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Spanning Trees with at most 5 Leaves and Branch Vertices in Total of $K_{1,5}$-free Graphs

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Abstract. In this paper, we prove that every n-vertex connected $K_{1,5}$-free graph G with $\sigma_4(G) \geq n-1$ contains a spanning tree with at most 5 leaves and branch vertices in total. Moreover, the degree sum condition "$\sigma_4(G) \geq n-1$" is best possible.

1. Introduction

In this paper, we only consider finite simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, we use $N_G(v)$ and $d_G(v)$ (or $N(v)$ and $d(v)$ if there is no ambiguity) to denote the set of neighbors of v and the degree of v in G, respectively. For any $X \subseteq V(G)$, we denote by $|X|$ the cardinality of X. We define $N(X) = \bigcup_{x \in X} N(x)$ and $d(X) = \sum_{x \in X} d(x)$. For an integer $k \geq 1$, we let $N_k(X) = \{x \in V(G) \mid |N(x) \cap X| = k\}$. We use $G - X$ to denote the graph obtained from G by deleting the vertices in X together with their incident edges. The subgraph of G induced by X is denoted by $G[X]$. We define $G - uv$ to be the graph obtained from G by deleting the edge $uv \in E(G)$, and $G + uv$ to be the graph obtained from G by adding an edge uv between two non-adjacent vertices u and v of G. We write $A := B$ to rename B as A.

A subset $X \subseteq V(G)$ is called an *independent set* of G if no two vertices of X are adjacent in G. The maximum cardinality of an independent set in G is denoted by $\alpha(G)$. For $k \geq 1$, we define $\sigma_k(G) = \min \left\{ \sum_{i=1}^{k} d(v_i) \mid \{v_1, \ldots, v_k\} \text{ is an independent set in } G \right\}$. For $r \geq 1$, a graph is said to be *$K_{1,r}$-free* if it does not contain $K_{1,r}$ as an induced subgraph. A $K_{1,3}$-free graph is also called a *claw-free* graph.

Let T be a tree. A vertex of degree one is a *leaf* of T and a vertex of degree at least three is a *branch vertex* of T. The set of leaves of T is denoted by $L(T)$ and the set of branch vertices of T is denoted by $B(T)$. For two distinct vertices u, v of T, we denote by $P_T[u,v]$ the unique path in T connecting u and v and denote by $d_T[u,v]$ the distance between u and v in T. We define the *orientation* of $P_T[u,v]$ is from u to v.

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There are many known results on the independence number conditions and the degree sum conditions to ensure that a connected graph G contains a spanning tree with a bounded number of leaves or branch vertices. Win [20] obtained a sufficient condition related to the independence number for k-connected graphs having a few leaves, which confirms a conjecture of Las Vergnas [14]. On the other hand, Broersma and Tuinstra [1] gave a degree sum condition for a connected graph to contain a spanning tree with a bounded number of leaves. Beside that, recently, the first named author [7] stated an improvement of Win's result by giving an independence number condition for a graph having a spanning tree which covers a certain subset of $V(G)$ and has at most l leaves.

In 2012, Kano et al. [11] presented a degree sum condition for a connected claw-free graph to have a spanning tree with at most l leaves, which generalizes a result of Matthews and Sumner [17] and a result of Gargano et al. [5]. Later, Chen et al. [2], Matsuda et al. [16] and Gould and Shull [6] also considered the sufficient conditions for a connected claw-free graph to have a spanning tree with few leaves or few branch vertices, respectively.

On the other hand, Kyaw [12,13] obtained the sharp sufficient conditions for connected $K_{1,4}$-free graphs to have a spanning tree with few leaves. After that, many researchers also studied sufficient conditions for existence of spanning trees with few leaves or few branch vertices in connected $K_{1,4}$-free graphs (see Chen et al. [3] and Ha [8] for examples).

For the $K_{1,5}$-free graphs, some results were obtained as follows.

Theorem 1.1. [4] *Let G be a connected $K_{1,5}$-free graph with n vertices. If $\sigma_5(G) \geq n-1$, then G contains a spanning tree with at most 4 leaves.*

Theorem 1.2. [10] *Let G be a connected $K_{1,5}$-free graph with n vertices. If $\sigma_6(G) \geq n-1$, then G contains a spanning tree with at most 5 leaves.*

Moreover, many researchers have also studied the degree sum conditions for graphs to have spanning trees with a bounded number of branch vertices and leaves.

Theorem 1.3. [18,19] *Let $k \geq 2$ be an integer. If a connected graph G satisfies $\deg_G(x) + \deg_G(y) \geq |G| - k + 1$ for every two non-adjacent vertices $x, y \in V(G)$, then G has a spanning tree T with $|L(T)| + |B(T)| \leq k + 1$.*

In 2019, Maezawa et al. improved the previous result by proving the following theorem.

Theorem 1.4. [15] *Let $k \geq 2$ be an integer. Suppose that a connected graph G satisfies $\max\{\deg_G(x), \deg_G(y)\} \geq (|G|-k+1)/2$ for every two non-adjacent vertices $x, y \in V(G)$, then G has a spanning tree T with $|L(T)| + |B(T)| \leq k + 1$.*

Recently, Hanh and the first named author also gave sharp results for the case of claw-free graphs and $K_{1,4}$-free graphs, respectively.

Theorem 1.5. [9] *Suppose that a connected claw-free graph G of order n satisfies $\sigma_5(G) \geq n-2$. Then G has a spanning tree T with $|B(T)| + |L(T)| \leq 5$.*

Theorem 1.6. [8] *Let k and m be two nonnegative integers with $m \leq k+1$ and let G be a connected $K_{1,4}$-free graph of order n. If $\sigma_{m+2}(G) \geq n-k$, then G has a spanning tree with at most $m+k+2$ leaves and branch vertices.*

In this paper, we further consider connected $K_{1,5}$-free graphs. We give a sufficient condition for a connected $K_{1,5}$-free graph to have a spanning tree with few leaves and branch vertices in total. More precisely, we prove the following theorem.

Theorem 1.7. *Let G be a connected $K_{1,5}$-free graph with n vertices. If $\sigma_4(G) \geq n-1$, then G contains a spanning tree with at most 5 leaves and branch vertices in total.*

It is easy to see that if a tree has at least 2 branch vertices then it has at least 4 leaves. Therefore, we immediately obtain the following corollary from Theorem 1.7.

Corollary 1.8. *Let G be a connected $K_{1,5}$-free graph with n vertices. If $\sigma_4(G) \geq n-1$, then G contains a spanning tree with at most 1 branch vertices.*

We end this section by constructing an example to show that the degree sum condition "$\sigma_4(G) \geq n-1$" in Theorem 1.7 is sharp. For an integer $m \geq 1$, let D_1, D_2, D_3, D_4 be vertex-disjoint copies of the complete graph K_m with m vertices. Let xy be an edge such that neither x nor y is contained in $\bigcup_{i=1}^{4} V(D_i)$. Join x to all the vertices in $V(D_1) \cup V(D_2)$ and join y to all the vertices in $V(D_3) \cup V(D_4)$. The resulting graph is denoted by G. Then it is easy to check that G is a connected $K_{1,5}$-free graph with $n = 4m+2$ vertices and $\sigma_4(G) = 4m = n-2$. However, every spanning tree of G contains at least 6 leaves and branch vertices in total.

2. Proof of the main result

In this section, we extend the idea of Chen–Ha–Hanh in [4] to prove Theorem 1.7. For this purpose, we need the following lemma.

Lemma 2.1. *Let G be a connected graph such that G does not have a spanning tree with at most 5 leaves and branch vertices in total, and let T be a maximal tree of G with $|L(T)| + |B(T)| \in \{6, 7\}$. Then there does not exist a tree T' in G such that $|L(T')| + |B(T')| \leq 5$ and $V(T') = V(T)$.*

Proof. Suppose for a contradiction that there exists a tree T' in G with at most 5 leaves and branch vertices in total and $V(T') = V(T)$. Since G has no spanning tree with at most 5 leaves and branch vertices in total, we see that $V(G) - V(T') \neq \emptyset$. Hence there must exist two vertices v and w in G such that $v \in V(T')$ and $w \in N(v) \cap (V(G) - V(T'))$. Let T_1 be the tree obtained from T' by adding the vertex w and the edge vw. Then $|L(T_1)| + |B(T_1)| - |L(T')| - |B(T')| \in \{0, 1, 2\}$.

If $|L(T_1)| + |B(T_1)| \in \{6, 7\}$, then T_1 contradicts the maximality of T (since $|V(T_1)| = |V(T)| + 1 > |V(T)|$). So we may assume that $|L(T_1)| + |B(T_1)| \leq 5$. By repeating this process, we can recursively construct a tree T_{i+1} from T_i for $i \geq 1$ in G such that $|L(T_i)| + |B(T_i)| \leq 5$ and $|V(T_{i+1})| = |V(T_i)| + 1$ for each $i \geq 1$. Since G has no spanning tree with at most 5 leaves and branch vertices in total and $|V(G)|$ is finite, the process must terminate after a finite number of steps, i.e., there exists some $k \geq 1$ such that T_{k+1} is a tree in G such that $|L(T_{k+1})| + |B(T_{k+1})| \in \{6, 7\}$. But this contradicts the maximality of T. So the lemma holds. □

Proof of Theorem 1.7. We prove the theorem by contradiction. Suppose to the contrary that G contains no spanning tree with at most 5 leaves and branch vertices in total. Then every spanning tree of G contains at least 6 leaves and branch vertices in total. We choose a maximal tree T of G with $|L(T)| + |B(T)| \in \{6, 7\}$.

In all such spanning trees, we choose T such that

(C) $|L(T)|$ is as small as possible.

We consider two cases according to the number of leaves in T.

Case 1*:* $|L(T)| \leq 4$. Since $|L(T)| \geq |B(T)| + 2$ and $|L(T)| + |B(T)| \in \{6, 7\}$ we obtain $|B(T)| = 2$ and $|L(T)| = 4$. Let s and t be two branch vertices in T and let $U = \{u_1, u_2, u_3, u_4\}$ be the set of leaves of T. Then $d_T(s) = d_T(t) = 3$. Moreover, by the maximality of T, we have $N(U) \subseteq V(T)$. For simplifying notation, let $[k]$ be the set of $\{1, 2, \ldots, k\}$ for some positive integer k.

For each $i \in [4]$, let B_i be the vertex set of the connected component of $T - \{s, t\}$ containing u_i and let v_i be the unique vertex in $B_i \cap N_T(\{s, t\})$. Without loss of generality, we may assume that $\{v_1, v_2\} \subseteq N_T(s)$ and $\{v_3, v_4\} \subseteq N_T(t)$. For each $1 \leq i \leq 4$ and $x \in B_i$, we use x^- and x^+ to denote the predecessor and the successor of x on $P_T[s, u_i]$ or $P_T[t, u_i]$, respectively (if such a vertex exists). Let s^+ be the successor of s on $P_T[s, t]$. Define $P := V(P_T[s, t]) - \{s, t\}$.

For this case, we further choose T such that

(C1) $d_T[s, t]$ is as small as possible.

Claim 2.2. *For all* $1 \leq i, j \leq 4$ *and* $i \neq j$*, if* $x \in N(u_j) \cap B_i$*, then* $x \neq u_i$*,* $x \neq v_i$ *and* $x^- \notin N(U - \{u_j\})$*.*

Proof. Suppose $x = u_i$ or $x = v_i$. Then $T' := T - v_i v_i^- + x u_j$ is a tree in G with 3 leaves and 1 branch vertex such that $V(T') = V(T)$, which contradicts Lemma 2.1. So we have $x \neq u_i$, $x \neq v_i$.

Next, assume $x^- \in N(U - \{u_j\})$. Then there exists some $k \in [4] - \{j\}$ such that $x^- u_k \in E(G)$. Now, $T' := T - \{v_i v_i^-, xx^-\} + \{xu_j, x^- u_k\}$ is a tree in G with 3 leaves and 1 branch vertex such that $V(T') = V(T)$, also contradicting Lemma 2.1. This proves Claim 2.2. □

By Claim 2.2, we know that U is an independent set in G.

Claim 2.3. *$N(u_i) \cap P = \emptyset$ for each $i \in [4]$.*

Proof. Suppose the assertion of the claim is false. Then there exists some vertex $x \in P$ such that $xu_i \in E(G)$ for some $i \in [4]$. Let $T' := T - v_i v_i^- + xu_i$, then T' is a tree in G such that $V(T') = V(T)$, T' has 4 leaves and 2 branch vertices s' and t' and $d_{T'}[s', t'] < d_T[s, t]$. But this contradicts the condition (C1). So the claim holds. □

Claim 2.4. *$N(u_i) \cap \{t\} = \emptyset$ for each $i \in [2]$.*

Proof. Suppose $su_i \in E(G)$ for some $i \in [2]$. Consider the tree $T' := T - v_i v_i^- + tu_i$ is a tree in G with 4 leaves and 1 branch vertex such that $V(T') = V(T)$, contradicting Lemma 2.1. This proves Claim 2.4. □

Similarly, we also have

Claim 2.5. *$N(u_i) \cap \{s\} = \emptyset$ for each $3 \leq i \leq 4$.*

Claim 2.6. *$N_2(U - u_i) \cap B_i = \emptyset$ for each $i \in [4]$. In particular, $N_3(U) = (N_2(U) - N(u_i)) \cap B_i = \emptyset$ for each $i \in [4]$.*

Proof. For the sake of convenience, we may assume by symmetry that $i \in [2]$.

Suppose this is false. Then there exists some vertex $x \in (N_2(U - u_i)) \cap B_i$ for some $i \in [2]$. By applying Claim 2.2, we have $x \neq u_i$ and $x \neq v_i$.

Since $x \in N_2(U - u_i) \cap B_i$ there must exist two distinct indices $j, k \in [4] - \{i\}$, $j < k$, such that $xu_j, xu_k \in E(G)$. Set

$$T' := \begin{cases} T - \{v_j v_j^-, v_k v_k^-\} + \{xu_j, xu_k\} & \text{if } j = 3 - i, \\ T - \{ss^+, v_k v_k^-\} + \{xu_j, xu_k\} & \text{if } 3 \leq j < k \leq 4. \end{cases}$$

Then T' is a tree in G with 1 branch vertex and 4 leaves such that $V(T') = V(T)$, contradicting Lemma 2.1. □

By Claims 2.2 and 2.6, $\{u_i\}$, $N(u_i) \cap B_i$, and $(N(U - \{u_i\}) \cap B_i)^-$ are pairwise disjoint subsets in B_i for each $i \in [4]$ (where $(N(U - \{u_i\}) \cap B_i)^- = \{x^- \mid x \in N(U - \{u_i\}) \cap B_i\}$)

and $N_3(U) = (N_2(U) - N(u_i)) \cap B_i = \emptyset$ for each $i \in [4]$. Then for each $i \in [4]$, we conclude that

$$\begin{aligned} |B_i| &\geq 1 + |N(u_i) \cap B_i| + |(N(U - \{u_i\}) \cap B_i)^-| \\ &= 1 + |N(u_i) \cap B_i| + |N(U - \{u_i\}) \cap B_i| \\ &= 1 + \sum_{j=1}^{4} |N(u_j) \cap B_i|. \end{aligned}$$

By applying Claim 2.3, we obtain

$$\sum_{i=1}^{4} |N(u_i) \cap P| = 0.$$

On the other hand, by Claims 2.4 and 2.5 we obtain that

$$\sum_{i=1}^{4} |N(u_i) \cap \{s\}| \leq 2, \quad \sum_{i=1}^{4} |N(u_i) \cap \{t\}| \leq 2.$$

Note that $N(U) \subseteq V(T)$. Now, we conclude that

$$\begin{aligned} |V(T)| &= \sum_{i=1}^{4} |B_i| + |V(P_T[s,t])| \\ &\geq \sum_{i=1}^{4} \left(\sum_{j=1}^{4} |N(u_j) \cap B_i| + 1 \right) \\ &\quad + \left(\sum_{i=1}^{4} |N(u_i) \cap \{s\}| + \sum_{i=1}^{4} |N(u_i) \cap \{t\}| - 2 + \sum_{i=1}^{4} |N(u_i) \cap P| \right) \\ &= 2 + \sum_{i=1}^{4} \sum_{j=1}^{4} |N(u_j) \cap B_i| + \sum_{i=1}^{4} |N(u_i) \cap \{s,t\}| + \sum_{i=1}^{4} |N(u_i) \cap P| \\ &= \sum_{j=1}^{4} |N(u_j) \cap V(T)| + 2 \\ &= \sum_{j=1}^{4} d(u_j) + 2 \\ &= d(U) + 2. \end{aligned}$$

Since U is an independent set in G, we have

$$n - 1 \leq \sigma_4(G) \leq d(U) \leq |V(T)| - 2 \leq n - 2,$$

a contradiction.

Case 2: $|L(T)| \geq 5$. Set $L(T) = \{u_i\}_{i=1}^{l}$, $l \geq 5$.

Claim 2.7. *$L(T)$ is an independent set in G.*

Proof. Suppose to the contrary that there exist 2 distinct indices i, j such that $u_iu_j \in E(G)$. Let c be the nearest branch vertex of u_i in T and c^- is the predecessor of c on $P_T[u_i, c]$. Let $T' := T - cc^- + u_iu_j$. Then T' is a tree in G with $V(T') = V(T)$, $|B(T')| \leq |B(T)|$ and $|L(T')| < |L(T)|$, a contradiction to either Lemma 2.1 or the condition (C). Then the claim holds. □

By Claim 2.7, we know that $\sigma_5(G)$ is non-trivially defined. Since $\sigma_4(G) \geq n - 1$, we have $\sigma_5(G) \geq \sigma_4(G) + 1 \geq n - 1 + 1 = n$. Thanks to Theorem 1.1, there exists a spanning tree T' in G such that $|L(T')| \leq 4$. Hence $|L(T')| + |B(T')| \leq 6$. By assumption, we obtain $|L(T')| + |B(T')| = 6$. Now, using the similar arguments as in the proof of Case 1, we can derive the contradiction. Therefore, the proof of Theorem 1.7 is completed. □

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