\infty)$ with initial data $u(x,0) = x^m$, $x \in (-\infty,\infty)$.

Case 2B: $m \in \mathbb{N}$ is even. The method in the odd case is no longer valid for the even case. For example, for $h(x) = x^2$, $g(t) \equiv 0$, the method will give rise to the function $u(x,t) = x^2 + 2t$, which satisfies the heat equation with $u(x,0) = x^2$, but with $u(0,t) = 2t$, which is not the desired $u(0,t) \equiv 0$. We note that for $m = 2k$, $k \in \mathbb{N}$, the last term $p_k(x)$ in (3.11) is now a constant, given by

$$p_k(x) = \frac{1}{k!} p_0^{(2k)}(x) = \frac{1}{k!}\frac{d^{2k}}{dx^{2k}} x^{2k} = \frac{(2k)!}{k!}$$

and the space-time polynomial solution $u(x,t)$, constructed by (3.11), satisfies

$$u(0,t) = \frac{(2k)!}{k!} t^k, \quad t \in (0,\infty). \tag{3.13}$$

Motivated by (3.13), let $v(x,t)$ be the solution of the ibvp (1.1) with

$$v(x,0) = 0, \quad v(0,t) = \frac{(2k)!}{k!} t^k, \quad \forall\, x \in (0,\infty),\ t \in (0,\infty).$$

By (3.5) in the previous example, we know that $v(x,t)$ is given by

$$v(x,t) = \frac{(2k)!}{k!}\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \left(t - \left(\frac{x}{2z}\right)^2\right)^k dz, \quad \text{where } t - \left(\frac{x}{2z}\right)^2 \in (0,t).$$

Hence the function

$$\begin{aligned} &w(x,t) = u(x,t) - v(x,t) \\ &= x^{2k} + (2k)(2k-1)x^{2k-2}t + \cdots + \frac{(2k)!}{k!}t^k - \frac{(2k)!}{k!}\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \left(t - \left(\frac{x}{2z}\right)^2\right)^k dz \end{aligned}$$

will satisfy the ibvp (1.1) with

$$w(x,0) = x^{2k}, \quad w(0,t) \equiv 0, \quad \forall\, x \in (0,\infty),\ t \in (0,\infty).$$

Note that by (3.7) (with n replaced by k), as $t \to \infty$, we have the asymptotic behavior for $v(x,t)$ as $t \to \infty$:

$$v(x,t) = \frac{(2k)!}{k!}\big(t^k + (-\widetilde{C} + o(1))x t^{k-1/2}\big), \quad \lim_{t\to\infty} o(1) = 0,$$

where $\widetilde{C} > 0$ is the value of the integral

$$\widetilde{C} = \frac{2k}{\sqrt{\pi}} \int_1^{\infty} \left(1 - \frac{1}{s^2}\right)^{k-1} \frac{1}{s^2}\, ds. \tag{3.14}$$

Similar to Lemma 3.4, we can conclude the following result.

Lemma 3.7. *Let $m = 2k$, $k \in \mathbb{N}$, be an* ***even*** *natural number. The solution $u(x,t)$ of the ibvp* (1.1) *with $h(x) = x^m$, $x \in (0,\infty)$, $g(t) \equiv 0$, $t \in (0,\infty)$, is given by*

$$u(x,t) = x^m + m(m-1)x^{m-2}t + \cdots + \frac{m!}{2!(k-1)!}x^2 t^{k-1} + \frac{m!}{k!}t^k - v(x,t), \tag{3.15}$$

where $(x,t) \in (0,\infty) \times (0,\infty)$ and

$$v(x,t) = \frac{m!}{k!}\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\left(t - \left(\frac{x}{2z}\right)^2\right)^k dz.$$

For fixed $t \in (0,\infty)$, $v(x,t)$ satisfies

$$\lim_{x\to\infty} v(x,t) = 0$$

and for fixed $x \in (0,\infty)$, as $t \to \infty$, it satisfies

$$v(x,t) = \frac{m!}{k!}\left(t^k + (-\widetilde{C} + o(1))x t^{k-1/2}\right), \quad \lim_{t\to\infty} o(1) = 0, \tag{3.16}$$

where $\widetilde{C} > 0$ is the constant from (3.14).

Remark 3.8. In case $m = 0$, we have $h(x) \equiv 1$, $g(t) \equiv 0$, and (3.15) is still correct with

$$u(x,t) = 1 - \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\,dz = \frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{x/\sqrt{4t}} e^{-z^2}\,dz \in (0,1),$$

where $(x,t) \in (0,\infty) \times (0,\infty)$.

Remark 3.9. For fixed $x \in (0,\infty)$, as $t \to \infty$, (3.16) says that the leading term in t for the solution in (3.15) is given by

$$\begin{aligned} \frac{m!}{k!}t^k - v(x,t) &= \frac{m!}{k!}(\widetilde{C} - o(1))x t^{k-1/2} \\ &= \frac{m!}{k!}(\widetilde{C} - o(1))x t^{(m-1)/2}, \quad m = 2k,\ k \in \mathbb{N}, \end{aligned} \tag{3.17}$$

where $\lim_{t\to\infty} o(1) = 0$. On the other hand, for $m = 2k+1$, as $t \to \infty$, the leading term in t for the solution in (3.12) is given by

$$\frac{m!}{k!}x t^k = \frac{m!}{k!}x t^{(m-1)/2}, \quad m = 2k+1,\ k \in \mathbb{N} \cup \{0\}. \tag{3.18}$$

In terms of m and for fixed $x \in (0,\infty)$, both (3.17) and (3.18) have the same order of exponent as $t \to \infty$, i.e., $t^{(m-1)/2}$.

As a consequence of Lemmas 3.1, 3.4 and 3.7, we can summarize the following theorem.

Theorem 3.10 (Solution with polynomial initial-boundary data)**.** *Let* $m, n \in \mathbb{N}$. *The solution* $u(x,t)$ *of the ibvp* (1.1) *with* $h(x) = x^m$, $x \in (0,\infty)$, $g(t) = t^n$, $t \in (0,\infty)$, *is given by*

$$\begin{aligned} u(x,t) &= x^m + m(m-1)x^{m-2}t + \cdots + \frac{m!}{3!(k-1)!}x^3t^{k-1} + \frac{m!}{k!}xt^k \\ &\quad + \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\left(t - \left(\frac{x}{2z}\right)^2\right)^n dz, \quad \text{if } m = 2k+1 \text{ is odd} \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} u(x,t) &= x^m + m(m-1)x^{m-2}t + \cdots + \frac{m!}{2!(k-1)!}x^2t^{k-1} + \frac{m!}{k!}t^k \\ &\quad - \frac{m!}{k!}\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\left(t - \left(\frac{x}{2z}\right)^2\right)^k dz \\ &\quad + \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\left(t - \left(\frac{x}{2z}\right)^2\right)^n dz, \quad \text{if } m = 2k \text{ is even,} \end{aligned} \tag{3.20}$$

where $(x,t) \in (0,\infty) \times (0,\infty)$.

Case 3: The general polynomial case. We now assume $h(x) = \sum_{i=0}^m a_i x^i$ and $g(t) = \sum_{j=0}^n b_j t^j$, $m, n \in \mathbb{N}$, where a_i, b_j are constant with $a_m \neq 0$, $b_n \neq 0$, then by Theorem 3.10 and the superposition principle for linear equations, one can determine the solution $u(x,t)$ of the ibvp (1.1) (given by the formula (1.3)) and its space-time asymptotic behavior. In particular, to determine the asymptotic behavior of $u(x,t)$ when $t \to \infty$ (for fixed $x \in (0,\infty)$), we need to compare the following quantities

$$\frac{m!}{k!}xt^{(m-1)/2} \;\; (m = 2k+1), \quad \frac{m!}{k!}(\widetilde{C} - o(1))xt^{(m-1)/2} \;\; (m = 2k), \quad t^n + (-C + o(1))xt^{n-1/2},$$

where $\lim_{t\to\infty} o(1) = 0$ and the constants C, $\widetilde{C}$ are from (3.8) and (3.14).

3.1.1. Restriction of $u(x,t)$ to the parabola $x/\sqrt{4t} = \lambda > 0$ for polynomial initial-boundary data

When we have polynomial initial-boundary data $h(x) = x^m$, $g(t) = t^n$, $m, n \in \mathbb{N}$, in Theorem 3.10, the solution $u(x,t) = u_h(x,t) + u_g(x,t)$ is no longer a polynomial due to the integral terms in (3.19) and (3.20). However, if we restrict the solution to the 1-parameter family of parabolas $P(\lambda) : x/\sqrt{4t} = \lambda$, where $\lambda \in (0,\infty)$ is a parameter, then, after suitable rewriting (write t as $x^2/4\lambda^2$ in $u_h(x,t)$ and write x as $\sqrt{4t}\lambda$ in $u_g(x,t)$), it can be expressed as the form

$$u(x,t) = A_m(\lambda)x^m + B_n(\lambda)t^n, \quad \forall\,(x,t) \in P(\lambda), \tag{3.21}$$

where $A_m(\lambda)$, $B_n(\lambda)$ are coefficient functions depending only on m, n, λ.

More precisely, for odd $m = 2k+1$, $k \in \mathbb{N} \cup \{0\}$, in (3.19) we have

$$\begin{aligned} u_h(x,t) &= x^m + m(m-1)x^{m-2}t + \cdots + \frac{m!}{k!}xt^k \\ &= \left(1 + \frac{m(m-1)}{4\lambda^2} + \cdots + \frac{m!}{k!(4\lambda^2)^k}\right)x^m := A_m(\lambda)x^m, \quad t = \frac{x^2}{4\lambda^2} \end{aligned}$$

and for even $m = 2k$, $k \in \mathbb{N}$, in (3.20) we have

$$\begin{aligned} u_h(x,t) &= x^m + m(m-1)x^{m-2}t + \cdots + \frac{m!}{2(k-1)!}x^2t^{k-1} + \frac{m!}{k!}t^k \\ &\quad - \frac{m!}{k!}\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\left(t - \left(\frac{x}{2z}\right)^2\right)^k dz \\ &= \left[1 + \frac{m(m-1)}{4\lambda^2} + \cdots + \frac{m!}{k!(4\lambda^2)^k}\left(1 - \frac{2}{\sqrt{\pi}}\int_{\lambda}^{\infty} e^{-z^2}\left(1 - \left(\frac{\lambda}{z}\right)^2\right)^k dz\right)\right]x^m \\ &:= A_m(\lambda)x^m. \end{aligned}$$

As for $u_g(x,t)$, by (3.1), we have

$$u_g(x,t) = \left(\frac{2}{\sqrt{\pi}}\int_{\lambda}^{\infty} e^{-z^2}\left(1 - \left(\frac{\lambda}{z}\right)^2\right)^n dz\right)t^n := B_n(\lambda)t^n, \quad x = \sqrt{4t}\lambda.$$

We note that

$$A_1(\lambda) = 1, \quad A_m(\lambda) \in (1,\infty), \quad B_n(\lambda) \in (0,1), \quad \forall\, m,n \in \mathbb{N},\ m > 1 \tag{3.22}$$

with

$$\lim_{\lambda\to\infty} A_m(\lambda) = 1, \quad \lim_{\lambda\to\infty} B_n(\lambda) = 0, \quad \forall\, m,n \in \mathbb{N},\ m > 1, \tag{3.23}$$

which matches with the initial data $h(x) = x^m$ since the parabola curve $t = x^2/4\lambda^2$ approaches the half-line $\{(x,0) : x > 0\}$ as $\lambda \to \infty$. On the other hand, as $\lambda \to 0^+$, we have

$$\lim_{\lambda\to 0^+} A_m(\lambda) = \infty, \quad \lim_{\lambda\to 0^+} A_m(\lambda)\lambda^m = 0, \quad \lim_{\lambda\to 0^+} B_n(\lambda) = 1, \quad \forall\, m,n \in \mathbb{N},\ m > 1, \tag{3.24}$$

regardless of whether m is odd or even.

Remark 3.11. For $m = 0$ and $n = 0$, we have

$$A_0(\lambda) = \frac{1}{\sqrt{\pi}}\int_{-\lambda}^{\lambda} e^{-z^2}\,dz, \quad B_0(\lambda) = \frac{2}{\sqrt{\pi}}\int_{\lambda}^{\infty} e^{-z^2}\,dz, \quad A_0(\lambda) + B_0(\lambda) = 1 \tag{3.25}$$

for all $\lambda \in (0,\infty)$.

Now we look at the general polynomial case. Assume $h(x) = \sum_{i=0}^{m} a_i x^i$ and $g(t) = \sum_{j=0}^{n} b_j t^j$, $m, n \in \mathbb{N}$, where a_i, b_j are constant with $a_m \neq 0$, $b_n \neq 0$. By the superposition principle and (3.21), we have

$$u(x,t) = \sum_{i=0}^{m} A_i(\lambda) a_i x^i + \sum_{j=0}^{n} B_j(\lambda) b_j t^j, \quad \forall\,(x,t) \in P(\lambda).$$

One can use (3.22), (3.23), (3.24) and (3.25) to know the asymptotic behavior of each $A_i(\lambda)$ and $B_j(\lambda)$.

3.2. Trigonometric initial-boundary data

The trigonometric functions which can be defined on $(0,\infty)$ are sine and cosine functions. For simplicity, we only look at the case $h(x) = \sin x$ and $g(t) = \sin t$ in the ibvp (1.1). The discussions of other combinations are similar. Both functions are bounded on $(0,\infty)$ satisfying the basic assumption (1.2). By (2.1), the solution $u(x,t)$ is given by

$$u(x,t) = u_h(x,t) + u_g(x,t) = e^{-t}\sin x + \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \sin\left(t - \left(\frac{x}{2z}\right)^2\right) dz, \tag{3.26}$$

which is smooth on $(0,\infty)\times(0,\infty)$ and satisfies (1.4), (1.5) and

$$\lim_{x\to\infty} |u(x,t) - e^{-t}\sin x| = 0 \quad \text{for fixed } t \in (0,\infty).$$

As for fixed $x \in (0,\infty)$ and $t \to \infty$, we first expand $\sin(t - x^2/(4z^2))$ to get

$$\begin{aligned} u_g(x,t) &= \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \sin\left(t - \frac{x^2}{4z^2}\right) dz \\ &= \left(\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \cos\left(\frac{x^2}{4z^2}\right) dz\right) \sin t - \left(\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \sin\left(\frac{x^2}{4z^2}\right) dz\right) \cos t, \end{aligned}$$

and conclude

$$\begin{aligned} &\lim_{t\to\infty} \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \cos\left(\frac{x^2}{4z^2}\right) dz = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2} \cos\left(\frac{x^2}{4z^2}\right) dz := A(x), \\ &\lim_{t\to\infty} \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \sin\left(\frac{x^2}{4z^2}\right) dz = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2} \sin\left(\frac{x^2}{4z^2}\right) dz := B(x). \end{aligned} \tag{3.27}$$

By Hölder inequality, the functions $A(x)$, $B(x)$ satisfy (decompose $e^{-z^2}\cos(x^2/(4z^2))$ as $\sqrt{e^{-z^2}} \cdot \sqrt{e^{-z^2}}\cos(x^2/(4z^2))$, etc.)

$$0 \le A^2(x) < \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2} \cos^2\left(\frac{x^2}{4z^2}\right) dz, \quad x \in (0,\infty)$$

and

$$0 \leq B^2(x) < \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} \sin^2\left(\frac{x^2}{4z^2}\right) dz, \quad x \in (0,\infty),$$

which implies $0 \leq A^2(x) + B^2(x) < 1$ for all $x \in (0,\infty)$. Since $u_h(x,t) = e^{-t}\sin x \to 0$ as $t \to \infty$, we can conclude

$$\lim_{t\to\infty} |u(x,t) - (A(x)\sin t - B(x)\cos t)| = 0 \quad \text{for fixed } x \in (0,\infty)$$

and so

$$\begin{aligned} -1 = \inf_{t\in(0,\infty)} g(t) < \liminf_{t\to\infty} u(x,t) = -\sqrt{A^2(x)+B^2(x)} \\ \leq \sqrt{A^2(x)+B^2(x)} = \limsup_{t\to\infty} u(x,t) < \sup_{t\in(0,\infty)} g(t) = 1. \end{aligned} \tag{3.28}$$

We also note that

$$\lim_{x\to 0^+} A(x) = 1, \quad \lim_{x\to 0^+} B(x) = 0,$$

which matches with the fact that $u(0,t) = g(t) = \sin t$. Also by the Riemann–Lebesgue Lemma, we have

$$\begin{aligned} \lim_{x\to\infty} A(x) &= \lim_{x\to\infty} \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} \cos\left(\frac{x^2}{4z^2}\right) dz \quad (\text{let } z = 1/\sqrt{s}) \\ &= \lim_{x\to\infty} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{s^{3/2}} e^{-\frac{1}{s}} \cos\left(\frac{x^2}{4}s\right) ds = 0 \end{aligned}$$

and similarly $\lim_{x\to\infty} B(x) = 0$.

Due to $h(0) = g(0) = 0$, $u(x,t)$ satisfies the 2-dimensional limit $\lim_{(x,t)\to(0^+,0^+)} u(x,t) = 0$ and lies in the space (1.9) if we define $u(0,0) = 0$.

We can summarize the main result in this section, i.e.,

Lemma 3.12 (Solution with trigonometric initial-boundary data)**.** *The solution $u(x,t)$ of the ibvp* (1.1) *with $h(x) = \sin x$, $x \in (0,\infty)$, $g(t) = \sin t$, $t \in (0,\infty)$, is given by* (3.26) *and it satisfies*

$$\lim_{x\to\infty} |u(x,t) - e^{-t}\sin x| = 0 \quad \textit{for fixed } t \in (0,\infty) \tag{3.29}$$

and

$$\lim_{t\to\infty} |u(x,t) - (A(x)\sin t - B(x)\cos t)| = 0 \quad \textit{for fixed } x \in (0,\infty), \tag{3.30}$$

where $A(x)$ and $B(x)$ are given by (3.27)*.*

Remark 3.13. We can write $A(x)$ on $(0,\infty)$ as

$$A(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty H(y)\cos(x^2 y)\, dy, \quad \text{where } H(y) = \frac{1}{4} y^{-3/2} e^{-1/4y} \in L^1(0,\infty).$$

By the uniqueness theorem for Fourier-cosine transform of functions in the space $L^1(0,\infty)$, the function $A(x)$ is not identically equal to zero on $x \in (0,\infty)$. Similarly, the function $B(x)$ is not identically equal to zero on $x \in (0,\infty)$. We think the identity $A^2(x)+B^2(x)=0$ will not happen for all $x \in (0,\infty)$, but do not know how to prove it.

3.3. Logarithmic initial-boundary data

In this section, we take $h(x) = \log(x+1)$, $x \in (0,\infty)$, $g(t) = \log(t+1)$, $t \in (0,\infty)$, in the ibvp (1.1). By (2.1), we have $u(x,t) = u_h(x,t) + u_g(x,t)$, where

$$\begin{aligned} &u_h(x,t) \\ &= \frac{1}{\sqrt{\pi}}\left(\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}\log(x+\sqrt{4t}z+1)\,dz - \int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\log(-x+\sqrt{4t}z+1)\,dz\right) \qquad (3.31)\\ &:= I(x,t) - II(x,t), \quad (x,t)\in(0,\infty)\times(0,\infty) \end{aligned}$$

and

$$u_g(x,t) = \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\log\left(t-\left(\frac{x}{2z}\right)^2+1\right)dz, \quad (x,t)\in(0,\infty)\times(0,\infty). \qquad (3.32)$$

We first look at $\lim_{x\to\infty} u_h(x,t)$ and $\lim_{x\to\infty} u_g(x,t)$ for fixed $t \in (0,\infty)$. For $u_g(x,t)$, we clearly have

$$\lim_{x\to\infty} u_g(x,t) = \lim_{x\to\infty}\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\log\left(t-\left(\frac{x}{2z}\right)^2+1\right)dz = 0 \quad \text{for fixed } t\in(0,\infty) \qquad (3.33)$$

due to $t-(x/2z)^2+1 \in (1,t+1)$ for $z \in (x/\sqrt{4t},\infty)$. As for $u_h(x,t)$, letting $\xi = -x+\sqrt{4t}z$ in $II(x,t)$, we obtain

$$\lim_{x\to\infty} II(x,t) = \lim_{x\to\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\int_0^{\infty} e^{-\frac{2x\xi}{4t}}e^{-\frac{\xi^2}{4t}}\log(\xi+1)\,d\xi = 0 \quad \text{for fixed } t\in(0,\infty), \qquad (3.34)$$

where we have applied the LDCT in (3.34) with

$$\left|e^{-\frac{2x\xi}{4t}}e^{-\frac{\xi^2}{4t}}\log(\xi+1)\right| \le e^{-\frac{\xi^2}{4t}}\log(\xi+1) \in L^1(0,\infty) \quad \text{for fixed } t\in(0,\infty)$$

for all $x \in (0,\infty)$. It remains to estimate $I(x,t)$. We look at the difference

$$\begin{aligned} &I(x,t) - \log x \\ &= \frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}\log(x+\sqrt{4t}z+1)\,dz - \left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}\,dz\right)\log x \\ &= \frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}[\log(x+\sqrt{4t}z+1)-\log x]\,dz - \left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{-x/\sqrt{4t}} e^{-z^2}\,dz\right)\log x \qquad (3.35)\\ &= \frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}\log\left(1+\frac{1}{x}+\frac{\sqrt{4t}z}{x}\right)dz - \left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{-x/\sqrt{4t}} e^{-z^2}\,dz\right)\log x, \end{aligned}$$

where, for fixed $t \in (0, \infty)$, the second integral in (3.35) will tend to zero as $x \to \infty$ due to the estimate (2.4) in Lemma 2.5.

For the first integral in (3.35), we note that for any fixed large number $M > 0$ (say $M > 1/\sqrt{4t}$), as long as $x > 0$ is sufficiently large (with $x > \sqrt{4t}M$), it can be decomposed as

$$\frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty}(*)\,dz = \frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{-M}(*)\,dz + \frac{1}{\sqrt{\pi}}\int_{-M}^{M}(*)\,dz + \frac{1}{\sqrt{\pi}}\int_{M}^{\infty}(*)\,dz, \tag{3.36}$$

and by the LDCT, the second integrals in (3.36) will tend to 0 as $x \to \infty$. As for the third integral in (3.36), for fixed $t \in (0, \infty)$, if $x > 0$ is large enough, we will have

$$0 < e^{-z^2}\log\left(1 + \frac{1}{x} + \frac{\sqrt{4t}z}{x}\right) < e^{-z^2}\log(2+z) \in L^1(M,\infty), \quad \forall\, z \in [M,\infty).$$

Hence the LDCT is applicable and we get

$$\lim_{x\to\infty}\left(\frac{1}{\sqrt{\pi}}\int_{M}^{\infty} e^{-z^2}\log\left(1 + \frac{1}{x} + \frac{\sqrt{4t}z}{x}\right)dz\right) = 0.$$

However, we need extra care on the first integral in (3.36) since the function $\log(1+1/x+\sqrt{4t}z/x)$ will tend to $-\infty$ as $x \to \infty$ and z is close to $-x/\sqrt{4t}$. We do the change of variables $z = (x/\sqrt{4t})s$ in the first integral to get

$$\begin{aligned} &\frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{-M} e^{-z^2}\log\left(1 + \frac{1}{x} + \frac{\sqrt{4t}z}{x}\right)dz \\ &= \frac{1}{\sqrt{\pi}}\int_{-1}^{-(\sqrt{4t}/x)M}\left(e^{-((x/\sqrt{4t})s)^2}\frac{x}{\sqrt{4t}}s\right)\frac{\log\left(1+s+\frac{1}{x}\right)}{s+\frac{1}{x}}\frac{s+\frac{1}{x}}{s}\,ds \end{aligned} \tag{3.37}$$

and note the following: (1) $e^{-\theta^2}\theta$ is a bounded function on $\theta \in (-\infty,\infty)$. (2) For large $M > 0$ and even larger $x \gg M$, we have

$$-1 + \frac{1}{x} \le s + \frac{1}{x} \le \frac{1 - \sqrt{4t}M}{x} < 0, \quad s + \frac{1}{x} \in (-1, 0), \quad \text{where } s \in [-1, -(\sqrt{4t}/x)M]$$

and

$$\left|\frac{s+\frac{1}{x}}{s}\right| \le 1 + \left|\frac{1}{sx}\right| \le 1 + \frac{1}{\sqrt{4t}M} \le 2, \quad \text{where } -x \le sx \le -\sqrt{4t}M < -1.$$

(3) The positive function $P(\theta)$, $\theta \in (-1,0)$, given by

$$P(\theta) = \frac{\log(1+\theta)}{\theta} > 0, \quad P'(\theta) = \frac{\theta - (1+\theta)\log(1+\theta)}{(1+\theta)\theta^2}, \quad \theta \in (-1,0)$$

is **strictly decreasing** on $\theta \in (-1,0)$ with

$$\lim_{\theta\to(-1)^+} P(\theta) = +\infty, \quad \lim_{\theta\to 0^-} P(\theta) = 1, \quad \lim_{\theta\to(-1)^+} P'(\theta) = -\infty, \quad \lim_{\theta\to 0^-} P'(\theta) = -1/2.$$

(4) The improper integral

$$\int_{-1}^{0} \frac{\log(1+s)}{s}\, ds$$

converges. By the above four properties, we can apply a slight modification of the LDCT to the integral in (3.37) and obtain, for fixed $t \in (0,\infty)$, the following

$$\begin{aligned}
&\lim_{x\to\infty} \frac{1}{\sqrt{\pi}} \int_{-1}^{-(\sqrt{4t}/x)M} \left(e^{-((x/\sqrt{4t})s)^2} \frac{x}{\sqrt{4t}} s\right) \frac{\log\left(1+\frac{1}{x}+s\right)}{s+\frac{1}{x}} \frac{s+\frac{1}{x}}{s}\, ds \\
&= \frac{1}{\sqrt{\pi}} \int_{-1}^{0} \lim_{x\to\infty} \left[\left(e^{-((x/\sqrt{4t})s)^2} \frac{x}{\sqrt{4t}} s\right) \frac{\log\left(1+\frac{1}{x}+s\right)}{s+\frac{1}{x}} \frac{s+\frac{1}{x}}{s}\right] ds = 0.
\end{aligned} \tag{3.38}$$

More precisely, for any sequence $x_n \to \infty$, where, without loss of generality, we may take $x_n = n \in \mathbb{N}$, let

$$F_n(s) = \begin{cases} \left(e^{-((n/\sqrt{4t})s)^2} \frac{n}{\sqrt{4t}} s\right) \frac{\log\left(1+\frac{1}{n}+s\right)}{s+\frac{1}{n}} \frac{s+\frac{1}{n}}{s}, & s \in \big(-1, -\frac{\sqrt{4t}}{n} M\big], \\ 0, & s \in \big[-\frac{\sqrt{4t}}{n} M, 0\big). \end{cases}$$

For each fixed $s \in (-1,0)$ it will lie on the integral $(-1, -(\sqrt{4t}/n)M]$ as long as $n \in \mathbb{N}$ is large enough, and we have

$$s + \frac{1}{n} \in \left(-1+\frac{1}{n}, -\frac{\sqrt{4t}}{n} M + \frac{1}{n}\right] \subset (-1,0) \quad \text{for all large } n \text{ and } \lim_{n\to\infty} F_n(s) = 0.$$

Also, by the decreasing property of $P(\theta) = (\log(1+\theta))/\theta$ on $\theta \in (-1,0)$, we have

$$|F_n(s)| \le \begin{cases} C_1 \cdot \frac{\log(1+s)}{s} \cdot C_2, & \forall\, s \in \big(-1, -\frac{\sqrt{4t}}{n} M\big] \subset (-1,0),\ \forall\, n \in \mathbb{N},\ n \gg 0, \\ 0, & s \in [-(\sqrt{4t}/n)M, 0), \end{cases}$$

where C_1, C_2 are some positive constants independent of s and n. Hence, as long as $n \in \mathbb{N}$ is large enough, we have

$$|F_n(s)| \le C_1 \cdot \frac{\log(1+s)}{s} \cdot C_2, \quad \forall\, s \in (-1,0),$$

where we know that $\int_{-1}^{0} \frac{1}{s} \log(1+s)\, ds$ converges. The LDCT can now be applied and we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_{-1}^{-(\sqrt{4t}/n)M} \left(e^{-((n/\sqrt{4t})s)^2} \frac{n}{\sqrt{4t}} s\right) \frac{\log\left(1+\frac{1}{n}+s\right)}{s+\frac{1}{n}} \frac{s+\frac{1}{n}}{s}\, ds \\
&= \lim_{n\to\infty} \int_{-1}^{0} F_n(s)\, ds = \int_{-1}^{0} \left(\lim_{n\to\infty} F_n(s)\right) ds = \int_{-1}^{0} 0\, ds = 0.
\end{aligned}$$

The same argument can be applied to any sequence $x_n \to \infty$. Therefore, the conclusion in (3.38) is verified and the three integrals in (3.36) all tend to 0 as $x \to \infty$.

As a result of (3.33), (3.34) and (3.35), we can conclude the following space asymptotic behavior (note that $u(x,t) = I(x,t) - II(x,t) + u_g(x,t)$)

$$\lim_{x\to\infty} |u(x,t) - \log x| = \lim_{x\to\infty} |u(x,t) - \log(x+1)| = 0 \quad \text{for fixed } t \in (0,\infty).$$

Next, we look at $\lim_{t\to\infty} u_h(x,t)$ and $\lim_{t\to\infty} u_g(x,t)$ for fixed $x \in (0,\infty)$. For $u_h(x,t)$, we can write $I(x,t)$ in (3.31) as

$$\begin{aligned} I(x,t) &= \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} \left(\log\sqrt{4t} + \log\left(\frac{x}{\sqrt{4t}} + z + \frac{1}{\sqrt{4t}} \right) \right) dz \\ &= \left(\frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}\, dz \right) \log\sqrt{4t} + \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} \log\left(\frac{x}{\sqrt{4t}} + z + \frac{1}{\sqrt{4t}} \right) dz \\ &:= I_A(x,t) + I_B(x,t) \end{aligned}$$

and similarly, we can write $II(x,t)$ in (3.31) as

$$\begin{aligned} II(x,t) &= \left(\frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\, dz \right) \log\sqrt{4t} + \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \log\left(-\frac{x}{\sqrt{4t}} + z + \frac{1}{\sqrt{4t}} \right) dz \\ &:= II_A(x,t) + II_B(x,t). \end{aligned}$$

Since the integral $\int_0^\infty e^{-z^2}(\log z)\, dz$ converges, one can modified the LDCT slightly to get

$$\lim_{t\to\infty} I_B(x,t) = \lim_{t\to\infty} II_B(x,t) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-z^2}(\log z)\, dz. \tag{3.39}$$

Hence, to find the limit $\lim_{t\to\infty} u_h(x,t)$, it suffices to look at

$$\lim_{t\to\infty} (I_A(x,t) - II_A(x,t)) = \lim_{t\to\infty} \left[\left(\frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{x/\sqrt{4t}} e^{-z^2}\, dz \right) \log\sqrt{4t} \right] = 0, \tag{3.40}$$

where the limit in (3.40) is easily seen by the L'Hospital rule. By (3.39) and (3.40), we conclude

$$\lim_{t\to\infty} u_h(x,t) = \lim_{t\to\infty} (I(x,t) - II(x,t)) = 0 \quad \text{for fixed } x \in (0,\infty). \tag{3.41}$$

Next, we look at $u_g(x,t)$ given by (3.32), which can be written as

$$u_g(x,t) = \left(\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\, dz \right) \log t + \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \log\left(1 - \frac{1}{t}\left(\frac{x}{2z}\right)^2 + \frac{1}{t} \right) dz. \tag{3.42}$$

Again we need extra care on the second integral in (3.42) since the function $\log\big(1-(x/2z)^2/t+1/t\big)$ will tend to $-\infty$ as $t\to\infty$ and z is close to $x/\sqrt{4t}$. We choose $\delta\in(0,1)$ to be a small fixed number and decompose the second integral in (3.42) as

(3.43)
$$\begin{aligned}&\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty}e^{-z^2}\log\left(1-\frac{1}{t}\left(\frac{x}{2z}\right)^2+\frac{1}{t}\right)dz\\&=\frac{2}{\sqrt{\pi}}\left(\int_{x/\sqrt{4t}}^{x/\sqrt{4\delta t}}+\int_{x/\sqrt{4\delta t}}^{\infty}\right)e^{-z^2}\log\left(1-\frac{1}{t}\left(\frac{x}{2z}\right)^2+\frac{1}{t}\right)dz,\quad (x,t)\in(0,\infty)\times(0,\infty)\end{aligned}$$

and do the change of variables $z=x/\sqrt{4st}$ for the first integral to get

(3.44)
$$\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{x/\sqrt{4\delta t}}e^{-z^2}\log\left(1-\frac{1}{t}\left(\frac{x}{2z}\right)^2+\frac{1}{t}\right)dz=\frac{1}{\sqrt{\pi}}\int_{\delta}^{1}e^{-\frac{x^2}{4st}}\frac{x}{\sqrt{4st}}\frac{\log\left(1-s+\frac{1}{t}\right)}{s}\,ds.$$

For large $t>0$ satisfying $1/t<\delta$, we have $1-s+\frac{1}{t}<1$ for all $s\in[\delta,1]$ and then

$$\left|\log\left(1-s+\frac{1}{t}\right)\right|\le\left|\log\left(\frac{1}{t}\right)\right|=\log t,\quad \forall\, s\in[\delta,1].$$

Therefore, the integrand function in (3.44) on the interval $s\in[\delta,1]$ satisfies

$$\left|e^{-\frac{x^2}{4st}}\frac{x}{\sqrt{4st}}\frac{\log\left(1-s+\frac{1}{t}\right)}{s}\right|\le\frac{x}{\sqrt{4\delta t}}\frac{\log t}{\delta},\quad \forall\, s\in[\delta,1],\ \frac{1}{t}<\delta,$$

which implies, for fixed $x\in(0,\infty)$ and fixed small $\delta\in(0,1)$, the limit

(3.45)
$$\lim_{t\to\infty}\left|\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{x/\sqrt{4\delta t}}e^{-z^2}\log\left(1-\frac{1}{t}\left(\frac{x}{2z}\right)^2+\frac{1}{t}\right)dz\right|\le\lim_{t\to\infty}\left|\frac{1}{\sqrt{\pi}}\int_{\delta}^{1}\frac{x}{\sqrt{4\delta t}}\frac{\log t}{\delta}\,ds\right|=0.$$

For the second integral in (3.43), we do not do change of variables. Instead, we restrict the integral on the parabola $x/\sqrt{4t}=\lambda$ and let $\lambda\to0$ (equivalent to $t\to\infty$ for fixed $x\in(0,\infty)$). More precisely, we have

(3.46)
$$\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4\delta t}}^{\infty}e^{-z^2}\log\left(1-\frac{1}{t}\left(\frac{x}{2z}\right)^2+\frac{1}{t}\right)dz=\frac{2}{\sqrt{\pi}}\int_{\lambda/\sqrt{\delta}}^{\infty}e^{-z^2}\log\left(1-\left(\frac{\lambda}{z}\right)^2+\frac{4\lambda^2}{x^2}\right)dz,$$

where by

$$1-\left(\frac{\lambda}{z}\right)^2+\frac{4\lambda^2}{x^2}\in\left(1-\delta+\frac{4\lambda^2}{x^2},1+\frac{4\lambda^2}{x^2}\right),\quad \forall\, z\in[\lambda/\sqrt{\delta},\infty)$$

we have for sufficiently small $\lambda>0$ the estimate

$$\begin{aligned}\left|\log\left(1-\left(\frac{\lambda}{z}\right)^2+\frac{4\lambda^2}{x^2}\right)\right|&\le\left|\log\left(1-\delta+\frac{4\lambda^2}{x^2}\right)\right|\\&\le|\log(1-\delta)|=-\log(1-\delta),\quad \forall\, z\in[\lambda/\sqrt{\delta},\infty)\end{aligned}$$

and so

$$e^{-z^2}\left|\log\left(1-\left(\frac{\lambda}{z}\right)^2+\frac{4\lambda^2}{x^2}\right)\right|\le e^{-z^2}(-\log(1-\delta)),\quad \forall\, z\in[\lambda/\sqrt{\delta},\infty)$$

for all sufficiently small $\lambda>0$.

Let $\lambda=1/n$, $n=1,2,3,\dots$ and let

$$F_n(z)=\begin{cases} e^{-z^2}\log\left(1-\left(\frac{1}{nz}\right)^2+\frac{4}{n^2x^2}\right), & z\in[1/(n\sqrt{\delta}),\infty),\\ 0, & z\in[0,1/(n\sqrt{\delta})).\end{cases}$$

It satisfies for large $n\in\mathbb{N}$ the estimate

$$|F_n(z)|\le e^{-z^2}(-\log(1-\delta)),\quad \forall\, n\gg 0,\ \forall\, z\in[0,\infty),$$

where $e^{-z^2}(-\log(1-\delta))\in L^1[0,\infty)$, and

$$\lim_{n\to\infty}F_n(z)=0\quad\text{for all fixed } z\in[0,\infty).$$

Hence the LDCT can be applied to the integral in (3.46), i.e., for fixed $x\in(0,\infty)$, we have

$$\begin{aligned}\lim_{n\to\infty}\int_{1/(n\sqrt{\delta})}^{\infty}e^{-z^2}\log\left(1-\left(\frac{1}{nz}\right)^2+\frac{4}{n^2x^2}\right)dz&=\lim_{n\to\infty}\int_0^\infty F_n(z)\,dz\\ &=\int_0^\infty\lim_{n\to\infty}F_n(z)\,dz=0.\end{aligned}$$

Therefore, we have

$$\lim_{\lambda\to0,\ \lambda=1/n}\frac{2}{\sqrt{\pi}}\int_{\lambda/\sqrt{\delta}}^{\infty}e^{-z^2}\log\left(1-\left(\frac{\lambda}{z}\right)^2+\frac{4\lambda^2}{x^2}\right)dz=0.$$

One can repeat the above argument along any sequence $\lambda=\lambda_n\to0$. Therefore, by (3.46), we conclude

$$\lim_{t\to\infty}\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4\delta t}}^{\infty}e^{-z^2}\log\left(1-\frac{1}{t}\left(\frac{x}{2z}\right)^2+\frac{1}{t}\right)dz=0. \tag{3.47}$$

Now (3.47) and (3.45) imply that the second integral in (3.42) converges to 0 as $t\to\infty$ and we obtain

$$\begin{aligned}\lim_{t\to\infty}|u_g(x,t)-\log t|&=\lim_{t\to\infty}\left|\left(\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty}e^{-z^2}\,dz-1\right)\log t\right|\\ &=\lim_{t\to\infty}\left|\left(-\frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{4t}}e^{-z^2}\,dz\right)\log t\right|=0,\end{aligned} \tag{3.48}$$

where the limit in (3.48) is due to the L'Hospital rule. As a result, we conclude

$$\lim_{t\to\infty} |u_g(x,t) - \log t| = \lim_{t\to\infty} |u_g(x,t) - \log(t+1)| = 0 \quad \text{for fixed } x \in (0,\infty)$$

and together with (3.41), we obtain

$$\lim_{t\to\infty} |u(x,t) - \log t| = \lim_{t\to\infty} |u(x,t) - \log(t+1)| = 0 \quad \text{for fixed } x \in (0,\infty).$$

Due to $h(0) = g(0) = 0$, $u(x,t)$ satisfies the 2-dimensional limit $\lim_{(x,t)\to(0^+,0^+)} u(x,t) = 0$ and lies in the space

$$u(x,t) \in C^\infty((0,\infty)\times(0,\infty)) \cap C^0([0,\infty)\times[0,\infty))$$

if we define $u(0,0) = 0$.

We can summarize the main result in this section, i.e.,

Lemma 3.14 (Solution with logarithmic initial-boundary data)**.** *The solution $u(x,t)$ of the ibvp* (1.1) *with $h(x) = \log(x+1)$, $x \in (0,\infty)$, $g(t) = \log(t+1)$, $t \in (0,\infty)$, satisfies*

$$\lim_{x\to\infty} |u(x,t) - \log(x+1)| = 0 \quad \textit{for fixed } t \in (0,\infty)$$

and

$$\lim_{t\to\infty} |u(x,t) - \log(t+1)| = 0 \quad \textit{for fixed } x \in (0,\infty).$$

Remark 3.15 (Exponential initial-boundary data)*.* As a comparison, if we take **exponential initial-boundary data** with $h(x) = e^x$, $x \in (0,\infty)$, $g(t) = e^t$, $t \in (0,\infty)$, then the solution $u(x,t)$ of the ibvp (1.1) is given by $u(x,t) = e^{x+t}$, $(x,t) \in (0,\infty)\times(0,\infty)$. The asymptotic behavior of $u(x,t)$ is clear.

4. Prescribing the oscillation limits of solutions of the ibvp (1.1)

4.1. Prescribing the oscillation limits of $u(x,t)$ using slow-oscillation initial-boundary data

In this section, we want to explore the space-time oscillation behavior of $u(x,t)$. In particular, we would like to prescribe the liminf and limsup values of $u(x,t)$ as $x\to\infty$ or as $t\to\infty$ respectively. The idea is to use certain slow-oscillation functions for $h(x)$ and for $g(t)$. For oscillation behavior of solutions to the heat equation on the entire space $x \in (-\infty,\infty)$ or $x \in \mathbb{R}^n$, one can see the two papers [3, 11].

We first take the following initial-boundary data in the ibvp (1.1):

$$h(x) = \sin(\log(x+1)), \quad g(t) = \sin(\log(t+1)), \quad x \in (0,\infty),\ t \in (0,\infty).$$

Both functions oscillate between -1 and $+1$, with asymptotically slow oscillation due to $h'(x) = O(1/(x+1))$, $x \to \infty$, and $g'(t) = O(1/(t+1))$, $t \to \infty$. In this case, we have

$$\begin{aligned} u_h(x,t) &= \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} \sin(\log(x + \sqrt{4t}z + 1))\, dz \\ &\quad - \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \sin(\log(-x + \sqrt{4t}z + 1))\, dz \\ &:= I(x,t) - II(x,t) \end{aligned}$$

and

$$u_g(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \sin\left(\log\left(t - \left(\frac{x}{2z}\right)^2 + 1\right)\right) dz. \tag{4.1}$$

For fixed $t \in (0,\infty)$ and $x \to \infty$, we clearly have

$$\lim_{x\to\infty} II(x,t) = 0 \quad \text{and} \quad \lim_{x\to\infty} u_g(x,t) = 0. \tag{4.2}$$

As for $I(x,t)$ in $u_h(x,t)$, we have

$$I(x,t) = \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} \sin\left(\log(x+1) + \log\left(1 + \frac{\sqrt{4t}z}{x+1}\right)\right) dz,$$

which, by the LDCT (note that $\sin(\cdot)$ is always bounded by 1), implies

$$\lim_{x\to\infty} |I(x,t) - \sin(\log(x+1))| = 0 \quad \text{for fixed } t \in (0,\infty). \tag{4.3}$$

Hence we conclude

$$\lim_{x\to\infty} |u(x,t) - \sin(\log(x+1))| = 0 \quad \text{for fixed } t \in (0,\infty).$$

On the other hand, for fixed $x \in (0,\infty)$ and $t \to \infty$, we first note that $u_h(0,t) = 0$ for all $t \in (0,\infty)$. Since $h(x)$ is a bounded function, the gradient estimate (2.19) holds and, for fixed $x \in (0,\infty)$, we have

$$|u_h(x,t)| = |u_h(x,t) - u_h(0,t)| \le \frac{Mx}{\sqrt{\pi t}} \to 0 \quad \text{as } t \to \infty. \tag{4.4}$$

For $u_g(x,t)$, we have

$$\begin{aligned} u_g(x,t) &= \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \sin\left[\log(t+1) + \log\left(1 - \frac{1}{t+1}\left(\frac{x}{2z}\right)^2\right)\right] dz \\ &= \left[\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \cos\left(\log\left(1 - \frac{1}{t+1}\left(\frac{x}{2z}\right)^2\right)\right) dz\right] \sin(\log(t+1)) \\ &\quad + \left[\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \sin\left(\log\left(1 - \frac{1}{t+1}\left(\frac{x}{2z}\right)^2\right)\right) dz\right] \cos(\log(t+1)), \end{aligned}$$

where by

$$\lim_{t\to\infty} \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \cos\left(\log\left(1 - \frac{1}{t+1}\left(\frac{x}{2z}\right)^2\right)\right)\, dz = 1,$$
$$\lim_{t\to\infty} \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-z^2} \sin\left(\log\left(1 - \frac{1}{t+1}\left(\frac{x}{2z}\right)^2\right)\right)\, dz = 0,$$

we obtain

$$\lim_{t\to\infty} |u_g(x,t) - \sin(\log(t+1))| = 0 \quad \text{for fixed } x \in (0,\infty). \tag{4.5}$$

Hence we conclude

$$\lim_{t\to\infty} |u(x,t) - \sin(\log(t+1))| = 0 \quad \text{for fixed } x \in (0,\infty).$$

In this example, we have (compare with (3.28))

$$-1 = \inf_{t\in(0,\infty)} g(t) = \liminf_{t\to\infty} u(x,t) < \limsup_{t\to\infty} u(x,t) = \sup_{t\in(0,\infty)} g(t) = 1.$$

Due to $h(0) = g(0) = 0$, $u(x,t)$ satisfies the 2-dimensional limit $\lim_{(x,t)\to(0^+,0^+)} u(x,t) = 0$ and lies in the space

$$u(x,t) \in C^\infty((0,\infty)\times(0,\infty)) \cap C^0([0,\infty)\times[0,\infty))$$

if we define $u(0,0) = 0$.

We can summarize the above as

Lemma 4.1. *The solution $u(x,t)$ of the ibvp* (1.1) *with*

$$h(x) = \sin(\log(x+1)), \quad g(t) = \sin(\log(t+1)), \quad x\in(0,\infty),\ t\in(0,\infty) \tag{4.6}$$

satisfies

$$\lim_{x\to\infty} |u(x,t) - \sin(\log(x+1))| = 0 \quad \textit{for fixed } t \in (0,\infty)$$

and

$$\lim_{t\to\infty} |u(x,t) - \sin(\log(t+1))| = 0 \quad \textit{for fixed } x \in (0,\infty).$$

4.1.1. Prescribing the space-time oscillation limits of $u(x,t)$

With the help of the above $h(x)$ and $g(t)$ in (4.6), we can establish the following prescribing result.

Lemma 4.2 (Prescribing the space-time oscillation limits; finite case)**.** *Consider the ibvp* (1.1)*. For any four finite numbers* $c_1 \le c_2$*,* $c_3 \le c_4$*, one can find a solution* $u(x,t)$ *of* (1.1) *lying in the space*

$$u(x,t) \in C^\infty((0,\infty)\times(0,\infty)) \cap C^0(([0,\infty)\times[0,\infty)) \setminus \{(0,0)\}) \tag{4.7}$$

and satisfies

$$c_1 = \liminf_{x\to\infty} u(x,t) \le \limsup_{x\to\infty} u(x,t) = c_2 \quad \textit{for fixed } t \in (0,\infty), \tag{4.8a}$$

$$c_3 = \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) = c_4 \quad \textit{for fixed } x \in (0,\infty). \tag{4.8b}$$

Proof. We take the initial-boundary data in (1.1) as

$$\begin{cases} h(x) = p\sin(\log(x+1)) + q, & x \in (0,\infty), \\ g(t) = \alpha\sin(\log(t+1)) + \beta, & t \in (0,\infty), \end{cases}$$

where α, β, p, q are constants with $p \ge 0$ and $\alpha \ge 0$. By (2.1), (4.2) and (4.3), we have

$$\lim_{x\to\infty} |u(x,t) - (p\sin(\log(x+1)) + q)| = 0 \quad \text{for fixed } t \in (0,\infty).$$

Similarly, by (2.1), (4.4) and (4.5), we have

$$\lim_{t\to\infty} |u(x,t) - (\alpha\sin(\log(t+1)) + \beta)| = 0 \quad \text{for fixed } x \in (0,\infty).$$

Hence we conclude

$$-p + q = \liminf_{x\to\infty} u(x,t) \le \limsup_{x\to\infty} u(x,t) = p + q \quad \text{for fixed } t \in (0,\infty)$$

and

$$-\alpha + \beta = \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) = \alpha + \beta \quad \text{for fixed } x \in (0,\infty).$$

Now (4.8a) will follow if we choose $q = (c_2 + c_1)/2$, $p = (c_2 - c_1)/2$. The proof for (4.8b) will also follow if we choose $\beta = (c_4 + c_3)/2$, $\alpha = (c_4 - c_3)/2$. Since we do not have $h(0) = g(0)$ in general, the solution lies in the space (4.7). It may not be continuous up to $(x,t) = (0,0)$. The proof is done. □

Remark 4.3. In case we have $c_1 + c_2 = c_3 + c_4$, it will imply $h(0) = g(0)$ and $u(x,t)$ can be continuous up to $(0,0)$ if we define $u(0,0) = h(0)$.

Another interesting prescribing result is the following

Lemma 4.4 (Prescribing the space-time oscillation limits; finite-infinite case)**.** *Consider the ibvp* (1.1)*. For any two finite numbers $c_1 \le c_2$, one can find a solution $u(x,t)$ of* (1.1) *lying in the space* (4.7) *and satisfies*

$$c_1 = \liminf_{x\to\infty} u(x,t) \le \limsup_{x\to\infty} u(x,t) = c_2 \quad \text{for fixed } t \in (0,\infty), \tag{4.9a}$$

$$-\infty = \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) = +\infty \quad \text{for fixed } x \in (0,\infty). \tag{4.9b}$$

Similarly, for any two finite numbers $c_3 \le c_4$, one can find a solution $u(x,t)$ of (1.1) *lying in the space* (4.7) *and satisfies*

$$-\infty = \liminf_{x\to\infty} u(x,t) \le \limsup_{x\to\infty} u(x,t) = +\infty \quad \text{for fixed } t \in (0,\infty), \tag{4.10a}$$

$$c_3 = \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) = c_4 \quad \text{for fixed } x \in (0,\infty). \tag{4.10b}$$

Proof. Take the initial-boundary data for (1.1) as

$$\begin{cases} h(x) = \frac{c_2-c_1}{2}\sin(\log(x+1)) + \frac{c_2+c_1}{2}, & x \in (0,\infty), \\ g(t) = t\sin(\log t), & t \in (0,\infty), \end{cases} \tag{4.11}$$

and obtain $u(x,t) = u_h(x,t) + u_g(x,t)$, where by the results in (2.1), (4.3) and (4.4), we have

$$\begin{aligned} &\lim_{x\to\infty}\left| u_h(x,t) - \left(\frac{c_2-c_1}{2}\sin(\log(x+1)) + \frac{c_2+c_1}{2}\right)\right| = 0 \quad \text{for fixed } t \in (0,\infty), \\ &\lim_{t\to\infty} u_h(x,t) = 0 \quad \text{for fixed } x \in (0,\infty). \end{aligned} \tag{4.12}$$

As for $u_g(x,t)$, we have

$$\begin{aligned} u_g(x,t) &= \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\left(t - \frac{x^2}{4z^2}\right)\sin\left(\log\left(t-\frac{x^2}{4z^2}\right)\right)dz \\ &= t\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\left(1 - \frac{1}{t}\frac{x^2}{4z^2}\right)\sin\left(\log t + \log\left(1-\frac{1}{t}\frac{x^2}{4z^2}\right)\right)dz \\ &= t\big[A(x,t)\sin(\log t) + B(x,t)\cos(\log t)\big], \quad (x,t) \in (0,\infty)\times(0,\infty), \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} A(x,t) &= \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\left(1 - \frac{1}{t}\frac{x^2}{4z^2}\right)\cos\left(\log\left(1-\frac{1}{t}\frac{x^2}{4z^2}\right)\right)dz, \\ B(x,t) &= \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\left(1 - \frac{1}{t}\frac{x^2}{4z^2}\right)\sin\left(\log\left(1-\frac{1}{t}\frac{x^2}{4z^2}\right)\right)dz. \end{aligned} \tag{4.14}$$

By

$$1 - \frac{1}{t}\frac{x^2}{4z^2} \in (0,1) \quad \text{for all } z \in (x/\sqrt{4t}, \infty),$$

we can apply a slight modification of the LDCT to the integral in (4.14) and obtain

$$\tag{4.15} \begin{aligned} &\lim_{x\to\infty} A(x,t) = \lim_{x\to\infty} B(x,t) = 0 \quad \text{for fixed } t\in(0,\infty),\\ &\lim_{t\to\infty} A(x,t) = 1, \quad \lim_{t\to\infty} B(x,t) = 0 \quad \text{for fixed } x\in(0,\infty), \end{aligned}$$

which implies

$$\tag{4.16} \begin{aligned} &\lim_{x\to\infty} u_g(x,t) = 0 \quad \text{for fixed } t\in(0,\infty),\\ &\liminf_{t\to\infty} u_g(x,t) = -\infty, \quad \limsup_{t\to\infty} u_g(x,t) = +\infty \quad \text{for fixed } x\in(0,\infty). \end{aligned}$$

Combining (4.12) and (4.16) will prove (4.9a) and (4.9b).

For (4.10), we take the initial-boundary data for (1.1) as

$$\begin{cases} h(x) = x\cos x, & x\in(0,\infty),\\ g(t) = \frac{c_4-c_3}{2}\sin(\log(t+1)) + \frac{c_4+c_3}{2}, & t\in(0,\infty). \end{cases}$$

The corresponding solution $u(x,t)$ is given by

$$\tag{4.17} \begin{aligned} u(x,t) &= \left(e^{-t}x\cos x - 2te^{-t}\sin x\right)\\ &\quad + \frac{c_4-c_3}{2}\left(\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\sin\left(\log\left(t-\left(\frac{x}{2z}\right)^2+1\right)\right)dz\right)\\ &\quad + \frac{c_4+c_3}{2}\left(\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\,dz\right)\\ &:= u_h(x,t) + \frac{c_4-c_3}{2}u_{g_1}(x,t) + \frac{c_4+c_3}{2}u_{g_2}(x,t), \quad (x,t)\in(0,\infty)\times(0,\infty). \end{aligned}$$

By (4.1) and (4.2), we know that

$$\lim_{x\to\infty} u_{g_1}(x,t) = \lim_{x\to\infty} u_{g_2}(x,t) = 0 \quad \text{for fixed } t\in(0,\infty).$$

Hence, for fixed $t\in(0,\infty)$, we conclude

$$\begin{aligned} \liminf_{x\to\infty} u(x,t) &= \liminf_{x\to\infty}\left(e^{-t}x\cos x - 2te^{-t}\sin x\right) = -\infty,\\ \limsup_{x\to\infty} u(x,t) &= \limsup_{x\to\infty}\left(e^{-t}x\cos x - 2te^{-t}\sin x\right) = +\infty \end{aligned}$$

which gives (4.10a). On the other hand, for fixed $x\in(0,\infty)$, we first have $\lim_{t\to\infty} u_h(x,t) = 0$. Also by (4.5), we have

$$\lim_{t\to\infty}\left|u_{g_1}(x,t) - \sin(\log(t+1))\right| = 0, \quad \lim_{t\to\infty} u_{g_2}(x,t) = 1 \quad \text{for fixed } x\in(0,\infty).$$

Therefore, by (4.17), we conclude

$$c_3 = \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) = c_4 \quad \text{for fixed } x\in(0,\infty),$$

which gives (4.10b). □

Our next prescribing result is the following

Lemma 4.5 (Prescribing the space-time oscillation limits; infinite case)**.** *Consider the ibvp* (1.1)*. One can find a solution* $u(x,t)$ *of* (1.1) *lying in the space* (4.7) *and satisfies*

$$
\begin{aligned}
&-\infty = \liminf_{x\to\infty} u(x,t) \le \limsup_{x\to\infty} u(x,t) = +\infty \quad \text{for fixed } t \in (0,\infty), \\
&-\infty = \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) = +\infty \quad \text{for fixed } x \in (0,\infty).
\end{aligned} \tag{4.18}
$$

Proof. This is comparatively easy. Let $\alpha \neq 0$, $\beta \neq 0$ be two arbitrary constants and let $p = \alpha^2 - \beta^2$, $q = 2\alpha\beta$. Then the function

$$u(x,t) = e^{\alpha x + pt} \sin(\beta x + qt), \quad (x,t) \in (0,\infty) \times (0,\infty)$$

is a solution of the heat equation on $(0,\infty) \times (0,\infty)$ with $u(x,0) = e^{\alpha x} \sin \beta x$, $u(0,t) = e^{pt} \sin qt$. We choose the initial-boundary data for (1.1) as $h(x) = e^{2x} \sin x$, $g(t) = e^{3t} \sin(4t)$ and obtain the solution

$$u(x,t) = e^{2x+3t} \sin(x + 4t), \quad (x,t) \in (0,\infty) \times (0,\infty),$$

which clearly satisfies (4.18). □

To end this section, we note that, without oscillation, it is much easier to prescribe the convergent limit of $u(x,t)$ as $x \to \infty$ or as $t \to \infty$. We have

Lemma 4.6 (Prescribing the space-time convergent limit)**.** *For any two numbers* $-\infty \le p, q \le +\infty$*, one can find a solution* $u(x,t)$ *of* (1.1) *lying in the space* (4.7) *and satisfies*

$$
\begin{aligned}
&\lim_{x\to\infty} u(x,t) = p \quad \text{for fixed } t \in (0,\infty), \\
&\lim_{t\to\infty} u(x,t) = q \quad \text{for fixed } x \in (0,\infty).
\end{aligned}
$$

Proof. If both p, q are finite numbers, we just take $h(x) = p$, $g(t) = q$. If p is finite and $q = \pm\infty$, by Lemma 3.14 we can take $h(x) = p$, $g(t) = \pm \log(t+1)$. Similarly, if $p = \pm\infty$ and q is finite, we can take $h(x) = \pm \log(x+1)$, $g(t) = q$. Finally, if $p = \pm\infty$ and $q = \pm\infty$, we can take $h(x) = \pm \log(x+1)$, $g(t) = \pm \log(t+1)$. □

4.1.2. Prescribing the space oscillation limits of $u(x,t)$

In Lemmas 4.2, 4.4 and 4.5, the liminf and limsup values of $u(x,t)$ are either both finite or both infinite. In this section, we focus on space oscillation limits and time oscillation limits respectively and construct examples in which one is finite and the other is infinite. See Lemmas 4.10 and 4.15 below.

To prove Lemma 4.10, we need to make use of the following interesting result.

Lemma 4.7. *The solution $u(x,t)$ of the ibvp* (1.1) *with*

$$h(x) = x\sin(\log x), \quad g(t) \equiv 0, \quad x \in (0,\infty),\ t \in (0,\infty)$$

satisfies

$$\lim_{x\to\infty} |u(x,t) - x\sin(\log x)| = 0 \quad \textit{for fixed } t \in (0,\infty) \tag{4.19}$$

and

$$\lim_{t\to\infty} \left|u(x,t) - (A\sin(\log\sqrt{4t}) + B\cos(\log\sqrt{4t}))x\right| = 0 \quad \textit{for fixed } x \in (0,\infty), \tag{4.20}$$

where A, B are constants given by the values of the integrals:

$$A = \frac{4}{\sqrt{\pi}}\int_0^\infty e^{-z^2} z^2 \cos(\log z)\,dz, \quad B = \frac{4}{\sqrt{\pi}}\int_0^\infty e^{-z^2} z^2 \sin(\log z)\,dz.$$

Remark 4.8. If we replace $h(x) = x\sin(\log x)$ by $h(x) = x\sin(\log(x+1))$, then the time limit remains the same, i.e., (4.20) is still correct. But now the space limit becomes

$$\lim_{x\to\infty} |u(x,t) - x\sin(\log(x+1))| = 0 \quad \text{for fixed } t \in (0,\infty).$$

Note that we have

$$\lim_{x\to\infty} [x\sin(\log(x+1)) - x\sin(\log x) - \cos(\log x)] = 0.$$

Remark 4.9. By Hölder inequality, we have

$$\begin{aligned}
0 < A^2 + B^2 &< \left(\frac{4}{\sqrt{\pi}}\int_0^\infty e^{-z^2} z^2 \cos^2(\log z)\,dz + \frac{4}{\sqrt{\pi}}\int_0^\infty e^{-z^2} z^2 \sin^2(\log z)\,dz\right) \\
&= \frac{4}{\sqrt{\pi}}\int_0^\infty e^{-z^2} z^2\,dz = 1.
\end{aligned}$$

Therefore, for fixed $x \in (0,\infty)$, the asymptotic time oscillation limit of $u(x,t)$ in (4.20) lies between $-x$ and $+x$. This matches with the maximum principle since the initial data of the ibvp (1.1) satisfies $-x \le x\sin(\log x) \le x$ for all $x \in (0,\infty)$ and by (1.3) we have the estimate

$$\begin{aligned}
|u(x,t)| &= \left|\frac{1}{\sqrt{4\pi t}}\int_0^\infty \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right)\xi\sin(\log\xi)\,d\xi\right| \\
&\le \frac{1}{\sqrt{4\pi t}}\int_0^\infty \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right)\xi\,d\xi = x, \quad \forall\,(x,t) \in (0,\infty)\times(0,\infty).
\end{aligned}$$

Proof of Lemma 4.7. By (2.1) we have $u(x,t) = u_h(x,t)$ with

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^\infty e^{-z^2}(x+\sqrt{4t}z)\sin(\log(x+\sqrt{4t}z))\,dz \\
&\quad - \frac{1}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^\infty e^{-z^2}(-x+\sqrt{4t}z)\sin(\log(-x+\sqrt{4t}z))\,dz \\
&:= I(x,t) - II(x,t), \quad (x,t) \in (0,\infty)\times(0,\infty).
\end{aligned} \tag{4.21}$$

By the LDCT and inequality (2.4), for fixed $t \in (0,\infty)$, we have

$$\lim_{x\to\infty} II(x,t) = \lim_{x\to\infty}\left(\frac{1}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}(-x+\sqrt{4t}z)\sin(\log(-x+\sqrt{4t}z))\,dz\right) = 0. \tag{4.22}$$

As for $I(x,t)$, we can write it as

$$\begin{aligned} I(x,t) &= \frac{\sin(\log x)}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}(x+\sqrt{4t}z)\cos\left(\log\left(1+\frac{\sqrt{4t}z}{x}\right)\right)dz \\ &\quad + \frac{\cos(\log x)}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}(x+\sqrt{4t}z)\sin\left(\log\left(1+\frac{\sqrt{4t}z}{x}\right)\right)dz \\ &:= I_A(x,t) + I_B(x,t) + \widetilde{I}_A(x,t) + \widetilde{I}_B(x,t), \end{aligned}$$

where, by the LDCT, we have

$$\lim_{x\to\infty} I_B(x,t) = \lim_{x\to\infty}\left\{\frac{\sqrt{4t}\sin(\log x)}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} z\cdot\cos\left(\log\left(1+\frac{\sqrt{4t}z}{x}\right)\right)dz\right\} = 0 \tag{4.23}$$

due to $\int_{-\infty}^{\infty} e^{-z^2} z\,dz = 0$. Similarly, we have

$$\lim_{x\to\infty} \widetilde{I}_B(x,t) = \lim_{x\to\infty}\left\{\frac{\sqrt{4t}\cos(\log x)}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} z\cdot\sin\left(\log\left(1+\frac{\sqrt{4t}z}{x}\right)\right)dz\right\} = 0. \tag{4.24}$$

It remains to find the limits of $I_A(x,t)$ and $\widetilde{I}_A(x,t)$ as $x\to\infty$. We will estimate the limits of $I_A(x,t) - x\sin(\log x)$ and $\widetilde{I}_A(x,t)$ as $x\to\infty$ respectively.

Estimate on $I_A(x,t) - x\sin(\log x)$: We first have

$$\begin{aligned} & I_A(x,t) - x\sin(\log x) \\ &= \frac{x\sin(\log x)}{\sqrt{\pi}}\left[\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}\cos\left(\log\left(1+\frac{\sqrt{4t}z}{x}\right)\right)dz - \int_{-\infty}^{\infty} e^{-z^2}\,dz\right] \\ &= \frac{x\sin(\log x)}{\sqrt{\pi}}\left\{\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}\left[\cos\left(\log\left(1+\frac{\sqrt{4t}z}{x}\right)\right)-1\right]dz - \int_{-\infty}^{-x/\sqrt{4t}} e^{-z^2}\,dz\right\} \\ &:= P(x,t) - Q(x,t) \end{aligned}$$

and by (2.4), we know that $\lim_{x\to\infty} Q(x,t) = 0$ for fixed $t\in(0,\infty)$. To estimate $P(x,t)$, we use the inequality

$$|\cos(\log(1+\theta)) - 1| \le 4|\theta|, \quad \forall\,\theta\in(-1,\infty). \tag{4.25}$$

Note that (4.25) is obvious for $\theta\in(-1,-1/2]$ since $4|\theta|\ge 2$. For $\theta\in(-1/2,\infty)$, by

$$\left|\frac{d}{d\theta}(\cos(\log(1+\theta))-1)\right| = \left|-\frac{\sin(\log(1+\theta))}{1+\theta}\right| \le \frac{1}{1+\theta} \le 2, \quad \forall\,\theta\in(-1/2,\infty),$$

(4.25) holds on $\theta \in (-1/2, \infty)$.

As long as $z \in (-x/\sqrt{4t}, \infty)$, we have $\sqrt{4t}z/x \subset (-1, \infty)$ and, by (4.25), the integrand of $P(x,t)$ satisfies

$$\begin{aligned}
&\left| \frac{x \sin(\log x)}{\sqrt{\pi}} e^{-z^2} \left[\cos\left(\log\left(1 + \frac{\sqrt{4t}z}{x}\right)\right) - 1\right]\right| \\
&\leq \frac{xe^{-z^2}}{\sqrt{\pi}} \left| \cos\left(\log\left(1 + \frac{\sqrt{4t}z}{x}\right)\right) - 1\right| \leq \frac{xe^{-z^2}}{\sqrt{\pi}} \cdot 4 \left| \frac{\sqrt{4t}z}{x} \right| \\
&= \frac{4e^{-z^2}}{\sqrt{\pi}} |\sqrt{4t}z| \in L^1(-\infty, \infty).
\end{aligned} \tag{4.26}$$

By (4.26), the LDCT can be applied and we obtain, for fixed $t \in (0, \infty)$, the following (denote $1/x$ as ρ)

$$\begin{aligned}
&\lim_{x\to\infty} \left(x \int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} \left[\cos\left(\log\left(1 + \frac{\sqrt{4t}z}{x}\right)\right) - 1\right] dz \right) \\
&= \int_{-\infty}^{\infty} e^{-z^2} \left[\lim_{\rho\to 0^+} \frac{\cos(\log(1+\sqrt{4t}z\rho)) - 1}{\rho} \right] dz = \int_{-\infty}^{\infty} e^{-z^2} \cdot 0 \, dz = 0,
\end{aligned}$$

which implies $\lim_{x\to\infty} P(x,t) = 0$ and we conclude

$$\lim_{x\to\infty} |I_A(x,t) - x\sin(\log x)| = 0 \quad \text{for fixed } t \in (0,\infty). \tag{4.27}$$

Estimate on $\widetilde{I}_A(x,t)$: We have

$$\widetilde{I}_A(x,t) = \frac{\cos(\log x)}{\sqrt{\pi}} \int_{-x/\sqrt{4t}}^{\infty} e^{-z^2} x \sin\left(\log\left(1 + \frac{\sqrt{4t}z}{x}\right)\right) dz.$$

To estimate $\widetilde{I}_A(x,t)$, we now use the inequality

$$|\sin(\log(1+\theta))| \leq 2|\theta|, \quad \forall\, \theta \in (-1, \infty). \tag{4.28}$$

Note that (4.28) is obvious for $\theta \in (-1, -1/2]$ since $2|\theta| \geq 1$. For $\theta \in (-1/2, \infty)$, by

$$\left| \frac{d}{d\theta} \sin(\log(1+\theta)) \right| = \left| \frac{\cos(\log(1+\theta))}{1+\theta} \right| \leq \frac{1}{1+\theta} \leq 2, \quad \forall\, \theta \in (-1/2, \infty),$$

(4.28) holds on $\theta \in (-1/2, \infty)$. As long as $z \in (-x/\sqrt{4t}, \infty)$, we have $\sqrt{4t}z/x \subset (-1, \infty)$ and by (4.28), we have

$$\begin{aligned}
\left| \frac{\cos(\log x)}{\sqrt{\pi}} e^{-z^2} x \sin\left(\log\left(1 + \frac{\sqrt{4t}z}{x}\right)\right) \right| &\leq \frac{xe^{-z^2}}{\sqrt{\pi}} \cdot 2 \left| \frac{\sqrt{4t}z}{x} \right| \\
&= \frac{2e^{-z^2}}{\sqrt{\pi}} |\sqrt{4t}z| \in L^1(-\infty, \infty).
\end{aligned} \tag{4.29}$$

By (4.29), the LDCT can be applied and we obtain, for fixed $t \in (0,\infty)$, the following (denote $1/x$ as ρ)

$$\lim_{x\to\infty}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}\left[x\sin\left(\log\left(1+\frac{\sqrt{4t}z}{x}\right)\right)\right]dz$$
$$=\int_{-\infty}^{\infty} e^{-z^2}\left[\lim_{\rho\to 0^+}\frac{\sin(\log(1+\sqrt{4t}z\rho))}{\rho}\right]dz=\int_{-\infty}^{\infty} e^{-z^2}(\sqrt{4t}z)\,dz=0,$$

which implies

$$\lim_{x\to\infty}\widetilde{I}_A(x,t)=0 \quad \text{for fixed } t\in(0,\infty). \tag{4.30}$$

By (4.22), (4.23), (4.24), (4.27) and (4.30), we conclude the interesting asymptotic behavior:

$$\lim_{x\to\infty}|u(x,t)-x\sin(\log x)|=0 \quad \text{for fixed } t\in(0,\infty).$$

The proof of (4.19) is done.

Next, we prove (4.20), which is more subtle. We decompose $u(x,t)$ in (4.21) as

$$u(x,t):=u^{(1)}(x,t)+u^{(2)}(x,t), \quad (x,t)\in(0,\infty)\times(0,\infty),$$

where

$$u^{(1)}(x,t)=\frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}x\sin\left(\log\sqrt{4t}+\log\left(\frac{x}{\sqrt{4t}}+z\right)\right)dz$$
$$-\frac{1}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}(-x)\sin\left(\log\sqrt{4t}+\log\left(\frac{-x}{\sqrt{4t}}+z\right)\right)dz$$

and

$$u^{(2)}(x,t)=\frac{1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{\infty} e^{-z^2}\sqrt{4t}z\sin(\log(x+\sqrt{4t}z))\,dz$$
$$-\frac{1}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\sqrt{4t}z\sin(\log(-x+\sqrt{4t}z))\,dz.$$

Expanding the sine functions in $u^{(1)}(x,t)$ and applying the LDCT, we can obtain

$$\lim_{t\to\infty}\left|u^{(1)}(x,t)-(A_1\sin(\log\sqrt{4t})+B_1\cos(\log\sqrt{4t}))x\right|=0 \quad \text{for fixed } x\in(0,\infty), \tag{4.31}$$

where

$$A_1=\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-z^2}\cos(\log z)\,dz, \quad B_1=\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-z^2}\sin(\log z)\,dz. \tag{4.32}$$

As for $u_2(x,t)$, we let $\lambda = x/\sqrt{4t}$ and note that, for fixed $x \in (0,\infty)$, $t \to \infty$ is equivalent to $\lambda \to 0^+$. We have

$$\begin{aligned} &u^{(2)}(x,t) \\ &= \frac{1}{\sqrt{\pi}}\frac{x}{\lambda}\left\{\int_{-\lambda}^{\infty} e^{-z^2} z \sin\left(\log\left(x\left(1+\frac{z}{\lambda}\right)\right)\right) dz - \int_{\lambda}^{\infty} e^{-z^2} z \sin\left(\log\left(x\left(-1+\frac{z}{\lambda}\right)\right)\right) dz\right\} \\ &= \frac{x}{\sqrt{\pi}}\int_0^{\infty} \frac{e^{-(\widetilde{z}-\lambda)^2}(\widetilde{z}-\lambda) - e^{-(\widetilde{z}+\lambda)^2}(\widetilde{z}+\lambda)}{\lambda} \sin\left(\log\left(x\frac{\widetilde{z}}{\lambda}\right)\right) d\widetilde{z}, \quad \lambda = \frac{x}{\sqrt{4t}} \in (0,\infty). \end{aligned}$$

Let $F(\theta) = e^{-\theta^2}\theta$, $\theta \in (-\infty,\infty)$. By the mean value theorem, we have

$$F(\widetilde{z}-\lambda) - F(\widetilde{z}+\lambda) = F'(\theta_{\widetilde{z},\lambda})(-2\lambda) = e^{-\theta^2_{\widetilde{z},\lambda}}(1-2\theta^2_{\widetilde{z},\lambda})(-2\lambda)$$

for some number $\theta_{\widetilde{z},\lambda}$ lying in the interval $(\widetilde{z}-\lambda, \widetilde{z}+\lambda)$. Hence we have

$$u^{(2)}(x,t) = \frac{x}{\sqrt{\pi}}\int_0^{\infty} e^{-\theta^2_{\widetilde{z},\lambda}}(4\theta^2_{\widetilde{z},\lambda}-2)\sin\left(\log\left(x\frac{\widetilde{z}}{\lambda}\right)\right) d\widetilde{z}, \quad \theta_{\widetilde{z},\lambda} \in (\widetilde{z}-\lambda, \widetilde{z}+\lambda)$$

and, without loss of generality, we may assume $\lambda \in (0,1)$ since we will let $\lambda \to 0^+$ eventually. By the estimate

$$\begin{aligned} &\left|e^{-\theta^2_{\widetilde{z},\lambda}}(4\theta^2_{\widetilde{z},\lambda}-2)\sin\left(\log\left(x\frac{\widetilde{z}}{\lambda}\right)\right)\right| \le e^{-(\widetilde{z}-\lambda)^2}(4(\widetilde{z}+\lambda)^2+2) \le e^{-\widetilde{z}^2+2\widetilde{z}}(4(\widetilde{z}+1)^2+2), \\ &\quad \theta_{\widetilde{z},\lambda} \in (\widetilde{z}-\lambda, \widetilde{z}+\lambda),\ \widetilde{z} \in [0,\infty),\ \lambda \in (0,1), \end{aligned}$$

where $e^{-\widetilde{z}^2+2\widetilde{z}}(4(\widetilde{z}+1)^2+2) \in L^1[0,\infty)$, we can apply the LDCT to $u^{(2)}(x,t)$.

Before applying the LDCT, we write $\sin\left(\log\left(x\frac{\widetilde{z}}{\lambda}\right)\right)$ as

(4.33)
$$\sin\left(\log\left(x\frac{\widetilde{z}}{\lambda}\right)\right) = \sin(\log(\sqrt{4t}\widetilde{z})) = \sin(\log\sqrt{4t})\cos(\log\widetilde{z}) + \cos(\log\sqrt{4t})\sin(\log\widetilde{z})$$

and note that

$$\lim_{\lambda\to 0}\frac{e^{-(\widetilde{z}-\lambda)^2}(\widetilde{z}-\lambda) - e^{-(\widetilde{z}+\lambda)^2}(\widetilde{z}+\lambda)}{\lambda} = (4\widetilde{z}^2-2)e^{-\widetilde{z}^2}. \tag{4.34}$$

By (4.33) and (4.34), we have for any fixed $\widetilde{z} \in (0,\infty)$ the following limits (note that $\lambda \to 0^+$ is equivalent to $t \to \infty$)

(4.35)
$$\lim_{\lambda\to 0^+}\left[\frac{e^{-(\widetilde{z}-\lambda)^2}(\widetilde{z}-\lambda) - e^{-(\widetilde{z}+\lambda)^2}(\widetilde{z}+\lambda)}{\lambda}\cos(\log\widetilde{z}) - (4\widetilde{z}^2-2)e^{-\widetilde{z}^2}\cos(\log\widetilde{z})\right] = 0$$

and

(4.36)
$$\lim_{\lambda\to 0^+}\left[\frac{e^{-(\widetilde{z}-\lambda)^2}(\widetilde{z}-\lambda) - e^{-(\widetilde{z}+\lambda)^2}(\widetilde{z}+\lambda)}{\lambda}\sin(\log\widetilde{z}) - (4\widetilde{z}^2-2)e^{-\widetilde{z}^2}\sin(\log\widetilde{z})\right] = 0.$$

Finally, by (4.33), (4.35), (4.36) and the LDCT, we can conclude

$$(4.37)\quad \lim_{t\to\infty} \left|u^{(2)}(x,t) - (A_2 \sin(\log\sqrt{4t}) + B_2\cos(\log\sqrt{4t}))x\right| = 0 \quad \text{for fixed } x \in (0,\infty),$$

where

(4.38)

$$A_2 = \frac{1}{\sqrt{\pi}}\int_0^\infty (4\widetilde{z}^2 - 2)e^{-\widetilde{z}^2}\cos(\log\widetilde{z})\,d\widetilde{z}, \quad B_2 = \frac{1}{\sqrt{\pi}}\int_0^\infty (4\widetilde{z}^2 - 2)e^{-\widetilde{z}^2}\sin(\log\widetilde{z})\,d\widetilde{z}.$$

The proof of (4.20) is done due to (4.31), (4.32), (4.37) and (4.38). □

As a consequence of Lemma 4.7, we have the following new result about prescribing the space oscillation limits.

Lemma 4.10 (Prescribing the space oscillation limits; finite-infinite case)**.** *Consider the ibvp* (1.1)*. For any finite number c_1, one can find a solution $u(x,t)$ of* (1.1) *lying in the space* (4.7) *and satisfies*

$$(4.39)\qquad c_1 = \liminf_{x\to\infty} u(x,t) \le \limsup_{x\to\infty} u(x,t) = +\infty \quad \textit{for fixed } t \in (0,\infty).$$

Similarly, for any finite number c_2, one can find a solution $u(x,t)$ of (1.1) *lying in the space* (4.7) *and satisfies*

$$(4.40)\qquad -\infty = \liminf_{x\to\infty} u(x,t) \le \limsup_{x\to\infty} u(x,t) = c_2 \quad \textit{for fixed } t \in (0,\infty).$$

Proof. We choose

$$(4.41)\qquad h(x) = x\sin(\log x) + x + c_1, \quad g(t) \equiv 0, \quad x \in (0,\infty),\ t \in (0,\infty).$$

The solution $U(x,t)$ of (1.1) with the initial-boundary data (4.41) is given by

$$(4.42)\qquad U(x,t) = u(x,t) + x + \frac{c_1}{\sqrt{\pi}}\int_{-x/\sqrt{4t}}^{x/\sqrt{4t}} e^{-z^2}\,dz, \quad (x,t) \in (0,\infty)\times(0,\infty),$$

where $u(x,t)$ in (4.42) is given by (4.21). Hence we have

$$\lim_{x\to\infty} |U(x,t) - (x\sin(\log x) + x + c_1)| = 0 \quad \text{for fixed } t \in (0,\infty),$$

where

$$\liminf_{x\to\infty} U(x,t) = \liminf_{x\to\infty} \left[x(\sin(\log x) + 1) + c_1\right] = c_1$$

and

$$\limsup_{x\to\infty} U(x,t) = \limsup_{x\to\infty} \left[x(\sin(\log x) + 1) + c_1\right] = +\infty.$$

The proof of (4.39) is done.

For (4.40), we just replace (4.41) by

$$h(x) = -x\sin(\log x) - x + c_2, \quad g(t) \equiv 0, \quad x \in (0,\infty),\ t \in (0,\infty).$$

The proof is done. □

Remark 4.11. For fixed $x \in (0,\infty)$, the solution $U(x,t)$ given by (4.42) also satisfies

$$\liminf_{t\to\infty} U(x,t) = \left(1 - \sqrt{A^2+B^2}\right) x, \quad \limsup_{t\to\infty} U(x,t) = \left(1 + \sqrt{A^2+B^2}\right) x.$$

4.1.3. Prescribing the time oscillation limits of $u(x,t)$

In this section, we focus on time oscillation limits and construct examples in which one is finite and the other is infinite. See Lemma 4.15 below. We first look at two examples with $g(t) = t\sin(\log t)$ and $g(t) = \sqrt{t}\sin(\log\sqrt{t})$ respectively. The first example can be viewed as a counterpart of Lemma 4.7.

Lemma 4.12. *The solution $u(x,t)$ of the ibvp* (1.1) *with*

$$h(x) \equiv 0, \quad g(t) = t\sin(\log t), \quad x \in (0,\infty),\ t \in (0,\infty)$$

satisfies

$$\lim_{x\to\infty} u(x,t) = 0 \quad \textit{for fixed } t \in (0,\infty) \tag{4.43}$$

and

$$\lim_{t\to\infty}\left|\frac{u(x,t) - t\sin(\log t)}{x\sqrt{t}} - \left(P\sin(\log t) + Q\cos(\log t)\right)\right| = 0 \quad \textit{for fixed } x \in (0,\infty),$$

where P, Q are constants given by the values of the integrals:

$$\begin{aligned} P &= -\frac{1}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{1-\theta}}\left[\cos(\log\theta) - \sin(\log\theta)\right] d\theta, \\ Q &= -\frac{1}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{1-\theta}}\left[\sin(\log\theta) + \cos(\log\theta)\right] d\theta. \end{aligned} \tag{4.44}$$

Remark 4.13. If we take $h(x) \equiv 0$ and $g(t) = t$, then by (3.6) and (3.7) in Lemma 3.1 with $n = 1$, we have

$$\lim_{x\to\infty} u(x,t) = 0 \quad \text{for fixed } t \in (0,\infty)$$

and

$$\lim_{t\to\infty}\left|\frac{u(x,t) - t}{x\sqrt{t}} + \frac{2}{\sqrt{\pi}}\right| = 0 \quad \text{for fixed } x \in (0,\infty).$$

Proof of Lemma 4.12. The function $g(t) = t\sin(\log t)$ has already appeared in (4.11), where by (4.13) we know that

$$u(x,t) = u_g(x,t) = t\big[A(x,t)\sin(\log t) + B(x,t)\cos(\log t)\big], \quad (x,t) \in (0,\infty)\times(0,\infty),$$

where $A(x,t)$ and $B(x,t)$ are given by (4.14) and satisfy (4.15). Therefore, we clearly have (4.43).

To find the precise asymptotic time limit for fixed $x \in (0,\infty)$, we let $\lambda = x/\sqrt{4t}$ (note that $\lambda \to 0^+$ is equivalent to $t \to \infty$) and write t as $x\sqrt{t}/(2\lambda)$ and look at the difference

$$u(x,t) - t\sin(\log t) = x\sqrt{t}\cdot\left(\frac{A(x,t)-1}{2\lambda}\sin(\log t) + \frac{B(x,t)}{2\lambda}\cos(\log t)\right), \quad \frac{x}{\sqrt{4t}} = \lambda,$$

where, by (4.14) and in terms of $\lambda = x/\sqrt{4t}$, we have

$$\frac{A(x,t)-1}{2\lambda} = \frac{\frac{2}{\sqrt{\pi}}\int_\lambda^\infty e^{-z^2}\big(1-\frac{\lambda^2}{z^2}\big)\cos\big(\log\big(1-\frac{\lambda^2}{z^2}\big)\big)\,dz - 1}{2\lambda}$$

and

$$\frac{B(x,t)}{2\lambda} = \frac{\frac{2}{\sqrt{\pi}}\int_\lambda^\infty e^{-z^2}\big(1-\frac{\lambda^2}{z^2}\big)\sin\big(\log\big(1-\frac{\lambda^2}{z^2}\big)\big)\,dz}{2\lambda}.$$

For fixed $x \in (0,\infty)$, as $t \to \infty$, by L'Hospital rule we have

$$\begin{aligned}\lim_{\lambda\to 0^+}\frac{A(x,t)-1}{2\lambda} = \lim_{\lambda\to 0^+}\Bigg\{&\frac{1}{\sqrt{\pi}}\int_\lambda^\infty e^{-z^2}\left(-\frac{2\lambda}{z^2}\right)\cos\left(\log\left(1-\frac{\lambda^2}{z^2}\right)\right)dz\\ &-\frac{1}{\sqrt{\pi}}\int_\lambda^\infty e^{-z^2}\left(-\frac{2\lambda}{z^2}\right)\sin\left(\log\left(1-\frac{\lambda^2}{z^2}\right)\right)dz\Bigg\}\end{aligned}$$

and if we let $z = \lambda s$, $s \in (1,\infty)$, the above becomes

(4.45)
$$\begin{aligned}\lim_{\lambda\to 0^+}\frac{A(x,t)-1}{2\lambda} &= \lim_{\lambda\to 0^+}\Bigg\{\frac{1}{\sqrt{\pi}}\int_1^\infty e^{-\lambda^2 s^2}\left(-\frac{2}{s^2}\right)\cos\left(\log\left(1-\frac{1}{s^2}\right)\right)ds\\ &\qquad -\frac{1}{\sqrt{\pi}}\int_1^\infty e^{-\lambda^2 s^2}\left(-\frac{2}{s^2}\right)\sin\left(\log\left(1-\frac{1}{s^2}\right)\right)ds\Bigg\}\\ &= -\frac{2}{\sqrt{\pi}}\int_1^\infty \frac{1}{s^2}\left[\cos\left(\log\left(1-\frac{1}{s^2}\right)\right) - \sin\left(\log\left(1-\frac{1}{s^2}\right)\right)\right]ds.\end{aligned}$$

Similarly, we have

$$\lim_{\lambda\to 0^+}\frac{B(x,t)}{2\lambda} = -\frac{2}{\sqrt{\pi}}\int_1^\infty \frac{1}{s^2}\left[\sin\left(\log\left(1-\frac{1}{s^2}\right)\right) + \cos\left(\log\left(1-\frac{1}{s^2}\right)\right)\right]ds. \tag{4.46}$$

Therefore, the ratio $(x\sqrt{t})^{-1}(u(x,t) - t\sin(\log t))$ approaches $P\sin(\log t) + Q\cos(\log t)$ as $t \to \infty$, where P and Q are the values of the two convergent integrals in (4.45) and (4.46). Finally, if we do the change of variables $\theta = 1 - 1/s^2$, we will get the two integrals in (4.44). The proof is done. □

Another interesting example similar to Lemma 4.12 is the following

Lemma 4.14. *The solution $u(x,t)$ of the ibvp* (1.1) *with*

$$h(x) \equiv 0, \quad g(t) = \sqrt{t}\sin(\log\sqrt{t}), \quad x \in (0,\infty),\ t \in (0,\infty)$$

satisfies

$$\lim_{x\to\infty} u(x,t) = 0 \quad \textit{for fixed } t \in (0,\infty)$$

and, for fixed $x \in (0,\infty)$, *the following*

$$\lim_{t\to\infty} \left|(u(x,t) - \sqrt{t}\sin(\log\sqrt{t})) - (M\sin(\log\sqrt{t}) + N\cos(\log\sqrt{t}))x\right| = 0, \tag{4.47}$$

where M, N *are constants given by the values of the integrals:*

$$\begin{aligned} M &= -\frac{1}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{1-\theta^2}}\left[\cos(\log\theta) - \sin(\log\theta)\right] d\theta, \\ N &= -\frac{1}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{1-\theta^2}}\left[\sin(\log\theta) + \cos(\log\theta)\right] d\theta. \end{aligned} \tag{4.48}$$

Proof. We have

$$\begin{aligned} u(x,t) = u_g(x,t) &= \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^\infty e^{-z^2}\sqrt{t-\left(\frac{x}{2z}\right)^2}\sin\left(\log\sqrt{t-\left(\frac{x}{2z}\right)^2}\right) dz \\ &= \sqrt{t}\left[\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^\infty e^{-z^2}\sqrt{1-\frac{1}{t}\frac{x^2}{4z^2}}\sin\left(\log\sqrt{t} + \log\sqrt{1-\frac{1}{t}\frac{x^2}{4z^2}}\right) dz\right] \\ &= \sqrt{t}\left[A(x,t)\sin(\log\sqrt{t}) + B(x,t)\cos(\log\sqrt{t})\right], \end{aligned}$$

where

$$\begin{aligned} A(x,t) &= \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^\infty e^{-z^2}\sqrt{1-\frac{1}{t}\frac{x^2}{4z^2}}\cos\left(\log\sqrt{1-\frac{1}{t}\frac{x^2}{4z^2}}\right) dz, \\ B(x,t) &= \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^\infty e^{-z^2}\sqrt{1-\frac{1}{t}\frac{x^2}{4z^2}}\sin\left(\log\sqrt{1-\frac{1}{t}\frac{x^2}{4z^2}}\right) dz \end{aligned}$$

with

$$\begin{aligned} &\lim_{x\to\infty} A(x,t) = \lim_{x\to\infty} B(x,t) = 0 \quad \text{for fixed } t \in (0,\infty), \\ &\lim_{t\to\infty} A(x,t) = 1, \quad \lim_{t\to\infty} B(x,t) = 0 \quad \text{for fixed } x \in (0,\infty), \end{aligned}$$

which implies

$$\lim_{x\to\infty} u(x,t) = 0 \quad \text{for fixed } t \in (0,\infty).$$

To find the precise asymptotic time limit for fixed $x \in (0,\infty)$, we let $\lambda = x/\sqrt{4t}$ (note that $\lambda \to 0^+$ is equivalent to $t \to \infty$) and look at the difference

$$u(x,t) - \sqrt{t}\sin(\log\sqrt{t}) = x\cdot\left(\frac{A(x,t)-1}{2\lambda}\sin(\log\sqrt{t}) + \frac{B(x,t)}{2\lambda}\cos(\log\sqrt{t})\right), \quad \sqrt{t} = \frac{x}{2\lambda}.$$

As $\lambda \to 0^+$, by L'Hospital rule, we have

$$\begin{aligned}
&\lim_{\lambda\to 0^+} \frac{A(x,t)-1}{2\lambda} \\
&= \lim_{\lambda\to 0^+} \frac{1}{2\lambda}\left[\frac{2}{\sqrt{\pi}}\int_\lambda^\infty e^{-z^2}\sqrt{1-\frac{\lambda^2}{z^2}}\cos\left(\log\sqrt{1-\frac{\lambda^2}{z^2}}\right)dz - 1\right] \\
&= \frac{1}{2}\lim_{\lambda\to 0^+}\frac{d}{d\lambda}\left[\frac{2}{\sqrt{\pi}}\int_\lambda^\infty e^{-z^2}\sqrt{1-\frac{\lambda^2}{z^2}}\cos\left(\log\sqrt{1-\frac{\lambda^2}{z^2}}\right)dz - 1\right] \\
&= \frac{1}{\sqrt{\pi}}\lim_{\lambda\to 0^+}\left\{\int_\lambda^\infty e^{-z^2}\frac{-\frac{\lambda}{z^2}}{\sqrt{1-\frac{\lambda^2}{z^2}}}\left[\cos\left(\log\sqrt{1-\frac{\lambda^2}{z^2}}\right)-\sin\left(\log\sqrt{1-\frac{\lambda^2}{z^2}}\right)\right]dz\right\}
\end{aligned}$$

and if we let $z = \lambda s$, $s \in (1,\infty)$, the above becomes

$$-\frac{1}{\sqrt{\pi}}\lim_{\lambda\to 0^+}\left\{\int_1^\infty e^{-\lambda^2 s^2}\frac{1}{s\sqrt{s^2-1}}\left[\cos\left(\log\sqrt{1-\frac{1}{s^2}}\right)-\sin\left(\log\sqrt{1-\frac{1}{s^2}}\right)\right]ds\right\}.$$

Since the integral

$$\int_1^\infty \frac{1}{s\sqrt{s^2-1}}\,ds = \frac{1}{2}\pi$$

converges, the LDCT can be applied and we have

$$\lim_{\lambda\to 0^+}\frac{A(x,t)-1}{2\lambda} = -\frac{1}{\sqrt{\pi}}\int_1^\infty \frac{1}{s\sqrt{s^2-1}}\left[\cos\left(\log\sqrt{1-\frac{1}{s^2}}\right)-\sin\left(\log\sqrt{1-\frac{1}{s^2}}\right)\right]ds.$$

Similarly, we have

$$\lim_{\lambda\to 0^+}\frac{B(x,t)}{2\lambda} = -\frac{1}{\sqrt{\pi}}\int_1^\infty \frac{1}{s\sqrt{s^2-1}}\left[\sin\left(\log\sqrt{1-\frac{1}{s^2}}\right)+\cos\left(\log\sqrt{1-\frac{1}{s^2}}\right)\right]ds.$$

Finally, if we do the change of variables $\theta = \sqrt{1-1/s^2}$, we will get the two integrals in (4.48). The proof of (4.47) is done. □

As a consequence of Lemma 4.14, we have the following new result about prescribing the time oscillation limits.

Lemma 4.15 (Prescribing the time oscillation limits; finite-infinite case)**.** *Consider the ibvp* (1.1)*. For any finite number c_1, one can find a solution $u(x,t)$ of* (1.1) *lying in the space* (4.7) *and satisfies*

$$c_1 = \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) = +\infty \quad \textit{for fixed } x \in (0,\infty). \tag{4.49}$$

Similarly, for any finite number c_2, one can find a solution $u(x,t)$ of (1.1) *lying in the space* (4.7) *and satisfies*

$$-\infty = \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) = c_2 \quad \textit{for fixed } x \in (0,\infty). \tag{4.50}$$

Proof. Unlike the proof of Lemma 4.10 where we have $g(t) \equiv 0$, here we need a nonzero $h(x)$ to help us achieve the goal. For (4.49), we take

$$h(x) = \left(M + \frac{\sqrt{\pi}}{2}\right)x, \quad g(t) = \sqrt{t}\sin(\log\sqrt{t}) + \sqrt{t} + c_1, \quad x \in (0,\infty),\ t \in (0,\infty), \tag{4.51}$$

where the constant M in (4.51) is given by (4.48). The solution $u(x,t)$ of (1.1) with the above data is given by $u(x,t) = u_h(x,t) + u_g(x,t)$, where

$$u_h(x,t) = \left(M + \frac{\sqrt{\pi}}{2}\right)x, \quad (x,t) \in (0,\infty)\times(0,\infty)$$

and

$$\begin{aligned} &u_g(x,t) \\ &= \sqrt{t}\left[\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\sqrt{1 - \frac{1}{t}\frac{x^2}{4z^2}}\sin\left(\log\sqrt{t} + \log\sqrt{1 - \frac{1}{t}\frac{x^2}{4z^2}}\right)dz\right] \\ &\quad + \sqrt{t}\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\sqrt{1 - \frac{1}{t}\frac{x^2}{4z^2}}\,dz + \frac{2c_1}{\sqrt{\pi}}\int_{x/\sqrt{4t}}^{\infty} e^{-z^2}\,dz, \quad (x,t) \in (0,\infty)\times(0,\infty). \end{aligned}$$

By Lemma 4.14 and (3.9) in Remark 3.3, we have for fixed $x \in (0,\infty)$ that

$$\lim_{t\to\infty} |u_g(x,t) - \Gamma(x,t)| = 0,$$

where

$$\Gamma(x,t) = \left(\sqrt{t}\sin(\log\sqrt{t}) + \sqrt{t} + c_1\right) + \left(M\sin(\log\sqrt{t}) + N\cos(\log\sqrt{t}) - \frac{\sqrt{\pi}}{2}\right)x.$$

Therefore, we have

$$\lim_{t\to\infty} |u(x,t) - \Lambda(x,t)| = \lim_{t\to\infty} |(u_h(x,t) + u_g(x,t)) - \Lambda(x,t)| = 0,$$

where now

$$\begin{aligned} \Lambda(x,t) &= \Gamma(x,t) + \left(M + \frac{\sqrt{\pi}}{2}\right)x \\ &= \left(M + \frac{\sqrt{\pi}}{2}\right)x + \left(\sqrt{t}\left[\sin(\log\sqrt{t}) + 1\right] + c_1\right) \\ &\quad + \left(M\sin(\log\sqrt{t}) + N\cos(\log\sqrt{t}) - \frac{\sqrt{\pi}}{2}\right)x, \quad (x,t) \in (0,\infty)\times(0,\infty). \end{aligned} \tag{4.52}$$

We clearly have

$$\limsup_{t\to\infty} \Lambda(x,t) = +\infty \quad \text{for fixed } x \in (0,\infty)$$

due to the term $\sqrt{t}[\sin(\log\sqrt{t}) + 1] \in [0, 2\sqrt{t}]$. For fixed $x \in (0,\infty)$, along the sequence $t_k \to \infty$ with $\sin(\log\sqrt{t_k}) = -1$ for all k, we have $\Lambda(x,t_k) = c_1$. Therefore, the limit

$\liminf_{t\to\infty} \Lambda(x,t) = \Lambda_0 \in (-\infty, c_1]$ exists and there exists a sequence $s_k \to \infty$ such that $\Lambda(x, s_k) \to \Lambda_0$ as $k \to \infty$. Clearly we must have $\sin(\log\sqrt{s_k}) + 1 \to 0$, otherwise we will get a contradiction. Therefore, along the sequence $\{s_k\}_{k=1}^{\infty}$ attaining Λ_0, we have

$$\lim_{k\to\infty} \sin(\log\sqrt{s_k}) = -1, \quad \lim_{k\to\infty} \cos(\log\sqrt{s_k}) = 0,$$

which, by (4.52), implies the convergence of the nonnegative sequence $\sqrt{s_k}(\sin(\log\sqrt{s_k}) + 1)$. Since $\Lambda_0 \le c_1$, the sequence must converge to 0 and we have $\Lambda_0 = c_1$. The proof of (4.49) is done.

For (4.50), we just replace (4.51) by

$$h(x) = -\left(M + \frac{\sqrt{\pi}}{2}\right) x, \quad g(t) = -\sqrt{t}\sin(\log\sqrt{t}) - \sqrt{t} + c_2, \quad x \in (0,\infty),\ t \in (0,\infty).$$

The proof is done. □

5. Space and time periodic solutions

In this section, we explore the existence of space and time-periodic solutions of the ibvp (1.1). For space and time-periodic solutions of the heat equation on the entire space $(x,t) \in (-\infty,\infty) \times (-\infty,\infty)$, one can see the paper [12]. A smooth solution $u(x,t)$ of the heat equation on $(0,\infty) \times (0,\infty)$ is called a **space-periodic solution** with period $a > 0$ if it satisfies

$$u(x + a, t) = u(x,t), \quad \forall\,(x,t) \in (0,\infty) \times (0,\infty).$$

Similarly, it is called a **time-periodic solution** with period $b > 0$ if it satisfies

$$u(x, t + b) = u(x,t), \quad \forall\,(x,t) \in (0,\infty) \times (0,\infty).$$

By parabolic scaling $(x,t) \to (\lambda x, \lambda^2 t)$, where $\lambda > 0$ is a constant, one can rescale a space-periodic solution $u(x,t)$ with period $a > 0$ into another solution with any period $p > 0$. Therefore, without loss of generality, one may assume $a = 2\pi$. Similarly, one can also assume $b = 2\pi$. Also note that a space-periodic solution is smooth on $[0,\infty) \times (0,\infty)$ and a time-periodic solution is smooth on $(0,\infty) \times [0,\infty)$.

In the following we would like to show that for a given space-periodic initial data $h(x)$, one can find a boundary data $g(t)$ so that the corresponding solution $u(x,t)$ of the ibvp (1.1) is space-periodic on $(0,\infty) \times (0,\infty)$. Similarly, for a given time-periodic boundary data $g(t)$, one can find an initial data $h(x)$ so that the solution $u(x,t)$ is time-periodic on $(0,\infty) \times (0,\infty)$.

Note that if both $h(x)$ and $g(t)$ are 2π-periodic on $[0,\infty)$, the solution $u(x,t)$ of the ibvp (1.1), given by the representation formula (1.3), may not have any periodic property at all. A simple example is when $h(x) = \sin x$ and $g(t) = \sin t$, then the solution $u(x,t)$ is given by (3.26), which is **neither space-periodic nor time-periodic**. However, for fixed $t \in (0,\infty)$ it is asymptotically space-periodic and for fixed $x \in (0,\infty)$ it is asymptotically time-periodic. See (3.29) and (3.30) in Lemma 3.12.

For our purpose of discussion, we need to use the following Fourier series result: Assume $h(x)$ is a 2π**-periodic** C^1 **function** defined on $x \in [0,\infty)$. Then the following series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad x \in [0,\infty)$$

converges **absolutely** and **uniformly** to $h(x)$ on $[0,\infty)$, where a_0, a_n, b_n are the Fourier series coefficients of $h(x)$, given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} h(x)\cos(nx)\,dx, \quad n = 0,1,2,3,\ldots, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} h(x)\sin(nx)\,dx, \quad n = 1,2,3,\ldots. \end{aligned} \tag{5.1}$$

For **space-periodic solution**, we have the following

Lemma 5.1 (Existence of space-periodic solution)**.** *Assume $h(x)$ is a given 2π-periodic C^1 function defined on $x \in [0,\infty)$. Then one can find a* ***unique*** *continuous bounded function $g(t)$ on $t \in [0,\infty)$ with $g(0) = h(0)$ such that the function $u(x,t)$ given by* (1.3) *is a bounded function lying in the space S in* (1.10)*, and is a* ***space-periodic solution*** *(with period 2π) of the ibvp* (1.1)*.*

Proof. Let

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t}, \quad t \in [0,\infty), \tag{5.2}$$

where $a_0, a_1, a_2, \ldots$ in (5.2) are the Fourier series coefficients of $h(x)$ in (5.1). Since the Fourier series for $h(x)$ converges absolutely for all $x \in [0,\infty)$, the series for $g(t)$ does converge and represents a bounded continuous function on $[0,\infty)$ with $g(0) = h(0) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n$. We claim that the bounded function $u(x,t)$ given by (1.3) is equal to the following, namely

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-n^2 t}(a_n \cos nx + b_n \sin nx), \quad (x,t) \in (0,\infty)\times(0,\infty). \tag{5.3}$$

To see this, note that one can commute any order of differentiation (with respect to x or t) with the summation sign $\sum$ in (5.3) on the domain $(0,\infty)\times(0,\infty)$. Therefore, the function $u(x,t)$ given by (5.3) is a bounded smooth solution of the heat equation on

$(0,\infty)\times(0,\infty)$ and it satisfies (1.4), (1.5) and lies in the space S in (1.10). Moreover, it is **space-periodic** with period 2π. Since the function $u(x,t)$ given by (5.3) is a bounded function, by Lemma 1.5 it is the same as the solution given by the formula (1.3).

As for the uniqueness of the function $g(t)$, if $u(x,t)\in S$ is a space-periodic solution (with period 2π) of the heat equation on $(0,\infty)\times(0,\infty)$ with $u(x,0)=h(x)$, $x\in(0,\infty)$, then by Fourier series expansion for the smooth solution $u(x,t)$ and the identity $u_t=u_{xx}$, it must be given by (5.3) (where $a_0,a_1,a_2,\ldots$ in (5.3) are the Fourier series coefficients of $h(x)$). Therefore, $u(0,t)$ must be equal to the function $g(t)$ in (5.2). The proof is done. □

Remark 5.2. If $u(x,t)\in S$ is a **space-periodic solution** (with period $2\pi>0$) of the heat equation on $(0,\infty)\times(0,\infty)$, then its **average space integral** $(2\pi)^{-1}\int_0^{2\pi}u(x,t)\,dx$ is a constant C independent of $t\in(0,\infty)$. Moreover, we also have we have $\lim_{t\to\infty}u(x,t)=C$ for all $x\in(0,\infty)$.

As for **time-periodic solution**, we have the following

Lemma 5.3 (Existence of time-periodic solution)**.** *Assume $g(t)$ is a given 2π-periodic C^1 function defined on $t\in[0,\infty)$. Then one can find a continuous function $h(x)$ on $x\in[0,\infty)$ with $h(0)=g(0)$ such that the function $u(x,t)$ given by* (1.3) *is a* ***time-periodic solution*** *(with period 2π) of the ibvp* (1.1) *lying in the space S in* (1.10).

Remark 5.4. The choice of $h(x)$ in Lemma 5.3 is not unique.

Proof of Lemma 5.3. Since $g(t)$ is a 2π-periodic C^1 function n $t\in[0,\infty)$, by the above Fourier series expansion, it is equal to

$$g(t)=\frac{\widetilde{a}_0}{2}+\sum_{n=1}^{\infty}(\widetilde{a}_n\cos nt+\widetilde{b}_n\sin nt),\quad t\in[0,\infty)$$

where

$$\begin{aligned}\widetilde{a}_n&=\frac{1}{\pi}\int_0^{2\pi}g(t)\cos(nt)\,dt,\quad n=0,1,2,3,\ldots,\\ \widetilde{b}_n&=\frac{1}{\pi}\int_0^{2\pi}g(t)\sin(nt)\,dt,\quad n=1,2,3,\ldots.\end{aligned}\tag{5.4}$$

Now we choose $h(x)$ as

$$h(x)=\frac{\widetilde{a}_0+cx}{2}+\sum_{n=1}^{\infty}\left(\widetilde{a}_ne^{-\sqrt{\frac{n}{2}}x}\cos\left(\sqrt{\frac{n}{2}}x\right)-\widetilde{b}_ne^{-\sqrt{\frac{n}{2}}x}\sin\left(\sqrt{\frac{n}{2}}x\right)\right),\quad x\in[0,\infty),\tag{5.5}$$

where $\widetilde{a}_0,\widetilde{a}_n,\widetilde{b}_n,\ldots$ in (5.5) are the Fourier series coefficients of $g(t)$ in (5.4) and $c\in(-\infty,\infty)$ is an arbitrary constant (which means that the choice of $h(x)$ is **not unique**).

Since the Fourier series for $g(t)$ converges absolutely for all $t \in [0, \infty)$, the series for $h(x)$ does converge and gives a continuous function on $[0, \infty)$ with $h(0) = g(0)$. We claim that the function $u(x,t)$ given by (1.3) is equal to the following, namely

(5.6)

$$u(x,t) = \frac{\widetilde{a}_0 + cx}{2} + \sum_{n=1}^{\infty} \left(\widetilde{a}_n e^{-\sqrt{\frac{n}{2}}x} \cos\left(\sqrt{\frac{n}{2}}x - nt \right) - \widetilde{b}_n e^{-\sqrt{\frac{n}{2}}x} \sin\left(\sqrt{\frac{n}{2}}x - nt \right) \right),$$

where $(x,t) \in (0,\infty) \times (0,\infty)$. To see the above identity, similar to the series (5.3), one can commute any order of differentiation (with respect to x or t) with the summation sign $\sum$ in (5.6) on the domain $(0,\infty) \times (0,\infty)$. Therefore, the function $u(x,t)$ given by (5.6) is a smooth solution of the heat equation on $(0,\infty) \times (0,\infty)$ and it satisfies (1.4) and (1.5) and lies in the space S in (1.10). Moreover, it is **time-periodic** with period 2π. Since the function $u(x,t)$ given by (5.6) is a continuous function satisfying the growth estimate (1.11), by Lemma 1.5 it is the same as the solution given by the formula (1.3). The proof is done. □

Remark 5.5. If we choose $c = 0$ in (5.5) and (5.6), the result of Lemma 5.3 is still correct. In such a case, the solution $u(x,t)$ in (5.6) is a bounded continuous function on $(0,\infty) \times (0,\infty)$.

Remark 5.6. This is to compare with Remark 5.2. If $u(x,t) \in S$ is a **time-periodic solution** (with period $2\pi > 0$) of the heat equation on $(0,\infty) \times (0,\infty)$, then its **average time integral** $(2\pi)^{-1} \int_0^{2\pi} u(x,t)\, dt$ is, in general, **not** a constant independent of $x \in (0,\infty)$. However, by the identity

$$\frac{d^2}{dx^2} \left(\frac{1}{2\pi} \int_0^{2\pi} u(x,t)\, dt \right) = \frac{1}{2\pi} \int_0^{2\pi} u_{xx}(x,t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} u_t(x,t)\, dt = 0, \quad x \in (0,\infty),$$

we must have

$$\frac{1}{2\pi} \int_0^{2\pi} u(x,t)\, dt = A + Bx, \quad x \in (0,\infty)$$

for some constants A, B. For example, for $u(x,t)$ given by (5.6), it satisfies

$$\frac{1}{2\pi} \int_0^{2\pi} u(x,t)\, dt = \frac{\widetilde{a}_0 + cx}{2}, \quad \forall\, x \in (0,\infty).$$

The following says that a solution $u(x,t)$ which is both space-periodic and time-periodic must be a constant.

Lemma 5.7. *Assume that $u(x,t)$ given by* (1.3) *is a solution of the ibvp* (1.1) *lying in the space S in* (1.10) *and moreover, it is both space-periodic with period $a > 0$ and time-periodic with period $b > 0$. Then it must be a **constant** function.*

Proof. Let $v(x,t) = u(\lambda x, \lambda^2 t)$, where $\lambda = a/(2\pi)$. Then $v(x,t)$ is a solution of the heat equation which is space-periodic with period $2\pi > 0$ and time-periodic with period $b/\lambda^2 > 0$. By Remark 5.2, we know that

$$\lim_{t\to\infty} v(x,t) = \frac{1}{2\pi}\int_0^{2\pi} v(x,0)\,dx, \quad \forall\, x\in(0,\infty).$$

Since $v(x,t)$ is also time-periodic with period b/λ^2, we have $v(x,t) = v(x, t+mb/\lambda^2)$ for all $m\in\mathbb{N}$ and all $(x,t)\in(0,\infty)\times(0,\infty)$. Letting $m\to\infty$, we obtain $v(x,t)\equiv$ const. and so is $u(x,t)$. □

6. Singular initial data

Until now, we always assume that the initial data function $h(x)$, $x\in(0,\infty)$, satisfies the basic assumption (1.2), which automatically implies that $h(x)$ is bounded near $x=0$. In this section, we look at one interesting example of $h(x)$ which is **singular** at $x=0$ and **may not be integrable** near $x=0$. In case $h(x)$ is not integrable near $x=0$, the space convolution formula in (1.3), given by

$$S(x,t) := \frac{1}{\sqrt{4\pi t}}\int_0^{\infty}\left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right)h(\xi)\,d\xi, \quad (x,t)\in(0,\infty)\times(0,\infty) \tag{6.1}$$

needs to be analyzed more. In particular, one cannot split (6.1) into the difference of two integrals since each one is divergent.

We have

Lemma 6.1 (Singular initial data)**.** *Assume $h(x)$ is **continuous** on $(0,\infty)$ and there exist positive constants M, ε, C_1, C_2 and $\alpha\in[0,1)$, $\beta\in[0,1)$, such that*

$$|h(x)| \le \begin{cases} \frac{M}{x^{1+\beta}} & \text{on } x\in(0,\varepsilon), \\ C_1 e^{C_2|x|^{1+\alpha}} & \text{on } x\in[\varepsilon,\infty), \end{cases} \tag{6.2}$$

*then the above function $S(x,t)$ is a **smooth** solution of the heat equation on $(0,\infty)\times(0,\infty)$ satisfying*

$$\lim_{(x,t)\to(x_0,0^+)} S(x,t) = h(x_0), \quad \forall\, x_0\in(0,\infty). \tag{6.3}$$

Proof. To see the convergence of the integral in (6.1), it suffices to prove the convergence of

$$\frac{1}{\sqrt{4\pi t}}\int_0^{\varepsilon}\left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}\right)|h(\xi)|\,d\xi \tag{6.4}$$

for each fixed $(x,t)\in(0,\infty)\times(0,\infty)$. However, by the assumption (6.2) and the limit

$$\lim_{\xi\to 0^+}\frac{1}{\xi}\left(e^{-\frac{(x-\xi)^2}{4t}}-e^{-\frac{(x+\xi)^2}{4t}}\right)=\frac{x}{t}e^{-\frac{x^2}{4t}},\quad (x,t)\in(0,\infty)\times(0,\infty),\tag{6.5}$$

we clearly have the convergence of (6.4). Together with the second condition in (6.2), the integral for $S(x,t)$ in (6.1) does converge for all $(x,t)\in(0,\infty)\times(0,\infty)$.

Next, we show that one can differentiate $S(x,t)$ with respect to x and t under the integral sign. Note that the fundamental solution $\Phi(x,t)$ of the heat equation, for any $m,k\in\mathbb{N}\cup\{0\}$ and fixed constant $0<\eta<1/4$, satisfies the estimate (see the book [8, p. 274])

$$\left|\frac{\partial^{m+k}}{\partial t^m\partial x^k}\Phi(x,t)\right|\le C(m,k,\eta)\cdot t^{-\left(m+\frac{k}{2}\right)-\frac{1}{2}}e^{-\eta\frac{x^2}{t}},\quad \forall\,(x,t)\in\mathbb{R}\times(0,\infty),\tag{6.6}$$

where $C(m,k,\eta)$ is a constant depending only on m, k, η. We now pick $\eta=1/8$ in (6.6) and get

$$\left|\frac{\partial^{m+k}}{\partial t^m\partial x^k}\Phi(x,t)\right|\le C(m,k)\cdot t^{-\left(m+\frac{k}{2}\right)-\frac{1}{2}}e^{-\frac{x^2}{8t}},\quad \forall\,(x,t)\in\mathbb{R}\times(0,\infty).\tag{6.7}$$

For fixed $t\in(0,\infty)$, let $Q(\theta)=e^{-\theta^2/8t}$, $\theta\in(-\infty,\infty)$. It satisfies

$$|Q'(\theta)|=\left|e^{-\frac{\theta^2}{8t}}\frac{\theta}{4t}\right|=\frac{\sqrt{2}}{2\sqrt{t}}\left|e^{-\frac{\theta^2}{8t}}\frac{\theta}{\sqrt{8t}}\right|\le\frac{\sqrt{2}}{2\sqrt{t}}\frac{1}{\sqrt{2e}}=\frac{1}{\sqrt{4e}}\frac{1}{\sqrt{t}},\quad \forall\,\theta\in(-\infty,\infty),\tag{6.8}$$

For a fixed but arbitrary point $(x_0,t_0)\in(0,\infty)\times(0,\infty)$, one can choose the open set $U=(x_0/2,2x_0)\times(t_0/2,2t_0)\subset(0,\infty)\times(0,\infty)$ containing (x_0,t_0) such that for any $m,k\in\mathbb{N}\cup\{0\}$ the improper integral

$$\int_0^{\varepsilon}\left(\frac{\partial^{m+k}}{\partial t^m\partial x^k}\Phi(x-\xi,t)-\frac{\partial^{m+k}}{\partial t^m\partial x^k}\Phi(x+\xi,t)\right)h(\xi)\,d\xi,\quad (x,t)\in U$$

converges uniformly on U. To see this, we can use (6.7) and mean value theorem and (6.8) to get

$$\begin{aligned}&\int_0^{\varepsilon}\left|\frac{\partial^{m+k}}{\partial t^m\partial x^k}\Phi(x-\xi,t)-\frac{\partial^{m+k}}{\partial t^m\partial x^k}\Phi(x+\xi,t)\right||h(\xi)|\,d\xi\\ &\le C(m,k)\cdot t^{-\left(m+\frac{k}{2}\right)-\frac{1}{2}}\int_0^{\varepsilon}\left|e^{-\frac{(x-\xi)^2}{8t}}-e^{-\frac{(x+\xi)^2}{8t}}\right||h(\xi)|\,d\xi\\ &\le C(m,k)\cdot t^{-\left(m+\frac{k}{2}\right)-\frac{1}{2}}\frac{1}{\sqrt{4e}}\frac{1}{\sqrt{t}}\int_0^{\varepsilon}2\xi\frac{M}{\xi^{1+\beta}}\,d\xi\le C(m,k,M,t_0,\beta)\varepsilon^{1-\beta},\quad\forall\,(x,t)\in U\end{aligned}$$

for some constant $C(m,k,M,t_0,\beta)$ depending only on m, k, M, t_0, β. Moreover, using the decay term $e^{-x^2/8t}$ in (6.7) and the assumption (6.2), for any $m,k\in\mathbb{N}\cup\{0\}$ the improper integral

$$\int_{\varepsilon}^{\infty}\left(\frac{\partial^{m+k}}{\partial t^m\partial x^k}\Phi(x-\xi,t)-\frac{\partial^{m+k}}{\partial t^m\partial x^k}\Phi(x+\xi,t)\right)h(\xi)\,d\xi,\quad (x,t)\in U$$

also **converges uniformly** on U. By standard theorem in analysis, one can differentiate $S(x,t)$ with respect to x and t under the integral sign at any point $(x_0,t_0)\in(0,\infty)\times(0,\infty)$ and we conclude that $S(x,t)$ is a **smooth** solution of the heat equation on $(0,\infty)\times(0,\infty)$ satisfying (6.3). The proof is done. □

Remark 6.2. The assumption $|h(x)|\le M/x^{1+\beta}$ on $x\in(0,\varepsilon)$ for $\beta\in[0,1)$ is **optimal**. If we have $\beta=1$ and take $h(x)=1/x^2$ on $(0,\varepsilon)$, then the integral

$$\frac{1}{\sqrt{4\pi t}}\int_0^{\varepsilon}\left(e^{-\frac{(x-\xi)^2}{4t}}-e^{-\frac{(x+\xi)^2}{4t}}\right)h(\xi)\,d\xi=\frac{1}{\sqrt{4\pi t}}\int_0^{\varepsilon}\left(e^{-\frac{(x-\xi)^2}{4t}}-e^{-\frac{(x+\xi)^2}{4t}}\right)\frac{1}{\xi^2}\,d\xi$$

will diverge for all $(x,t)\in(0,\infty)\times(0,\infty)$ due to the limit (6.5).

6.1. Singular initial-boundary value problem; an example with $h(x)=1/x$ and $g(t)=1/\sqrt{4\pi t}$

Consider the ibvp (1.1) with singular initial-boundary data $h(x)=1/x$, $x\in(0,\infty)$ and $g(t)=1/\sqrt{4\pi t}$, $t\in(0,\infty)$. We first note that the function $g(t)=1/\sqrt{4\pi t}$ still satisfies the growth condition in (1.2). For $h(x)=1/x$, as proved in Lemma 6.1, the representation formula (6.1) still makes sense and the solution formula in (1.3) is now given by

$$\begin{aligned}u(x,t)&=u_h(x,t)+u_g(x,t)\\&=\frac{1}{\sqrt{4\pi t}}\int_0^{\infty}\left(e^{-\frac{(x-\xi)^2}{4t}}-e^{-\frac{(x+\xi)^2}{4t}}\right)\frac{1}{\xi}\,d\xi+\frac{x}{\sqrt{4\pi}}\int_0^t\frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}\frac{1}{\sqrt{4\pi s}}\,ds,\end{aligned}$$

where $(x,t)\in(0,\infty)\times(0,\infty)$.

We first claim the following

Lemma 6.3. *We have the identity*

$$\frac{x}{\sqrt{4\pi}}\int_0^t\frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}\frac{1}{\sqrt{4\pi s}}\,ds=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\tag{6.9}$$

for all $(x,t)\in(0,\infty)\times(0,\infty)$.

Proof. Both sides of (6.9) are solutions to the heat equation $u_t=u_{xx}$ on $(0,\infty)\times(0,\infty)$ with the same initial-boundary limits:

$$\lim_{(x,t)\to(x_0,0^+)}u(x,t)=0,\;\;x_0\in(0,\infty);\qquad\lim_{(x,t)\to(0^+,t_0)}ku(x,t)=\frac{1}{\sqrt{4\pi t_0}},\;\;t_0\in(0,\infty).$$

However, since both solutions blow up near the origin $(0,0)$, we cannot apply the familiar uniqueness property as in Lemma 1.5. Here we can give a direct quick proof if we make use of the 1-parameter family of parabolas $P(\lambda):x/\sqrt{4t}=\lambda$, $\lambda\in(0,\infty)$. Note that, if we multiply both sides of (6.9) by $x\in(0,\infty)$ and restrict them to the parabola $P(\lambda)$, and use the formula (2.2) for $u_g(x,t)$, then (6.9) is equivalent to the identity

$$\frac{2}{\sqrt{\pi}}\int_{\lambda}^{\infty}e^{-z^2}\frac{\lambda}{\sqrt{\pi\left(1-\left(\frac{\lambda}{z}\right)^2\right)}}\,dz=\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2},\quad\forall\,\lambda\in(0,\infty). \tag{6.10}$$

We can prove (6.10) easily if we cancel $\lambda/\sqrt{\pi}$ first and then do the change of variables $\theta=\sqrt{z^2-\lambda^2}$, $\theta\in[0,\infty)$. The proof is done. □

Next, we claim the following

Lemma 6.4. *We have the identity*

$$\frac{1}{\sqrt{4\pi t}}\int_0^{\infty}\left(e^{-\frac{(x-\xi)^2}{4t}}-e^{-\frac{(x+\xi)^2}{4t}}\right)\frac{1}{\xi}\,d\xi=\left(\frac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}}\right)\int_0^{x/\sqrt{4t}}e^{z^2}\,dz \tag{6.11}$$

for all $(x,t)\in(0,\infty)\times(0,\infty)$.

Proof. It is easy to check that the right-hand side of (6.11) is a solution of the heat equation on $(0,\infty)\times(0,\infty)$. Moreover, for fixed $x\in(0,\infty)$, by L'Hospital rule we have

$$\lim_{t\to0^+}\frac{\int_0^{x/\sqrt{4t}}e^{z^2}\,dz}{\sqrt{t}e^{x^2/(4t)}}=\lim_{t\to0^+}\frac{e^{x^2/(4t)}\left(-\frac{x}{4t^{3/2}}\right)}{\frac{1}{2t^{1/2}}e^{x^2/(4t)}+\sqrt{t}e^{x^2/(4t)}\left(-\frac{x^2}{4t^2}\right)}=\frac{1}{x},\quad\forall\,x\in(0,\infty). \tag{6.12}$$

Therefore, both sides of (6.11) are solutions to the heat equation $u_t=u_{xx}$ on $(0,\infty)\times(0,\infty)$ with the same initial-boundary limits:

$$\lim_{(x,t)\to(x_0,0^+)}u(x,t)=\frac{1}{x_0},\quad x_0\in(0,\infty);\qquad\lim_{(x,t)\to(0^+,t_0)}u(x,t)=0,\quad t_0\in(0,\infty).$$

Unfortunately, unlike the proof in Lemma 6.3, here we cannot restrict both sides of (6.11) to the parabola $P(\lambda)$ and use the formula (2.2) for $u_h(x,t)$. This is because each integral in (2.2) for h diverges when $h(x)=1/x$. Therefore, we need to use a different trick here. Note that (6.11) is the same as the identity

$$\int_0^{\infty}\left(e^{-\frac{(x-\xi)^2}{4t}}-e^{-\frac{(x+\xi)^2}{4t}}\right)\frac{1}{\xi}\,d\xi=\left(\sqrt{4\pi}e^{-\frac{x^2}{4t}}\right)\int_0^{x/\sqrt{4t}}e^{z^2}\,dz,\quad(x,t)\in(0,\infty)\times(0,\infty)$$

and we let

$$\Gamma(x,t)=\int_0^{\infty}\left(e^{-\frac{(x-\xi)^2}{4t}}-e^{-\frac{(x+\xi)^2}{4t}}\right)\frac{1}{\xi}\,d\xi,\quad\Lambda(x,t)=\left(\sqrt{4\pi}e^{-\frac{x^2}{4t}}\right)\int_0^{x/\sqrt{4t}}e^{z^2}\,dz.$$

We first have

$$\Gamma(0,t)=\Lambda(0,t)=0,\quad\forall\,t\in(0,\infty). \tag{6.13}$$

Next, for fixed $t \in (0, \infty)$ we can compute

$$\begin{aligned}
\frac{d}{dx}\Gamma(x,t) &= \int_0^\infty \left[e^{-\frac{(x-\xi)^2}{4t}} \left(-\frac{x-\xi}{2t} \right) - e^{-\frac{(x+\xi)^2}{4t}} \left(-\frac{x+\xi}{2t} \right) \right] \frac{1}{\xi}\, d\xi \\
&= -\frac{x}{2t}\Gamma(x,t) + \frac{1}{2t} \int_0^\infty \left(e^{-\frac{(x-\xi)^2}{4t}} + e^{-\frac{(x+\xi)^2}{4t}} \right) d\xi \\
&= -\frac{x}{2t}\Gamma(x,t) + \frac{1}{2t}\sqrt{4t} \left(\int_{-x/\sqrt{4t}}^\infty e^{-z^2}\, dz + \int_{x/\sqrt{4t}}^\infty e^{-z^2}\, dz \right) \\
&= -\frac{x}{2t}\Gamma(x,t) + \sqrt{\frac{\pi}{t}}, \quad \forall\, x \in (0,\infty)
\end{aligned}$$

and similarly we also have

$$\frac{d}{dx}\Lambda(x,t) = -\frac{x}{2t}\Lambda(x,t) + \sqrt{\frac{\pi}{t}}, \quad \forall\, x \in (0,\infty).$$

Therefore, for each fixed $t \in (0, \infty)$, $\Gamma(x, t)$ and $\Lambda(x, t)$ satisfy the same linear ODE in $x \in (0, \infty)$ with the same initial condition (6.13). Uniqueness theorem implies $\Gamma(x, t) = \Lambda(x, t)$ for all $(x, t) \in (0, \infty) \times (0, \infty)$ and (6.11) follows. □

As a consequence of Lemmas 6.3 and 6.4, we can conclude a solution formula for the following **singular initial-boundary value problem**:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & (x,t) \in (0,\infty)\times(0,\infty), \\ u(x,0) = h(x) = \frac{1}{x}, & x \in (0,\infty), \\ u(0,t) = g(t) = \frac{1}{\sqrt{4\pi t}}, & t \in (0,\infty). \end{cases} \tag{6.14}$$

The following result is now clear.

Theorem 6.5 (Solution of (6.14))**.** *The function*

$$\begin{aligned}
u(x,t) &= u_h(x,t) + u_g(x,t) \\
&= \left(\frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \right) \int_0^{x/\sqrt{4t}} e^{z^2}\, dz + \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad (x,t) \in (0,\infty)\times(0,\infty)
\end{aligned} \tag{6.15}$$

is a solution of the singular initial-boundary value problem (6.14) *and it lies in the space* (1.6)*. In particular, along each fixed parabola $P(\lambda) : x/\sqrt{4t} = \lambda$, $\lambda \in (0, \infty)$, it can be expressed as a linear combination of $h(x) = 1/x$ and $g(t) = 1/\sqrt{4\pi t}$, given by*

$$u(x,t) = \left(2\lambda e^{-\lambda^2} \int_0^\lambda e^{z^2}\, dz \right) \cdot \frac{1}{x} + e^{-\lambda^2} \cdot \frac{1}{\sqrt{4\pi t}}, \quad (x,t) \in P(\lambda), \tag{6.16}$$

where

$$\lim_{\lambda\to\infty} \left(2\lambda e^{-\lambda^2} \int_0^\lambda e^{z^2}\, dz \right) = 1, \qquad \lim_{\lambda\to\infty} e^{-\lambda^2} = 0,$$

$$\lim_{\lambda\to 0^+} \left(2\lambda e^{-\lambda^2} \int_0^\lambda e^{z^2}\, dz \right) = 0, \qquad \lim_{\lambda\to 0^+} e^{-\lambda^2} = 1.$$

Remark 6.6. Note that both (6.16) and (3.21) have similar form when we restrict solution $u(x,t)$ to the parabola $P(\lambda)$. In (3.21) we have $m, n \in \mathbb{N}$, but now we have $m = -1$, $n = -1/2$ in (6.16).

Corollary 6.7. *The function* $u_h(x,t)$ *in* (6.15) *satisfies*

$$u_h(x,t) = O\left(\frac{1}{x}\right) \quad \text{as } x \to \infty \text{ for fixed } t \in (0,\infty), \tag{6.17a}$$

$$u_h(x,t) = O\left(\frac{1}{t}\right) \quad \text{as } t \to \infty \text{ for fixed } x \in (0,\infty), \tag{6.17b}$$

which implies

$$u(x,t) = \begin{cases} O\left(\frac{1}{x}\right) & \text{as } x \to \infty \text{ for fixed } t \in (0,\infty), \\ O\left(\frac{1}{\sqrt{t}}\right) & \text{as } t \to \infty \text{ for fixed } x \in (0,\infty). \end{cases} \tag{6.18}$$

Proof. For fixed $t \in (0,\infty)$, by the L'Hospital rule, we have

$$\begin{aligned} \lim_{x\to\infty} u_h(x,t) &= \lim_{x\to\infty} \frac{x \int_0^{x/\sqrt{4t}} e^{z^2}\, dz}{\sqrt{t} e^{x^2/(4t)}} = \lim_{x\to\infty} \frac{\int_0^{x/\sqrt{4t}} e^{z^2}\, dz + x e^{x^2/(4t)} \frac{1}{\sqrt{4t}}}{\sqrt{t} e^{x^2/(4t)} \frac{x}{2t}} \\ &= \lim_{x\to\infty} \frac{x e^{x^2/(4t)} \frac{1}{\sqrt{4t}}}{\sqrt{t} e^{x^2/(4t)} \frac{x}{2t}} = 1, \end{aligned}$$

which implies (6.17a). The proof of (6.17b) is similar. By (6.17), we have (6.18). □

An interesting observation of the singular solution (6.15) is the following

Corollary 6.8. *One can write the singular solution* (6.15) *as*

$$u(x,t) = \left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\right) \left(\sqrt{4\pi} \int_0^{x/\sqrt{4t}} e^{z^2}\, dz + 1\right) := \Phi(x,t) \cdot \Psi(x,t),$$

where $\Phi(x,t)$ *is the **fundamental solution** of the heat equation and* $\Psi(x,t)$ *is a **solution of the backward heat equation**, i.e.,*

$$\Psi_t(x,t) + \Psi_{xx}(x,t) = 0, \quad (x,t) \in (0,\infty) \times (0,\infty).$$

Remark 6.9. A solution $H(x,t)$ of the heat equation times a solution $B(x,t)$ of the backward heat equation will be a solution of the heat equation if and only if the function $H(x,t)B_x(x,t)$ is independent of x.

6.2. Self-similar solutions

Another interesting property we have discovered is that (6.15) is a **self-similar solution** of the heat equation on $(0,\infty)\times(0,\infty)$. We recall the following definition for a self-similar solution of the heat equation on $\mathbb{R}^n$.

Definition 6.10. Let $u(x,t)$ be a solution of the heat equation on $\mathbb{R}^n\times(0,\infty)$. Then for any constant $k>0$ the function $k^n u(kx,k^2t)$ is also a solution of the heat equation on $\mathbb{R}^n\times(0,\infty)$. We say $u(x,t)$ is a **self-similar solution** of the heat equation $u_t=\triangle u$ on $\mathbb{R}^n\times(0,\infty)$ if it satisfies

$$u(x,t)=k^n u(kx,k^2t) \tag{6.19}$$

for all $(x,t)\in\mathbb{R}^n\times(0,\infty)$ and all constant $k>0$.

Remark 6.11. In the one-dimensional case, by differentiating (6.19) with respect to k and let $k=1$, a self-similar solution of the heat equation $u_t=u_{xx}$ on $\mathbb{R}\times(0,\infty)$ will satisfy the identity

$$u(x,t)+xu_x(x,t)+2tu_t(x,t)=0,\quad \forall\,(x,t)\in\mathbb{R}\times(0,\infty).$$

For $n=1$, the above definition still makes sense if we confine the domain of $u(x,t)$ to $(0,\infty)\times(0,\infty)$. Therefore, the solution (6.15) is indeed a self-similar solution of the heat equation on $(0,\infty)\times(0,\infty)$.

The following result says that (6.15) is essentially the only self-similar solution on $(0,\infty)\times(0,\infty)$.

Lemma 6.12 (Self-similar solution)**.** *Assume $u(x,t)$ is a **self-similar solution** of the heat equation on $(0,\infty)\times(0,\infty)$. Then it has the form*

$$u(x,t)=C_1\left(\frac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}}\right)\int_0^{x/\sqrt{4t}}e^{z^2}\,dz+C_2\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\right),\quad (x,t)\in(0,\infty)\times(0,\infty) \tag{6.20}$$

for some constants C_1, C_2. In particular, when $C_1=1$ and $C_2=1$, it is the solution of the singular initial-boundary value problem (6.14)*.*

Proof. Clearly, if $u(x,t)$ satisfies (6.19) on $(0,\infty)\times(0,\infty)$, then, by taking $k=1/\sqrt{t}$, it must have the form

$$u(x,t)=\frac{1}{\sqrt{t}}v\left(\frac{x}{\sqrt{t}}\right),\quad (x,t)\in(0,\infty)\times(0,\infty) \tag{6.21}$$

for some single-variable function $v(z)$ defined on $(0,\infty)$. Hence we can just focus on (6.21). Since $u(x,t)$ is a solution of the heat equation on $(0,\infty)\times(0,\infty)$, $v(z)$ must satisfy the equation

$$v''(z)+\frac{1}{2}zv'(z)+\frac{1}{2}v(z)=\frac{d}{dz}\left(v'(z)+\frac{1}{2}zv(z)\right)=0,\quad \forall\,z\in(0,\infty),$$

and so there exists a constant c_1 such that $v'(z) + zv(z)/2 = c_1$ for all $z \in (0, \infty)$, which gives the general solution

$$v(z) = \left(c_1 \int_0^z e^{\theta^2/4}\, d\theta + c_2 \right) e^{-\frac{z^2}{4}}, \quad z = \frac{x}{\sqrt{t}} \in (0, \infty)$$

for some constants c_1, c_2. Plugging the above $v(z)$ into (6.21) will give us the formula (6.20). The proof is done. □

To end this section, we note the following interesting **improper integral representations** for the two self-similar solutions in (6.20).

Lemma 6.13. *We have the identities*

$$\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} = \frac{1}{\pi} \int_0^\infty e^{-ty^2} \cos(xy)\, dy, \quad \forall\, x \in (-\infty, \infty),\ t \in (0, \infty) \tag{6.22}$$

and

$$\left(\frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \right) \int_0^{x/\sqrt{4t}} e^{z^2}\, dz = \int_0^\infty e^{-ty^2} \sin(xy)\, dy, \quad \forall\, x \in (-\infty, \infty),\ t \in (0, \infty). \tag{6.23}$$

Remark 6.14. By (6.22), (6.23) and (6.12), we have the following two interesting limits which are not so obvious:

$$\lim_{t \to 0^+} \frac{1}{\pi} \int_0^\infty e^{-ty^2} \cos(xy)\, dy = 0, \quad \lim_{t \to 0^+} \int_0^\infty e^{-ty^2} \sin(xy)\, dy = \frac{1}{x} \quad \text{for fixed } x \in (0, \infty).$$

The remaining limits (as $t \to \infty$, $x \to 0^+$, and $x \to \infty$) are obvious due to the LDCT and Riemann–Lebesgue Lemma.

Proof of Lemma 6.13. For (6.22), we first consider the function

$$F(x) = \int_0^\infty e^{-y^2} \cos(xy)\, dy, \ \ x \in (-\infty, \infty), \quad F(0) = \frac{\sqrt{\pi}}{2}$$

with

$$F'(x) = -\int_0^\infty y e^{-y^2} \sin(xy)\, dy, \ \ x \in (-\infty, \infty), \quad F'(0) = 0.$$

For $x \neq 0$, integration by parts gives

$$F(x) = \frac{2}{x} \int_0^\infty y e^{-y^2} \sin(xy)\, dy, \quad x \in (-\infty, \infty),\ x \neq 0,$$

which yields the ODE for $F(x)$ on $x \in (-\infty, \infty)$:

$$F'(x) + \frac{x}{2} F(x) = 0, \ \ x \in (-\infty, \infty), \quad F(0) = \frac{\sqrt{\pi}}{2}$$

and its unique solution is given by

$$F(x)=\int_0^\infty e^{-y^2}\cos(xy)\,dy=\frac{\sqrt{\pi}}{2}e^{-\frac{x^2}{4}},\quad x\in(-\infty,\infty).$$

After a change of variables, we can obtain

$$\frac{1}{\pi}\int_0^\infty e^{-ty^2}\cos(xy)\,dy=\frac{1}{\pi}\int_0^\infty e^{-z^2}\cos\left(\frac{x}{\sqrt{t}}z\right)\frac{1}{\sqrt{t}}\,dz=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$$

for all $x\in(-\infty,\infty)$ and $t\in(0,\infty)$.

For the proof of (6.23), we let

$$G(x)=\int_0^\infty e^{-y^2}\sin(xy)\,dy,\ \ x\in(-\infty,\infty),\quad G(0)=0$$

with

$$G'(x)=\int_0^\infty ye^{-y^2}\cos(xy)\,dy,\ \ x\in(-\infty,\infty),\quad G'(0)=\frac{1}{2}$$

and by similar computations to the above, we obtain the ODE

$$G'(x)+\frac{x}{2}G(x)=\frac{1}{2},\ \ \forall\, x\in(-\infty,\infty),\quad G(0)=0,$$

with unique solution

$$G(x)=\int_0^\infty e^{-y^2}\sin(xy)\,dy=\frac{1}{2}e^{-\frac{x^2}{4}}\int_0^x e^{z^2/4}\,dz,\quad x\in(-\infty,\infty).$$

After a change of variables, we can obtain

$$\begin{aligned}\int_0^\infty e^{-ty^2}\sin(xy)\,dy&=\frac{1}{\sqrt{t}}\int_0^\infty e^{-z^2}\sin\left(\frac{x}{\sqrt{t}}z\right)dz\\&=\frac{1}{\sqrt{t}}G\left(\frac{x}{\sqrt{t}}\right)=\left(\frac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}}\right)\int_0^{x/\sqrt{4t}}e^{z^2}\,dz\end{aligned}$$

for all $x\in(-\infty,\infty)$ and $t\in(0,\infty)$. The proof is done. □

7. Generalization of the ibvp (1.1) to a linear equation with constant coefficients

It is not difficult to generalize the problem (1.1) to a linear equation with constant coefficients, namely

$$\text{(7.1)}\qquad\begin{cases}u_t(x,t)=u_{xx}(x,t)+Au_x(x,t)+Bu(x,t), & (x,t)\in(0,\infty)\times(0,\infty),\\ u(x,0)=h(x), & x\in(0,\infty),\\ u(0,t)=g(t), & t\in(0,\infty),\end{cases}$$

where A, B are given constants and $h(x)$, $g(t)$ are given continuous functions on $(0,\infty)$ satisfying the growth condition (1.2).

For the linear equation in (7.1) on the entire space $(x,t)\in\mathbb{R}^2$, it is closely related to the standard heat equation on $\mathbb{R}^2$. More precisely, we have

Lemma 7.1. *Let A, B be any two constants. If $u(x,t)$ is a solution of the linear equation $u_t = u_{xx} + Au_x + Bu$ on $\mathbb{R}^2$, then the two functions*

$$v(x,t) = e^{-Bt}u(x - At, t), \quad w(x,t) = e^{\frac{A}{2}x + \frac{1}{4}(A^2 - 4B)t}u(x,t), \quad (x,t) \in \mathbb{R}^2 \tag{7.2}$$

are both solutions to the ***heat equation*** *on* $\mathbb{R}^2$.

Proof. This is a straightforward verification. □

Using the second identity in (7.2), we see that $u(x,t)$ is a solution of the problem (7.1) on $(0,\infty) \times (0,\infty)$ if and only if $w(x,t)$ in (7.2) is a solution of the problem

$$\begin{cases} w_t(x,t) = w_{xx}(x,t), & (x,t) \in (0,\infty) \times (0,\infty), \\ w(x,0) = e^{\frac{A}{2}x}h(x), & x \in (0,\infty), \\ w(0,t) = e^{\frac{1}{4}(A^2-4B)t}g(t), & t \in (0,\infty). \end{cases} \tag{7.3}$$

Therefore, to solve (7.1) one can solve the problem (7.3) first and then obtain $u(x,t)$ by the second identity in (7.2). By (1.3) (or (2.1)), we can obtain a **solution formula** for $u(x,t)$ of the initial-boundary value problem (7.1). The details are left to the readers.

8. Oblique initial-boundary value problem for the heat equation

Next, we will use the first identity in (7.2) and consider the function $v(x,t) = e^{-Bt}u(x - At, t)$, where we assume $u(x,t)$ is a solution of the problem (7.1) on $(0,\infty) \times (0,\infty)$. Now $v(x,t)$ is defined on the domain $D = \{x > At\} \cap \{t > 0\} \subset \mathbb{R}^2$ and it satisfies the following **oblique initial-boundary value problem**:

$$\begin{cases} v_t(x,t) = v_{xx}(x,t), & (x,t) \in \{x > At\} \cap \{t > 0\} \\ v(x,0) = u(x,0) = h(x), & x \in (0,\infty) \\ v(At,t) = e^{-Bt}u(0,t) = e^{-Bt}g(t), & t \in (0,\infty), \end{cases} \tag{8.1}$$

where A and B in (8.1) are two arbitrary given constants. For given continuous functions $h(x)$, $g(t)$ on $(0,\infty)$, the problem (8.1) has a solution formula due to the relation between $v(x,t)$ and $w(x,t)$ in (7.2), given by

$$v(x,t) = e^{-Bt}\left[e^{-\frac{A}{2}(x-At) - \frac{1}{4}(A^2-4B)t}w(x - At, t)\right].$$

Since there is a solution formula for $w(x,t)$ in the problem (7.3), there is a solution formula for $v(x,t)$ in (8.1). If we use the formula (2.1) for $w(x,t)$, we can obtain

Lemma 8.1. *Let A, B be any two constants and $h(x)$, $g(t)$ are given continuous functions on $(0,\infty)$ satisfying the growth condition* (1.2)*. Then the function*

$$\begin{aligned} v(x,t) &= e^{-Bt}u(x-At,t) = e^{-Bt}\left[e^{-\frac{A}{2}(x-At)-\frac{1}{4}(A^2-4B)t}w(x-At,t)\right] \\ &= e^{-\frac{A}{2}(x-At)-\frac{A^2}{4}t}\cdot\left(\frac{1}{\sqrt{\pi}}\int_{-(x-At)/\sqrt{4t}}^{\infty} e^{-z^2}e^{\frac{A}{2}((x-At)+\sqrt{4t}z)}h((x-At)+\sqrt{4t}z)\,dz\right. \\ &\quad -\frac{1}{\sqrt{\pi}}\int_{(x-At)/\sqrt{4t}}^{\infty} e^{-z^2}e^{\frac{A}{2}(-(x-At)+\sqrt{4t}z)}h(-(x-At)+\sqrt{4t}z)\,dz \\ &\quad \left.+\frac{2}{\sqrt{\pi}}\int_{(x-At)/\sqrt{4t}}^{\infty} e^{-z^2}e^{\frac{1}{4}(A^2-4B)\left(t-\left(\frac{x-At}{2z}\right)^2\right)}g\left(t-\left(\frac{x-At}{2z}\right)^2\right)dz,\right) \end{aligned}$$

where $(x,t)\in D=\{x>At\}\cap\{t>0\}$, gives a solution of the ***oblique initial-boundary value problem*** (8.1) *on D and it lies in the space*

$$u(x,t)\in C^\infty(D)\cap C^0(\overline{D}\setminus\{(0,0)\}).$$

Moreover, if we have $h(0)=g(0)$, then $u(x,t)\in C^\infty(D)\cap C^0(\overline{D})$.

Proof. This is a direct consequence of the known properties stated in the Introduction section, together with Lemmas 2.1 and 7.1. □

Remark 8.2. Note that the constant A in Lemma 8.1 can be either positive or negative ($A=0$ is trivial and we ignore it). For $A>0$, the domain $D\subset(0,\infty)\times(0,\infty)$ and for $A<0$, the domain $D\supset(0,\infty)\times(0,\infty)$.

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