On Symmetric Plane Quartic Curves

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Abstract. In the present paper, we study the geometry of plane quartics with large automorphism groups. We show results devoted to smooth plane quartics that are invariant under the action of the elementary Abelian group of type $[2, 2, 2]$. Then we study geometric properties of the smooth plane quartic having automorphism group of order 48.

1. Introduction

In the present paper, we study some geometric and combinatorial properties of irreducible quartic curves $C \subset \mathbb{P}^2_{\mathbb{C}}$. Following the lines of $[1,7]$, we say that an irreducible and reduced curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ of genus $g \geq 2$ has **large automorphism group** if

$$\# \operatorname{Aut}(C) > 4(g - 1).$$

In the case of irreducible plane quartics, the above condition means that $\# \operatorname{Aut}(C) > 8$. It is worth recalling that smooth plane quartic curves in $\mathbb{P}^2_{\mathbb{C}}$ with $\# \operatorname{Aut}(C) > 8$ were classified in $[1]$ with respect to the order of automorphism groups, namely

- The Klein quartic $C_{168} : x^3 y + y^3 z + z^3 x = 0$ with $\# \operatorname{Aut}(C_{168}) = 168$,

- the Dyck quartic $C_{96} : x^4 + y^4 + z^4 = 0$ with $\# \operatorname{Aut}(C_{96}) = 96$,

- quartic $C_{48} : x^4 + y^4 + xz^3 = 0$ with $\# \operatorname{Aut}(C_{48}) = 48$,

- quartics in one-parameter family $C_{24,a} : x^4 + y^4 + z^4 + 3a(x^2y^2 + y^2z^2 + y^2z^2) = 0$ with $\# \operatorname{Aut}(C_{24,a}) = 24$, where $a \notin \{0, (-1 \pm \sqrt{-7})/2\}$,

- quartics in one parameter family $C_{16,\delta} : x^4 + y^4 + z^4 + \delta z^2 y^2 = 0$ with $\# \operatorname{Aut}(C_{16,\delta}) = 16$, where $\delta \notin \{0, \pm 2, \pm 6, \pm 2\sqrt{-3}\}$,

- quartic $C_9 : z^4 + zy^3 + yx^3 = 0$ with $\# \operatorname{Aut}(C_9) = 9$.

In the joint work with Pokora and Zieliński [6] we studied combinatorial properties of line arrangements determined by bitangents to smooth plane quartics having large automorphism groups and, in particular, we described the weak combinatorics of such arrangements. It turns out that the arrangements consisting of smooth plane quartics and their bitangents allow to construct non-trivial free and plus-one generated arrangements. In the light of these results, we focus on the geometry of the smooth plane quartic C_{48} and we find effectively all the 28 bitangent lines, the weak combinatorics of the arrangement of bitangents, and then we construct new examples of free plane curve arrangements. In this way, we fill the gap in our knowledge about of arrangements consisting of bitangents to symmetric plane quartics. The second result of the paper is devoted to plane quartics such that they are invariant under the action of $G_8 = \mathbb{Z}_2^3$ and such that their Hessian is very symmetric. In that setting, our result tells us that if C is an irreducible plane quartic which is G_8-invariant having Hessian of the form $x^2y^2z^2$, then C is given by $Ax^4 + By^4 + Cz^4$ with $ABC = 1/1728$.

The structure of the paper goes as follows. In Section 2, we present our classification result for G_8-invariant irreducible plane quartics with a prescribed Hessian. In Section 3, we study the geometry of the plane quartic C_{48} and its 28 bitangents. In particular, we provide a classification result on free arrangements consisting of C_{48} and its bitangents.

We work only over the complex numbers, and our computations are supported by SINGULAR [2].

2. G_8-invariant plane quartic curves

Here we want to study plane quartics that are invariant under the action of the elementary Abelian group of type $[2, 2, 2]$ of order 8, and we denote this group by G_8. In our studies we will use the following faithful matrix representation generated in $\mathrm{GL}(3, \mathbb{Z})$ by the three matrices:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Recall that if $\rho\colon G \to \mathrm{GL}(n, \mathbb{C})$ is a faithful representation of a finite group G, then we have the following standard action on the graded ring of polynomials $S := \mathbb{C}[x, y, z]$, namely

$$G \times S \ni (g, f) \mapsto f(\rho(g)^{-1} \cdot (x, y, z)^t) \in S.$$

Since our representation of G_8 is generated by the diagonal matrices and for every $g \in G_8$ one has $\rho(g)^{-1} = \rho(g)$, the action on S can be described as

$$G_8 \times S \ni (g, f) = f(ax, by, cz) \in S,$$

where $\rho(g) = \mathrm{Diag}(a, b, c)$ with $a, b, c \in \{-1, 1\}$. Our first observation is

Proposition 2.1. *One has* $\mathbb{C}[x, y, z]^{G_8} = \mathbb{C}[x^2, y^2, z^2]$.

Proof. See for example [9, p. 57]. $\hspace{6cm}$ $\square$

From now on, we study only these reduced plane quartics C that are G_8-invariant. Such a quartic has the following defining polynomial

$$(2.1) \qquad Q(x, y, z) = Ax^4 + By^4 + Cz^4 + Dx^2y^2 + Ex^2z^2 + Fy^2z^2$$

with $A, B, C, D, E, F \in \mathbb{C}$. In our further investigations, we are going to work with the Hessians associated with plane quartics. In that context, we have the following crucial folkloric result, see [8, p. 115].

Proposition 2.2. *Let* $\rho \colon G \to \mathrm{GL}(n, \mathbb{C})$ *be a faithful representation of a finite group* G. *Let* $f \in \mathbb{C}[x_1, \ldots, x_n]^G$ *and assume that for every* $g \in G$ *one has* $(\det(\rho(g)))^2 = 1$, *then the Hessian of* f *defined as*

$$\mathrm{Hess}(f) = \det \left[\frac{\partial^2 f}{\partial_{x_i} \partial_{x_j}} \right]_{i,j}$$

is G-*invariant.*

In the light of the above result, if we have a G_8-invariant quartic $C : f = 0$, then its associated Hessian curve $C_H : \mathrm{Hess}(f) = 0$ is also G_8-invariant.

Next, for a general G_8-invariant quartic given by (2.1) we compute its Hessian, namely

$$\begin{aligned}
\mathrm{Hess}(Q) = {}& 48ADEx^6 + 48BDFy^6 + 48CEFz^6 \\
& + (288ABE + 48ADF - 24D^2E)x^4y^2 + (288ABF + 48BDE - 24D^2F)x^2y^4 \\
& + (288BCE + 48CDF - 24EF^2)y^2z^4 + (288BCD + 48BEF - 24DF^2)y^4z^2 \\
& + (288ACD + 48AEF - 24DE^2)x^4z^2 + (288ACF + 48CDE - 24E^2F)x^2z^4 \\
& + (1728ABC - 144AF^2 - 144BE^2 - 144CD^2 + 144DEF)x^2y^2z^2.
\end{aligned}$$

Now we are ready to present our result devoted to G_8-invariant quartics having a very special Hessian.

Theorem 2.3. *Let* $C : f = 0$ *be an irreducible* G_8-*invariant quartic in* $\mathbb{P}^2_{\mathbb{C}}$ *such that* $\mathrm{Hess}(f) = x^2y^2z^2$. *Then* C *is given by the equation of the form*

$$Q(x, y, z) = Ax^4 + By^4 + Cz^4,$$

where A, B, C *such that* $ABC = 1/1728$.

Proof. Let C be a G_8-invariant quartic given by (2.1). Our problem boils down to solving the following system of equations

(2.2) $\mathrm{Hess}(Q) = x^2 y^2 z^2.$

The equation (2.2) leads us to the following system of equations:

$$\begin{cases} 48ADE = 0, \quad 48BDF = 0, \quad 48CEF = 0, \\ 288ABE + 48ADF - 24D^2E = 0, \\ 288ABF + 48BDE - 24D^2F = 0, \\ 288BCE + 48CDF - 24EF^2 = 0, \\ 288BCD + 48BEF - 24DF^2 = 0, \\ 288ACD + 48AEF - 24DE^2 = 0, \\ 288ACF + 48CDE - 24E^2F = 0, \\ 1728ABC - 144AF^2 - 144BE^2 - 144CD^2 + 144DEF = 1. \end{cases}$$

Using the `SINGULAR` script from Appendix, we can verify that the above system of equations has the following solutions:

$$\begin{cases} D = 0, \\ E = 0, \\ F = 0, \\ 1728ABC = 1, \end{cases} \quad \begin{cases} A = 0, \\ B = 0, \\ E = 0, \\ F = 0, \\ 144CD^2 = -1, \end{cases} \quad \begin{cases} A = 0, \\ C = 0, \\ D = 0, \\ F = 0, \\ 144BE^2 = -1, \end{cases} \quad \begin{cases} B = 0, \\ C = 0, \\ D = 0, \\ E = 0, \\ 144AF^2 = -1. \end{cases}$$

In order to finish our proof, we need to observe that the last three solutions lead to reducible plane quartics. Indeed, if $A = B = E = F = 0$ and $144CD^2 = -1$, then we get

$$Q(x, y, z) = \frac{-z^4}{144D^2} + Dx^2y^2 = \frac{-1}{144D^2}\left(z^2 + 12\sqrt{D^3}xy\right)\left(z^2 - 12\sqrt{D^3}xy\right).$$

In the same way, we can show that in the last two remaining cases our quartics are also reducible, and this completes the proof.   □

Remark 2.4. Observe that the quartic $C : Ax^4 + By^4 + Cz^4 = 0$ is projectively equivalent to the Dyck quartic C_{96}, we need to use the following obvious map:

$$(x, y, z) \mapsto (ax, by, cz),$$

where $a^4 = 1/A$, $b^4 = 1/B$, $c^4 = 1/C$.

3. On the geometry of the smooth plane quartic with automorphism group of order 48

In this section we want to focus on a very particular symmetric smooth plane quartic $C_{48} \subset \mathbb{P}^2_{\mathbb{C}}$ with the automorphism group of order 48 that is given by

$$F(x, y, z) = x^4 + y^4 + xz^3.$$

As it is explained in [1], our quartic curve C_{48} is unique, and it has the automorphism group $\mathrm{Aut}(C_{48}) = C_4 \odot A_4$, i.e., this is the central extension by C_4 of A_4, and this group has the following presentation:

$$\mathrm{Aut}(C_{48}) = \langle a,\, b,\, c,\, d \mid a^4 = d^3 = 1, b^2 = c^2 = a^2, ab = ba, ac = ca, ad = da,$$
$$cbc^{-1} = a^2 b, dbd^{-1} = a^2 bc, dcd^{-1} = b \rangle.$$

Our aim here is to understand the geometry of the 28 bitangents to the quartic C_{48}, and in order to do so we will use a geometric description provided by Wall in [10]. First of all, we have the following important description of singular points of the curve dual to the smooth plane quartic.

Proposition 3.1. [10, Proposition in Section 1] *Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be an irreducible quartic curve and let $C^\vee$ be the dual curve to C. Then $C^\vee$ has the same list of singularities as C, except as follows. The nodes, ordinary cusps and triple points of C do not contribute to singularities of $C^\vee$. Conversely, $C^\vee$ may have nodes, ordinary cusps and singularities of type E_6 not arising from singularities of C.*

From this perspective, one can ask what we can learn about the geometry of smooth plane quartics by looking at the singularities of the dual curve of degree 12. We focus here on the situation when the singularities of the dual curves are E_6 singularities. For such a description, let Γ be a curve germ at P with a unique tangent line $P^\vee$, $\Gamma^\vee$ the dual germ at $P^\vee$ with tangent P, and denote by $i(\Gamma)$ the local intersection number of Γ and $P^\vee$.

Proposition 3.2. [10] *For a smooth plane quartic curve $C \subset \mathbb{P}^2_{\mathbb{C}}$, a E_6 singularity of $C^\vee$ dualizes to a smooth germ with a hyperflex tangent, i.e., the local intersection number $i(\Gamma) = 4$, and vice versa.*

In other words, if the dual curve $C^\vee$ of a smooth plane quartic C has exactly k singularities of type E_6, then C admits exactly k hyperosculating lines, i.e., lines tangent to C with $i(\Gamma) = 4$. Furthermore, such a tangent point, considered as a singularity created by C and the associated hyperosculating line, is a A_7 singularity.

Let us come back to C_{48}. The dual curve to C_{48}, which will be denoted by $C_{48}^{\vee}$, is given by the following polynomial

$$G(x, y, z) = x^8 y^4 + 2x^4 y^8 + y^{12} + \frac{256}{27} x^9 z^3 + 16x^5 y^4 z^3$$

$$+ 16xy^8 z^3 - \frac{256}{9} x^6 z^6 + 64x^2 y^4 z^6 + \frac{256}{9} x^3 z^9 - \frac{256}{27} z^{12}.$$

As we can check directly, using `SINGULAR`, the curve $C_{48}^{\vee}$ has exactly 16 ordinary cusps, 24 nodes, and 4 singularities of type E_6. In the light of the above propositions, our curve C_{48} admits exactly 4 hyperosculating lines. It is worth recalling here that hyperosculating lines are considered as (degenerate) bitangent lines, so altogether we have 24 (classical) bitangent lines and 4 hyperosculating lines to C_{48}.

Here we present equations of the 28 bitangents to C_{48}, namely

$\ell_1 : 468x + (16r^{13} + 78r^9 + 546r^5 - 990r)y + (3r^{12} - 639)z = 0,$

$\ell_2 : 468x - (16r^{13} + 78r^9 + 546r^5 - 990r)y + (3r^{12} - 639)z = 0,$

$\ell_3 : 468x + (50r^{13} + 312r^9 + 2028r^5 - 198r)y + (3r^{12} - 639)z = 0,$

$\ell_4 : 468x - (50r^{13} + 312r^9 + 2028r^5 - 198r)y + (3r^{12} - 639)z = 0,$

$\ell_5 : 468x + (16r^{13} + 78r^9 + 546r^5 - 990r)y + (30r^{12} + 195r^8 + 1287r^4 + 45)z = 0,$

$\ell_6 : 468x + (16r^{13} + 78r^9 + 546r^5 - 990r)y + (-33r^{12} - 195r^8 - 1287r^4 + 594)z = 0,$

$\ell_7 : 468x + (50r^{13} + 312r^9 + 2028r^5 - 198r)y + (30r^{12} + 195r^8 + 1287r^4 + 45)z = 0,$

$\ell_8 : 468x + (-16r^{13} - 78r^9 - 546r^5 + 990r)y + (-33r^{12} - 195r^8 - 1287r^4 + 594)z = 0,$

$\ell_9 : 468x + (-16r^{13} - 78r^9 - 546r^5 + 990r)y + (30r^{12} + 195r^8 + 1287r^4 + 45)z = 0,$

$\ell_{10} : 468x + (50r^{13} + 312r^9 + 2028r^5 - 198r)y + (-33r^{12} - 195r^8 - 1287r^4 + 594)z = 0,$

$\ell_{11} : 468x + (-50r^{13} - 312r^9 - 2028r^5 + 198r)y + (-33r^{12} - 195r^8 - 1287r^4 + 594)z = 0,$

$\ell_{12} : 468x + (-50r^{13} - 312r^9 - 2028r^5 + 198r)y + (30r^{12} + 195r^8 + 1287r^4 + 45)z = 0,$

$\ell_{13} : 468x + (10r^{13} + 78r^9 + 546r^5 + 288r)y + (-3r^{12} + 171)z = 0,$

$\ell_{14} : 468x + (-10r^{13} - 78r^9 - 546r^5 - 288r)y + (-3r^{12} + 171)z = 0,$

$\ell_{15} : 468x + (-10r^{13} - 78r^9 - 546r^5 - 288r)y + (9r^{12} + 39r^8 + 351r^4 - 162)z = 0,$

$\ell_{16} : 468x + (-10r^{13} - 78r^9 - 546r^5 - 288r)y + (-6r^{12} - 39r^8 - 351r^4 - 9)z = 0,$

$\ell_{17} : 468x + (10r^{13} + 78r^9 + 546r^5 + 288r)y + (9r^{12} + 39r^8 + 351r^4 - 162)z = 0,$

$\ell_{18} : 468x + (10r^{13} + 78r^9 + 546r^5 + 288r)y + (-6r^{12} - 39r^8 - 351r^4 - 9)z = 0,$

$\ell_{19} : 156x + (-8r^{13} - 52r^9 - 312r^5 + 144r)y + (-r^{12} + 57)z = 0,$

$\ell_{20} : 156x + (8r^{13} + 52r^9 + 312r^5 - 144r)y + (-r^{12} + 57)z = 0,$

$\ell_{21} : 156x + (-8r^{13} - 52r^9 - 312r^5 + 144r)y + (-2r^{12} - 13r^8 - 117r^4 - 3)z = 0,$

$\ell_{22} : 156x + (-8r^{13} - 52r^9 - 312r^5 + 144r)y + (3r^{12} + 13r^8 + 117r^4 - 54)z = 0,$

$\ell_{23} : 156x + (8r^{13} + 52r^9 + 312r^5 - 144r)y + (-2r^{12} - 13r^8 - 117r^4 - 3)z = 0,$

$\ell_{24} : 156x + (8r^{13} + 52r^9 + 312r^5 - 144r)y + (3r^{12} + 13r^8 + 117r^4 - 54)z = 0,$

$\ell_{25} : 39x + (2r^{12} + 13r^8 + 78r^4 - 36)z = 0,$

$\ell_{26} : 39x - (2r^{12} + 13r^8 + 78r^4 + 3)z = 0,$

$\ell_{27} : x = 0,$

$\ell_{28} : x + z = 0,$

where $r^{16} + 6r^{12} + 39r^8 - 18r^4 + 9 = 0$. Let us notice that the lines $\ell_{25}, \dots, \ell_{28}$ are hyperosculating.

In order to formulate our result devoted to the weak-combinatorics of the 28 bitangents to C_{48} lines, we need the following notation. For a given line arrangement $\mathcal{L}$, let us denote by $n_i = n_i(\mathcal{L})$ the number of i-fold intersection points among the lines in $\mathcal{L}$.

Proposition 3.3. *The arrangement $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ consisting of the 28 bitangents to C_{48} has the following intersections*

$$n_2 = 240, \quad n_3 = 32, \quad n_4 = 7.$$

Proof. This is a standard check that can be done using SINGULAR. Let us denote by f be the defining equation of the arrangement $\mathcal{L}$, and denote by $J_f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$ the Jacobian ideal. Recall that

$$\deg(J_f) = \tau(\mathcal{L}) = n_2 + 4n_3 + 9n_4,$$

where $\tau(\mathcal{L})$ denotes the total Tjurina number of $\mathcal{L}$ (see the precise definition below). Using SINGULAR, we can check that $\deg(J_f) = 431$. Since

(3.1) $$\binom{28}{2} = n_2 + 3n_3 + 6n_4$$

we obtain

$$53 = n_3 + 3n_4.$$

Define by T_f the ideal generated by all the partial derivatives of order three. Using SINGULAR we can check that $\deg(T_f) = 7$, which gives us that

$$n_3 = 53 - 3n_4 = 32.$$

Finally, using (3.1) we get $n_2 = 240$, and this completes the proof. $\qquad \square$

Now we would like to detect some free arrangements that are determined by the quartic C_{48} and its bitangents. We need to recall some fundamental definitions.

Let $S := \mathbb{C}[x, y, z]$ be the graded ring of polynomials with complex coefficients, and for a homogeneous polynomial $f \in S$ let J_f be the Jacobian ideal given by f.

Definition 3.4. Consider the graded S-module of Jacobian syzygies of f, namely

$$AR(f) = \{(a, b, c) \in S^3 : af_x + bf_y + cf_z = 0\}.$$

The minimal degree of non-trivial Jacobian relations for f is defined as

$$\mathrm{mdr}(f) := \min\{r : AR(f)_r \neq (0)\}.$$

Recall that for a reduced plane curve $C : f = 0$ we denote by $\tau(C)$ its total Tjurina number, i.e.,

$$\tau(C) = \sum_{p \in \mathrm{Sing}(C)} \tau_p,$$

the sum goes over all singular points of C and τ_p denotes the local Tjurina number. Now we can define free plane curves using a result due to du Plessis and Wall [5].

Definition 3.5. Let $C : f = 0$ be a reduced curve in $\mathbb{P}_{\mathbb{C}}^2$ of degree d. Then the curve C with $r := \mathrm{mdr}(f) \leq (d-1)/2$ is free if and only if

(3.2) $(d-1)^2 - r(d-r-1) = \tau(C).$

Definition 3.6. If $C : f = 0$ is a reduced free plane curve of degree d in $\mathbb{P}_{\mathbb{C}}^2$, then the exponents of C is the pair defined as

$$\exp(C) = (\mathrm{mdr}(f), d - 1 - \mathrm{mdr}(f)).$$

Observe that our arrangements $\mathcal{EL}$ consisting of the quartic curve C_{48} and its bitangents admit only n_2 nodes, n_3 ordinary triple points, n_4 ordinary quadruple points, t_3 tacnodes, and t_7 singularities of type A_7, and the total Tjurina number of $\mathcal{EL}$ is equal to

$$\tau(\mathcal{EL}) = n_2 + 3t_3 + 4n_3 + 7t_7 + 9n_4.$$

Theorem 3.7. *The quartic curve C_{48} and its bitangents admits exactly one free arrangement of degree 8, namely the arrangement admits one ordinary quadruple point and four singularities of type A_7.*

Proof. Assume that $\mathcal{EL}$ given by $f = 0$ is a free arrangement consisting of the quartic C_{48} and 4 bitangents. We want to find some numerical constraints on such free arrangements. First of all, recall that the arrangement $\mathcal{EL}$ has to satisfy the following system of Diophantine equations:

(3.3) $\begin{cases} n_2 + 2t_3 + 3n_3 + 4t_7 + 6n_4 = 4 \cdot 4 + \binom{4}{2} = 22, \\ n_2 + 3t_3 + 4n_3 + 7t_7 + 9n_4 = \tau(\mathcal{EL}), \end{cases}$

wherein $\tau(\mathcal{EL}) \in \{37, 39\}$. Indeed, the first equation follows from Bézout's theorem, and for the second equation we need to observe that by [4, Theorem 2.1] and the fact that $\mathrm{mdr}(f) \leq (d-1)/2$ one has

$$\frac{7}{2} \geq \mathrm{mdr}(f) \geq \frac{1}{2} \cdot 8 - 2 = 2,$$

which implies that $\mathrm{mdr}(f) \in \{2, 3\}$. Then, using (3.2), we get $\tau(\mathcal{EL}) \in \{37, 39\}$. It turns out that for $\tau(\mathcal{EL}) = 39$ our system (3.3) does not have any non-negative integer solution, and for $\tau(\mathcal{EL}) = 37$ we have exactly three solutions, namely

$$(n_2, n_3, n_4, t_3, t_7) \in \{(0,0,0,3,4), (2,0,0,0,5), (0,0,1,0,4)\}.$$

Observe that the first two weak-combinatorics cannot be realized geometrically using bitangents and quartic C_{48}, i.e., the first one cannot be realized due to Bézout's theorem (too many intersections), and the second combinatorics cannot be realized since having 4 bitangents we can produce at most 4 singularities of type A_7. The last combinatorics can be realized geometrically. Consider the following arrangement $\mathcal{EL}$ given by

$$Q(x, y, z) = (x^4 + y^4 + xz^3) \cdot \ell_{25} \cdot \ell_{26} \cdot \ell_{27} \cdot \ell_{28}.$$

We can check directly that the lines ℓ_{25}, ℓ_{26}, ℓ_{27}, ℓ_{28} are concurrent, which means that they intersect at a quadruple points. Moreover, each line ℓ_i is tangent to quartic C_{48} at exactly one point, so the local intersection index is equal to 4, and we have a singular point of type A_7. Summing up, we have $n_4 = 1$ and $t_7 = 4$. We can check using SINGULAR that $\mathrm{mdr}(Q) = 3$, and

$$37 = r^2 - r(d-1) + (d-1)^2 = \tau(\mathcal{EL}) = 7t_7 + 9n_4 = 37,$$

hence $\mathcal{EL}$ is free with exponents $(3, 4)$. Since in our arrangement of bitangents we have exactly 4 hyperosculating lines, our arrangement is unique (up to the labeling of lines). $\quad\square$

Remark 3.8. In light of [3, Theorem 1.1], if we delete one (and any) line from $\mathcal{EL}$, we get either a free or plus-one generated arrangement, and it turns out that we end up with the second scenario, namely we get a plus-one generated curve.

A. Appendix

Here we present our SINGULAR script that allows to verify the claim in Theorem 2.3.

```
option(redSB);
LIB "primdec.lib";
```

```
ring R = 0, (A,B,C,D,E,F), dp;
ideal si =
A*D*E,
B*D*F,
C*E*F,
12*A*B*E + 2*A*D*F - D2*E,
12*A*B*F + 2*B*D*E - D2*F,
12*B*C*E + 2*C*D*F - E*F2,
12*B*C*D + 2*B*E*F - D*F2,
12*A*C*D + 2*A*E*F - D*E2,
12*A*C*F + 2*C*D*E - E2*F,
1728*A*B*C - 144*A*F2 - 144*B*E2 - 144*C*D2 + 144*D*E*F - 1;
primdecGTZ(si);
```

Acknowledgments

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https://mat.uab.cat/~francesc/mates/autgen3.pdf.

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