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A Well-balanced and Positivity-preserving Godunov-type Numerical Scheme for a Spray Model

Dao Huy Cuong and Mai Duc Thanh*

Abstract. Computational exact Riemann solvers are constructed for a spray model. The model is characterized by a system of balance laws with nonconservative terms in the equations for conservations of momentum and energy. The proposed scheme is combined from an upwind scheme for the equation of the volume fraction evolution and a Godunov-type scheme for the governing equations. The numerical results show that the scheme provides us with the convergence and it possesses a good accuracy, even when resonant phenomena occur. It is interesting to see that the proposed scheme is well-balanced, and preserves the positivity of density, of volume fraction, and the C-property.

1. Introduction

In this paper, we build a well-balanced and positivity-preserving scheme for the numerical approximation of weak solutions to the Cauchy problem associated with the following spray model, which consists of the conservation law of mass, balance equation of momentum, balance equation of energy, and a convection equation:

$$
\begin{aligned}
&\partial_t(\alpha\rho) + \partial_x(\alpha\rho u) = 0,\\
&\partial_t(\alpha\rho u) + \partial_x(\alpha(\rho u^2 + p)) = p\partial_x\alpha,\\
&\partial_t(\alpha\rho E) + \partial_x(\alpha u(\rho E + p)) = pw\partial_x\alpha,\\
&\partial_t\alpha + w\partial_x\alpha = 0, \quad x \in \mathbb{R},\ t > 0,
\end{aligned}
\tag{1.1}
$$

where $\alpha(x,t)$, $\rho(x,t)$, $u(x,t)$, $p(x,t)$, $E(x,t)$ denote the volume fraction, density, velocity, pressure and total energy in the first phase of flow, respectively; and $w(x,t)$ denotes the given velocity in the second phase of flow. This model is based on the Baer–Nunziato model of two-phase flow, see [4, 6], in which the dynamics of the fluid in the first phase is the subject to be studied, and the fluid in the second phase is pumped into the first phase

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*Corresponding author.

with a speed w. The fluid in the first phase is assumed to be a stiffed gas, which has an equation of state of the form

$$p = (\gamma - 1)\rho\varepsilon - \gamma\pi,$$

where $\gamma, \pi > 0$ are constants. The total energy E defined by

$$E = \varepsilon + \frac{u^2}{2},$$

where ε is the internal energy.

The system (1.1) has two nonconservative terms on the right-hand side, so weak solutions of this model can be viewed in term of *nonconservative products*, see [15, 25]. Numerical approximation of systems of balance laws in nonconservative form has been a very challenging topic for many authors during the past decades.

Recently, the Riemann problem for a spray model with constant w has been considered in [36]. Motivated by that work, in order to solve the system (1.1) numerically, we first compute the volume fraction by an upwind scheme, then we employ the local Riemann solutions of (1.1) at interfaces $x_{j+1/2}$ corresponding to constant speeds $w^n_{j+1/2}$ to build up a Godunov-type scheme for computing the first phase. We will show that the proposed scheme can keep some families of steady solutions, which means it is well-balanced and preserve C-property. Moreover, we will prove the positivity of density and volume fraction preserving property of the proposed scheme in this paper.

Note that numerical schemes for a nonconservative single conservation law were presented in [3, 7, 8]. The Riemann problem for fluid flows in a nozzle with discontinuous cross-section was presented in [26]. The Riemann problem for two-phase flow models were constructed in [34, 35]. The Riemann problem for other nonconservative hyperbolic systems were considered in [19, 22–24, 29, 32, 33]. A Godunov-type scheme for the isentropic model of a fluid flow in a nozzle with variable cross-section was built in [11]. Godunov-type schemes for hyperbolic systems of balance laws in nonconservative forms were presented in [2, 28, 31]. A Godunov-type scheme for two-phase flow models were presented in [14]. Godunov-type schemes for multi-phase flow models and other hyperbolic systems of balance laws in nonconservative forms were studied in [2, 12, 28, 30]. Numerical schemes for two-phase flow models were built in [1, 5, 10, 38]. The Riemann problem and numerical schemes for shallow water equations were constructed in [9, 13, 16–18, 20, 21, 27, 28, 37]. See also the references therein.

The organization of this paper is as follows. Section 2 presents the basic properties of the system (1.1). Algorithms for computing the Riemann solution are revisited in Section 3. Then, Section 4 is devoted to the construction of a well-balanced and positivity of density preserving numerical scheme based on the exact solutions of local Riemann problems stated in the previous section. Numerical tests are given in Section 5. Finally, we draw several conclusions in Section 6.

2. Backgrounds

2.1. Characteristic fields

For smooth solutions, the system (1.1) can be re-written as a system of balance laws in non-conservative form as

$$\partial_t \mathbf{U} + \mathbf{A}(\mathbf{U})\partial_x \mathbf{U} = 0,$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ u \\ p \\ \alpha \end{bmatrix}, \quad \mathbf{A}(\mathbf{U}) = \begin{bmatrix} u & \rho & 0 & \rho(u-w)/\alpha \\ 0 & u & 1/\rho & 0 \\ 0 & \rho c^2 & u & \rho c^2(u-w)/\alpha \\ 0 & 0 & 0 & w \end{bmatrix},$$

and c denotes sound speed of the fluid in the first phase

$$c = \sqrt{\frac{\gamma(p+\pi)}{\rho}}.$$

The matrix $\mathbf{A}(\mathbf{U})$ has four real eigenvalues

$$\lambda_1(\mathbf{U}) = u - c, \quad \lambda_2(\mathbf{U}) = u, \quad \lambda_3(\mathbf{U}) = u + c, \quad \lambda_4(\mathbf{U}) = w. \tag{2.1}$$

Moreover, 2- and 4-characteristic fields are linearly degenerate, while 1- and 3-characteristic fields are genuinely nonlinear, see [36].

2.2. Elementary waves of (1.1) with w constant

Recall that though k-waves $W_k(\mathbf{U}_L, \mathbf{U}_R)$, $k = 1, 2, 3$, the volume fraction α remains constant, see [36]. A k-shock wave $S_k(\mathbf{U}_L, \mathbf{U}_R)$ $(k = 1, 3)$ is defined by

$$\mathbf{U}(x,t) = \begin{cases} \mathbf{U}_L & \text{if } x/t < \sigma_k(\mathbf{U}_L, \mathbf{U}_R), \\ \mathbf{U}_R & \text{if } x/t > \sigma_k(\mathbf{U}_L, \mathbf{U}_R), \end{cases}$$

where

$$\sigma_k(\mathbf{U}_L, \mathbf{U}_R) = \frac{\rho_R u_R - \rho_L u_L}{\rho_R - \rho_L}.$$

A k-rarefaction wave $R_k(\mathbf{U}_L, \mathbf{U}_R)$ $(k = 1, 3)$ is defined by

$$\mathbf{U}(x,t) = \begin{cases} \mathbf{U}_L & \text{if } x/t \le \lambda_k(\mathbf{U}_L), \\ \mathrm{Fan}_k(x/t; \mathbf{U}_L, \mathbf{U}_R) & \text{if } \lambda_k(\mathbf{U}_L) < x/t < \lambda_k(\mathbf{U}_R), \\ \mathbf{U}_R & \text{if } x/t \ge \lambda_k(\mathbf{U}_R), \end{cases}$$

where

$$\mathrm{Fan}_k(\xi; \mathbf{U}_L, \mathbf{U}_R) := [\rho(\xi), u(\xi), p(\xi), \alpha_L]^{\mathrm{T}},$$

$$p(\xi) = \left[(p_L+\pi)^{\frac{\gamma-1}{2\gamma}} + (-1)^{\frac{k+1}{2}} \frac{(\gamma-1)\sqrt{\rho_L}}{(\gamma+1)\sqrt{\gamma(p_L+\pi)^{\frac{1}{\gamma}}}}(\xi - \lambda_k(\mathbf{U}_L))\right]^{\frac{2\gamma}{\gamma-1}} - \pi,$$

$$u(\xi) = u_L + \frac{2}{\gamma+1}(\xi - \lambda_k(\mathbf{U}_L)), \quad \rho(\xi) = \rho_L \left(\frac{p(\xi)+\pi}{p_L+\pi}\right)^{\frac{1}{\gamma}}.$$

A 2-contact wave $W_2(\mathbf{U}_L, \mathbf{U}_R)$ is defined by

$$\mathbf{U}(x,t) = \begin{cases} \mathbf{U}_L & \text{if } x/t < \lambda_2(\mathbf{U}_L), \\ \mathbf{U}_R & \text{if } x/t > \lambda_2(\mathbf{U}_L). \end{cases}$$

Fix $\mathbf{U}_0 = [\rho_0, u_0, p_0, \alpha_0]^{\mathrm{T}}$, the forward 1-wave curve issuing from $\mathbf{U}_0$ is defined by

$$\mathcal{W}_1(\mathbf{U}_0) = \mathcal{S}_1(\mathbf{U}_0) \cup \mathcal{R}_1(\mathbf{U}_0), \tag{2.2}$$

where

$$\mathcal{S}_1(\mathbf{U}_0) = \Bigg\{[\rho, u, p, \alpha_0]^{\mathrm{T}} : p > p_0, u = u_0 - \sqrt{(p-p_0)\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)}, \\ \frac{\rho}{\rho_0} = \frac{(\gamma+1)(p+\pi) + (\gamma-1)(p_0+\pi)}{(\gamma+1)(p_0+\pi) + (\gamma-1)(p+\pi)}\Bigg\},$$

$$\mathcal{R}_1(\mathbf{U}_0) = \Bigg\{[\rho, u, p, \alpha_0]^{\mathrm{T}} : p \le p_0, \frac{\rho}{\rho_0} = \left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}, \\ u = u_0 - \frac{2\sqrt{\gamma(p_0+\pi)^{\frac{1}{\gamma}}}}{(\gamma-1)\sqrt{\rho_0}}\left((p+\pi)^{\frac{\gamma-1}{2\gamma}} - (p_0+\pi)^{\frac{\gamma-1}{2\gamma}}\right)\Bigg\}.$$

If $\lambda_1(\mathbf{U}_0) > w$, then there exists a unique state $\mathbf{U}_0^{\#} \in \mathcal{S}_1(\mathbf{U}_0)$ such that

$$\sigma_1(\mathbf{U}_0, \mathbf{U}_0^{\#}) = w, \quad \sigma_1(\mathbf{U}_0, \mathbf{U}) \begin{cases} < w & \text{for } \rho > \rho_0^{\#}, \\ > w & \text{for } \rho_0 < \rho < \rho_0^{\#}. \end{cases} \tag{2.3}$$

If $\lambda_1(\mathbf{U}_0) < w$, then $\sigma_1(\mathbf{U}_0, \mathbf{U}) < w$ for all $\mathbf{U} \in \mathcal{S}_1(\mathbf{U}_0)$.

The backward 3-wave curve issuing from $\mathbf{U}_0$ is defined by

$$\mathcal{W}_{3B}(\mathbf{U}_0) = \mathcal{S}_{3B}(\mathbf{U}_0) \cup \mathcal{R}_{3B}(\mathbf{U}_0), \tag{2.4}$$

where

$$\mathcal{S}_{3B}(\mathbf{U}_0) = \left\{ [\rho, u, p, \alpha_0]^{\mathrm{T}} : p > p_0, u = u_0 + \sqrt{(p - p_0)\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)}, \right.$$
$$\left. \frac{\rho}{\rho_0} = \frac{(\gamma+1)(p+\pi) + (\gamma-1)(p_0+\pi)}{(\gamma+1)(p_0+\pi) + (\gamma-1)(p+\pi)} \right\},$$
$$\mathcal{R}_{3B}(\mathbf{U}_0) = \left\{ [\rho, u, p, a_0]^{\mathrm{T}} : p \le p_0, \frac{\rho}{\rho_0} = \left(\frac{p+\pi}{p_0+\pi}\right)^{\frac{1}{\gamma}}, \right.$$
$$\left. u = u_0 + \frac{2\sqrt{\gamma(p_0+\pi)^{\frac{1}{\gamma}}}}{(\gamma-1)\sqrt{\rho_0}}\left((p+\pi)^{\frac{\gamma-1}{2\gamma}} - (p_0+\pi)^{\frac{\gamma-1}{2\gamma}}\right) \right\}.$$

Only 4-constant-speed wave involves the jump of the volume fraction α. Let us fix $\mathbf{U}_0 = [\rho_0, u_0, p_0, \alpha_0]^{\mathrm{T}}$ and given $\alpha \neq \alpha_0$. Any state $\mathbb{U} = [\rho, u, p, \alpha]^{\mathrm{T}}$ that can be connected with $\mathbb{U}_0$ by a 4-constant-speed wave must satisfy the equations

(2.5)
$$\alpha\rho(u-w) = \alpha_0\rho_0(u_0-w), \quad (u-w)^2 + \frac{2\gamma(p+\pi)}{(\gamma-1)\rho} = (u_0-w)^2 + \frac{2\gamma(p_0+\pi)}{(\gamma-1)\rho_0}, \quad \frac{p+\pi}{p_0+\pi} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}.$$

If $u_0 = w$, then (2.5) admits a unique root

$$(\rho, u, p) = (\rho_0, u_0, p_0). \tag{2.6}$$

Consider the case $u_0 \neq w$. By substituting 1st and 3rd equations into 2nd equation of (2.5), we obtain the following equation of ρ:

$$\frac{2\gamma(p_0+\pi)}{(\gamma-1)\rho_0^{\gamma}}\rho^{\gamma+1} - \left[(u_0-w)^2 + \frac{2\gamma(p_0+\pi)}{(\gamma-1)\rho_0}\right]\rho^2 + \left(\frac{\alpha_0\rho_0(u_0-w)}{\alpha}\right)^2 = 0. \tag{2.7}$$

So we can resolve (ρ, u, p) in terms of $\mathbf{U}_0$, α. As showed in [36], the equation (2.7) admits a root ρ if and only if $\alpha \ge \alpha_{\min}$, where

$$\alpha_{\min} := \frac{\alpha_0\rho_0|u_0-w|}{\sqrt{\frac{\gamma(p_0+\pi)}{\rho_0^{\gamma}}}(\rho_{\min})^{\frac{\gamma+1}{2}}} \quad \text{and} \quad \rho_{\min} := \left(\frac{(\gamma-1)\rho_0^{\gamma}(u_0-w)^2}{\gamma(\gamma+1)(p_0+\pi)} + \frac{2\rho_0^{\gamma-1}}{\gamma+1}\right)^{\frac{1}{\gamma-1}}.$$

If $\alpha = \alpha_{\min}$, then (2.7) admits a unique root $\rho = \rho_{\min}$. If $\alpha > \alpha_{\min}$, then (2.7) admits two distinct roots denoted by ρ_0^s and ρ_0^b such that

$$\rho_0^s < \rho_{\min} < \rho_0^b.$$

Accordingly, the system (2.5) has two distinct roots (ρ_0^s, u_0^s, p_0^s) and (ρ_0^b, u_0^b, p_0^b). Moreover, for $\lambda_2(\mathbf{U}_0) > w$, we can prove that

$$\lambda_1(\rho_0^s, u_0^s, p_0^s) > w \quad \text{and} \quad \lambda_1(\rho_0^b, u_0^b, p_0^b) < w < \lambda_2(\rho_0^b, u_0^b, p_0^b). \tag{2.8}$$

Let us denote

$$\Upsilon(\mathbf{U}_0;\alpha) = [\rho, u, p, \alpha]^{\mathrm{T}}, \tag{2.9}$$

where (ρ, u, p) is the root of (2.5) satisfying the admissible criterion. Thus, (2.6) and (2.8) imply that

(i) If $\lambda_1(\mathbf{U}_0) \geq w$, and $\alpha > \alpha_{\min}$, then $\Upsilon(\mathbf{U}_0;\alpha) = [\rho_0^s, u_0^s, p_0^s, \alpha]^{\mathrm{T}}$.

(ii) If $\lambda_1(\mathbf{U}_0) < w < \lambda_2(\mathbf{U}_0)$, and $\alpha > \alpha_{\min}$, then $\Upsilon(\mathbf{U}_0;\alpha) = [\rho_0^b, u_0^b, p_0^b, \alpha]^{\mathrm{T}}$.

(iii) If $\lambda_2(\mathbf{U}_0) = w$, then $\Upsilon(\mathbf{U}_0;\alpha) = [\rho_0, u_0, p_0, \alpha]^{\mathrm{T}}$.

3. Algorithms for computing all configurations of Riemann solution

Recently, a complete set of configurations of Riemann solution has been presented in [36]. In this section, given a left-hand state $\mathbf{U}_L = [\rho_L, u_L, p_L, \alpha_L]^{\mathrm{T}}$ and a right-hand state $\mathbf{U}_R = [\rho_R, u_R, p_R, \alpha_R]^{\mathrm{T}}$ such that $|\alpha_R - \alpha_L|$ is sufficiently small, our objective is to detail the algorithms for computing intermediate states in all configurations of Riemann solution of (1.1) with w constant (denoted by $\mathbf{U}^{\mathrm{Rie}}(x/t; \mathbf{U}_L, \mathbf{U}_R, w)$).

3.1. Construction A1

Assume that

$$\lambda_1(\mathbf{U}_L) \geq w,\ \mathcal{W}_{3B}(\mathbf{U}_R) \cap \mathcal{L}_1(\mathbf{U}_L) \neq \emptyset \text{ in projecting onto } (p,u) \text{ plane},$$

where $\lambda_1(\mathbf{U}_L)$, $\mathcal{W}_{3B}(\mathbf{U}_R)$ are defined as (2.1), (2.4), and $\mathcal{L}_1(\mathbf{U}_L)$ is defined by

$$\begin{aligned} \mathcal{L}_1(\mathbf{U}_L) &= \{\mathbf{U} \in \mathcal{W}_1(\mathbf{U}_L^s) : p < p_L^{s\#}\}, \\ \mathbf{U}_L^s &= \Upsilon(\mathbf{U}_L; \alpha_R), \end{aligned} \tag{3.1}$$

where $\Upsilon(\mathbf{U}_0;\alpha)$, $\mathcal{W}_1(\mathbf{U}_0)$, $\mathbf{U}_0^{\#}$ are defined as (2.9), (2.2), (2.3). Then, the Riemann solution has the form

$$W_4(\mathbf{U}_L, \mathbf{U}_L^s) \oplus W_1(\mathbf{U}_L^s, \mathbf{U}_*) \oplus W_2(\mathbf{U}_*, \mathbf{U}_{**}) \oplus W_3(\mathbf{U}_{**}, \mathbf{U}_R), \tag{3.2}$$

where the states $\mathbf{U}_L^s$, $\mathbf{U}_*$, $\mathbf{U}_{**}$ are computed as follows:

- Compute $\mathbf{U}_L^s = \Upsilon(\mathbf{U}_L; \alpha_R)$.
- Compute the intersection point $(p_*, u_*) = \mathcal{W}_1(\mathbf{U}_L^s) \cap \mathcal{W}_{3B}(\mathbf{U}_R)$ in projecting onto (p,u)-plane.
- Compute $\mathbf{U}_* = [\rho_*, u_*, p_*, a_R]^{\mathrm{T}} \in \mathcal{W}_1(\mathbf{U}_L^s)$.

- Compute $\mathbf{U}_{**} = [\rho_{**}, u_*, p_*, a_R]^{\mathrm{T}} \in \mathcal{W}_{3B}(\mathbf{U}_R)$.

Lemma 3.1. *For the Riemann solution of Construction A1 as shown in* (3.2)*, we have*

$$(\alpha\rho u)^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) \leq \max\{\alpha_L\rho_L u_L, \alpha_L\rho_L(u_L - w)\} \tag{3.3}$$

and

$$(\alpha\rho u)^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) \geq \min\{\alpha_R\rho_R u_R, \alpha_R\rho_R(u_R - \lambda_3(\mathbf{U}_R))\}. \tag{3.4}$$

Proof. If $w \geq 0$, then

$$\mathbf{U}^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_L, \quad \lambda_2(\mathbf{U}^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w)) \geq w,$$

so

$$(\alpha\rho u)^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) = \alpha_L\rho_L u_L$$

and

$$(\alpha\rho u)^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) \geq 0 \geq \alpha_R\rho_R(u_R - \lambda_3(\mathbf{U}_R)).$$

Consider the case $w < 0$. For $u_* = 0$, we have

$$\mathbf{U}^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_*, \quad \mathbf{U}^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_{**}.$$

Therefore

$$(\alpha\rho u)^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) = 0 \leq \alpha_L\rho_L(u_L - w)$$

and

$$(\alpha\rho u)^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) = 0 \geq \alpha_R\rho_R(u_R - \lambda_3(\mathbf{U}_R)).$$

For $u_* < 0$ and $W_3(\mathbf{U}_{**}, \mathbf{U}_R)$ is the 3-shock wave with a shock speed σ_3, we have three cases as follows.

- If $\sigma_3 = 0$, then

$$\mathbf{U}^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_{**}, \quad \mathbf{U}^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_R.$$

  So, we obtain

$$(\alpha\rho u)^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) < 0, \quad (\alpha\rho u)^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) = \alpha_R\rho_R u_R.$$

- If $\sigma_3 < 0$, then

$$\mathbf{U}^{\mathrm{Rie}}(0\pm; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_R.$$

  Since $u_R < u_*$, we have

$$(\alpha\rho u)^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) < 0, \quad (\alpha\rho u)^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) = \alpha_R\rho_R u_R.$$

- If $\sigma_3 > 0$, then
$$\mathbf{U}^{\text{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w)=\mathbf{U}_{**}.$$
Since $\rho_{**}u_* = \rho_R u_R + \sigma_3(\rho_{**} - \rho_R)$ and $\rho_{**} > \rho_R$, we have
$$\rho_{**}u_* > \rho_R u_R,$$
so
$$(\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < 0, \quad (\alpha\rho u)^{\text{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) > \alpha_R\rho_R u_R.$$

For $u_* < 0$ and $W_3(\mathbf{U}_{**},\mathbf{U}_R)$ is the 3-rarefaction wave, we also have three cases as follows.

- If $0 \le \lambda_3(\mathbf{U}_{**})$, then
$$\mathbf{U}^{\text{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w)=\mathbf{U}_{**}.$$
Since $0 < \rho_{**} < \rho_R$, $u_* < 0$, $u_* = u_R + \frac{2}{\gamma+1}(\lambda_3(\mathbf{U}_{**}) - \lambda_3(\mathbf{U}_R))$, and $\gamma > 1$, we have
$$\rho_{**}u_* > \rho_R\left(u_R + \frac{2}{\gamma+1}(\lambda_3(\mathbf{U}_{**}) - \lambda_3(\mathbf{U}_R))\right) \ge \rho_R(u_R - \lambda_3(\mathbf{U}_R)),$$
this implies that
$$(\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < 0, \quad (\alpha\rho u)^{\text{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) > \alpha_R\rho_R(u_R - \lambda_3(\mathbf{U}_R)).$$

- If $\lambda_3(\mathbf{U}_{**}) < 0 < \lambda_3(\mathbf{U}_R)$, then
$$\mathbf{U}^{\text{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w)=\mathbf{U} := \text{Fan}_3(0;\mathbf{U}_{**},\mathbf{U}_R).$$
Since $u = u_* + \frac{2}{\gamma+1}(0 - \lambda_3(\mathbf{U}_{**})) = \frac{\gamma-1}{\gamma+1}u_* - \frac{2}{\gamma+1}c(\mathbf{U}_{**})$, and $u_* < 0$, we have $u < 0$. Moreover, since $0 < \rho < \rho_R$, $u = u_R + \frac{2}{\gamma+1}(0 - \lambda_3(\mathbf{U}_R))$, and $\gamma > 1$, we infer that
$$\rho u > \rho_R\left(u_R + \frac{2}{\gamma+1}(0 - \lambda_3(\mathbf{U}_R))\right) \ge \rho_R(u_R - \lambda_3(\mathbf{U}_R)),$$
so
$$(\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < 0, \quad (\alpha\rho u)^{\text{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) > \alpha_R\rho_R(u_R - \lambda_3(\mathbf{U}_R)).$$

- If $\lambda_3(\mathbf{U}_R) \le 0$, then
$$\mathbf{U}^{\text{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w)=\mathbf{U}_R.$$
Since $\lambda_3(\mathbf{U}_R) \le 0$, we have $u_R < 0$, so
$$(\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < 0, \quad (\alpha\rho u)^{\text{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) = \alpha_R\rho_R u_R.$$

For $u_* > 0$ and $W_1(\mathbf{U}_L^s,\mathbf{U}_*)$ is the 1-shock wave with a shock speed σ_1, we have three cases as follows.

- If $\sigma_1 = 0$, then

$$\mathbf{U}^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_L^s, \quad \mathbf{U}^{\mathrm{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_*.$$

Since $\alpha_R\rho_L^s(u_L^s - w) = \alpha_L\rho_L(u_L - w)$, we have

$$\alpha_R\rho_L^s u_L^s = \alpha_L\rho_L u_L - \alpha_L\rho_L w + \alpha_R\rho_L^s w < \alpha_L\rho_L(u_L - w),$$

so

$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < \alpha_L\rho_L(u_L - w), \quad (\alpha\rho u)^{\mathrm{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) > 0.$$

- If $\sigma_1 < 0$, then

$$\mathbf{U}^{\mathrm{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_*.$$

Since $\alpha_R\rho_*(u_* - \sigma_1) = \alpha_R\rho_L^s(u_L^s - \sigma_1)$, $\alpha_R\rho_L^s(u_L^s - w) = \alpha_L\rho_L(u_L - w)$, and $\sigma_1 \geq w$, we have

$$\alpha_R\rho_* u_* = \alpha_L\rho_L(u_L - w) + \alpha_R\rho_*\sigma_1 + \alpha_R\rho_L^s(w - \sigma_1) < \alpha_L\rho_L(u_L - w),$$

so

$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < \alpha_L\rho_L(u_L - w), \quad (\alpha\rho u)^{\mathrm{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) > 0.$$

- If $\sigma_1 > 0$, then

$$\mathbf{U}^{\mathrm{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_L^s.$$

Since $\rho_L^s(u_L^s - \sigma_1) = \rho_*(u_* - \sigma_1)$, and $\sigma_1 \leq u_*$, we infer that $u_L^s \geq \sigma_1 > 0$, so

$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < \alpha_L\rho_L(u_L - w), \quad (\alpha\rho u)^{\mathrm{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) > 0.$$

For $u_* > 0$ and $W_1(\mathbf{U}_L^s,\mathbf{U}_*)$ is the 1-rarefaction wave, we have three cases as follows.

- If $0 \leq \lambda_1(\mathbf{U}_L^s)$, then

$$\mathbf{U}^{\mathrm{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_L^s.$$

Since $0 \leq \lambda_1(\mathbf{U}_L^s)$, we have $u_L^s > 0$, so

$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < \alpha_L\rho_L(u_L - w), \quad (\alpha\rho u)^{\mathrm{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) > 0.$$

- If $\lambda_1(\mathbf{U}_L^s) < 0 < \lambda_1(\mathbf{U}_*)$, then

$$\mathbf{U}^{\mathrm{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U} := \mathrm{Fan}_1(0;\mathbf{U}_L^s,\mathbf{U}_*).$$

Since $u = u_* + \frac{2}{\gamma+1}(0 - \lambda_1(\mathbf{U}_*)) = \frac{\gamma-1}{\gamma+1}u_* + \frac{2}{\gamma+1}c(\mathbf{U}_*)$, and $u_* > 0$, we have $u > 0$, so

$$(\alpha\rho u)^{\text{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) > 0.$$

Moreover, since $0 < \rho < \rho_L^s$, $u = u_L^s + \frac{2}{\gamma+1}(0 - \lambda_1(\mathbf{U}_L^s))$, $\lambda_1(\mathbf{U}_L^s) > w$, and $\gamma > 1$, we infer that

$$\rho u < \rho_L^s \left(u_L^s + \frac{2}{\gamma+1}(0 - \lambda_1(\mathbf{U}_L^s)) \right) < \rho_L^s(u_L^s - w),$$

so

$$(\alpha\rho u)^{\text{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) < \alpha_L \rho_L(u_L - w).$$

- If $\lambda_1(\mathbf{U}_*) \le 0$, then

$$\mathbf{U}^{\text{Rie}}(0\pm; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_*.$$

Since $0 < \rho_* < \rho_L^s$, $u_* = u_L^s + \frac{2}{\gamma+1}(\lambda(\mathbf{U}_*) - \lambda_1(\mathbf{U}_L^s))$, $\lambda_1(\mathbf{U}_L^s) > w$, and $\gamma > 1$, we infer that

$$\rho_* u_* < \rho_L^s \left(u_L^s + \frac{2}{\gamma+1}(\lambda(\mathbf{U}_*) - \lambda_1(\mathbf{U}_L^s)) \right) < \rho_L^s(u_L^s - w),$$

so

$$(\alpha\rho u)^{\text{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) < \alpha_L \rho_L(u_L - w), \quad (\alpha\rho u)^{\text{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) > 0.$$

This terminates the proof. □

3.2. Construction A2

Assume that

$$\lambda_1(\mathbf{U}_L) \ge w,\ \mathcal{W}_{3B}(\mathbf{U}_R) \cap \mathcal{L}_2(\mathbf{U}_L) \ne \emptyset \text{ in projecting onto } (p, u) \text{ plane},$$

where $\lambda_1(\mathbf{U}_L)$, $\mathcal{W}_{3B}(\mathbf{U}_R)$ are defined as (2.1), (2.4), and $\mathcal{L}_2(\mathbf{U}_L)$ is defined by

$$\mathcal{L}_2(\mathbf{U}_L) = \left\{ \mathbf{U}_L^{s\#b}(\alpha_M) : \alpha_M \text{ varies between } \alpha_L \text{ and } \alpha_R \right\}, \tag{3.5}$$
$$\mathbf{U}_L^s(\alpha_M) = \Upsilon(\mathbf{U}_L; \alpha_M), \quad \mathbf{U}_L^{s\#}(\alpha_M) = (\mathbf{U}_L^s(\alpha_M))^\#, \quad \mathbf{U}_L^{s\#b}(\alpha_M) = \Upsilon(\mathbf{U}_L^{s\#}(\alpha_M); \alpha_R),$$

where $\Upsilon(\mathbf{U}_0; \alpha)$, $\mathbf{U}_0^\#$ are defined as (2.9), (2.3). Then, the Riemann solution has the form

$$W_4(\mathbf{U}_L, \mathbf{U}_L^s) \oplus S_1(\mathbf{U}_L^s, \mathbf{U}_L^{s\#}) \oplus W_4(\mathbf{U}_L^{s\#}, \mathbf{U}_L^{s\#b}) \oplus W_2(\mathbf{U}_L^{s\#b}, \mathbf{U}_*) \oplus W_3(\mathbf{U}_*, \mathbf{U}_R), \tag{3.6}$$

where the states $\mathbf{U}_L^s$, $\mathbf{U}_L^{s\#}$, $\mathbf{U}_L^{s\#b}$, $\mathbf{U}_*$ are computed as follows:

- Compute the intersection point $(p_*, u_*) = \mathcal{L}_2(\mathbf{U}_L) \cap \mathcal{W}_{3B}(\mathbf{U}_R)$ in projecting onto (p, u)-plane, and get the parameter α_M corresponding to this point.

- Compute $\mathbf{U}_L^s = \Upsilon(\mathbf{U}_L; \alpha_M)$.
- Compute $\mathbf{U}_L^{s\#} \in \mathcal{S}_1(\mathbf{U}_L^s)$ such that $\sigma_1(\mathbf{U}_L^s, \mathbf{U}_L^{s\#}) = w$.
- Compute $\mathbf{U}_L^{s\#b} = \Upsilon(\mathbf{U}_L^{s\#}; \alpha_R)$.
- Compute $\mathbf{U}_* = [\rho_*, u_*, p_*, \alpha_R]^{\mathrm{T}} \in \mathcal{W}_{3B}(\mathbf{U}_R)$.

Lemma 3.2. *For the Riemann solution of Construction A2 as shown in* (3.6)*, we have* (3.3) *and* (3.4)*.*

Proof. If $w \geq 0$, then

$$\mathbf{U}^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_L, \quad \lambda_2(\mathbf{U}^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w)) \geq w,$$

so

$$(\alpha\rho u)^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) = \alpha_L \rho_L u_L, \quad (\alpha\rho u)^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) \geq 0.$$

Consider the case $w < 0$. For $u_L^{s\#b} = 0$, we have

$$\mathbf{U}^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_L^{s\#b}, \quad \mathbf{U}^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_*.$$

Therefore

$$(\alpha\rho u)^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) = 0, \quad (\alpha\rho u)^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) = 0.$$

For $u_L^{s\#b} < 0$ and $W_3(\mathbf{U}_*, \mathbf{U}_R)$ is the 3-shock wave with a shock speed σ_3, we have three cases as follows.

- If $\sigma_3 = 0$, then

  $$\mathbf{U}^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_*, \quad \mathbf{U}^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_R.$$

  So, we obtain

  $$(\alpha\rho u)^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) < 0, \quad (\alpha\rho u)^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) = \alpha_R \rho_R u_R.$$

- If $\sigma_3 < 0$, then

  $$\mathbf{U}^{\mathrm{Rie}}(0\pm; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_R.$$

  Since $u_R < u_* = u_L^{s\#b}$, we have

  $$(\alpha\rho u)^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) < 0, \quad (\alpha\rho u)^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) = \alpha_R \rho_R u_R.$$

- If $\sigma_3 > 0$, then
$$\mathbf{U}^{\text{Rie}}(0\pm; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_*.$$
Since $\rho_* u_* = \rho_R u_R + \sigma_3(\rho_* - \rho_R)$ and $\rho_* > \rho_R$, we have $\rho_* u_* > \rho_R u_R$, so
$$(\alpha\rho u)^{\text{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) < 0, \quad (\alpha\rho u)^{\text{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) > \alpha_R \rho_R u_R.$$

For $u_L^{s\#b} < 0$ and $W_3(\mathbf{U}_*, \mathbf{U}_R)$ is the 3-rarefaction wave, we also have three cases as follows.

- If $0 \le \lambda_3(\mathbf{U}_*)$, then
$$\mathbf{U}^{\text{Rie}}(0\pm; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_*.$$
Since $0 < \rho_* < \rho_R$, $u_* < 0$, $u_* = u_R + \frac{2}{\gamma+1}(\lambda_3(\mathbf{U}_*) - \lambda_3(\mathbf{U}_R))$, and $\gamma > 1$, we have
$$\rho_* u_* > \rho_R \left(u_R + \frac{2}{\gamma+1}(\lambda_3(\mathbf{U}_*) - \lambda_3(\mathbf{U}_R)) \right) \ge \rho_R(u_R - \lambda_3(\mathbf{U}_R)),$$
this implies that
$$(\alpha\rho u)^{\text{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) < 0, \quad (\alpha\rho u)^{\text{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) > \alpha_R \rho_R (u_R - \lambda_3(\mathbf{U}_R)).$$

- If $\lambda_3(\mathbf{U}_*) < 0 < \lambda_3(\mathbf{U}_R)$, then
$$\mathbf{U}^{\text{Rie}}(0\pm; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U} := \text{Fan}_3(0; \mathbf{U}_*, \mathbf{U}_R).$$
Since $u = u_* + \frac{2}{\gamma+1}(0 - \lambda_3(\mathbf{U}_*)) = \frac{\gamma-1}{\gamma+1} u_* - \frac{2}{\gamma+1} c(\mathbf{U}_*)$, and $u_* < 0$, we have $u < 0$. Moreover, since $0 < \rho < \rho_R$, $u = u_R + \frac{2}{\gamma+1}(0 - \lambda_3(\mathbf{U}_R))$, and $\gamma > 1$, we have
$$\rho u > \rho_R \left(u_R + \frac{2}{\gamma+1}(0 - \lambda_3(\mathbf{U}_R)) \right) \ge \rho_R(u_R - \lambda_3(\mathbf{U}_R)),$$
so
$$(\alpha\rho u)^{\text{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) < 0, \quad (\alpha\rho u)^{\text{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) > \alpha_R \rho_R (u_R - \lambda_3(\mathbf{U}_R)).$$

- If $\lambda_3(\mathbf{U}_R) \le 0$, then
$$\mathbf{U}^{\text{Rie}}(0\pm; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_R.$$
Since $\lambda_3(\mathbf{U}_R) \le 0$, we have $u_R < 0$, so
$$(\alpha\rho u)^{\text{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) < 0, \quad (\alpha\rho u)^{\text{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) = \alpha_R \rho_R u_R.$$

For $u_L^{s\#b} > 0$, we have

$$\mathbf{U}^{\mathrm{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_L^{s\#b}.$$

Since

$$\alpha_R\rho_L^{s\#b}(u_L^{s\#b}-w) = \alpha_M\rho_L^{s\#}(u_L^{s\#}-w) = \alpha_M\rho_L^{s}(u_L^{s}-w) = \alpha_L\rho_L(u_L-w),$$

we have

$$\alpha_R\rho_L^{s\#b}u_L^{s\#b} = \alpha_L\rho_L(u_L-w) + \alpha_R\rho_L^{s\#b}w < \alpha_L\rho_L(u_L-w),$$

so

$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < \alpha_L\rho_L(u_L-w), \quad (\alpha\rho u)^{\mathrm{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) = 0.$$

This terminates the proof. □

3.3. Construction A3

Assume that

$$\lambda_1(\mathbf{U}_L) \geq w,\ \mathcal{W}_{3B}(\mathbf{U}_R) \cap \mathcal{L}_3(\mathbf{U}_L) \neq \emptyset \text{ in projecting onto } (p,u) \text{ plane},$$

where $\lambda_1(\mathbf{U}_L)$, $\mathcal{W}_{3B}(\mathbf{U}_R)$ are defined as (2.1), (2.4), and $\mathcal{L}_3(\mathbf{U}_L)$ is defined by

$$\mathcal{L}_3(\mathbf{U}_L) = \big\{\Upsilon(\mathbf{U}_*;\alpha_R) : \mathbf{U}_* \in \mathcal{W}_1(\mathbf{U}_L), p_L^{\#} \leq p_* \leq p_L^{\flat}\big\},$$

where $\mathcal{W}_1(\mathbf{U}_L)$, $\mathbf{U}_L^{\#}$ are defined as (2.2), (2.3), and $\mathbf{U}_L^{\flat}$ is defined by

$$\mathbf{U}_L^{\flat} \in \mathcal{W}_1(\mathbf{U}_L) \text{ such that } u_L^{\flat} = w. \tag{3.7}$$

Then, the Riemann solution has the form

$$S_1(\mathbf{U}_L,\mathbf{U}_*) \oplus W_4(\mathbf{U}_*,\mathbf{U}_*^b) \oplus W_2(\mathbf{U}_*^b,\mathbf{U}_{**}) \oplus W_3(\mathbf{U}_{**},\mathbf{U}_R), \tag{3.8}$$

where the states $\mathbf{U}_*$, $\mathbf{U}_*^b$, $\mathbf{U}_{**}$ are computed as follows:

- Compute the intersection point $(p_{**},u_{**}) = \mathcal{L}_3(\mathbf{U}_L) \cap \mathcal{W}_{3B}(\mathbf{U}_R)$ in projecting onto (p,u)-plane, and get the parameter p_* corresponding to this point.
- Compute $\mathbf{U}_* = (\rho_*,u_*,p_*,\alpha_L) \in \mathcal{W}_1(\mathbf{U}_L)$.
- Compute $\mathbf{U}_*^b = \Upsilon(\mathbf{U}_*;\alpha_R)$.
- Compute $\mathbf{U}_{**} = (\rho_{**},u_{**},p_{**},\alpha_R) \in \mathcal{W}_{3B}(\mathbf{U}_R)$.

Lemma 3.3. *For the Riemann solution of Construction A3 as shown in* (3.8)*, we have* (3.3) *and* (3.4)*.*

Proof. If $w \geq 0$, then

$$\mathbf{U}^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_L \quad \text{or} \quad \mathbf{U}^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_*.$$

Since $\rho_* u_* = \rho_L u_L + \sigma_1(\rho_* - \rho_L)$, $\sigma_1 \leq w$, and $\rho_* > \rho_L$, we obtain

$$(\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < \alpha_L \rho_L u_L.$$

Moreover, since $\lambda_2(\mathbf{U}^{\text{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w)) \geq w \geq 0$, we have

$$(\alpha\rho u)^{\text{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) \geq 0.$$

For $w < 0$, we have three cases as follows.

- For $u_*^b = 0$, we have

$$\mathbf{U}^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_*^b, \quad \mathbf{U}^{\text{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_{**}.$$

  Since $u_{**} = u_*^b$, we have

$$(\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) = 0, \quad (\alpha\rho u)^{\text{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) = 0.$$

- For $u_*^b < 0$, as in the proof in Lemma 3.1, we obtain (3.4) and

$$(\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) \leq 0.$$

- For $u_*^b > 0$, we have

$$\mathbf{U}^{\text{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_*^b.$$

  Since

$$\alpha_R \rho_*^b (u_*^b - w) = \alpha_L \rho_* (u_* - w), \quad \rho_* u_* = \rho_L u_L + \sigma_1(\rho_* - \rho_L),$$

  we get

$$\alpha_R \rho_*^b u_*^b = \alpha_L \rho_L u_L + w(\alpha_R \rho_*^b - \alpha_L \rho_*) + \alpha_L \sigma_1 (\rho_* - \rho_L).$$

  Moreover, since $\alpha_R \rho_*^b > \alpha_L \rho_*$ and $\rho_* > \rho_L$, we have

$$(\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < \alpha_L \rho_L u_L, \quad (\alpha\rho u)^{\text{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) > 0.$$

This terminates the proof. □

3.4. Construction B1

Given a left-hand state $\mathbf{U}_L = (\rho_L, u_L, p_L, \alpha_L)$ and a right-hand state $\mathbf{U}_R = (\rho_R, u_R, p_R, \alpha_R)$ such that $|\alpha_R - \alpha_L|$ is sufficiently small. Assume that

$$\lambda_1(\mathbf{U}_L) < w \le \lambda_2(\mathbf{U}_L),\ \mathcal{W}_{3B}(\mathbf{U}_R) \cap \mathcal{L}_1(\mathbf{U}_L^\dagger) \neq \emptyset \text{ in projecting onto } (p,u) \text{ plane},$$

where $\mathbf{U}_L^\dagger$ is defined by

$$\mathbf{U}_L^\dagger \in \mathcal{W}_1(\mathbf{U}_L) \text{ such that } \lambda_1(\mathbf{U}_L^\dagger) = w, \tag{3.9}$$

and $\lambda_1(\mathbf{U}_L)$, $\lambda_2(\mathbf{U}_L)$, $\mathcal{W}_1(\mathbf{U}_L)$, $\mathcal{W}_{3B}(\mathbf{U}_R)$, $\mathcal{L}_1(\mathbf{U}_L^\dagger)$ are defined as (2.1), (2.2), (2.4), (3.1). Then, the Riemann solution has the form

$$R_1(\mathbf{U}_L, \mathbf{U}_L^\dagger) \oplus W_4(\mathbf{U}_L^\dagger, \mathbf{U}_L^{\dagger s}) \oplus W_1(\mathbf{U}_L^{\dagger s}, \mathbf{U}_*) \oplus W_2(\mathbf{U}_*, \mathbf{U}_{**}) \oplus W_3(\mathbf{U}_{**}, \mathbf{U}_R), \tag{3.10}$$

where the states $\mathbf{U}_L^\dagger$, $\mathbf{U}_L^{\dagger s}$, $\mathbf{U}_*$, $\mathbf{U}_{**}$ are computed as follows:

- Compute $\mathbf{U}_L^\dagger \in \mathcal{W}_1(\mathbf{U}_L)$ such that $\lambda_1(\mathbf{U}_L^\dagger) = w$.
- Compute $\mathbf{U}_L^{\dagger s} = \Upsilon(\mathbf{U}_L^\dagger; \alpha_R)$.
- Compute the intersection point $(p_*, u_*) = \mathcal{W}_1(\mathbf{U}_L^{\dagger s}) \cap \mathcal{W}_{3B}(\mathbf{U}_R)$ in projecting onto (p,u)-plane.
- Compute $\mathbf{U}_* = (\rho_*, u_*, p_*, a_R) \in \mathcal{W}_1(\mathbf{U}_L^{\dagger s})$.
- Compute $\mathbf{U}_{**} = (\rho_{**}, u_*, p_*, a_R) \in \mathcal{W}_{3B}(\mathbf{U}_R)$.

Lemma 3.4. *If $\lambda_1(\mathbf{U}_L) < w \le \lambda_2(\mathbf{U}_L)$, then we have*

$$\rho_L^\dagger(u_L^\dagger - w) \le \rho_L(u_L - \lambda_1(\mathbf{U}_L)),$$

where $\mathbf{U}_L^\dagger$ is defined as (3.9).

Proof. Since $\rho_L^\dagger < \rho_L$ and $\lambda_1(\mathbf{U}_L^\dagger) = u_L^\dagger - \sqrt{\frac{\gamma(p_L^\dagger + \pi)}{\rho_L^\dagger}} = w$, we have

$$\begin{aligned}
\rho_L^\dagger(u_L^\dagger - w) &= \rho_L^\dagger \sqrt{\frac{\gamma(p_L^\dagger + \pi)}{\rho_L^\dagger}} < \rho_L \sqrt{\frac{\gamma(p_L^\dagger + \pi)}{\rho_L^\dagger}} = \rho_L \sqrt{\frac{\gamma(p_L^\dagger + \pi)}{(\rho_L^\dagger)^\gamma} \cdot (\rho_L^\dagger)^{\gamma-1}} \\
&= \rho_L \sqrt{\frac{\gamma(p_L + \pi)}{(\rho_L)^\gamma} \cdot (\rho_L^\dagger)^{\gamma-1}} = \rho_L \sqrt{\frac{\gamma(p_L + \pi)}{\rho_L} \cdot \frac{(\rho_L^\dagger)^{\gamma-1}}{(\rho_L)^{\gamma-1}}} \\
&< \rho_L \sqrt{\frac{\gamma(p_L + \pi)}{\rho_L}} = \rho_L(u_L - \lambda_1(\mathbf{U}_L)).
\end{aligned}$$

This terminates the proof. □

Lemma 3.5. *For the Riemann solution of Construction B1 as shown in* (3.10)*, we have* (3.4) *and*

$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) \le \max\{\alpha_L\rho_L u_L, \alpha_L\rho_L(u_L-\lambda_1(\mathbf{U}_L))\}. \tag{3.11}$$

Proof. Since $\lambda_2(\mathbf{U}^{\mathrm{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w)) \ge w$, we always have

$$(\alpha\rho u)^{\mathrm{Rie}}(0+;\mathbf{U}_L,\mathbf{U}_R,w) \ge 0 \quad \text{for } w \ge 0.$$

For $w = 0$, we have $\mathbf{U}^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_L^\dagger$. It is implied from Lemma 3.4 that

$$\rho_L^\dagger u_L^\dagger \le \rho_L(u_L-\lambda_1(\mathbf{U}_L)),$$

so

$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) \le \alpha_L\rho_L(u_L-\lambda_1(\mathbf{U}_L)).$$

For $w > 0$, we have two cases as follows.

- If $0 \le \lambda_1(\mathbf{U}_L)$, then
$$\mathbf{U}^{\mathrm{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_L,$$
so
$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) = \alpha_L\rho_L u_L.$$

- If $\lambda_1(\mathbf{U}_L) < 0$, then
$$\mathbf{U}^{\mathrm{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U} := \mathrm{Fan}_1(0;\mathbf{U}_L,\mathbf{U}_L^\dagger).$$
Since $u = u_L + \frac{2}{\gamma+1}(0-\lambda_1(\mathbf{U}_L))$, $\gamma > 1$, and $\rho < \rho_L$, we have
$$\rho u < \rho_L\left(u_L + \frac{2}{\gamma+1}(0-\lambda_1(\mathbf{U}_L))\right) < \rho_L(u_L-\lambda_1(\mathbf{U}_L)),$$
so
$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) < \alpha_L\rho_L(u_L-\lambda_1(\mathbf{U}_L)).$$

For $w < 0$, we have two cases as follows.

- For $u_* \le 0$, as in the proof in Lemma 3.1, we obtain (3.4) and
$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) \le 0.$$

- For $u_* > 0$, also as in the proof in Lemma 3.1, we have (3.4) and
$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) \le \alpha_L\rho_L^\dagger(u_L^\dagger - w).$$
Moreover, by Lemma 3.4, we have $\rho_L^\dagger(u_L^\dagger - w) \le \rho_L(u_L-\lambda_1(\mathbf{U}_L))$, so
$$(\alpha\rho u)^{\mathrm{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) \le \alpha_L\rho_L(u_L-\lambda_1(\mathbf{U}_L)).$$

This terminates the proof. □

3.5. Construction B2

Given a left-hand state $\mathbf{U}_L = (\rho_L, u_L, p_L, \alpha_L)$ and a right-hand state $\mathbf{U}_R = (\rho_R, u_R, p_R, \alpha_R)$ such that $|\alpha_R - \alpha_L|$ is sufficiently small. Assume that

$$\lambda_1(\mathbf{U}_L) < v_* < \lambda_2(\mathbf{U}_L),\ \mathcal{W}_{3B}(\mathbf{U}_R) \cap \mathcal{L}_2(\mathbf{U}_L^\dagger) \neq \emptyset \text{ in projecting onto } (p,u) \text{ plane},$$

where $\lambda_1(\mathbf{U}_L)$, $\lambda_2(\mathbf{U}_L)$, $\mathcal{W}_{3B}(\mathbf{U}_R)$, $\mathbf{U}_L^\dagger$, $\mathcal{L}_2(\mathbf{U}_L^\dagger)$ are defined as (2.1), (2.4), (3.9), (3.5). Then, the Riemann solution has the form

$$\begin{aligned} &R_1(\mathbf{U}_L, \mathbf{U}_L^\dagger) \oplus W_4(\mathbf{U}_L^\dagger, \mathbf{U}_L^{\dagger s}) \oplus S_1(\mathbf{U}_L^{\dagger s}, \mathbf{U}_L^{\dagger s\#}) \\ &\oplus W_4(\mathbf{U}_L^{\dagger s\#}, \mathbf{U}_L^{\dagger s\# b}) \oplus W_2(\mathbf{U}_L^{\dagger s\# b}, \mathbf{U}_*) \oplus W_3(\mathbf{U}_*, \mathbf{U}_R), \end{aligned} \tag{3.12}$$

where the states $\mathbf{U}_L^\dagger$, $\mathbf{U}_L^{\dagger s}$, $\mathbf{U}_L^{\dagger s\#}$, $\mathbf{U}_L^{\dagger s\# b}$, $\mathbf{U}_*$ are computed as follows:

- Compute $\mathbf{U}_L^\dagger \in \mathcal{W}_1(\mathbf{U}_L)$ such that $\lambda_1(\mathbf{U}_L^\dagger) = w$.
- Compute the intersection point $(p_*, u_*) = \mathcal{L}_2(\mathbf{U}_L^\dagger) \cap \mathcal{W}_{3B}(\mathbf{U}_R)$ in projecting onto (p,u)-plane, and get the parameter α_M corresponding to this point.
- Compute $\mathbf{U}_L^{\dagger s} = \Upsilon(\mathbf{U}_L^\dagger; \alpha_M)$.
- Compute $\mathbf{U}_L^{\dagger s\#} \in \mathcal{W}_1(\mathbf{U}_L^{\dagger s})$ such that $\sigma_1(\mathbf{U}_L^{\dagger s}, \mathbf{U}_L^{\dagger s\#}) = w$.
- Compute $\mathbf{U}_L^{\dagger s\# b} = \Upsilon(\mathbf{U}_L^{\dagger s\#}; \alpha_R)$.
- Compute $\mathbf{U}_* = (\rho_*, u_*, p_*, \alpha_R) \in \mathcal{W}_{3B}(\mathbf{U}_R)$.

Lemma 3.6. *For the Riemann solution of Construction B2 as shown in* (3.12)*, we have* (3.4) *and* (3.11)*.*

Proof. For $w \geq 0$, as in the proof of Lemma 3.5, we obtain (3.11) and

$$(\alpha\rho u)^{\text{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) \geq 0.$$

For $w < 0$ and $u_* \leq 0$, as in the proof in Lemma 3.1, we obtain (3.4) and

$$(\alpha\rho u)^{\text{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) \leq 0.$$

For $w < 0$ and $u_* > 0$, also as in the proof in Lemma 3.1, we have the following results:

$$(\alpha\rho u)^{\text{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) \leq \alpha_L \rho_L^\dagger (u_L^\dagger - w),$$

and

$$(\alpha\rho u)^{\text{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) \geq 0.$$

We infer from Lemma 3.4 that

$$(\alpha\rho u)^{\text{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) \leq \alpha_L \rho_L (u_L - \lambda_1(\mathbf{U}_L)).$$

This terminates the proof. □

3.6. Construction B3

Given a left-hand state $\mathbf{U}_L = (\rho_L, u_L, p_L, \alpha_L)$ and a right-hand state $\mathbf{U}_R = (\rho_R, u_R, p_R, \alpha_R)$ such that $|\alpha_R - \alpha_L|$ is sufficiently small. Assume that

$$\lambda_1(\mathbf{U}_L) < w \le \lambda_2(\mathbf{U}_L),\ \mathcal{W}_{3B}(\mathbf{U}_R) \cap \mathcal{L}_4(\mathbf{U}_L) \neq \emptyset \text{ in projecting onto } (p,u) \text{ plane},$$

where $\lambda_1(\mathbf{U}_L)$, $\lambda_2(\mathbf{U}_L)$, $\mathcal{W}_{3B}(\mathbf{U}_R)$ are defined as (2.1), (2.4), and $\mathcal{L}_4(\mathbf{U}_L)$ is defined by

$$\mathcal{L}_4(\mathbf{U}_L) = \{\Upsilon(\mathbf{U}_*; \alpha_R) : \mathbf{U}_* \in \mathcal{W}_1(\mathbf{U}_L), p_L^\dagger \le p_* \le p_L^\flat\},$$

where $\mathcal{W}_1(\mathbf{U}_L)$, $\mathbf{U}_L^\flat$, $\mathbf{U}_L^\dagger$ are defined as (2.2), (3.7), (3.9). Then, the Riemann solution has the form

$$W_1(\mathbf{U}_L, \mathbf{U}_*) \oplus W_4(\mathbf{U}_*, \mathbf{U}_*^b) \oplus W_2(\mathbf{U}_*^b, \mathbf{U}_{**}) \oplus W_3(\mathbf{U}_{**}, \mathbf{U}_R), \tag{3.13}$$

where the states $\mathbf{U}_*$, $\mathbf{U}_*^b$, $\mathbf{U}_{**}$ are computed as follows:

- Compute the intersection point $(p_{**}, u_{**}) = \mathcal{L}_4(\mathbf{U}_L) \cap \mathcal{W}_{3B}(\mathbf{U}_R)$ in projecting onto (p,u)-plane, and get the parameter p_* corresponding to this point.
- Compute $\mathbf{U}_* = (\rho_*, u_*, p_*, \alpha_L) \in \mathcal{W}_1(\mathbf{U}_L)$.
- Compute $\mathbf{U}_*^b = \Upsilon(\mathbf{U}_*; \alpha_R)$.
- Compute $\mathbf{U}_{**} = (\rho_{**}, u_{**}, p_{**}, \alpha_R) \in \mathcal{W}_{3B}(\mathbf{U}_R)$.

Lemma 3.7. *For the Riemann solution of Construction B3 as shown in* (3.13)*, we have* (3.4) *and* (3.11)*.*

Proof. Consider the case $w \ge 0$. Since $\lambda_2(\mathbf{U}^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w)) \ge w$, we always have

$$(\alpha\rho u)^{\mathrm{Rie}}(0+; \mathbf{U}_L, \mathbf{U}_R, w) \ge 0.$$

If $W_1(\mathbf{U}_L, \mathbf{U}_*)$ is a 1-shock wave, as in the proof in Lemma 3.3, we obtain

$$(\alpha\rho u)^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) \le \alpha_L\rho_L u_L.$$

If $W_1(\mathbf{U}_L, \mathbf{U}_*)$ is a 1-rarefaction wave, we have three cases as follows.

- If $0 \le \lambda_1(\mathbf{U}_L)$, then $\mathbf{U}^{\mathrm{Rie}}(0\pm; \mathbf{U}_L, \mathbf{U}_R, w) = \mathbf{U}_L$, so

$$(\alpha\rho u)^{\mathrm{Rie}}(0-; \mathbf{U}_L, \mathbf{U}_R, w) = \alpha_L\rho_L u_L.$$

- If $\lambda_1(\mathbf{U}_L) < 0 < \lambda_1(\mathbf{U}_*)$, then $\mathbf{U}^{\text{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U} := \text{Fan}_1(0;\mathbf{U}_L,\mathbf{U}_L^\dagger)$. Since

$$u = u_L + \frac{2}{\gamma+1}(0-\lambda_1(\mathbf{U}_L)), \quad \gamma > 1, \quad \rho < \rho_L,$$

we have

$$\rho u < \rho_L \left(u_L + \frac{2}{\gamma+1}(0-\lambda_1(\mathbf{U}_L)) \right) < \rho_L(u_L - \lambda_1(\mathbf{U}_L)),$$

so

$$(\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) \le \alpha_L\rho_L(u_L - \lambda_1(\mathbf{U}_L)).$$

- If $\lambda_1(\mathbf{U}_*) \le 0$, then $\mathbf{U}^{\text{Rie}}(0\pm;\mathbf{U}_L,\mathbf{U}_R,w) = \mathbf{U}_*$. Since

$$u_* = u_L + \frac{2}{\gamma+1}(\lambda_1(\mathbf{U}_*)-\lambda_1(\mathbf{U}_L)), \quad \gamma > 1, \quad \rho_* < \rho_L,$$

we have

$$\rho_* u_* < \rho_L \left(u_L + \frac{2}{\gamma+1}(\lambda_1(\mathbf{U}_*)-\lambda_1(\mathbf{U}_L)) \right) < \rho_L(u_L - \lambda_1(\mathbf{U}_L)),$$

so

$$(\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) \le \alpha_L\rho_L(u_L - \lambda_1(\mathbf{U}_L)).$$

For $w < 0$, as in the proof of Lemma 3.3, we have (3.4) and

$$(\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) \le 0 \quad \text{or} \quad (\alpha\rho u)^{\text{Rie}}(0-;\mathbf{U}_L,\mathbf{U}_R,w) \le \alpha_L\rho_L u_L.$$

This terminates the proof. □

4. Numerical scheme based on Riemann solvers

4.1. Building a numerical scheme based on Riemann solvers

In this section, we revisit the numerical scheme based on Riemann solvers for approximating the weak solutions of the system (1.1) with the initial condition

$$\mathbf{U}(x,0) = \mathbf{U}_0(x), \quad x \in \mathbb{R}.$$

Consider a 1D uniform grid Δx. Suppose that the approximation $\{\mathbf{U}_j^n\}_{j\in\mathbb{Z}}$ of $\mathbf{U}(x,t)$ at the time $t = t_n$ is known. We compute the approximation $\{\mathbf{U}_j^{n+1}\}_{j\in\mathbb{Z}}$ at the time $t = t_n + \Delta t$ by the algorithm as follows.

- Firstly, we compute sequences $\{\alpha_j^{n+1}\}_{j\in\mathbb{Z}}$ by the upwind scheme

$$\alpha_j^{n+1} = \alpha_j^n - \frac{\Delta t}{\Delta x}\big((w_j^n)^+(\alpha_j^n - \alpha_{j-1}^n) + (w_j^n)^-(\alpha_{j+1}^n - \alpha_j^n)\big), \quad j \in \mathbb{Z}, \tag{4.1}$$

where

$$w_j^n := w(x_j,t_n), \quad (w_j^n)^+ := \max\{w_j^n, 0\}, \quad (w_j^n)^- := \min\{w_j^n, 0\}, \quad j \in \mathbb{Z}.$$

- Secondly, we compute the sequence $\{(\rho, u, p)_j^{n+1}\}_{j\in\mathbb{Z}}$ by the scheme

$$\begin{aligned}\mathbf{W}_j^{n+1} &= \mathbf{W}_j^n - \frac{\Delta t}{\Delta x}(\mathbf{F}_{j+1/2,-}^n - \mathbf{F}_{j-1/2,+}^n)\\ &\quad + \frac{1}{2}\frac{\Delta t}{\Delta x}(\alpha_{j+1/2,-}^n - \alpha_{j-1/2,+}^n)(\mathbf{H}_{j+1/2,-}^n + \mathbf{H}_{j-1/2,+}^n), \quad j \in \mathbb{Z},\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}\mathbf{W} &:= [\alpha\rho, \alpha\rho u, \alpha\rho E]^{\mathrm{T}},\\ \mathbf{F}(\mathbf{U}) &:= [\alpha\rho u, \alpha(\rho u^2 + p), \alpha u(\rho E + p)]^{\mathrm{T}},\\ \mathbf{H}(\mathbf{U}, w) &:= [0, p, pw]^{\mathrm{T}},\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}_{j+1/2,-}^n &:= \mathbf{F}(\mathbf{U}^{\mathrm{Rie}}(0-; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n, w_{j+1/2}^n)),\\ \mathbf{F}_{j-1/2,+}^n &:= \mathbf{F}(\mathbf{U}^{\mathrm{Rie}}(0+; \mathbf{U}_{j-1}^n, \mathbf{U}_j^n, w_{j-1/2}^n)),\\ \alpha_{j+1/2,-}^n &:= \alpha^{\mathrm{Rie}}(0-; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n, w_{j+1/2}^n),\\ \alpha_{j-1/2,+}^n &:= \alpha^{\mathrm{Rie}}(0+; \mathbf{U}_{j-1}^n, \mathbf{U}_j^n, w_{j-1/2}^n),\\ \mathbf{H}_{j+1/2,-}^n &:= \mathbf{H}(\mathbf{U}^{\mathrm{Rie}}(0-; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n, w_{j+1/2}^n), w_{j+1/2}^n),\\ \mathbf{H}_{j-1/2,+}^n &:= \mathbf{H}(\mathbf{U}^{\mathrm{Rie}}(0+; \mathbf{U}_{j-1}^n, \mathbf{U}_j^n, w_{j-1/2}^n), w_{j-1/2}^n),\\ w_{j+1/2}^n &:= w(x_{j+1/2}, t_n), \quad j \in \mathbb{Z},\end{aligned}$$

and Δt must satisfy the CFL condition

$$\frac{\Delta t}{\Delta x}\max\left\{|\lambda_k(\mathbf{U}_j^n)|, |w_j^n|, |w_{j+1/2}^n| : j \in \mathbb{Z}, k = 1, 2, 3\right\} \le \frac{1}{2}. \tag{4.3}$$

4.2. Positivity preserving

Theorem 4.1. *The scheme* (4.1), (4.2) *preserve the positivity of the volume fraction and density in the first phase of flow.*

Proof. Assume that $\alpha_j^n > 0$ and $\rho_j^n > 0$ for all $j \in \mathbb{Z}$. We first prove that $\alpha_j^{n+1} > 0$ for all $j \in \mathbb{Z}$. Indeed, the upwind scheme (4.1) says that

$$\alpha_j^{n+1} = \begin{cases}\alpha_j^n & \text{if } w_j^n = 0,\\ \left(1 - \frac{\Delta t}{\Delta x}w_j^n\right)\alpha_j^n + \frac{\Delta t}{\Delta x}w_j^n\alpha_{j-1}^n & \text{if } w_j^n > 0,\\ \left(1 + \frac{\Delta t}{\Delta x}w_j^n\right)\alpha_j^n - \frac{\Delta t}{\Delta x}w_j^n\alpha_{j+1}^n & \text{if } w_j^n < 0,\end{cases} \quad j \in \mathbb{Z}.$$

By (4.3), it follows that $\alpha_j^{n+1} > 0$ for all $j \in \mathbb{Z}$. Now, we will prove that $(\alpha\rho)_j^{n+1} > 0$ for all $j \in \mathbb{Z}$. Indeed, the first equation of the scheme (4.2) leads to

$$(\alpha\rho)_j^{n+1} = (\alpha\rho)_j^n - \frac{\Delta t}{\Delta x}\left((\alpha\rho u)_{j+1/2,-}^n - (\alpha\rho u)_{j-1/2,+}^n\right), \quad j \in \mathbb{Z}.$$

Consider the case $\lambda_2(\mathbf{U}_j^n) \geq w_{j+1/2}^n$ for all $j \in \mathbb{Z}$. By Lemmas 3.1, 3.2, 3.3, 3.5, 3.6 and 3.7, it follows that

$$\begin{aligned}(\alpha\rho u)_{j+1/2,-}^n &\leq \max\left\{(\alpha\rho u)_j^n, (\alpha\rho)_j^n(u_j^n - w_{j+1/2}^n), (\alpha\rho)_j^n(u_j^n - \lambda_1(\mathbf{U}_j^n))\right\},\\ (\alpha\rho u)_{j-1/2,+}^n &\geq \min\left\{(\alpha\rho u)_j^n, (\alpha\rho)_j^n(u_j^n - \lambda_3(\mathbf{U}_j^n))\right\}.\end{aligned} \tag{4.4}$$

Consider the case $\lambda_2(\mathbf{U}_j^n) < w_{j+1/2}^n$ for all $j \in \mathbb{Z}$. By the transformation $x \mapsto -x$, $u \mapsto -u$, $w \mapsto -w$, the left-hand state (right-hand state, respectively) $\mathbf{U}_j^n = (\rho_j^n, u_j^n, p_j^n, \alpha_j^n)$ will become the right-hand state (left-hand state, respectively) $\mathbf{V}_j^n = (\rho_j^n, -u_j^n, p_j^n, \alpha_j^n)$ and $\lambda_2(\mathbf{V}_j^n) > -w_{j+1/2}^n$, so

$$\begin{aligned}(\alpha\rho u)_{j+1/2,-}^n &\leq \max\left\{(\alpha\rho u)_j^n, (\alpha\rho)_j^n(u_j^n - \lambda_1(\mathbf{U}_j^n))\right\},\\ (\alpha\rho u)_{j-1/2,+}^n &\geq \min\left\{(\alpha\rho u)_j^n, (\alpha\rho)_j^n(u_j^n - w_{j-1/2}^n), (\alpha\rho)_j^n(u_j^n - \lambda_3(\mathbf{U}_j^n))\right\}.\end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we conclude that

$$(\alpha\rho u)_{j+1/2,-}^n - (\alpha\rho u)_{j-1/2,+}^n \leq 2(\alpha\rho)_j^n u_{\max},$$

where

$$u_{\max} := \max\left\{|\lambda_k(\mathbf{U}_j^n)|, |w_{j+1/2}^n| : j \in \mathbb{Z}, k = 1, 2, 3\right\}.$$

We thus get

$$(\alpha\rho)_j^{n+1} \geq (\alpha\rho)_j^n - \frac{\Delta t}{\Delta x} 2(\alpha\rho)_j^n u_{\max} = (\alpha\rho)_j^n \left(1 - \frac{\Delta t}{\Delta x} 2u_{\max}\right) > 0,$$

since $\frac{\Delta t}{\Delta x} u_{\max} \leq 1/2$ by (4.3). This completes the proof. □

4.3. C-property

Theorem 4.2. *The scheme* (4.1), (4.2) *preserve all the steady states such that*

$$u = 0, \quad p = \textit{constant}, \quad \alpha = \textit{constant}. \tag{4.6}$$

Proof. Assume that the sequence $\{\mathbf{U}_j^n\}_{j\in\mathbb{Z}}$ satisfies (4.6). Since $u = 0$, $\alpha =$ constant and $p =$ constant, according to 2-contact wave, we obtain

$$\mathbf{U}_{j+1/2,-}^n = \mathbf{U}_j^n = [\rho_j^n, u_j^n = 0, p_j^n, \alpha_j^n]^{\mathrm{T}}, \quad \mathbf{U}_{j-1/2,+}^n = [(\rho_j^n)^*, u_{j-1}^n = 0, p_{j-1}^n, \alpha_{j-1}^n]^{\mathrm{T}}, \quad j \in \mathbb{Z}.$$

Since $\alpha_j^n = \alpha_{j-1}^n$ and $p_j^n = p_{j-1}^n$ for all $j \in \mathbb{Z}$, we have

$$\mathbf{F}_{j+1/2,-}^n - \mathbf{F}_{j-1/2,+}^n = \begin{bmatrix} 0 \\ \alpha_j^n p_j^n \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \alpha_{j-1}^n p_{j-1}^n \\ 0 \end{bmatrix} = \mathbf{0},$$

$$\alpha_{j+1/2,-}^0 - \alpha_{j-1/2,+}^0 = \alpha_j^0 - \alpha_{j-1}^0 = 0, \quad j \in \mathbb{Z},$$

so the scheme (4.1), (4.2) yield

$$\alpha_j^{n+1} = \alpha_j^n, \quad \mathbf{W}_j^{n+1} = \mathbf{W}_j^n, \quad j \in \mathbb{Z},$$

or $\mathbf{U}_j^{n+1} = \mathbf{U}_j^n$ for all $j \in \mathbb{Z}$. This terminates the proof. □

Theorem 4.3. *The scheme* (4.1), (4.2) *preserve all the steady states such that*

$$w = 0, \quad u = 0, \quad p = constant. \tag{4.7}$$

Proof. Assume that the sequence $\{\mathbf{U}_j^n\}_{j\in\mathbb{Z}}$ satisfies (4.7). Since $w = 0$, according to Riemann solution in Construction B3 as shown in (3.13), we obtain

$$\begin{aligned}
\mathbf{U}_{j+1/2,-}^n &= \mathbf{U}_j^n = [\rho_j^n, u_j^n = 0, p_j^n, \alpha_j^n]^{\mathrm{T}},\\
\mathbf{U}_{j-1/2,+}^n &= \Upsilon(\mathbf{U}_{j-1}^n; \alpha_j^n) = [\rho_{j-1}^n, u_{j-1}^n = 0, p_{j-1}^n, \alpha_j^n]^{\mathrm{T}}, \quad j \in \mathbb{Z}.
\end{aligned}$$

Since $p_j^0 = p_{j-1}^0$ for all $j \in \mathbb{Z}$, we have

$$\begin{aligned}
\mathbf{F}_{j+1/2,-}^n - \mathbf{F}_{j-1/2,+}^n &= \begin{bmatrix} 0 \\ \alpha_j^n p_j^n \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \alpha_j^n p_{j-1}^n \\ 0 \end{bmatrix} = \mathbf{0},\\
\alpha_{j+1/2,-}^n - \alpha_{j-1/2,+}^n &= \alpha_j^n - \alpha_j^n = 0, \quad j \in \mathbb{Z},
\end{aligned}$$

so the scheme (4.1), (4.2) yield

$$\alpha_j^{n+1} = \alpha_j^n, \quad \mathbf{W}_j^{n+1} = \mathbf{W}_j^n, \quad j \in \mathbb{Z},$$

or $\mathbf{U}_j^{n+1} = \mathbf{U}_j^n$ for all $j \in \mathbb{Z}$. This terminates the proof. □

4.4. Partially well-balanced scheme

Theorem 4.4. *Consider* (1.1) *with* $w = 0$. *The scheme* (4.1), (4.2) *preserve all the steady states* $\mathbf{U}(x)$ *such that*

$$\alpha\rho u = constant, \quad u^2 + \frac{2\gamma(p+\pi)}{(\gamma-1)\rho} = constant, \quad \frac{p+\pi}{\rho^\gamma} = constant, \tag{4.8}$$

and $\lambda_1(\mathbf{U}(x)) \geq 0$ *for all* $x \in \mathbb{R}$ *(or* $\lambda_1(\mathbf{U}(x)) \leq 0 \leq \lambda_2(\mathbf{U}(x))$ *for all* $x \in \mathbb{R}$*).*

Proof. Assume that the sequence $\{\mathbf{U}_j^n\}_{j\in\mathbb{Z}}$ satisfies (4.8). According to 4-constant-speed wave, we have

$$\mathbf{U}_{j+1/2,-}^n = \mathbf{U}_j^n, \quad \mathbf{U}_{j-1/2,+}^n = \mathbf{U}_j^n, \quad j \in \mathbb{Z}.$$

So, the scheme (4.1), (4.2) yield

$$\alpha_j^{n+1} = \alpha_j^n, \quad \mathbf{W}_j^{n+1} = \mathbf{U}_j^n, \quad j \in \mathbb{Z}.$$

This terminates the proof. □

5. Numerical experiments

5.1. Test for C-property

Test 1. Consider the system (1.1) where w is given by

$$w(x,t) = \sin(x+t), \quad x \in \mathbb{R}, \geq 0,$$

and the parameters is given by

$$\gamma = 1.4, \quad \pi = 0. \tag{5.1}$$

Consider the initial data given by

$$\mathbf{U}_0(x) = [0.5\exp(x^2), 0, 0.5, 0.3]^{\mathrm{T}}, \quad x \in \mathbb{R},$$

The exact solution of this problem is just the smooth stationary wave

$$\mathbf{U}(x,t) = \mathbf{U}_0(x), \quad x \in \mathbb{R}, \ t \geq 0.$$

We compute the numerical solution by the scheme (4.1), (4.2) with TOL $= 10^{-14}$ and Neumann boundary condition. Figure 5.1 displays this steady state and the numerical solution for the mesh size $\Delta x = 1/80$ at the time $t = 0.1$ in the computational domain $x \in [-1, 1]$ of Test 1. The L^1 errors and CPU times are reported by Table 5.1, which show that the scheme (4.1), (4.2) can preserve the C-property perfectly.

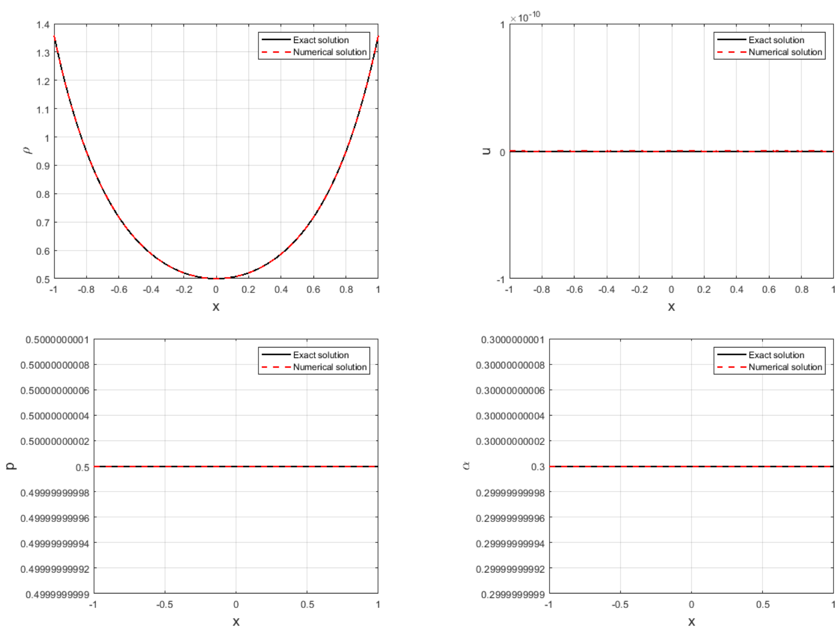

Figure 5.1: Exact solution and numerical solution by the scheme (4.1), (4.2) for the mesh size $\Delta x = 1/80$ at the time $t = 0.1$ of Test 1.

Table 5.1: Errors and CPU times of Test 1.

Δx	L^1 error	CPU time
1/10	1.67E − 14	0.7
1/20	2.30E − 14	1.1
1/40	3.31E − 14	3.6
1/80	5.68E − 14	13.1
1/160	9.27E − 14	52.1
1/320	1.39E − 13	205.3
1/640	1.88E − 13	748.5
1/1280	2.63E − 13	3146.3

Test 2. Consider the steady state

$$\mathbf{U}(x) = [0.5\exp(x^2), 0, 1, 0.7\exp(-x^2)]^{\mathrm{T}}, \quad x \in \mathbb{R}$$

for the system (1.1) with $w = 0$. The numerical solution is computed by the scheme (4.1), (4.2) with TOL $= 10^{-14}$, and Neumann boundary condition, where the parameters are given as (5.1). Figure 5.2 displays this steady state and the numerical solution for the mesh size $\Delta x = 1/80$ at the time $t = 0.1$ in the computational domain $x \in [-1, 1]$ of Test 2. The L^1 errors and CPU times are reported by Table 5.2. Again, the C-property of our scheme is confirmed.

Table 5.2: Errors and CPU times of Test 2.

Δx	L^1 error	CPU time
1/10	8.94E − 16	2.3
1/20	4.55E − 15	2.9
1/40	6.74E − 14	10.4
1/80	9.80E − 14	35.7
1/160	1.07E − 13	144.6
1/320	1.31E − 13	565.3
1/640	1.38E − 13	2326.3
1/1280	1.53E − 13	10533.0

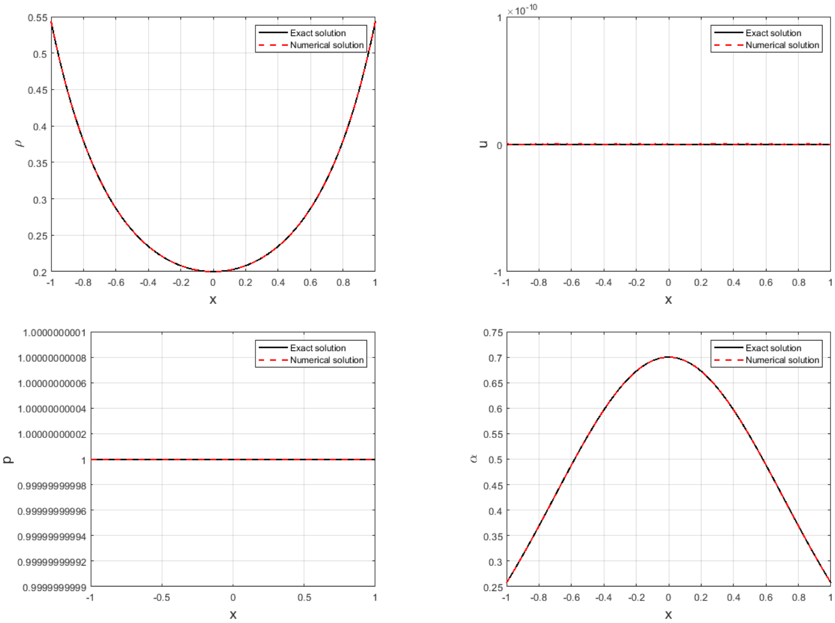

Figure 5.2: Exact solution and numerical solution by the scheme (4.1), (4.2) for the mesh size $\Delta x = 1/80$ at the time $t = 0.1$ of Test 2.

5.2. Test for partially well-balanced property

Test 3. Consider the steady state

$$\mathbf{U}(x) = [\rho(x), u(x), p(x), \alpha(x)]^{\mathrm{T}}, \quad x \in [-1, 1],$$

where

$$\rho(x) = \left(\frac{\gamma - 1}{2\gamma}(5 - \exp(-x^2))\right)^{\frac{1}{\gamma-1}}, \qquad u(x) = \exp(-0.5x^2),$$
$$p(x) = \left(\frac{\gamma - 1}{2\gamma}(5 - \exp(-x^2))\right)^{\frac{\gamma}{\gamma-1}} - \pi, \quad \alpha(x) = \frac{0.1}{\left(\frac{\gamma-1}{2\gamma}(5 - \exp(-x^2))\right)^{\frac{1}{\gamma-1}} \exp(-0.5x^2)}$$

for the system (1.1) with $w = 0$. The numerical solution is computed by the scheme (4.1), (4.2) with TOL $= 10^{-14}$, and Neumann boundary condition, where the parameters are given as (5.1). Figure 5.3 displays this steady state and the numerical solution for the mesh size $\Delta x = 1/80$ at the time $t = 0.1$ in the computational domain $x \in [-1, 1]$ of Test 3. The L^1 errors, orders of convergence and CPU times are reported by Table 5.3. The main tendency of L^1 errors is downward to zero with the order of 1.93, so the partially well-balanced property of our scheme is verified.

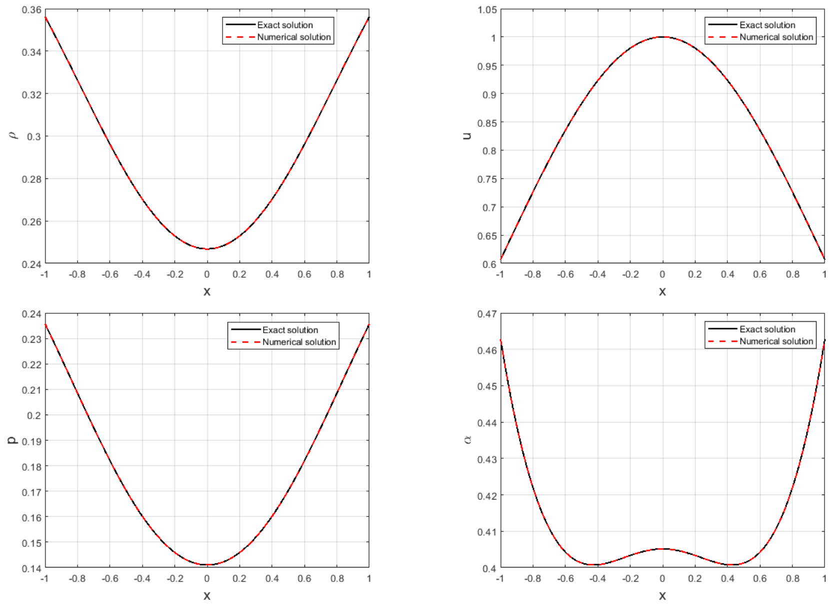

Figure 5.3: Exact solution and numerical solution by the scheme (4.1), (4.2) for the mesh size $\Delta x = 1/80$ at the time $t = 0.1$ of Test 3.

Table 5.3: Errors, orders of convergence, and CPU times of Test 3.

Δx	L^1 error	Order	CPU time
$1/10$	$1.58\text{E}-4$	–	0.5
$1/20$	$1.59\text{E}-5$	3.32	1.5
$1/40$	$1.55\text{E}-5$	0.04	6.0
$1/80$	$3.10\text{E}-6$	2.32	20.1
$1/160$	$5.06\text{E}-7$	2.62	80.3
$1/320$	$3.65\text{E}-8$	3.79	330.4
$1/640$	$3.61\text{E}-8$	0.02	1378.2
$1/1280$	$3.51\text{E}-9$	3.36	5560.8
$1/2560$	$3.50\text{E}-9$	0.01	21636.0

5.3. Test for a smooth constant speed contact wave

Test 4. Consider the smooth constant speed contact wave

$$\mathbf{U}(x,t) = [1, 2, 5, 0.5\exp(-(x-2t)^2)]^{\mathrm{T}}, \quad x \in \mathbb{R},\ t \geq 0 \tag{5.2}$$

for the system (1.1) with $w = 2$. Initial and boundary condition are set according to (5.2). The numerical solution is computed by the scheme (4.1), (4.2) with $\mathrm{TOL} = 10^{-6}$.

Figure 5.3 displays the exact solution and the numerical solution for the mesh size $\Delta x = 1/80$ at the time $t = 0.1$ in the computational domain $x \in [-1, 1]$ of Test 4. The L^1 errors, orders of convergence, and CPU times are reported by Table 5.4. This confirms the accuracy and convergence of our scheme toward a smooth constant speed contact wave.

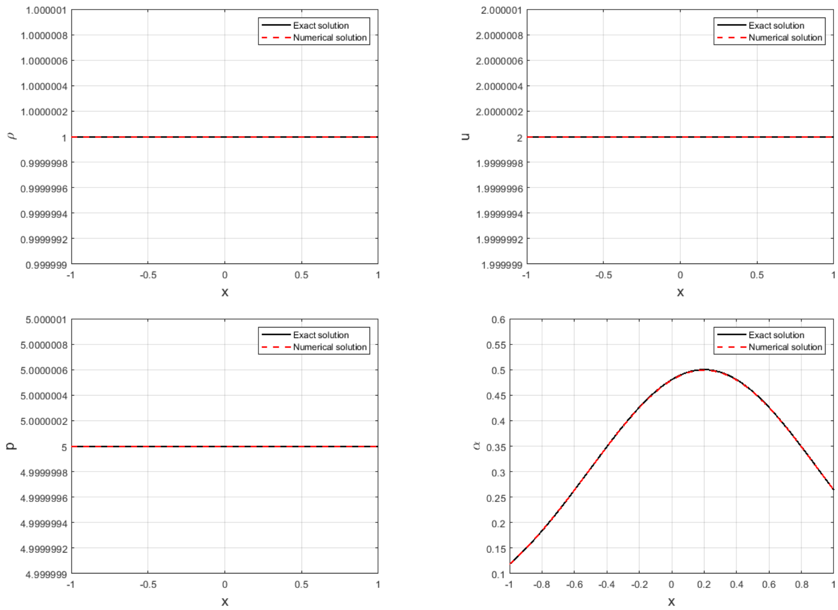

Figure 5.4: Exact solution and numerical solution by the scheme (4.1), (4.2) for the mesh size $\Delta x = 1/80$ at the time $t = 0.1$ of Test 4.

Table 5.4: Errors of Test 4.

Δx	L^1 error	Order	CPU time
1/10	7.35E − 3	–	0.7
1/20	3.73E − 3	0.98	2.1
1/40	1.88E − 3	0.99	7.4
1/80	9.44E − 4	0.99	28.3
1/160	4.73E − 4	0.99	100.1
1/320	2.40E − 4	0.98	418.3
1/640	1.26E − 4	0.93	1658.1
1/1280	6.83E − 5	0.88	6898.5
1/2560	3.93E − 5	0.80	27693.0

5.4. Test for a Riemann problem with w constant

Test 5. Consider the system (1.1) with $w = 1.5$ and the parameters is given as (5.1). Consider the initial data given by

$$\mathbf{U}_0(x) = \begin{cases} \mathbf{U}_L = [1, 3, 4, 0.2]^{\mathrm{T}} & \text{if } x < 0, \\ \mathbf{U}_R = [2.439569, 3.654866, 7, 0.3]^{\mathrm{T}} & \text{if } x > 0. \end{cases}$$

The exact solution of this problem is Construction B2 as shown in (3.12), where

$$\begin{aligned} \mathbf{U}_L^{\dagger} &= [0.729921, 3.722027, 2.574225, 0.2]^{\mathrm{T}}, \\ \mathbf{U}_L^{\dagger s} &= [0.384725, 4.742883, 1.050198, 0.26]^{\mathrm{T}}, \\ \mathbf{U}_L^{\dagger s\#} &= [0.819435, 3.022536, 3.196536, 0.26]^{\mathrm{T}}, \\ \mathbf{U}_L^{\dagger s\# b} &= [0.881960, 2.725986, 3.543134, 0.3]^{\mathrm{T}}, \\ \mathbf{U}_* &= [1.5, 2.725986, 3.543134, 0.3]^{\mathrm{T}}. \end{aligned}$$

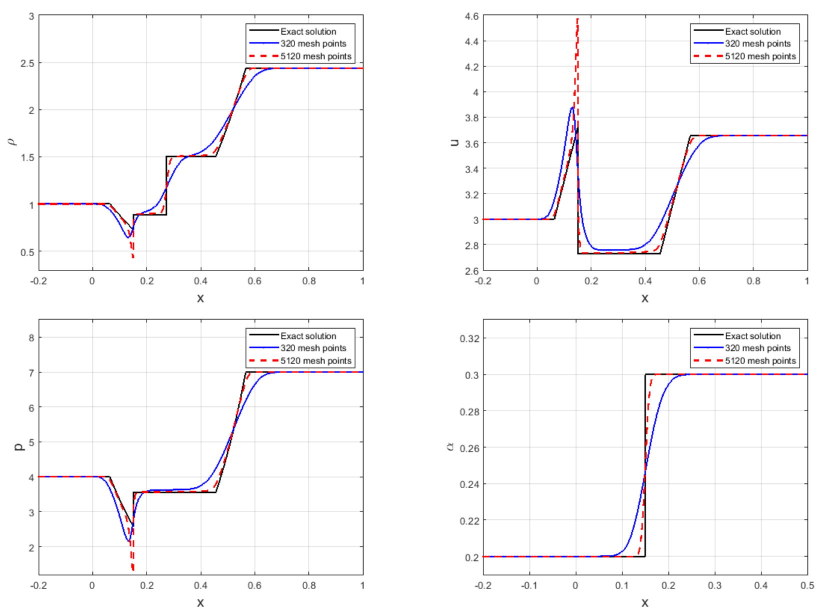

Figure 5.5: Exact solution and numerical solutions by the scheme (4.1), (4.2) for the mesh sizes $\Delta x = 1/160$, $\Delta x = 1/2560$ at the time $t = 0.1$ of Test 5.

An interesting aspect of this problem is the resonance of three waves with the same speed w: $W_4(\mathbf{U}_L^{\dagger}, \mathbf{U}_L^{\dagger s})$, $S_1(\mathbf{U}_L^{\dagger s}, \mathbf{U}_L^{\dagger s\#})$, and $W_4(\mathbf{U}_L^{\dagger s\#}, \mathbf{U}_L^{\dagger s\# b})$. We compute the numerical solution by the scheme (4.1), (4.2) with $\mathrm{TOL} = 10^{-6}$ and Neumann boundary condition. Figure 5.5 displays the exact solution and the numerical solutions for the mesh sizes $\Delta x = 1/160$, $\Delta x = 1/2560$ at the time $t = 0.1$ in the computational domain $x \in [-1, 1]$

of Test 5. The L^1 errors, L^1 relative errors, orders of convergence, and CPU times are reported by Table 5.5. These results show that our scheme performs well toward the proposed Riemann solution with w constant. However, it appears that the scheme given by equations (4.1) and (4.2) does not accurately capture state $\mathbf{U}_L^\dagger$, instead it captures state $\mathbf{U}_L^{\dagger s}$. This discrepancy can be explained by the inaccuracy of the quantity α calculated from the upwind scheme in equation (4.1).

Table 5.5: Errors, orders of convergence, and CPU times of Test 5.

Δx	L^1 error	L^1 relative error	Order	CPU time
1/10	6.84E − 1	3.45%	–	0.6
1/20	6.27E − 1	3.25%	0.13	2.3
1/40	4.73E − 1	2.49%	0.41	7.1
1/80	3.67E − 1	1.94%	0.37	27.3
1/160	2.85E − 1	1.52%	0.36	117.7
1/320	2.21E − 1	1.17%	0.37	447.6
1/640	1.67E − 1	0.89%	0.40	1981.4
1/1280	1.21E − 1	0.65%	0.46	7775.6
1/2560	8.60E − 2	0.46%	0.50	31521.0

6. Conclusions

In this paper, a numerical scheme for the spray model (1.1) is constructed. This scheme is based on the exact solution of local Riemann problems at each grid cell with w constant. The well-balanced and C-property of this scheme is proved in the sense that it can capture exactly some families of steady solutions. Moreover, the proposed scheme can preserve the positivity of density and volume fraction. The method is shown by numerical experiments to possess a good accuracy.

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References

[1] A. Ambroso, C. Chalons, F. Coquel and T. Galié, *Relaxation and numerical approximation of a two-fluid two-pressure diphasic model*, M2AN Math. Model. Numer. Anal. **43** (2009), no. 6, 1063–1097.

[2] A. Ambroso, C. Chalons and P.-A. Raviart, *A Godunov-type method for the seven-equation model of compressible two-phase flow*, Comput. & Fluids **54** (2012), 67–91.

[3] E. Audusse, F. Bouchut, M.-O. Bristeau, R. Klein and B. Perthame, *A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows*, SIAM J. Sci. Comput. **25** (2004), no. 6, 2050–2065.

[4] M. R. Baer and J. W. Nunziato, *A two-phase mixture theory for the deflagration-to-detonation transition (DDT) in reactive granular materials*, Int. J. Multiph. Flow **12** (1986), no. 6, 861–889.

[5] M. Baudin, F. Coquel and Q.-H. Tran, *A semi-implicit relaxation scheme for modeling two-phase flow in a pipeline*, SIAM J. Sci. Comput. **27** (2005), no. 3, 914–936.

[6] J. B. Bdzil, R. Menikoff, S. F. Son, A. K. Kapila and D. S. Steward, *Two-phase modeling of deflagration-to-detonation transition in granular materials: A critical examination of modeling issues*, Phys. Fluids **11** (1999), 378–402.

[7] R. Botchorishvili, B. Perthame and A. Vasseur, *Equilibrium schemes for scalar conservation laws with stiff sources*, Math. Comp. **72** (2003), no. 241, 131–157.

[8] R. Botchorishvili and O. Pironneau, *Finite volume schemes with equilibrium type discretization of source terms for scalar conservation laws*, J. Comput. Phys. **187** (2003), no. 2, 391–427.

[9] A. Chinnayya, A.-Y. LeRoux and N. Seguin, *A well-balanced numerical scheme for the approximation of the shallow-water equations with topography: the resonance phenomenon*, Int. J. Finite Vol. **1** (2004), no. 1, 33 pp.

[10] F. Coquel, J.-M. Hérard, K. Saleh and N. Seguin, *Two properties of two-velocity two-pressure models for two-phase flows*, Commun. Math. Sci. **12** (2014), no. 3, 593–600.

[11] D. H. Cuong and M. D. Thanh, *A Godunov-type scheme for the isentropic model of a fluid flow in a nozzle with variable cross-section*, Appl. Math. Comput. **256** (2015), 602–629.

[12] ______, *A high-resolution van Leer-type scheme for a model of fluid flows in a nozzle with variable cross-section*, J. Korean Math. Soc. **54** (2017), no. 1, 141–175.

[13] ______, *A well-balanced van Leer-type numerical scheme for shallow water equations with variable topography*, Adv. Comput. Math. **43** (2017), no. 5, 1197–1225.

[14] ______, *Building a Godunov-type numerical scheme for a model of two-phase flows*, Comput. & Fluids **148** (2017), 69–81.

[15] G. Dal Maso, P. G. Lefloch and F. Murat, *Definition and weak stability of nonconservative products*, J. Math. Pures Appl. (9) **74** (1995), no. 6, 483–548.

[16] U. S. Fjordholm, S. Mishra and E. Tadmor, *Well-balanced and energy stable schemes for the shallow water equations with discontinuous topography*, J. Comput. Phys. **230** (2011), no. 14, 5587–5609.

[17] J. M. Gallardo, C. Parés and M. Castro, *On a well-balanced high-order finite volume scheme for shallow water equations with topography and dry areas*, J. Comput. Phys. **227** (2007), no. 1, 574–601.

[18] T. Gallouët, J.-M. Hérard and N. Seguin, *Some approximate Godunov schemes to compute shallow-water equations with topography*, Comput. & Fluids **32** (2003), no. 4, 479–513.

[19] P. Goatin and P. G. LeFloch, *The Riemann problem for a class of resonant hyperbolic systems of balance laws*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **21** (2004), no. 6, 881–902.

[20] J. M. Greenberg and A. Y. Leroux, *A well-balanced scheme for the numerical processing of source terms in hyperbolic equations*, SIAM J. Numer. Anal. **33** (1996), no. 1, 1–16.

[21] T. Y. Hou and P. LeFloch, *Why nonconservative schemes converge to wrong solutions: Error analysis*, Math. Comp. **62** (1994), no. 206, 497–530.

[22] E. Isaacson and B. Temple, *Nonlinear resonance in systems of conservation laws*, SIAM J. Appl. Math. **52** (1992), no. 5, 1260–1278.

[23] ______, *Convergence of the* 2×2 *Godunov method for a general resonant nonlinear balance law*, SIAM J. Appl. Math. **55** (1995), no. 3, 625–640.

[24] B. L. Keyfitz, R. Sander and M. Sever, *Lack of hyperbolicity in the two-fluid model for two-phase incompressible flow*, Discrete Contin. Dyn. Syst. Ser. B **3** (2003), no. 4, 541–563.

[25] P. LeFloch, *Entropy weak solutions to nonlinear hyperbolic systems under nonconservative form*, Comm. Partial Differential Equations **13** (1988), no. 6, 669–727.

[26] P. G. Lefloch and M. D. Thanh, *The Riemann problem for fluid flows in a nozzle with discontinuous cross-section*, Commun. Math. Sci. **1** (2003), no. 4, 763–797.

[27] ______, *The Riemann problem for the shallow water equations with discontinuous topography*, Commun. Math. Sci. **5** (2007), no. 4, 865–885.

[28] ______, *A Godunov-type method for the shallow water equations with discontinuous topography in the resonant regime*, J. Comput. Phys. **230** (2011), no. 20, 7631–7660.

[29] D. Marchesin and P. J. Paes-Leme, *A Riemann problem in gas dynamics with bifurcation: Hyperbolic partial differential equations III*, Comput. Math. Appl. Part A **12** (1986), no. 4-5, 433–455.

[30] R. Saurel and R. Abgrall, *A multiphase Godunov method for compressible multifluid and multiphase flows*, J. Comput. Phys. **150** (1999), no. 2, 425–467.

[31] D. W. Schwendeman, C. W. Wahle and A. K. Kapila, *The Riemann problem and a high-resolution Godunov method for a model of compressible two-phase flow*, J. Comput. Phys. **212** (2006), no. 2, 490–526.

[32] M. D. Thanh, *The Riemann problem for a nonisentropic fluid in a nozzle with discontinuous cross-sectional area*, SIAM J. Appl. Math. **69** (2009), no. 6, 1501–1519.

[33] ______, *A phase decomposition approach and the Riemann problem for a model of two-phase flows*, J. Math. Anal. Appl. **418** (2014), no. 2, 569–594.

[34] M. D. Thanh and D. H. Cuong, *Existence of solutions to the Riemann problem for a model of two-phase flows*, Electron. J. Differential Equations **2015**, No. 32, 18 pp.

[35] M. D. Thanh, D. H. Cuong and D. X. Vinh, *The resonant cases and the Riemann problem for a model of two-phase flows*, J. Math. Anal. Appl. **494** (2021), no. 1, Paper No. 124578, 28 pp.

[36] M. D. Thanh and D. X. Vinh, *The Riemann problem for a spray model*, Appl. Anal. **102** (2023), no. 7, 1854–1873.

[37] N. X. Thanh, M. D. Thanh and D. H. Cuong, *Dimensional splitting well-balanced schemes on Cartesian mesh for 2D shallow water equations with variable topography*, Bull. Iranian Math. Soc. **48** (2022), no. 5, 2321–2348.

[38] B. Tian, E. F. Toro and C. E. Castro, *A path-conservative method for a five-equation model of two-phase flow with an HLLC-type Riemann solver*, Comput. & Fluids **46** (2011), 122–132.

Dao Huy Cuong
Department of Mathematics and Computer Science, Ho Chi Minh City University of Education, 280 An Duong Vuong str., District 5, Ho Chi Minh City, Vietnam
E-mail address: `cuongdh@hcmue.edu.vn`

Mai Duc Thanh
Department of Mathematics, International University, Quarter 6, Linh Trung Ward, Thu Duc City, Ho Chi Minh City, Vietnam
and
Vietnam National University, Ho Chi Minh City, Vietnam
E-mail address: `mdthanh@hcmiu.edu.vn`