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Nonnegative Holomorphic Sectional Curvature on Compact Almost Hermitian Manifolds

Masaya Kawamura

Abstract. We study nonnegative holomorphic sectional curvature on a compact almost Hermitian manifold. In the positive case, we show some geometric conditions for negative Kodaira dimension. In the zero case, we give some conditions of the Chern–Yamabe problem for zero Chern scalar curvature.

1. Introduction

The holomorphic sectional curvature plays an important role not only in differential geometry but also in algebraic geometry. In the early 1990s, Yau [26] asked whether a compact Kähler manifold with positive holomorphic sectional curvature has negative Kodaira dimension. Yang answered Yau's question in a more general setting, stating that if the holomorphic sectional curvature on a compact Hermitian manifold is positive, then the manifold has negative Kodaira dimension (see [25]). In this paper, we generalize this problem to almost Hermitian geometry.

Let (M^{2n}, J) be an almost complex manifold of real dimension $2n$ with $n \geq 3$ and let g be an almost Hermitian metric on M. Let $\{e_r\}$ be an arbitrary local $(1,0)$-frame around a fixed point $p \in M$ and let $\{\theta^r\}$ be the associated coframe. Then the associated real $(1,1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1} g_{r\overline{k}} \theta^r \wedge \theta^{\overline{k}}$. We will also refer to ω as to an almost Hermitian metric in the present paper. We define a Gauduchon metric and a k-th Gauduchon metric on an almost Hermitian manifold in the following.

Definition 1.1. [19, Definition 1.1] Let (M^{2n}, J, ω) be a real $2n$-dimensional almost Hermitian manifold. An almost Hermitian metric ω is called Gauduchon if ω satisfies that

$$\partial\overline{\partial}\omega^{n-1} = 0.$$

For an integer k such that $1 \leq k \leq n-1$, an almost Hermitian metric ω is called k-th Gauduchon if the metric ω satisfies that

$$\partial\overline{\partial}\omega^k \wedge \omega^{n-k-1} = 0.$$

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From the definition, we see that $(n-1)$-th Gauduchon metrics are the usual Gauduchon metrics. Fino and Ugarte have shown that for each $k = 1, \dots, [n/2]-1$, a Hermitian metric is k-th Gauduchon if and only if it is $(n-k-1)$-th Gauduchon on a complex nilmanifold (see [7, Lemma 4.7]). Latorre and Ugarte have investigated the k-th Gauduchon condition on homogeneous compact complex manifolds (see [20]).

One has the following well-known result.

Proposition 1.2. [9] *Let (M^{2n}, J, ω) be a compact almost Hermitian manifold with $n \geq 2$. Then there exists a smooth function u, unique up to addition of a constant, such that the conformal almost Hermitian metric $e^u\omega$ is Gauduchon.*

We have characterized the k-th Gauduchon condition on a compact almost Hermitian manifold as follows:

Proposition 1.3. [19, Theorem 1.1] *Let (M^{2n}, J, ω) be compact almost Hermitian manifold with $n \geq 3$ and let k be an integer such that $1 \leq k \leq n-1$. Then the following are equivalent.*

(i) *ω is k-th Gauduchon;*

(ii) $s_\omega - \widehat{s}_\omega = \frac{k-1}{n-2}|\partial^*\omega|^2 + \frac{n-k-1}{n-2}|\partial\omega|^2 + T^{\bar{r}}_{ij}T^{i}_{\bar{r}\bar{j}}$,

where s_ω is the Chern scalar curvature and $\widehat{s}_\omega$ is the Riemannian type scalar curvature of the metric ω with respect to the Chern connection (see (2.7)), $\partial^* = -*\overline{\partial}*$ is the adjoint operator (see Lemma 2.3), $|\partial\omega|^2 = \frac{1}{2!}g^{i\bar{j}}g^{p\bar{q}}g^{k\bar{l}}(\partial\omega)_{ip\bar{l}}\overline{(\partial\omega)_{jq\bar{k}}}$, $|\partial^*\omega|^2 = g^{i\bar{j}}(\partial^*\omega)_{\bar{j}}\overline{(\partial^*\omega)_{\bar{i}}}$, and $T^{k}_{\bar{i}\bar{j}}$'s are components of the torsion (see Section 2 for more detail). Note that $T^{\bar{r}}_{ij}T^{i}_{\bar{r}\bar{j}}$ means that we sum over repeated indices i, j and r with respect to the metric ω, that is, $T^{\bar{r}}_{ij}T^{i}_{\bar{r}\bar{j}} = g^{j\bar{k}}g^{i\bar{s}}g^{p\bar{q}}T_{ijp}T_{\bar{q}\bar{k}\bar{s}}$, where $T_{ijp} = T^{\bar{l}}_{ij}g_{p\bar{l}}$, $T_{\bar{q}\bar{k}\bar{s}} = T^{r}_{\bar{q}\bar{k}}g_{r\bar{s}}$.

Remark 1.4. From Proposition 1.13, we have that for a Gauduchon metric (i.e., $(n-1)$-th Gauduchon) ω,

$$s_\omega - \widehat{s}_\omega = |\partial^*\omega|^2 + T^{\bar{r}}_{ij}T^{i}_{\bar{r}\bar{j}} \tag{1.1}$$

on a real $2n$-dimensional compact almost Hermitian manifold (M^{2n}, J, ω) with $n \geq 3$. Especially, if the manifold (M^{2n}, J, ω) is quasi-Kähler and satisfies that $T^{\bar{r}}_{ij}T^{i}_{\bar{r}\bar{j}} = 0$, then we have that $s_\omega = \widehat{s}_\omega$ from the formula (1.1). Here, recall that a quasi-Kähler structure is an almost Hermitian structure whose real $(1,1)$-form ω satisfies $(d\omega)^{(1,2)} = \overline{\partial}\omega = 0$, which is equivalent to the original definition of quasi-Kählerianity: $D_XJ(Y) + D_{JX}J(JY) = 0$ for all vector fields X, Y, where D is the Levi-Civita connection (see [11]). The quasi Kählerity is equivalent to that $T^{k}_{ij} = 0$ for all i, j, k on an almost Hermitian manifold.

The difference appears in the formula (ii) in Proposition 1.3 compared to the one in the complex case [4, Proposition 1.7] is obviously the term $T_{ij}^{\bar{r}}T_{\bar{r}\bar{j}}^{i}$. We investigate this term in some special cases below.

We denote by Ω the curvature of the Chern connection ∇ on an almost Hermitian manifold. We can regard Ω as a section of $\Lambda^2 M\otimes \mathrm{End}(T^{1,0}M)$, $\Omega\in\Gamma(\Lambda^2 M\otimes\mathrm{End}(T^{1,0}M))$ and Ω splits in

$$\Omega=\Omega^{(2,0)}+\Omega^{(1,1)}+\Omega^{(0,2)}=H+R+\overline{H},$$

where $R\in\Gamma(\Lambda^{1,1}M\otimes\mathrm{End}(T^{1,0}M))$, $H\in\Gamma(\Lambda^{2,0}M\otimes\mathrm{End}(T^{1,0}M))$ and $\overline{H}\in\Gamma(\Lambda^{0,2}M\otimes\mathrm{End}(T^{1,0}M))$. Note that since the Chern connection has torsion, R and H do not satisfy the first Bianchi identity and do not satisfy $R(X,Y,Z,W)=R(Z,W,X,Y)$, $H(X,Y,Z,W)=H(Z,W,X,Y)$ in general. By choosing a local unitary $(1,0)$-frame $\{e_i\}$ with respect to g, we have that (see Lemma 2.2)

$$\begin{aligned}
R_{i\bar{j}k\bar{l}}&=g(\nabla_i\nabla_{\bar{j}}e_k-\nabla_{\bar{j}}\nabla_i e_k-\nabla_{[e_i,e_{\bar{j}}]}e_k,e_{\bar{l}}),\\
H_{ijk\bar{l}}&=g(\nabla_i\nabla_j e_k-\nabla_j\nabla_i e_k-\nabla_{[e_i,e_j]}e_k,e_{\bar{l}}),\\
H_{\bar{i}\bar{j}k\bar{l}}&=g(\nabla_{\bar{i}}\nabla_{\bar{j}}e_k-\nabla_{\bar{j}}\nabla_{\bar{i}}e_k-\nabla_{[e_{\bar{i}},e_{\bar{j}}]}e_k,e_{\bar{l}}).
\end{aligned}$$

We define the curvature operator by

$$R^L(X,Y)Z=D_XD_YZ-D_YD_XZ-D_{[X,Y]}Z,$$

where D is the Levi-Civita connection, and define the curvatuer tensor

$$R^L(X,Y,Z,W):=g(R^L(X,Y)Z,W),$$

which satisfies the following symmetries:

$$\begin{gathered}
R^L(X,Y,Z,W)=-R^L(Y,X,Z,W),\quad R^L(X,Y,Z,W)=-R^L(X,Y,W,Z),\\
R^L(X,Y,Z,W)+R^L(Y,Z,X,W)+R^L(Z,X,Y,W)=0,\\
R^L(X,Y,Z,W)=R^L(Z,W,X,Y).
\end{gathered}$$

We define the Kähler-likeness and the G-Kähler-likeness on almost Hermitian manifolds as follows.

Definition 1.5. [2], [15, Definition 1.2] Given an almost Hermitian manifold (M^{2n},J,g), almost Hermitian metric g will be called Kähler-like, if $R_{X\overline{Y}Z\overline{W}}=R_{Z\overline{Y}X\overline{W}}$ for any $(1,0)$-tangent vectors X, Y, Z and W. When the almost Hermitian metric g is Kähler-like, the triple (M^{2n},J,g) will be called a Kähler-like almost Hermitian manifold. Similarly, if $R^L_{XY\overline{ZW}}=R^L_{XYZ\overline{W}}=0$ for any type $(1,0)$ tangent vectors X, Y, Z and W, we will say that g is G-Kähler-like. When the almost Hermitian metric g is G-Kähler-like, the triple (M^{2n},J,g) will be called a G-Kähler-like almost Hermitian manifold.

The Kähler-likeness can be restated by using following notations: The curvature R^L of the Levi-Civita connection D satisfies the first Bianchi identity:

$$\sum_{\sigma\in\mathfrak{S}} R^L(\sigma X, \sigma Y)\sigma Z = 0, \tag{1Bnc}$$

sum over circular permutation. The curvature $\Omega^{(1,1)} = R$ of the Chern connection ∇ satisfies

$$R(X,Y,Z,W) = R(X,Y,JZ,JW) = R(JX,JY,Z,W). \tag{Cplx}$$

This condition (Cplx) for the curvature R^L is referred to as the AH_1-condition and an almost Hermitian manifold satisfies (Cplx) for R^L is called an AH_1-manifold.

We define the Kähler-likeness in the way of [2, Definition 4] as follows.

Definition 1.6. Let (M, J, ω) be an almost Hermitian manifold. Let $\widetilde{\nabla}$ be a metric connection on this manifold. We say that the curvature of the connection $\widetilde{\nabla}$ is Kähler-like if it satisfies both (1Bnc) and (Cplx).

We see the following equivalences which are similar to the ones in [2, Remark 5].

Lemma 1.7. [2, Remark 5]

(i) *An almost Hermitian manifold (M, J, ω) is Kähler-like in the sense of Definition* 1.5 *if and only if the curvature $\Omega^{(1,1)} = R$ of the Chern connection is Kähler-like in the sense of Definition* 1.6.

(ii) *An almost Hermitian manifold (M, J, ω) is G-Kähler-like in the sense of Definition* 1.5 *if and only if the curvature R^L of the Levi-Civita connection is Kähler-like in the sense of Definition* 1.6.

Remark 1.8. Notice that since the curvature $\Omega^{(1,1)} = R$ with respect to the Chern connection automatically satisfies (Cplx), the condition the curvature has to satisfy is only (1Bnc) to be Kähler-like in the sense of Definition 1.5. From Lemma 1.7(ii), a G-Kähler-like almost Hermitian manifold coincides with an AH_1-manifold.

Remark 1.9. On a compact Kähler-like k-th Gauduchon manifold (M^{2n}, J, ω) with $n \geq 3$ for some integer $1 \leq k \leq n-1$, we have that $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \leq 0$ since we have $s_\omega = \widehat{s}_\omega$ from the Kähler-likeness.

Definition 1.10. [28] An almost Hermitian manifold (M^{2n}, J, ω) is called almost Kähler if $d\omega = 0$. When an almost Hermitian metric ω is almost Kähler, the triple (M^{2n}, J, ω) is called an almost Kähler manifold.

Lemma 1.11. [28] *The almost Kählerity is equivalent to*

$$T_{ij}^{k}=0,\quad T_{ij}^{\overline{k}}+T_{ki}^{\overline{j}}+T_{jk}^{\overline{i}}=0\quad \textit{for all } i,j,k=1,\dots,n.$$

From Lemma 1.11, if assuming that a manifold is almost Kähler, we can compute that

$$\begin{aligned} &T_{ij}^{\overline{r}}T_{\overline{r}\overline{j}}^{i}=-T_{ij}^{\overline{r}}T_{\overline{i}\overline{r}}^{j}-T_{ij}^{\overline{r}}T_{\overline{j}\overline{i}}^{r}=-T_{ji}^{\overline{r}}T_{\overline{r}\overline{i}}^{j}+T_{ij}^{\overline{r}}T_{\overline{i}\overline{j}}^{r}=-T_{ij}^{\overline{r}}T_{\overline{r}\overline{j}}^{i}+|T''|_g^2 \\ \Longleftrightarrow\quad &T_{ij}^{\overline{r}}T_{\overline{r}\overline{j}}^{i}=\frac{1}{2}|T''|_g^2, \end{aligned} \tag{1.2}$$

where g is the associated almost Hermitian metric with respect to the real $(1,1)$-form ω and $|T''|_g^2:=g^{j\overline{k}}g^{i\overline{s}}g_{r\overline{l}}T_{ij}^{\overline{l}}T_{\overline{s}\overline{k}}^{r}$. The computation (1.2) implies that assuming $T_{ij}^{\overline{r}}T_{\overline{r}\overline{j}}^{i}=0$ on an almost Kähler manifold, we must have $T''\equiv 0$ and the manifold must be Kähler.

Lemma 1.12. *Let (M^{2n},J,ω) be an almost Kähler manifold. Then we have that*

$$T_{ij}^{\overline{r}}T_{\overline{r}\overline{j}}^{i}\geq 0.$$

The equality $T_{ij}^{\overline{r}}T_{\overline{r}\overline{j}}^{i}=0$ holds if and only if the almost Kähler manifold is Kähler.

From the formula (1.1), we have that $T_{ij}^{\overline{r}}T_{\overline{r}\overline{j}}^{i}\leq 0$ on a compact Kähler-like manifold (M^{2n},J,ω) with $n\geq 3$ since we have $s_\omega=\widehat{s}_\omega$ under the Kähler-likeness. On a real $2n$-dimensional compact Kähler-like almost Kähler manifold with $n\geq 3$, we obtain that $T_{ij}^{\overline{r}}T_{\overline{r}\overline{j}}^{i}=0$, which implies that the manifold must be Kähler from Lemma 1.12.

Proposition 1.13. *A real $2n$-dimensional compact Kähler-like almost Kähler manifold with $n\geq 3$ is Kähler.*

For the case of $n=2$, on a real 4-dimensional compact Kähler-like almost Hermitian manifold, since it is almost Kähler (see [14, Theorem 1.1]) and we have $s_\omega=\widehat{s}_\omega$ under the Kähler-likeness, by applying the formula (4.9), we have that $T_{ij}^{\overline{r}}T_{\overline{r}\overline{j}}^{i}=0$, which implies that the manifold must be Kähler from Lemma 1.12. It is known that a real 4-dimensional compact Kähler-like AH_3-manifold is Kähler in [14, Corollary 1.1]. We obtain the following improved result.

Proposition 1.14. *A real 4-dimensional compact Kähler-like almost Hermitian manifold is Kähler.*

The condition $T_{ij}^{\overline{r}}T_{\overline{r}\overline{j}}^{i}\geq 0$ appears for instance, on a Kähler-like and G-Kähler-like almost Hermitian manifold (M^{2n},J,ω) with $n\geq 2$, if $T_{ij}^{\overline{r}}T_{\overline{r}\overline{j}}^{i}\geq 0$, then the metric ω is semi-Kähler (see Definition 1.39) and the almost complex structure J is integrable (i.e., the metric ω is balanced) (see [15, Corollary 1.2]). Note that it is known that if an AH_1-manifold is almost Kähler, it must be Kähler (see [13, Theorem 5.1]).

Lemma 1.15. [28, Corollary 3.4] *Let (M^{2n}, J, ω) be an almost Kähler manifold. Then we have that*

$$R^L_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} + T^{\bar{k}}_{ri} T^{l}_{\bar{r}\bar{j}}.$$

On an almost Kähler manifold, we have that by applying Lemmas 1.11, 1.15 and (2.6),

$$T^{\bar{r}}_{ki} T^{l}_{\bar{r}\bar{j}} = R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} = R^L_{i\bar{j}k\bar{l}} - R^L_{k\bar{j}i\bar{l}} - T^{\bar{k}}_{ri} T^{l}_{\bar{r}\bar{j}} + T^{\bar{i}}_{rk} T^{l}_{\bar{r}\bar{j}} = R^L_{ik\bar{j}\bar{l}} - T^{\bar{r}}_{ki} T^{l}_{\bar{r}\bar{j}},$$

which implies that

$$R^L_{ik\bar{j}\bar{l}} = 2T^{\bar{r}}_{ki} T^{l}_{\bar{r}\bar{j}}. \tag{1.3}$$

Proposition 1.16. *A G-Kähler-like almost Kähler manifold is Kähler.*

Proof. The G-Kähler-likeness implies that $R^L_{ik\bar{j}\bar{l}} = 0$ for all i, j, k, l. Hence, from (1.3), we obtain that $T^{\bar{r}}_{ki} T^{l}_{\bar{r}\bar{j}} = 0$ for all i, j, k, l, which gives us that $T^{\bar{r}}_{ki} T^{k}_{\bar{r}\bar{i}} = 0$ and then the manifold must be Kähler from Lemma 1.12. □

The compactness in Proposition 1.13 can be actually omitted.

Proposition 1.17. *A Kähler-like almost Kähler manifold is Kähler.*

Proof. If a real $2n$-dimensional almost Kähler manifold (M^{2n}, J, ω) is Kähler-like, we obtain that $s_\omega = \widehat{s}_\omega$, and then we have that $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ from (4.9) for $n \geq 2$ since we have that $\langle \overline{\partial \partial^*} \omega, \omega \rangle = 0$ from the almost Kählerity. □

Definition 1.18. [28] An almost Hermitian manifold (M^{2n}, J, ω) is called nearly Kähler if $(D_X J)X = 0$ for any tangent vector field X, where D denotes the Levi-Civita connection. When an almost Hermitian metric ω is nearly Kähler, the triple (M^{2n}, J, ω) is called a nearly Kähler manifold.

Lemma 1.19. [28, Lemma 2.4] *The nearly Kählerity is equivalent to*

$$T^{k}_{ij} = 0, \quad T^{\bar{k}}_{ij} = T^{\bar{i}}_{jk} \quad \textit{for all } i, j, k = 1, \ldots, n.$$

From Lemma 1.19, if assuming that a manifold is nearly Kähler, we compute that

$$T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = T^{\bar{r}}_{ij} T^{r}_{\bar{j}\bar{i}} = -T^{\bar{r}}_{ij} T^{r}_{\bar{i}\bar{j}} = -|T''|^2_g, \tag{1.4}$$

which implies we have that assuming $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ on a nearly Kähler manifold, we must have $T'' \equiv 0$ and the manifold must be Kähler.

Lemma 1.20. *Let (M^{2n}, J, ω) be a nearly Kähler manifold. Then we have that*

$$T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \leq 0. \tag{1.5}$$

The equality $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ holds if and only if the nearly Kähler manifold is Kähler.

Combining (1.5) with (4.10), we have that $T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i} = 0$ on a real 4-dimensional nearly Kähler manifold, which implies that $T'' \equiv 0$ from the computation (1.4) on a real 4-dimensional nearly Kähler manifold and the manifold must be Kähler as it has been already proven in [12].

Lemma 1.21. [28, Corollary 3.5] *Let (M^{2n}, J, ω) be a nearly Kähler manifold. Then we have that*

$$R^L_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} + \frac{1}{4} T_{ik}^{\overline{r}} T_{\bar{j}\bar{l}}^{r}.$$

On a nearly Kähler manifold, we have that by applying Lemmas 1.19, 1.21 and (2.8), computing as in (1.3),

$$T_{ki}^{\overline{r}} T_{\overline{r}\overline{j}}^{l} = R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} = R^L_{i\bar{j}k\bar{l}} - R^L_{k\bar{j}i\bar{l}} - \frac{1}{4} T_{ik}^{\overline{r}} T_{\bar{j}\bar{l}}^{r} + \frac{1}{4} T_{ki}^{\overline{r}} T_{\bar{j}\bar{l}}^{r} = R^L_{ik\bar{j}\bar{l}} + \frac{1}{2} T_{ki}^{\overline{r}} T_{\overline{r}\overline{j}}^{l},$$

which implies that

$$R^L_{ik\overline{j}\overline{l}} = \frac{1}{2} T_{ki}^{\overline{r}} T_{\overline{r}\overline{j}}^{l}. \tag{1.6}$$

Proposition 1.22. *A G-Kähler-like nearly Kähler manifold is Kähler.*

Proof. The G-Kähler-likeness implies that $R^L_{ik\overline{j}\overline{l}} = 0$ for all i, j, k, l. Hence, from (1.6), we obtain that $T_{ki}^{\overline{r}} T_{\overline{r}\overline{j}}^{l} = 0$ for all i, j, k, l, which gives us that $T_{ki}^{\overline{r}} T_{\overline{r}\overline{i}}^{k} = 0$ and then the manifold must be Kähler from Lemma 1.20. □

We also have the following result which has been alresdy given in [18, Theorem 1.1].

Proposition 1.23. *A Kähler-like nearly Kähler manifold is Kähler.*

Proof. The Kähler-likeness implies that $s_\omega = \widehat{s}_\omega$, and then we have $T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i} = 0$ from (4.9) for $n \geq 2$ since the nearly Kählerity is included in the quasi-Kählerity and then we have $\langle \overline{\partial \partial}^* \omega, \omega \rangle = 0$. □

Since we know that if a manifold is almost Kähler and nearly Kähler simultaneously, then it is Kähler, i.e., the almost complex structure J is integrable, which is equivalent to that $T_{ij}^{\overline{k}} = 0$ for all $i, j, k = 1, \ldots, n$. This implies that especially $T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i} = 0$ as we see from (1.2) and (1.4).

We recall and introduce the definition of Kodaira dimension on an almost complex manifold by following [5]. Let (M, J) be a compact $2n$-dimensional smooth manifold equipped with an almost complex structure J. Let $\pi^{p,q}$ be the projection to the set of smooth section of $\Lambda^{p,q}M$: $\Gamma(M, \Lambda^{p,q}M)$, where $\Lambda^{p,q}M$ is the bundle of (p, q)-forms on M. The $\overline{\partial}$ and ∂ operator can be defined by

$$\overline{\partial} := \pi^{p,q+1} \circ d \colon \Gamma(M, \Lambda^{p,q}M) \to \Gamma(M, \Lambda^{p,q+1}M),$$
$$\partial := \pi^{p+1,q} \circ d \colon \Gamma(M, \Lambda^{p,q}M) \to \Gamma(M, \Lambda^{p+1,q}M),$$

where d is the exterior differential. Both $\overline{\partial}$ and ∂ satisfy the Leibniz rule, but in general $\overline{\partial}^2$ and ∂^2 may not be zero. Applying $\overline{\partial}$ to a smooth section of the canonical line bundle $\mathcal{K}_M := \Lambda^n(\Lambda^{1,0}M) = \Lambda^{n,0}M$ (see (2.1)), we have

$$\overline{\partial}\colon \Gamma(M, \mathcal{K}_M) \to \Gamma(M, \Lambda^{n,1}M) \cong \Gamma(M, (T^*M)^{0,1} \otimes \mathcal{K}_M).$$

We can extend the $\overline{\partial}$ to an operator $\overline{\partial}_m\colon \Gamma(M, \mathcal{K}_M^{\otimes m}) \to \Gamma(M, (T^*M)^{0,1} \otimes \mathcal{K}_M^{\otimes m})$, $\overline{\partial}_1 := \overline{\partial}$, inductively by the product rule for $m \in \mathbb{Z}_{\geq 2}$, $s_1 \in \Gamma(M, \mathcal{K}_M)$ and $s_2 \in \Gamma(M, \mathcal{K}_M^{\otimes(m-1)})$,

$$\overline{\partial}_m(s_1 \otimes s_2) = \overline{\partial} s_1 \otimes s_2 + s_1 \otimes \overline{\partial}_{m-1} s_2.$$

Then, the operator $\overline{\partial}_m$ satisfies the Leibniz rule

$$\overline{\partial}_m(fs) = \overline{\partial} f \otimes s + f \overline{\partial}_m s$$

for any smooth function $f \in C^\infty(M, \mathbb{R})$ and any smooth section $s \in \Gamma(M, \mathcal{K}_M)$ of $\mathcal{K}_M^{\otimes m}$. Hence, $\overline{\partial}_m$ is a pseudoholomorphic structure on $\mathcal{K}_M^{\otimes m}$. For $m \in \mathbb{Z}_{\geq 1}$, the space of pseudo-holomorphic sections of $\mathcal{K}_M^{\otimes m}$ is defined to be (see [5, Definition 2.1])

$$H^0(M, \mathcal{K}_M^{\otimes m}) = \left\{ s \in \Gamma(M, \mathcal{K}_M^{\otimes m}) : \overline{\partial}_m s = 0 \right\}.$$

The Kodaira dimension on an almost complex manifold (M, J) is defined as follows.

Definition 1.24. [5, Definition 1.2] We define the m^{th}-plurigenus of (M, J) by

$$P_m(M, J) := \dim_{\mathbb{C}} H^0(M, \mathcal{K}_M^{\otimes m}).$$

The Kodaira dimension of (M, J) is defined by

$$\kappa(M) := \begin{cases} -\infty & \text{if } P_m(M, J) = 0 \text{ for any } m \geq 1, \\ \limsup_{m \to \infty} \frac{\log P_m(M,J)}{\log m} & \text{otherwise.} \end{cases}$$

By taking direct products of the Kodaira–Thurston surface $X = S^1 \times (\Gamma \setminus \mathrm{Nil}^3)$ with copies of 2-torus T^2, we have compact $2n$-manifolds with non-integrable almost complex structure and $\kappa = -\infty$ or 0.

By taking direct products of the 4-manifold $X = T^2 \times S$ with copies of 2-torus T^2 or a compact Riemann surface S with genus $g \geq 2$, we get compact $2n$-manifolds with non-integrable almost complex structures and $\kappa = 1, 2, \ldots, n-1$.

Proposition 1.25. [5, Theorem 6.10] *There are examples of compact $2n$-dimensional non-integral almost complex manifolds (M^{2n}, J) with Kodaira dimension $\kappa(M)$ lying among $\{-\infty, 0, 1, \ldots, n-1\}$ for $n \geq 2$.*

We now introduce the following result.

Proposition 1.26. [6, Theorem 4.3] *Let (M^{2n}, J) be a real $2n$-dimensional compact almost complex manifold with $n \geq 2$. If one of the following is satisfied:*

(i) *M admits an almost Hermitian metric with positive Chern scalar curvature everywhere,*

(ii) *M admits a Gauduchon metric with positive total scalar curvature,*

then $\kappa^J(M) = -\infty$.

In 1990s, Yau proposed the following question (see [26, Problem 67]). Let $\mathrm{HCF}(\omega)$ denote the holomorphic sectional curvature of the metric ω (see (2.9) for its definition).

Question 1.27. If (M, ω) is a compact Kähler manifold with $\mathrm{HSC}(\omega) > 0$, does M have negative Kodaira dimension, i.e., $\kappa(M) = -\infty$?

Yang has given an answer for Yau's question in a general setting.

Proposition 1.28. [25, Theorem 1.2] *Let (M, ω) be a compact Hermitian manifold with semipositive holomorphic sectional curvature. If the holomorphic sectional curvature is not identically zero, then M has Kodaira dimension $-\infty$. In particular, if (M, ω) has* $\mathrm{HSC}(\omega) > 0$, *then $\kappa(M) = -\infty$.*

Question 1.29. What about the almost Hermitian case?

Applying the formula (4.9), we have the following proposition.

Proposition 1.30. *Let (M^{2n}, J, ω) be a compact almost Hermitian manifold with $n \geq 2$, $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \geq 0$ and* $\mathrm{HCF}(\omega) > 0$. *Then, we have that $\kappa(M) = -\infty$.*

Combining Proposition 1.30 with Lemma 1.12 for the almost Kähler case, we have the following result.

Theorem 1.31. *Let (M^{2n}, J, ω) be a compact almost Kähler manifold with $n \geq 2$ and* $\mathrm{HCF}(\omega) > 0$. *Then, $\kappa(M) = -\infty$.*

Combining Proposition 1.30 with Lemma 4.6 for the case of $n = 2$, we have the following result.

Theorem 1.32. *Let (M^4, J, ω) be a real 4-dimensional compact almost Hermitian manifold with* $\mathrm{HCF}(\omega) > 0$. *Then, $\kappa(M) = -\infty$.*

Note that in [21, Theorem 1.1], it has shown that if a compact Hermitian manifold has $\mathrm{HSC}(\omega) > 0$, then the Kodaira dimension is negative. Since one has $T_{ij}^{\overline{r}} = 0$ for all $i, j, r = 1, \dots, n$ in the complex case, the result of Theorem 1.31 can be considered as a generalization of [21, Theorem 1.1].

From Lemmas 4.4 and 4.6, we have the following corollary.

Corollary 1.33. *If $\widehat{s}_\omega > 0$ on a real 4-dimensional compact quasi-Kähler manifold (M^4, J, ω), then $\kappa^J(M^4) = -\infty$.*

Note that the quasi-Kählerity implies $\alpha_\omega = J\delta\omega = 0$, where $\delta := -*d*$, since we have $d*\omega = \frac{1}{(n-1)!} d\omega^{n-1} = \frac{1}{(n-1)!}(\partial + \overline{\partial})\omega^{n-1} = 0$, where we used $A\omega^{n-1} = \overline{A}\omega^{n-1} = 0$. Hence, we have the following lemma.

Lemma 1.34. [8, Corollary 4.5] *Let (M^4, J, ω) be a real 4-dimensional quasi-Kähler (equivalently almost Kähler or semi-Kähler) manifold. Then,*

$$\widehat{s}_\omega = \frac{1}{2}s + \frac{1}{32}|N|^2 \geq \frac{1}{2}s, \tag{1.7}$$

where s is the Riemannian scalar curvature with respect to the Levi-Civita connection, and N is the Nijenhuis tensor of the almost complex structure J.

Combining Corollary 1.33 with (1.7), we obtain

Corollary 1.35. *If $s > 0$ on a real 4-dimensional compact quasi-Kähler manifold (M^4, J, ω), then $\kappa^J(M^4) = -\infty$.*

Since we have ${}^cT_{ri}^{\overline{k}}\,{}^cT_{\overline{ki}}^{r} \geq 0$ on an almost Kähler manifold, we have the following result.

Corollary 1.36. *If $\widehat{s}_\omega > 0$ on a compact almost Kähler manifold (M^{2n}, J, ω) with $n \geq 2$, then $\kappa^J(M) = -\infty$.*

Since we have $(d\omega)^- = 0$ and $\alpha_\omega = 0$ on an almost Kähler manifold, where $(d\omega)^-$ is the sum of $(3,0)$ and $(0,3)$ components of $d\omega$, we have the following lemma.

Lemma 1.37. [8, Theorem 4.3] *Let (M^{2n}, J, ω) be an almost Kähler manifold of real dimension $2n$. Then, for $t = 0$,*

$$\widehat{s}_\omega = \frac{1}{2}s + \frac{1}{32}|N^0|^2 \geq \frac{1}{2}s, \tag{1.8}$$

where $N^0 := N - \mathfrak{b}N$, $\mathfrak{b}N$ is the skew-symmetric part of the Nijenhuis tensor N.

Combining Corollary 1.36 with (1.8), we have the following result.

Corollary 1.38. *If $s > 0$ on a compact almost Kähler manifold (M^{2n}, J, ω) with $n \geq 2$, then $\kappa^J(M) = -\infty$.*

We define a semi-Kähler metric on almost complex manifolds. Note that when a manifold is complex, a semi-Kähler metric is called a balanced metric.

Definition 1.39. [11] Let (M^{2n}, J) be an almost complex manifold. An almost Hermitian metric ω is called semi-Kähler if the metric ω satisfies $d\omega^{n-1} = 0$. When an almost Hermitian metric ω is semi-Kähler, the triple (M^{2n}, J, ω) is called a semi-Kähler manifold.

We have shown the following characterization of the semi-Kählerity on compact Kähler-like almost Hermitian manifolds.

Proposition 1.40. [16, Theorem 1.1] *Let (M^{2n}, J, ω) be a compact Kähler-like almost Hermitian manifold with $n \geq 2$. Then (M^{2n}, J, ω) is semi-Kähler if and only if $T^{\bar{q}}_{ik} T^{i}_{\bar{q}\bar{l}} = 0$ for all $k, l = 1, \ldots, n$.*

Note that from Proposition 1.30, we have that if a compact Kähler-like almost Hermitian manifold is semi-Kähler, then $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$. Since a real 4-dimensional compact Kähler-like manifold is almost Kähler (i.e., semi-Kähler) (see [14, Theorem 1.1]), we have $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ on a real 4-dimensional compact Kähler-like manifold. In fact, we see that a real 4-dimensional compact Kähler-like manifold is Kähler (see Proposition 1.14). Here we note that $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ is equivalent to that $T^{\bar{q}}_{ik} T^{i}_{\bar{q}\bar{l}} = 0$ for all $k, l = 1, 2$ in the case of $n = 2$.

Let (M, J, g) be a quasi-Kähler manifold. Choose and fix a local unitary $(1,0)$-frame $\{e_i\}$ around a point $p_0 \in M$ with respect to g such that $g_{i\bar{j}}(p_0) = \delta_{ij}$ and $\nabla e_i(p_0) = 0$. Then we have that $[e_k, e_{\bar{l}}](p_0) = 0$. On a quasi-Kähler manifold (M, J, g), we have that from (2.12), [28, Theorem 3.2], since we have $T^{k}_{ij} = 0$ for all i, j, k, computing at p_0,

$$
\begin{aligned}
R^{L}_{ijk\bar{l}} &= R^{L}_{k\bar{l}ij} \\
&= g(D_{e_k} D_{e_{\bar{l}}} e_i - D_{e_{\bar{l}}} D_{e_k} e_i, e_j) \\
&= \frac{1}{2}(\nabla_{\bar{l}} T^{\bar{k}}_{ji} - \nabla_{\bar{l}} T^{\bar{i}}_{kj} - \nabla_{\bar{l}} T^{\bar{j}}_{ik}) + \frac{1}{2} \nabla_k T^{l}_{ij} \\
&\quad + \frac{1}{4}(T^{l}_{jr} T^{r}_{ik} - T^{l}_{ir} T^{r}_{jk}) - \frac{1}{4} T^{\bar{i}}_{\bar{l}\bar{r}} (T^{\bar{r}}_{jk} + T^{\bar{j}}_{kr} - T^{\bar{k}}_{rj}) + \frac{1}{4} T^{\bar{j}}_{\bar{l}\bar{r}} (T^{\bar{r}}_{ik} + T^{\bar{i}}_{kr} - T^{\bar{k}}_{ri}) \\
&= \frac{1}{2}(H^{l}_{ijk} - H^{l}_{jki} - H^{l}_{kij})
\end{aligned}
$$

and we also have that from Lemma 2.1,

$$
\begin{aligned}
H^{l}_{ijk} + H^{l}_{jki} + H^{l}_{kij} &= T^{r}_{ij} T^{l}_{rk} + T^{r}_{jk} T^{l}_{ri} + T^{r}_{ki} T^{l}_{rj} + \nabla_i T^{l}_{jk} + \nabla_j T^{l}_{ki} + \nabla_k T^{l}_{ij} \\
&= 0.
\end{aligned}
$$

Combining these, we have the following lemma.

Lemma 1.41. [28, Corollary 3.7] *Let (M^{2n}, J, ω) be a quasi-Kähler manifold and fix a local unitary $(1,0)$-frame. Then we have that*

$$R^L_{ijk\bar{l}} = H^l_{ijk}.$$

Note that if a quasi-Kähler manifold satisfies both Kähler-like and G-Kähler-like conditions, then it must be Kähler (see [17, Theorem 1.1]). On an almost Kähler manifold or on a nearly Kähler manifold, the Kähler-likeness is equivalent to that $T^{\bar{r}}_{ki} T^{l}_{\bar{r}\bar{j}} = 0$ for all i, j, k, l, which is also equivalent to that $R^L_{ik\bar{j}\bar{l}} = 0$ for all i, j, k, l from (1.3) in the almost Kähler case and from (1.6) in the nearly Kähler case. On a quasi-Kähler manifold, since we have $T^k_{ij} = 0$ for all i, j, k, applying Lemma 1.34 and (2.7), we see that $R^L_{ikj\bar{l}} = 0$ for all i, j, k, l is equivalent to that $H = \Omega^{(2,0)} \equiv 0$, which is also equivalent to that $\nabla_{\bar{j}} T^{\bar{i}}_{kl} = 0$ for all i, j, k, l. Since the almost Kählerity and the nearly Kählerity are included in the quasi-Kählerity, these equivalences hold on an almost Kähler manifold, or on a nearly Kähler manifold. Since we have $\overline{\nabla} T'' = 0$ on a nearly Kähler manifold, which implies that we have $H \equiv 0$ and $R^L_{ikj\bar{l}} = 0$ for all i, j, k, l on a nearly Kähler manifold. We also note that the Kähler-likeness is included in the G-Kähler-likeness on an almost Kähler manifold or a nearly Kähler manifold. Note that the Kähler-likeness is equivalent to the G-Kähler-likeness on a nearly Kähler manifold (see [18, Proposition 1.1]). Since it is shown that a Kähler-like nearly Kähler manifold is Kähler (see [18, Theorem 1.1]), we find that a nearly Kähler manifold with $T^{\bar{r}}_{ki} T^{l}_{\bar{r}\bar{j}} = 0$ for all i, j, k, l is Kähler. Notice that $T^{\bar{r}}_{ki} T^{l}_{\bar{r}\bar{j}} = 0$ for all $i, j, k, l = 1, 2$ implies $T'' \equiv 0$ on a real 4-dimensional nearly Kähler manifold.

Gauduchon introduced one parameter family of canonical connection ∇^t on a compact almost Hermitian manifold (M^{2n}, J, ω) with $n \geq 2$ and with the associated almost Hermitian metric g with respect to the real $(1,1)$-form ω as follows (see [10]):

$$\begin{aligned} g(\nabla^t_X Y, Z) &= g\left(D_X Y - \frac{1}{2} J(D_X J)Y, Z\right) \\ &\quad + \frac{t}{4} g((D_{JY}J)Z + J(D_Y J)Z, X) - \frac{t}{4} g((D_{JZ}J)Y + J(D_Z J)Y, X), \end{aligned}$$

where D is the Levi-Civita connection, X, Y, Z are smooth vector fields on M and $t \in \mathbb{R}$. Note that ∇^1 is the Chern connection.

Let K^t be the curvature tensor and define the Gauduchon scalar curvature by

$$s(t) := \sum_{i,j} K^t(e_i, e_{\bar{i}}, e_j, e_{\bar{j}}),$$

where $\{e_i\}$ is a local unitary $(1,0)$-frame.

We introduce the prescribed Gauduchon scalar curvature problem, which is known as the Gauduchon–Yamabe problem:

Question 1.42. For a given smooth function $\widehat{s}(t)$ on an almost Hermitian manifold (M, J, h), does M admit a conformal almost Hermitian metric $e^u g$ with Chern scalar curvature $\widehat{s}(t)$?

Define

$$c(t) := \frac{2}{nt - t + 1} \int_M s(t)\omega^n.$$

In [22], Li, Zhou and Zhou have solved the Gauduchon–Yamabe problem for zero Gauduchon scalar curvature.

Proposition 1.43. [22, Theorem 1.3] *If $c(t) = 0$, then there are almost Hermitian metrics conformal to g with zero Gauduchon scalar curvature.*

Question 1.42 for $t = 1$ is especially called the Chern–Yamabe problem. We restate the case of $t = 1$ as follows for the later use.

Proposition 1.44. [22, Theorem 1.3] *Let (M^{2n}, J, ω) be a real $2n$-dimensional compact almost Hermitian manifold with $n \geq 2$. If $\int_M s_\omega \omega^n = 0$, then there are almost Hermitian metrics conformal to ω with zero Chern scalar curvature.*

In the case of $\mathrm{HSC}(\omega) = 0$, since we obtain $s_\omega - \widehat{s}_\omega = 0$ from the Kähler-likeness, we have the following proposition (see Lemma 3.1).

Proposition 1.45. *If (M^{2n}, J, ω) is a compact Kähler-like almost Hermitian manifold with $n \geq 2$ and $\mathrm{HSC}(\omega) = 0$, then we have that $\int_M s_\omega \omega^n = 0$.*

By combining Proposition 1.45 with Proposition 1.44, we have a condition of the Chern–Yamabe problem for zero Chern scalar curvature (see [1, Theorem 3.1], [22, Theorem 1.3]).

Theorem 1.46. *Let (M^{2n}, J, ω) be a compact Kähler-like almost Hermitian manifold with $n \geq 2$ and $\mathrm{HSC}(\omega) = 0$. Then, there are almost Hermitian metrics conformal to ω with zero Chern scalar curvature.*

For $n = 2$, the statement of Theorem 1.46 becomes the Kähler case from Proposition 1.14. Since we have $s_\omega = \widehat{s}_\omega$ under the quasi-Kählerity with $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ from the formula (4.9), we also obtian the following condition for having zero Chern scalar curvature.

Theorem 1.47. *Let (M^{2n}, J, ω) be a compact quasi-Kähler manifold with $n \geq 2$, and $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$, $\mathrm{HSC}(\omega) = 0$. Then, there are almost Hermitian metrics conformal to ω with zero Chern scalar curvature.*

Note that for $n = 2$, the statement of Theorem 1.47 becomes the Kähler case since the quasi-Kählerity implies the almost Kählerity for $n = 2$ and the almost Kählerity with $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ implies the Kählerity from Lemma 1.12.

We show Proposition 1.45 and the following proposition as a proof of Theorems 1.46 and 1.47.

Proposition 1.48. *Let (M^{2n}, J, ω) be a compact quasi Kähler manifold with $n \geq 2$ and $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$, $\mathrm{HSC}(\omega) = 0$. Then, we have that $\int_M s_\omega \omega^n = 0$.*

Proofs of Propositions 1.45 *and* 1.48. Under the assumptions: the Kähler-likeness in Proposition 1.48, or the quasi-Kählerity with $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ in Proposition 1.51 with (4.9), we have that $s_\omega = \widehat{s}_\omega$.

We compute that

$$\int_M s_\omega \omega^n = \frac{1}{2} \int_M (s_\omega + \widehat{s}_\omega) \omega^n + \frac{1}{2} \int_M (s_\omega - \widehat{s}_\omega) \omega^n = 0,$$

where we have used that $\mathrm{HSC}(\omega) = 0$ implies $s_\omega + \widehat{s}_\omega = 0$ from Lemma 3.1. □

Combining Theorem 1.47 with Lemma 4.6, we have the following corollary.

Corollary 1.49. *Let (M^4, J, ω) be a real 4-dimensional compact quasi-Kähler manifold with $T^{\bar{1}}_{12} = T^{\bar{2}}_{12}$, $\mathrm{HSC}(\omega) = 0$. Then, there are almost Hermitian metrics conformal to ω with zero Chern scalar curvature.*

Remark 1.50. We compute on a real 4-dimensional quasi-Kähler manifold,

$$\begin{aligned} 2T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} &= 2(T^{\bar{1}}_{12} T^{1}_{\bar{1}\bar{2}} + T^{\bar{2}}_{12} T^{1}_{\bar{2}\bar{1}} + T^{\bar{1}}_{21} T^{2}_{\bar{1}\bar{2}} + T^{\bar{2}}_{21} T^{2}_{\bar{2}\bar{1}}) \\ &= R^L_{21\bar{2}\bar{1}} + R^L_{21\bar{1}\bar{1}} + R^L_{12\bar{2}\bar{2}} + R^L_{12\bar{1}\bar{2}} \\ &= 2R^L_{12\bar{1}\bar{2}}, \end{aligned}$$

where we have used that $R^L_{21\bar{1}\bar{1}} = R^L_{12\bar{2}\bar{2}} = 0$, $R^L_{21\bar{2}\bar{1}} = R^L_{12\bar{1}\bar{2}}$. Hence, $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = R^L_{12\bar{1}\bar{2}}$, and $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ is equivalent to that $R^L_{12\bar{1}\bar{2}} = 0$ on a real 4 dimensional quasi-Kähler manifold. On the other hand, on an almost Kähler manifold, the Kähler-likeness is equivalent to that $R^L_{ij\bar{k}\bar{l}} = 0$ for all i, j, k, l. Since $R^L_{ij\bar{k}\bar{l}} = 0$ for all $i, j, k, l = 1, 2$ is equivalent to that $R^L_{12\bar{1}\bar{2}} = 0$ and the quasi-Kählerity is equivalent to the almost Kählerity in the case of $n = 2$, and also since a real 4-dimensional compact Kähler-like manifold must be quasi-Kähler and have $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ from Proposition 1.30, we conclude that the quasi Kählerity with $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ is equivalent to the Kähler-likeness on a real 4-dimensional compact almost Hermitian manifold. On the other hand, we see that a real 4-dimensional quasi-Kähler (i.e., almost Kähler) manifold with $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ is Kähler from Lemma 1.12. Combining these results, we again conclude that a real 4-dimensional compact Kähler-like almost Hermitian manifold is Kähler as we have seen in Proposition 1.14.

Note that since we have $T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} \geq 0$ from Lemma 4.6, a real 4-dimensional Kähler-like and G-Kähler-like almost Hermitian manifold is Kähler (see [17, Corollary 1.2]). We restate the equivalence obtained in Remark 1.50 as follows.

Proposition 1.51. *On a real 4-dimensional compact almost Hermitian manifold, the quasi Kählerity with $T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} = 0$ (i.e., $T_{12}^{\bar{1}} = T_{12}^{\bar{2}}$) is equivalent to the Kähler-likeness.*

This paper is organized as follows: in Section 2, we recall some basic definitions and computations in almost Hermitian geometry. In Section 3, we introduce some lemmas whose proofs can be given as in the corresponding lemmas of [19]. In Section 4, we give proofs of Proposition 1.30, Theorems 1.31 and 1.32. Notice that we assume the Einstein convention omitting the symbol of sum over repeated indices in all this paper.

2. Preliminaries

2.1. The Chern connection

An almost complex structure on M is an endomorphism J of TM, $J \in \Gamma(\mathrm{End}(TM))$, satisfying $J^2 = -\mathrm{Id}_{TM}$, where TM is the real tangent vector bundle of M. The pair (M, J) is called an almost complex manifold. Let (M, J) be an almost complex manifold. A Riemannian metric g on M is called J-invariant if J is compatible with g. In this case, the pair (J, g) is called an almost Hermitian structure. The complexified tangent vector bundle is given by $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ for the real tangent vector bundle TM. By extending J $\mathbb{C}$-linearly and g $\mathbb{C}$-bilinearly to $T^{\mathbb{C}}M$, they are also defined on $T^{\mathbb{C}}M$ and we observe that the complexified tangent vector bundle $T^{\mathbb{C}}M$ can be decomposed as $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$, $T^{0,1}M$ are the eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively:

$$T^{1,0}M = \{X - \sqrt{-1}JX \mid X \in TM\}, \quad T^{0,1}M = \{X + \sqrt{-1}JX \mid X \in TM\}.$$

Let $\Lambda^1 M$ denote the dual of the real tangent vector bundle TM. We have that

$$\Lambda^1 M \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M,$$

where

$$\Lambda^{1,0}M = \{\iota + \sqrt{-1}J\iota \mid \forall\, \iota \in \Lambda^1 M\}, \quad \Lambda^{0,1}M = \{\iota - \sqrt{-1}J\iota \mid \forall\, \iota \in \Lambda^1 M\}. \tag{2.1}$$

It can be seen that $(T^{1,0}M)^* = \Lambda^{1,0}M$, $(T^{0,1}M)^* = \Lambda^{0,1}M$. Now let us define

$$\Lambda^{p,q}M := \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M).$$

Then we have $\Lambda^r M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}M$.

Notice that on an almost complex manifold M, we can split the exterior differential operator $d\colon \Lambda^p M\otimes_{\mathbb{R}}\mathbb{C}\to \Lambda^{p+1}M\otimes_{\mathbb{R}}\mathbb{C}$, into four components

$$d=A+\partial+\overline{\partial}+\overline{A}$$

with

$$\partial\colon \Lambda^{p,q}M\to\Lambda^{p+1,q}M,\qquad \overline{\partial}\colon \Lambda^{p,q}M\to\Lambda^{p,q+1}M,$$
$$A\colon \Lambda^{p,q}M\to\Lambda^{p+2,q-1}M,\qquad \overline{A}\colon \Lambda^{p,q}M\to\Lambda^{p-1,q+2}M.$$

In terms of these components, the condition $d^2=0$ can be written as

$$\begin{gathered}A^2=0,\quad \partial A+A\partial=0,\quad \overline{\partial A}+\overline{A\partial}=0,\quad \overline{A}^2=0,\\ A\overline{\partial}+\partial^2+\overline{\partial}A=0,\quad A\overline{A}+\partial\overline{\partial}+\overline{\partial}\partial+\overline{A}A=0,\quad \partial\overline{A}+\overline{\partial}^2+\overline{A}\partial=0.\end{gathered}\tag{2.2}$$

For any p-form ψ, there holds that

$$\begin{aligned}d\psi(X_1,\dots,X_{p+1})&=\sum_{i=1}^{p+1}(-1)^{i+1}X_i(\psi(X_1,\dots,\widehat{X_i},\dots,X_{p+1}))\\&\quad+\sum_{i<j}(-1)^{i+j}\psi([X_i,X_j],X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{p+1})\end{aligned}$$

for any vector fields $X_1,\dots,X_{p+1}$ on M (see [29]). We directly compute that

$$d\theta^s=-\frac{1}{2}B^s_{kl}\theta^k\wedge\theta^l-B^s_{k\bar{l}}\theta^k\wedge\theta^{\bar{l}}-\frac{1}{2}B^s_{\bar{k}\bar{l}}\theta^{\bar{k}}\wedge\theta^{\bar{l}}.$$

Let (M,g,J) be an almost Hermitian manifold. There exists a unique affine connection ∇ preserving g and J on M whose torsion has vanishing $(1,1)$-part (see [10]), which is called the Chern connection. Now let ∇ be the Chern connection on M.

Let $\{e_r\}$ be a local $(1,0)$-frame with respect to an almost Hermitian metric g and let $\{\theta^r\}$ be a local associated coframe with respect to $\{e_r\}$, i.e., $\theta^i(e_j)=\delta^i_j$ for $i,j=1,\dots,n$. We write $g_{i\bar{j}}:=g(e_i,e_{\bar{j}})$. The fundamental $(1,1)$-form ω associated to g is locally given by $\omega=\sqrt{-1}g_{i\bar{j}}\theta^i\wedge\theta^{\bar{j}}$. We denote the structure coefficients of Lie bracket by

$$[e_i,e_j]=:B^r_{ij}e_r+B^{\bar{r}}_{ij}e_{\bar{r}},\quad [e_i,e_{\bar{j}}]=:B^r_{i\bar{j}}e_r+B^{\bar{r}}_{i\bar{j}}e_{\bar{r}},\quad [e_{\bar{i}},e_{\bar{j}}]=:B^r_{\bar{i}\bar{j}}e_r+B^{\bar{r}}_{\bar{i}\bar{j}}e_{\bar{r}}.$$

Notice that J is integrable if and only if the $B^{\bar{r}}_{ij}$'s vanish.

For any $u\in C^\infty(M,\mathbb{R})$, we have since we have $\overline{A}\partial u=0$,

$$d\overline{\partial}u=\partial\overline{\partial}u+\overline{\partial}^2u+A\overline{\partial}u.\tag{2.3}$$

By taking the conjugate of (2.3), and by adding together, we get that

$$d(\partial+\overline{\partial})u=(\partial\overline{\partial}+\overline{\partial}\partial)u+(\partial^2+A\overline{\partial})u+(\overline{\partial}^2+\overline{A}\partial)u.$$

Since we have $Au = \overline{A}u = 0$, from the relations in (1.1), $(\partial^2 + A\overline{\partial})u = -\overline{\partial}Au = 0$, $(\overline{\partial}^2 + \overline{A}\partial)u = -\partial\overline{A}u = 0$. And since $(\partial + \overline{\partial})u = du$, we get that $d(\partial + \overline{\partial})u = d^2u = 0$. Therefore, we obtain that

$$\partial\overline{\partial}u = -\overline{\partial}\partial u,$$

which implies that $\sqrt{-1}\partial\overline{\partial}u$ is a smooth real $(1,1)$-form. A direct computation yields for any $u \in C^\infty(M, \mathbb{R})$,

$$\begin{aligned}(dJdu)(e_i, e_{\bar{j}}) &= e_i(Jdu(e_{\bar{j}})) - e_{\bar{j}}(Jdu(e_i)) - Jdu([e_i, e_{\bar{j}}]) \\ &= -e_i(du(Je_{\bar{j}})) + e_{\bar{j}}(du(Je_i)) + du(J[e_i, e_{\bar{j}}]) \\ &= \sqrt{-1}e_ie_{\bar{j}}(u) + \sqrt{-1}e_{\bar{j}}e_i(u) + J[e_i, e_{\bar{j}}](u) \\ &= 2\sqrt{-1}e_ie_{\bar{j}}(u) - \sqrt{-1}([e_i, e_{\bar{j}}] + \sqrt{-1}J[e_i, e_{\bar{j}}])u \\ &= 2\sqrt{-1}(e_ie_{\bar{j}} - [e_i, e_{\bar{j}}]^{(0,1)})u,\end{aligned} \tag{2.4}$$

which tells us that

$$\sqrt{-1}\partial\overline{\partial}u = \frac{1}{2}(dJdu)^{(1,1)} = \sqrt{-1}(e_ie_{\bar{j}} - [e_i, e_{\bar{j}}]^{(0,1)})u\theta^i \wedge \theta^{\bar{j}},$$

so we write locally

$$\partial_i\partial_{\bar{j}}u = (e_ie_{\bar{j}} - [e_i, e_{\bar{j}}]^{(0,1)})u.$$

2.2. The curvature on almost complex manifolds

Since the Chern connection ∇ preserves J, we have

$$\nabla_i e_j := \nabla_{e_i} e_j = \Gamma_{ij}^r e_r, \quad \nabla_i e_{\bar{j}} = \Gamma_{i\bar{j}}^{\bar{r}} e_{\bar{r}},$$

where $\Gamma_{ij}^r = g^{r\bar{s}}e_i(g_{j\bar{s}}) - g^{r\bar{s}}g_{j\bar{l}}B_{i\bar{s}}^{\bar{l}}$. Note that the mixed derivatives $\nabla_i e_{\bar{j}}$ do not depend on the metric g, which means that $\Gamma_{i\bar{j}}^{\bar{r}} = B_{i\bar{j}}^{\bar{r}}$'s do not depend on g (see [23]). Let $\{\gamma_j^i\}$ be the connection form, which is defined by $\gamma_j^i = \Gamma_{sj}^i\theta^s + \Gamma_{\bar{s}j}^i\theta^{\bar{s}}$. The torsion T of the Chern connection ∇ is given by $T^i = d\theta^i - \theta^p \wedge \gamma_p^i$, $T^{\bar{i}} = d\theta^{\bar{i}} - \theta^{\bar{p}} \wedge \gamma_{\bar{p}}^{\bar{i}}$, which has no $(1,1)$-part and the only non-vanishing components are as follows:

$$\begin{aligned}T_{ij}^s &= T^s(e_i, e_j) = -\theta^s([e_i, e_j]) - (\Gamma_{qp}^s\theta^p \wedge \theta^q + \Gamma_{\bar{q}p}^s\theta^p \wedge \theta^{\bar{q}})(e_i, e_j) = -B_{ij}^s - \Gamma_{ji}^s + \Gamma_{ij}^s, \\ T_{\bar{i}\bar{j}}^s &= T^s(e_{\bar{i}}, e_{\bar{j}}) = d\theta^s(e_{\bar{i}}, e_{\bar{j}}) = -\theta^s([e_{\bar{i}}, e_{\bar{j}}]) = -B_{\bar{i}\bar{j}}^s = N_{\bar{i}\bar{j}}^s.\end{aligned}$$

These tell us that $T = (T^i)$ splits into $T = T' + T''$, where $T' \in \Gamma(\Lambda^{2,0}M \otimes T^{1,0}M)$, $T'' \in \Gamma(\Lambda^{0,2}M \otimes T^{1,0}M)$. Since the torsion T of the Chern connection ∇ has no $(1,1)$-part;

$$0 = T_{k\bar{l}}^{\bar{i}} = T^{\bar{i}}(e_k, e_{\bar{l}}) = -\theta^{\bar{i}}([e_k, e_{\bar{l}}]) - (\Gamma_{s\bar{p}}^{\bar{i}}\theta^{\bar{p}} \wedge \theta^s + \Gamma_{\bar{s}\bar{p}}^{\bar{i}}\theta^{\bar{p}} \wedge \theta^{\bar{s}})(e_k, e_{\bar{l}}) = -B_{k\bar{l}}^{\bar{i}} + \Gamma_{k\bar{l}}^{\bar{i}},$$

we obtain that

$$\Gamma^{\bar{r}}_{i\bar{j}} = B^{\bar{r}}_{i\bar{j}}.$$

By taking complex conjugate, we have that

$$\Gamma^{k}_{\bar{j}i} = \overline{\Gamma^{\bar{k}}_{j\bar{i}}} = \overline{B^{\bar{k}}_{j\bar{i}}} = B^{k}_{\bar{j}i}.$$

Since we have

$$\nabla_i \nabla_{\bar{j}} u = \nabla_{e_i} \nabla_{e_{\bar{j}}} u = e_i e_{\bar{j}}(u) - \Gamma^{\bar{s}}_{i\bar{j}} e_{\bar{s}}(u) = e_i e_{\bar{j}}(u) - B^{\bar{s}}_{i\bar{j}} e_{\bar{s}}(u)$$

and since $[e_i, e_{\bar{j}}]^{(0,1)} u = B^{\bar{s}}_{i\bar{j}} e_{\bar{s}}(u)$, we obtain that

$$\partial_i \partial_{\bar{j}} u = \nabla_i \nabla_{\bar{j}} u.$$

The curvature Ω of the Chern connection ∇ splits in $\Omega = H + R + \overline{H}$ (see Section 1) and the curvature form can be expressed by $\Omega^i_j = d\gamma^i_j + \gamma^i_s \wedge \gamma^s_j$.

In terms of e_r's, we have

$$(2.5) \qquad R^r_{i\bar{j}k} = \Omega^r_k(e_i, e_{\bar{j}}) = e_i(\Gamma^r_{\bar{j}k}) - e_{\bar{j}}(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{ik} - B^s_{i\bar{j}}\Gamma^r_{sk} + B^{\bar{s}}_{\bar{j}i}\Gamma^r_{\bar{s}k},$$

$$H^r_{ijk} = \Omega^r_k(e_i, e_j) = e_i(\Gamma^r_{jk}) - e_j(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik} - B^s_{ij}\Gamma^r_{sk} - B^{\bar{s}}_{ij}\Gamma^r_{\bar{s}k},$$

$$(2.6) \qquad H^r_{\bar{i}\bar{j}k} = \Omega^r_k(e_{\bar{i}}, e_{\bar{j}}) = e_{\bar{i}}(\Gamma^r_{\bar{j}k}) - e_{\bar{j}}(\Gamma^r_{\bar{i}k}) + \Gamma^r_{\bar{i}s}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{\bar{i}k} - B^s_{\bar{i}\bar{j}}\Gamma^r_{sk} - B^{\bar{s}}_{\bar{i}\bar{j}}\Gamma^r_{\bar{s}k}.$$

We define that

$$R_{i\bar{j}k\bar{l}} := R^r_{i\bar{j}k} g_{r\bar{l}}, \quad H_{ijk\bar{l}} := H^r_{ijk} g_{r\bar{l}}, \quad H_{\bar{i}\bar{j}k\bar{l}} := H^r_{\bar{i}\bar{j}k} g_{r\bar{l}}.$$

We define the Chern scalar curvature s_ω and the Riemannian type scalar curvature $\widehat{s}_\omega$ of the metric ω with respect to the Chern connection:

$$(2.7) \qquad s_\omega := g^{i\bar{j}} g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = g^{i\bar{j}} P_{i\bar{j}}(\omega) = g^{i\bar{j}} S_{i\bar{j}}(\omega), \quad \widehat{s}_\omega := g^{i\bar{l}} g^{k\bar{j}} R_{i\bar{j}k\bar{l}},$$

where P, S denote the first and second Chern–Ricci curvature respectively locally given by $P_{i\bar{j}} := g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$, $S_{i\bar{j}} := g^{k\bar{l}} R_{k\bar{l}i\bar{j}}$.

Lemma 2.1 (The first Bianchi identity for the Chern curvature)**.** *For any $X, Y, Z \in T^{\mathbb{C}}M$,*

$$\sum \Omega(X,Y)Z = \sum \big(T(T(X,Y),Z) + \nabla_X T(Y,Z)\big),$$

where the sum is taken over all cyclic permutations.

This identity induces the following formulae:

$$(2.8) \qquad R^l_{i\bar{j}k} - R^l_{k\bar{j}i} = \nabla_{\bar{j}} T^l_{ki} + T^{\bar{r}}_{ki} T^l_{\bar{r}\bar{j}},$$

where used that $R_{ij\overline{kl}} = R_{\overline{ij}kl} = 0$.

For a point $p \in M$ and a non-zero $(1,0)$-vector $\xi \in T^{1,0}_p M$, the holomorphic sectional curvature $\mathcal{H}$ of the metric ω (HSC(ω) for short) at the point p and the direction ξ is define by

$$\mathcal{H}_p(\xi) := \frac{1}{|\xi|^4_{g_p}} R(\xi, \overline{\xi}, \xi, \overline{\xi})|_p = \frac{1}{|\xi|^4_{g_p}} R_{i\overline{j}k\overline{l}}|_p \xi^i \xi^{\overline{j}} \xi^k \xi^{\overline{l}}, \tag{2.9}$$

where $|\xi|^2_{g_p} := g_p(\xi, \overline{\xi})$. We write HSC($\omega$) > 0 (resp. $= 0$) when we have that $\mathcal{H}_p(\xi) > 0$ (resp. $= 0$) for any point $p \in M$ and any non-zero $(1,0)$-vector $\xi \in T^{1,0}_p M$.

Let $\{e_r\}$ be a local unitary $(1,0)$-frame with respect to g around a fixed point $p \in M$. Note that unitary frames always exist locally since we can take any frame and apply the Gram-Schmidt process. Then with respect to a local g-unitary frame around a point p_0, we have $g_{i\overline{j}}(p_0) = \delta_{ij}$ for any $i, j, k = 1, \ldots, n$, and the Christoffel symbols satisfy at p_0,

$$\Gamma^k_{ij} = -\Gamma^{\overline{j}}_{i\overline{k}}, \quad \Gamma^{\overline{k}}_{\overline{i}\overline{j}} = -\Gamma^{j}_{\overline{i}k},$$

since we have at p_0,

$$\begin{aligned}
\Gamma^k_{ij} &= g(\nabla_i e_j, e_{\overline{k}}) = e_i(g_{j\overline{k}}) - g(e_j, \nabla_i e_{\overline{k}}) = -\Gamma^{\overline{j}}_{i\overline{k}},\\
\Gamma^{\overline{k}}_{\overline{i}\overline{j}} &= g(e_k, \nabla_{\overline{i}} e_{\overline{j}}) = e_{\overline{i}}(g_{k\overline{j}}) - g(\nabla_{\overline{i}} e_k, e_{\overline{j}}) = -\Gamma^{j}_{\overline{i}k}.
\end{aligned}$$

Then we have that

$$\begin{aligned}
H^r_{ijk} &= e_i(\Gamma^r_{jk}) - e_j(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik} - B^s_{ij}\Gamma^r_{sk} - B^{\overline{s}}_{ij}\Gamma^r_{\overline{s}k}\\
&= -e_i(\Gamma^{\overline{k}}_{j\overline{r}}) - e_j(\Gamma^{\overline{k}}_{i\overline{r}}) + \Gamma^{\overline{s}}_{i\overline{r}}\Gamma^{\overline{k}}_{j\overline{s}} - \Gamma^{\overline{s}}_{j\overline{r}}\Gamma^{\overline{k}}_{i\overline{s}} + B^s_{ij}\Gamma^{\overline{k}}_{s\overline{r}} + B^{\overline{s}}_{ij}\Gamma^{\overline{k}}_{\overline{s}\overline{r}}\\
&= -H^{\overline{k}}_{ij\overline{r}}.
\end{aligned} \tag{2.10}$$

From the first Bianch identity in Lemma 2.1, we obtain that

$$\begin{aligned}
H^{\overline{k}}_{ij\overline{l}} &= H^{\overline{k}}_{ij\overline{l}} + H^{\overline{k}}_{j\overline{l}i} + H^{\overline{k}}_{\overline{l}ij}\\
&= \nabla_i T^{\overline{k}}_{j\overline{l}} + \nabla_j T^{\overline{k}}_{\overline{l}i} + \nabla_{\overline{l}} T^{\overline{k}}_{ij} + T^{\overline{r}}_{ij} T^{\overline{k}}_{\overline{r}\overline{l}} + T^{r}_{j\overline{l}} T^{\overline{k}}_{ri} + T^{r}_{\overline{l}i} T^{\overline{k}}_{rj}\\
&= \nabla_{\overline{l}} T^{\overline{k}}_{ij} + T^{\overline{r}}_{ij} T^{\overline{k}}_{\overline{r}\overline{l}},
\end{aligned} \tag{2.11}$$

where used that $H_{j\overline{l}ik} = H_{\overline{l}ijk} = 0$. Therefore, combining (2.10) with (2.11), we have

$$H^l_{ijk} = \nabla_{\overline{l}} T^{\overline{k}}_{ji} + T^{\overline{r}}_{ji} T^{\overline{k}}_{\overline{r}\overline{l}}. \tag{2.12}$$

Note that we have the following formula.

Lemma 2.2. *Fix a local unitary $(1,0)$-frame with respect to g. One has*

$$\begin{aligned}
R_{i\overline{j}k\overline{l}} &= g(\nabla_i \nabla_{\overline{j}} e_k - \nabla_{\overline{j}} \nabla_i e_k - \nabla_{[e_i, e_{\overline{j}}]} e_k, e_{\overline{l}}),\\
H_{ijk\overline{l}} &= g(\nabla_i \nabla_j e_k - \nabla_j \nabla_i e_k - \nabla_{[e_i, e_j]} e_k, e_{\overline{l}}),\\
H_{\overline{i}\overline{j}k\overline{l}} &= g(\nabla_{\overline{i}} \nabla_{\overline{j}} e_k - \nabla_{\overline{j}} \nabla_{\overline{i}} e_k - \nabla_{[e_{\overline{i}}, e_{\overline{j}}]} e_k, e_{\overline{l}}).
\end{aligned}$$

Proof. Using a local unitary $(1,0)$-frame $\{e_i\}$ around a point p_0 with respect to g, we have that at p_0,

$$\begin{aligned}
&g(\nabla_i\nabla_{\bar{j}}e_k - \nabla_{\bar{j}}\nabla_i e_k - \nabla_{[e_i,e_{\bar{j}}]}e_k, e_{\bar{l}})\\
&= e_i(g(\nabla_{\bar{j}}e_k, e_{\bar{l}})) - g(\nabla_{\bar{j}}e_k, \nabla_i e_{\bar{l}}) - e_{\bar{j}}(g(\nabla_i e_k, e_{\bar{l}})) + g(\nabla_i e_k, \nabla_{\bar{j}}e_{\bar{l}})\\
&\quad - B^r_{i\bar{j}}g(\nabla_r e_k, e_{\bar{l}}) - B^{\bar{r}}_{i\bar{j}}g(\nabla_{\bar{r}}e_k, e_{\bar{l}})\\
&= e_i(\Gamma^s_{\bar{j}k}g_{s\bar{l}}) - \Gamma^s_{\bar{j}k}\Gamma^{\bar{r}}_{i\bar{l}}g_{s\bar{r}} - e_{\bar{j}}(\Gamma^s_{ik}g_{s\bar{l}}) + \Gamma^s_{ik}\Gamma^{\bar{r}}_{\bar{j}\bar{l}}g_{s\bar{r}} - B^s_{i\bar{j}}\Gamma^r_{sk}g_{r\bar{l}} - B^{\bar{r}}_{i\bar{j}}\Gamma^s_{\bar{r}k}g_{s\bar{l}}\\
&= e_i(\Gamma^l_{\bar{j}k}) - e_{\bar{j}}(\Gamma^l_{ik}) + \Gamma^l_{is}\Gamma^s_{\bar{j}k} - \Gamma^l_{\bar{j}s}\Gamma^s_{ik} - B^s_{i\bar{j}}\Gamma^l_{sk} - B^{\bar{s}}_{i\bar{j}}\Gamma^l_{\bar{s}k}\\
&= R^r_{i\bar{j}k}g_{r\bar{l}}\\
&= R_{i\bar{j}k\bar{l}}.
\end{aligned}$$

Similarly, we compute that at p_0,

$$\begin{aligned}
&g(\nabla_i\nabla_j e_k - \nabla_j\nabla_i e_k - \nabla_{[e_i,e_j]}e_k, e_{\bar{l}})\\
&= e_i(\Gamma^l_{jk}) - e_j(\Gamma^l_{ik}) - \Gamma^{\bar{r}}_{i\bar{l}}\Gamma^r_{jk} + \Gamma^{\bar{r}}_{j\bar{l}}\Gamma^r_{ik} - B^r_{ij}\Gamma^l_{rk} - B^{\bar{r}}_{ij}\Gamma^l_{\bar{r}k}\\
&= H_{ijk\bar{l}},\\
&g(\nabla_{\bar{i}}\nabla_{\bar{j}}e_k - \nabla_{\bar{j}}\nabla_{\bar{i}}e_k - \nabla_{[e_{\bar{i}},e_{\bar{j}}]}e_k, e_{\bar{l}})\\
&= e_{\bar{i}}(\Gamma^l_{\bar{j}k}) - e_{\bar{j}}(\Gamma^l_{\bar{i}k}) - \Gamma^{\bar{r}}_{\bar{i}\bar{l}}\Gamma^r_{\bar{j}k} + \Gamma^{\bar{r}}_{\bar{j}\bar{l}}\Gamma^r_{\bar{i}k} - B^r_{\bar{i}\bar{j}}\Gamma^l_{rk} - B^{\bar{r}}_{\bar{i}\bar{j}}\Gamma^l_{\bar{r}k}\\
&= H_{\bar{i}\bar{j}k\bar{l}},
\end{aligned}$$

where we have used that $\Gamma^{\bar{r}}_{i\bar{l}} = -\Gamma^l_{ir}$, $\Gamma^{\bar{r}}_{\bar{i}\bar{l}} = -\Gamma^l_{\bar{i}r}$. □

As in [27], we can choose a local g-unitary frame $\{e_i\}$ around an arbitrary chosen point $p_0 \in M$ such that

$$g_{i\bar{j}}(p_0) = \delta_{ij}, \quad \nabla e_i(p_0) = 0. \tag{2.13}$$

Then we have

$$\Gamma^k_{ij}(p_0) = 0 \quad \text{for all } i,j,k = 1,\dots,n \tag{2.14}$$

since $\nabla_i e_j(p_0) = \Gamma^k_{ij}(p_0)e_k = 0$, also we obtain that

$$[e_i, e_{\bar{j}}](p_0) = \nabla_i e_{\bar{j}}(p_0) - \nabla_{\bar{j}}e_i(p_0) - T(e_i, e_{\bar{j}})(p_0) = 0 \quad \text{for all } i,j = 1,\dots,n.$$

Then we have that $0 = [e_i, e_{\bar{j}}](p_0) = B^k_{i\bar{j}}(p_0)e_k + B^{\bar{k}}_{i\bar{j}}(p_0)e_{\bar{k}}$, which gives that

$$B^k_{i\bar{j}}(p_0) = 0, \quad B^{\bar{k}}_{i\bar{j}}(p_0) = 0 \quad \text{for all } i,j,k = 1,\dots,n. \tag{2.15}$$

Using such a local unitary frame $\{e_r\}$ with respect to g around p_0, we compute that at the point p_0, by applying (2.14) and (2.15) to (2.5),

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= R^r_{i\bar{j}k} g_{r\bar{l}} \\ &= \big\{e_i(\Gamma^r_{\bar{j}k}) - e_{\bar{j}}(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{ik} - B^s_{i\bar{j}}\Gamma^r_{sk} + B^{\bar{s}}_{\bar{j}i}\Gamma^r_{\bar{s}k}\big\} g_{r\bar{l}} \\ &= \big\{e_i(\Gamma^r_{\bar{j}k}) - e_{\bar{j}}(\Gamma^r_{ik})\big\} g_{r\bar{l}}. \end{aligned} \tag{2.16}$$

2.3. The bundle of real k-forms and the interior product

Let M be a real $2n$-dimensional smooth differentiable manifold and let h be a Riemannian metric on M. In a local coordinate $(x^1, x^2, \ldots, x^{2n})$ on M, we write $h = h_{ij}\, dx^i dx^j$. Denote (h^{ij}) the inverse matrix of (h_{ij}), $1 \le i, j \le 2n$. Then the metric h induces an inner product $\langle \cdot, \cdot \rangle$ on the cotangent bundle T^*M by $\langle dx^i, dx^j \rangle = h^{ij}$. Let $\Lambda^k T^*M$ be the bundle of real k-forms for $1 \le k \le 2n$. The inner product induced by h on $\Lambda^k T^*M$ is given by

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k \rangle = \det(\langle \alpha_i, \beta_j \rangle), \tag{2.17}$$

for $\alpha_i, \beta_j \in T^*M$. For $\varphi = \frac{1}{k!}\varphi_{i_1\cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, $\psi = \frac{1}{k!}\psi_{j_1\cdots j_k}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}$, where $\varphi_{i_1\cdots i_k}$ is skew symmetric in $i_1, \ldots, i_k$ and $\psi_{j_1\cdots j_k}$ is skew symmetric in $j_1, \ldots, j_k$,

$$\langle \varphi, \psi \rangle = \frac{1}{k!} h^{i_1 j_1} \cdots h^{i_k j_k} \varphi_{i_1\cdots i_k} \psi_{j_1\cdots j_k}. \tag{2.18}$$

We define the interior product $\iota_X \varphi \in \Lambda^{k-1}T^*M$ for vector fields $X, X_1, \ldots, X_{k-1}$ on M and $\varphi \in \Lambda^k T^*M$ by

$$\iota_X \varphi(X_1, \ldots, X_{k-1}) := \varphi(X, X_1, \ldots, X_{k-1}).$$

Note that we have

$$\iota_X(\alpha_1 \wedge \cdots \wedge \alpha_k) = \sum_{i=1}^k (-1)^{i-1} \alpha_i(X) \alpha_1 \wedge \cdots \wedge \alpha_{i-1} \wedge \widehat{\alpha_i} \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_k. \tag{2.19}$$

Define $\widetilde{X} := h(X, \cdot) \in T^*M$, then we obtain that for $\varphi \in \Lambda^{k+1}T^*M$ and $\psi \in \Lambda^k T^*M$ (see [4, (2.3)]),

$$\langle \iota_X \varphi, \psi \rangle = \langle \varphi, \widetilde{X} \wedge \psi \rangle. \tag{2.20}$$

2.4. The Hodge $*$-operator and the adjoint operators

We extend the inner product $\langle \cdot, \cdot \rangle$ given in (2.17), (2.18) on the bundle of real k-forms $\Lambda^k T^*M$ for $1 \le k \le 2n$ to the space of (p, q)-forms $\Lambda^{p,q}M$ defined in (2.1), $1 \le p, q \le n$, for $b, c \in \mathbb{C}$ and $\varphi_i, \psi_i \in \Lambda^k T^*M$, $i = 1, 2$, by

$$\langle b\varphi_1 + c\varphi_2, \psi\rangle = b\langle\varphi_1, \psi\rangle + c\langle\varphi_2, \psi\rangle,$$
$$\langle\varphi, b\psi_1 + c\psi_2\rangle = \overline{b}\langle\varphi, \psi_1\rangle + \overline{c}\langle\varphi, \psi_2\rangle.$$

Locally, (p, q)-forms $\varphi, \psi \in \Lambda^{p,q}M$ are given by

$$\varphi = \frac{1}{p!q!}\varphi_{i_1\cdots i_p\overline{l_1}\cdots\overline{l_q}}\theta^{i_1} \wedge \cdots \wedge \theta^{i_p} \wedge \theta^{\overline{l_1}} \wedge \cdots \wedge \theta^{\overline{l_q}},$$
$$\psi = \frac{1}{p!q!}\psi_{j_1\cdots j_p\overline{k_1}\cdots\overline{k_q}}\theta^{j_1} \wedge \cdots \wedge \theta^{j_p} \wedge \theta^{\overline{k_1}} \wedge \cdots \wedge \theta^{\overline{k_q}},$$

where $\varphi_{i_1\cdots i_p\overline{l_1}\cdots\overline{l_q}}$ is skew symmetric in $i_1, \ldots, i_p$ and skew symmetric in $l_1, \ldots, l_q$, $\psi_{j_1\cdots j_p\overline{l_1}\cdots\overline{l_q}}$ is skew symmetric in $j_1, \ldots, j_p$ and skew symmetric in $k_1, \ldots, k_q$. Then we have that

$$\langle\varphi, \psi\rangle = \frac{1}{p!q!}g^{i_1\overline{j_1}} \cdots g^{i_p\overline{j_p}}g^{k_1\overline{l_1}} \cdots g^{k_q\overline{l_q}}\varphi_{i_1\cdots i_p\overline{l_1}\cdots\overline{l_q}}\overline{\psi_{j_1\cdots j_p\overline{k_1}\cdots\overline{k_q}}}.$$

We define the total inner product by

$$(\varphi, \psi) := \int_M \langle\varphi, \psi\rangle\, dV_g,$$

where dV_g is the volume form defined by $dV_g := \frac{\omega^n}{n!}$. The Hodge $*$ operator is the unique operator determined by the metric g satisfying that for $\varphi, \psi \in \Lambda^k T^*M$,

$$*\colon \Lambda^k T^*M \to \Lambda^{2n-k}T^*M, \quad \varphi \wedge *\psi = \langle\varphi, \psi\rangle\, dV_g,$$

which can be extended $\mathbb{C}$-linearly satisfying that for $\varphi, \psi \in \Lambda^{p,q}M$,

$$*\colon \Lambda^{p,q}M \to \Lambda^{n-q,n-p}M, \quad \varphi \wedge *\overline{\psi} = \langle\varphi, \psi\rangle\, dV_g.$$

We have that $\overline{*\varphi} = *\overline{\varphi}$, $**\varphi = (-1)^{p+q}\varphi$, $\langle*\varphi, *\psi\rangle = \langle\varphi, \psi\rangle$. The adjoint operators ∂^*, $\overline{\partial}^*$ are given by

$$(\partial\varphi, \psi) = (\varphi, \partial^*\psi), \quad (\overline{\partial}\varphi, \psi) = (\varphi, \overline{\partial}^*\psi). \tag{2.21}$$

We define that

$$|\varphi|^2 := \langle\varphi, \varphi\rangle.$$

Lemma 2.3. [19, Lemma 2.3] *Let (M, J, ω) be a compact almost Hermitian manifold. One has that*

$$\partial^* = - * \overline{\partial} *, \quad \overline{\partial}^* = - * \partial * .$$

We write $\iota_i\varphi := \iota_{Z_i}\varphi$, $\iota_{\bar{j}}\varphi := \iota_{Z_{\bar{j}}}\varphi$. It follows from (2.20),

$$\langle \varphi, \zeta^i \wedge \psi\rangle = \langle g^{j\bar{i}}\iota_j\varphi, \psi\rangle, \quad \langle \varphi, \zeta^{\bar{i}} \wedge \psi\rangle = \langle g^{i\bar{j}}\iota_{\bar{j}}\varphi, \psi\rangle. \tag{2.22}$$

We define the Lefschetz operator $L\colon \Lambda^{p,q}M \to \Lambda^{p+1,q+1}M$ and its adjoint operator $\Lambda\colon \Lambda^{p+1,q+1}M \to \Lambda^{p,q}M$ by

$$L\varphi = \omega \wedge \varphi, \quad \langle L\varphi, \psi\rangle = \langle \varphi, \Lambda\psi\rangle.$$

Locally, we obtain that from (2.22),

$$\Lambda = \sqrt{-1}g^{i\bar{j}}\iota_i\iota_{\bar{j}}. \tag{2.23}$$

For a (p,q)-form $\varphi \in \Lambda^{p,q}M$ with $p+q = k \leq n$, we have that $[L,\Lambda]\varphi = (k-n)\varphi$. Applying this repeatedly, we obtain that

$$[L^r, \Lambda]\varphi = [L^{r-s}, \Lambda]L^s\varphi + s(k-n+s-1)L^{r-1}\varphi.$$

Especially, we have that for $s = r$,

$$[L^r, \Lambda]\varphi = r(k-n+r-1)L^{r-1}\varphi. \tag{2.24}$$

Definition 2.4. We call a (p,q)-form φ primitive if $\Lambda\varphi = 0$.

For a primitive (p,q)-form φ with $p+q = k \leq n$, we have that

$$\Lambda(\omega \wedge \varphi) = (n-k)\varphi \quad \text{and} \quad \Lambda L^r\varphi = r(n-k-r+1)L^{r-1}\varphi.$$

3. Key lemmas

Let (M^{2n}, J, ω) be a $2n$-dimensional compact almost Hermitian manifold with $n \geq 2$. Let g be the almost Hermitian metric associated to the real $(1,1)$-form ω. Let $\{e_r\}$ be a local $(1,0)$-frame with respect to the metric g around a point $p_0 \in M$ and let $\{\theta^r\}$ be a local associated coframe with respect to $\{e_r\}$.

We introduce the following lemma.

Lemma 3.1. [21, Lemma 4.1] *Let (M^{2n}, J, ω) be a compact almost Hermitian manifold of real dimension $2n$ with $n \geq 2$. Then,* $\mathrm{HSC}(\omega) > 0$ *(resp.* $\mathrm{HSC}(\omega) = 0$*) implies that* $s_\omega + \widehat{s}_\omega > 0$ *(resp.* $s_\omega + \widehat{s}_\omega = 0$*).*

Define the set of the conformal class of ω as follows:

$$\{\omega\} := \{e^u\omega \mid u \in C^\infty(M;\mathbb{R})\}.$$

From Proposition 1.2, we may take a Gauduchon metric ω_0 in the conformal class of ω such that $\omega_0 = f_0^{\frac{1}{n-1}}\omega \in \{\omega\}$, where f_0 is a positive smooth function. Let g_0 be the associated almost Hermitian metric with respect to the real $(1,1)$-form ω_0.

Lemma 3.2. [3, (1.7)] *Let (M^{2n}, J, ω) be a real $2n$-dimensional compact almost Hermitian manifold with $n \geq 2$. One has that*

$$\int_M s_{\omega_0}\omega_0^n = \int_M f_0 s_\omega \omega^n, \quad \int_M \widehat{s}_{\omega_0}\omega_0^n = \int_M f_0 \widehat{s}_\omega \omega^n.$$

Proof. Let $\Gamma(g)$, $\Gamma(g_0)$ denote the Christoffel symbols of g, g_0 respectively. Just writing Γ or B means that they do not depend on any metrics. Note that choosing an arbitrarily chosen local $(1,0)$-frame $\{e_i\}$, since we have $\Gamma(g)^k_{ij} = g^{r\bar{s}}e_i(g_{j\bar{s}}) - g^{r\bar{s}}g_{j\bar{l}}B^{\bar{l}}_{i\bar{s}}$, we compute for the Gaudhuchon metric $g_0 = f^{\frac{1}{n-1}}g$,

$$\begin{aligned}
\Gamma(g_0)^k_{ij} &= g_0^{r\bar{s}}e_i((g_0)_{j\bar{s}}) - g_0^{r\bar{s}}(g_0)_{j\bar{l}}B^{\bar{l}}_{i\bar{s}} \\
&= f^{-\frac{1}{n-1}}g^{r\bar{s}}e_i(f^{\frac{1}{n-1}}g_{j\bar{s}}) - f^{-\frac{1}{n-1}}g^{r\bar{s}}f^{\frac{1}{n-1}}g_{j\bar{l}}B^{\bar{l}}_{i\bar{s}} \\
&= g^{r\bar{s}}e_i(g_{j\bar{s}}) - g^{r\bar{s}}g_{j\bar{l}}B^{\bar{l}}_{i\bar{s}} + f^{-\frac{1}{n-1}}g^{r\bar{s}}e_i(f^{\frac{1}{n-1}})g_{j\bar{s}} \\
&= \Gamma(g)^k_{ij} + e_i\{\log(f^{\frac{1}{n-1}})\}\delta_{rj}.
\end{aligned} \tag{3.1}$$

Now, we choose a local unitary $(1,0)$-frame $\{e_i\}$ with respect to g around an arbitrary chosen point $p_0 \in M$ satisfying (2.13). Note that $\{e_i\}$ is a local $(1,0)$-frame with respect to the metric g_0 from the construction. From (2.16) and (3.1), at p_0,

$$\begin{aligned}
R(\omega)_{i\bar{j}k\bar{l}} &= R(\omega)^r_{i\bar{j}k}g_{r\bar{l}} \\
&= \{e_i(\Gamma^r_{\bar{j}k}) - e_{\bar{j}}(\Gamma(g)^r_{ik})\}g_{r\bar{l}} \\
&= \{e_i(\Gamma^r_{\bar{j}k}) - e_{\bar{j}}(\Gamma(g_0)^r_{ik}) + \Gamma(g_0)^r_{is}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma(g_0)^s_{ik} - B^s_{i\bar{j}}\Gamma(g_0)^r_{sk} + B^{\bar{s}}_{\bar{j}i}\Gamma^r_{\bar{s}k} \\
&\quad + e_{\bar{j}}e_i\{\log(f^{\frac{1}{n-1}})\}\delta_{kr}\}g_{r\bar{l}} \\
&= R(\omega_0)^r_{i\bar{j}k}(g_0)_{r\bar{l}}f_0^{-\frac{1}{n-1}} + e_ie_{\bar{j}}\{\log(f^{\frac{1}{n-1}})\}g_{k\bar{l}} \\
&= f_0^{-\frac{1}{n-1}}\{R(\omega_0)_{i\bar{j}k\bar{l}} + \partial_i\partial_{\bar{j}}\{\log(f^{\frac{1}{n-1}})\}(g_0)_{k\bar{l}}\},
\end{aligned} \tag{3.2}$$

where we have used $[e_i, e_{\bar{j}}](p_0) = 0$ and that $B^{\bar{s}}_{\bar{j}i}$, $B^s_{i\bar{j}}$, $\Gamma^r_{\bar{j}k}$ and $\Gamma^{\bar{k}}_{i\bar{r}}$ do not depend on metrics and $B^{\bar{s}}_{\bar{j}i}(p_0) = B^s_{i\bar{j}}(p_0) = \Gamma^r_{\bar{j}k}(p_0) = \Gamma^{\bar{k}}_{i\bar{r}}(p_0) = 0$. From the relation between the curvatures of ω and ω_0 in (3.2), we obtain that at p_0,

$$\begin{aligned}
s_{\omega_0} &= g_0^{i\bar{j}}g_0^{k\bar{l}}R(\omega_0)_{i\bar{j}k\bar{l}} \\
&= g_0^{i\bar{j}}g_0^{k\bar{l}}\big(f_0^{\frac{1}{n-1}}R(\omega)_{i\bar{j}k\bar{l}} - \partial_i\partial_{\bar{j}}\{\log(f^{\frac{1}{n-1}})\}(g_0)_{k\bar{l}}\big) \\
&= f_0^{-\frac{1}{n-1}}g^{i\bar{j}}f_0^{-\frac{1}{n-1}}g^{k\bar{l}}f_0^{\frac{1}{n-1}}R(\omega)_{i\bar{j}k\bar{l}} - ng_0^{i\bar{j}}\partial_i\partial_{\bar{j}}\{\log(f^{\frac{1}{n-1}})\} \\
&= f_0^{-\frac{1}{n-1}}s_\omega - n\Delta_0\log(f^{\frac{1}{n-1}}),
\end{aligned} \tag{3.3}$$

where Δ_0 is the Laplacian with respect to the metric g_0. Since the point p_0 is chosen arbitrary, we have that from (3.3),

$$s_{\omega_0} = f_0^{-\frac{1}{n-1}}s_\omega - n\Delta_0\log(f^{\frac{1}{n-1}}) \quad \text{on whole } M.$$

Similarly, we compute at p_0,

$$
\begin{aligned}
\widehat{s}_{\omega_0} &= g_0^{i\bar{l}} g_0^{k\bar{j}} R(\omega_0)_{i\bar{j}k\bar{l}} \\
&= g_0^{i\bar{l}} g_0^{k\bar{j}} \big(f_0^{\frac{1}{n-1}} R(\omega)_{i\bar{j}k\bar{l}} - \partial_i \partial_{\bar{j}} \{ \log(f^{\frac{1}{n-1}}) \} (g_0)_{k\bar{l}} \big) \\
&= f_0^{-\frac{1}{n-1}} g^{i\bar{l}} f_0^{-\frac{1}{n-1}} g^{k\bar{j}} f_0^{\frac{1}{n-1}} R(\omega)_{i\bar{j}k\bar{l}} - g_0^{i\bar{l}} \delta_{jl} \partial_i \partial_{\bar{j}} \{ \log(f^{\frac{1}{n-1}}) \} \\
&= f_0^{-\frac{1}{n-1}} \widehat{s}_\omega - \Delta_0 \log(f^{\frac{1}{n-1}}).
\end{aligned} \tag{3.4}
$$

Since the point p_0 is arbitrary, we have that $\widehat{s}_{\omega_0} = f_0^{-\frac{1}{n-1}} \widehat{s}_\omega - \Delta_0 \log(f^{\frac{1}{n-1}})$ on whole M. By applying the Stokes' theorem, we obtain that

$$
\begin{aligned}
\int_M \Delta_0 \log(f^{\frac{1}{n-1}}) \omega_0^n &= \int_M n \frac{\partial\overline{\partial} \log(f^{\frac{1}{n-1}}) \wedge \omega_0^{n-1}}{\omega_0^n} \omega_0^n \\
&= n \int_M \partial\overline{\partial} \log(f^{\frac{1}{n-1}}) \wedge \omega_0^{n-1} \\
&= n \int_M d(\overline{\partial} \log(f^{\frac{1}{n-1}}) \wedge \omega_0^{n-1}) + \int_M \overline{\partial} \log(f^{\frac{1}{n-1}}) \wedge \partial \omega_0^{n-1} \\
&= \int_M d(\log(f^{\frac{1}{n-1}}) \partial \omega_0^{n-1}) - \int_M \log(f^{\frac{1}{n-1}}) \overline{\partial} \partial \omega_0^{n-1} \\
&= 0,
\end{aligned} \tag{3.5}
$$

where we used that $(\overline{\partial} + A + \overline{A})(\overline{\partial} \log(f^{\frac{1}{n-1}}) \wedge \omega_0^{n-1}) = 0$, $(\partial + A + \overline{A})(\log(f^{\frac{1}{n-1}}) \partial \omega_0^{n-1}) = 0$, and $\overline{\partial}\partial\omega_0^{n-1} = -(\partial\overline{\partial} + A\overline{A} + \overline{A}A)\omega_0^{n-1} = -\partial\overline{\partial}\omega_0^{n-1} = 0$ since ω_0 is Gauduchon and $A\omega_0^{n-1} = \overline{A}\omega_0^{n-1} = 0$. Integrating (3.3), we have that from (3.5),

$$
\int_M s_{\omega_0} \omega_0^n = \int_M f_0^{-\frac{1}{n-1}} s_\omega f_0^{\frac{n}{n-1}} \omega^n = \int_M f_0 s_\omega \omega^n.
$$

Similarly, by integrating (3.4) and from (3.5), we have that $\int_M \widehat{s}_{\omega_0} \omega_0^n = \int_M f_0 \widehat{s}_\omega \omega^n$. $\square$

4. Proofs of Proposition 1.30, Theorems 1.31 and 1.32

We first investigate the case of $n = 2$. Let (M^4, J, ω) be a real 4-dimensional compact Kähler-like almost Hermitian manifold and let g be the associated almost Hermitian metric of ω. Let $\{e_r\}$ be an arbitrary chosen local $(1,0)$-frame around a point $p_0 \in M^4$ with respect to the metric g and let $\{\theta^r\}$ be a local associated coframe with respect to $\{e_r\}$ in this section.

We define the torsion $(1,0)$-form (see [14]) by

$$
w_i := T_{ik}^k = g^{k\bar{l}} T_{ik\bar{l}}, \quad \eta := -w_i \theta^i.
$$

Note that for any real $(1,1)$-form $\sigma = \sqrt{-1}\sigma_{i\bar{j}}\theta^i \wedge \theta^{\bar{j}}$, we have

$$\partial\sigma = \frac{\sqrt{-1}}{2}\big(e_i(\sigma_{j\bar{k}}) - e_j(\sigma_{i\bar{k}}) - B^s_{ij}\sigma_{s\bar{k}} - B^{\bar{s}}_{i\bar{k}}\sigma_{j\bar{s}} + B^{\bar{s}}_{j\bar{k}}\sigma_{i\bar{s}}\big)\theta^i \wedge \theta^j \wedge \theta^{\bar{k}}, \tag{4.1}$$

Then we have from (2.19), (2.23) and (4.1),

$$\eta = -\Lambda\partial\omega. \tag{4.2}$$

Lemma 4.1. [19, Lemma 3.1] *Let (M^4, J, ω) be a real 4-dimensional almost Hermitian manifold. Then one has that*

$$\partial\omega = -\eta \wedge \omega.$$

Proof. From (2.24) for $r = k = 1$, we have that $\Lambda(\partial\omega - L(\Lambda\partial\omega)) = 0$. Since Λ is injective (see [24, Lemma 6.24]), we obtain that

$$\partial\omega - L(\Lambda\partial\omega) = 0. \tag{4.3}$$

Combining (4.3) with (4.2), we have

$$\partial\omega = L(\Lambda\partial\omega) = -\eta \wedge \omega.$$ □

We restate the following lemma combining Lemma 4.1 with the case of $n \geq 3$ in [19, Lemma 3.1].

Lemma 4.2. *Let (M^{2n}, J, ω) be an almost Hermitian manifold with $n \geq 2$. Then one has that*

$$\partial\omega^{n-1} = -\eta \wedge \omega^{n-1}.$$

We also have the following lemma. We can give a proof in the same manner by taking $n = 2$ and using $*\omega = \omega$ in the case of $n = 2$ as well.

Lemma 4.3. [19, Lemma 3.2] *Let (M^{2n}, J, ω) be a real $2n$-dimensional compact almost Hermitian manifold with $n \geq 2$. Then, one has that*

$$\eta = -\Lambda\partial\omega = \sqrt{-1}\partial^*\omega. \tag{4.4}$$

For any $(1,0)$-form α, we have that $\overline{\partial}\alpha = \partial_{\bar{j}}\alpha_i\theta^{\bar{j}} \wedge \theta^i$, and by using (2.2) and (2.4),

$$\nabla_{\bar{j}}\alpha_i = e_{\bar{j}}(\alpha_i) - \Gamma^k_{\bar{j}i}\alpha_k = e_{\bar{j}}(\alpha_i) - B^k_{\bar{j}i}\alpha_k = \partial_{\bar{j}}\alpha_i. \tag{4.5}$$

We compute by applying (4.4) and (4.5),

$$\overline{\partial}\partial^*\omega = -\sqrt{-1}\overline{\partial}\eta = \sqrt{-1}\overline{\partial}(w_i\theta^i) = \sqrt{-1}\partial_{\bar{j}}w_i\theta^{\bar{j}} \wedge \theta^i = -\sqrt{-1}\nabla_{\bar{j}}w_i\theta^i \wedge \theta^{\bar{j}}, \tag{4.6}$$

where ∇ is the Chern connection.

We have from (2.8),

$$P_{i\bar{j}} - g^{k\bar{l}} R_{k\bar{j}i\bar{l}} = -\nabla_{\bar{j}} w_i + T^{\bar{r}}_{si} T^{s}_{\bar{r}\bar{j}}, \tag{4.7}$$

where $P_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$ is the first Chern–Ricci curvature, and we used that $T^s_{si} = -T^s_{is} = -w_i$ and $B^{\bar{r}}_{is} = -B^{\bar{r}}_{si} = -T^{\bar{r}}_{si}$. By combining (4.6) with (4.7), and by summing over indices i, j with respect to the metric ω, we obtain that

$$s_\omega - \widehat{s}_\omega = \langle \overline{\partial\partial}^* \omega, \omega \rangle + T^{\bar{r}}_{si} T^{s}_{\bar{r}\bar{i}}. \tag{4.8}$$

Since the formula (4.8) holds for $n \geq 3$ as well (see [19, (3.7)]), we restate the following statement for $n \geq 2$.

Lemma 4.4. *Let (M^{2n}, J, ω) be a real $2n$-dimensional compact almost Hermitian manifold with $n \geq 2$. Then we have*

$$s_\omega - \widehat{s}_\omega = \langle \overline{\partial\partial}^* \omega, \omega \rangle + T^{\bar{r}}_{si} T^{s}_{\bar{r}\bar{i}}, \tag{4.9}$$

where $\langle \overline{\partial\partial}^* \omega, \omega \rangle = g^{i\bar{j}} g^{p\bar{q}} (\overline{\partial\partial}^* \omega)_{i\bar{q}} \overline{\omega_{j\bar{p}}} = g^{i\bar{j}} g^{p\bar{q}} (-\sqrt{-1} \nabla_{\bar{q}} w_i) \overline{\sqrt{-1} g_{j\bar{p}}} = -g^{i\bar{j}} \nabla_{\bar{j}} w_i$.

We consider the general dimension $n \geq 2$ in the following computations. We have the following proposition, which implies that Theorem 1.31 holds from Lemma 1.12.

Proposition 4.5. *Let (M^{2n}, J, ω) be a compact almost Hermitian manifold with $n \geq 2$ and* $\mathrm{HSC}(\omega) > 0$*. Assume that $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \geq 0$, then $\int_M s_{\omega_0} \omega_0^n > 0$.*

Proof. We may take a Gauduchon metric ω_0 in the conformal class of ω such that $\omega_0 = f_0^{\frac{1}{n-1}} \omega \in \{\omega\}$, where f_0 is a positive smooth function. Let g_0 be the associated almost Hermitian metric with respect to ω_0. Define $dV_{g_0} := \frac{\omega_0^n}{n!}$. By integrating the formula (4.9) for ω_0, assuming $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \geq 0$, we obtain that from (2.21),

$$\begin{aligned}
\int_M (s_{\omega_0} - \widehat{s}_{\omega_0}) \, dV_{g_0} &= \int_M \langle \overline{\partial\partial}^* \omega_0, \omega_0 \rangle \, dV_{g_0} + \int_M T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \, dV_{g_0} \\
&= (\overline{\partial\partial}^* \omega_0, \omega_0) + \int_M T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \, dV_{g_0} \\
&= (\overline{\partial}^* \omega_0, \overline{\partial}^* \omega_0) + \int_M T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \, dV_{g_0} \\
&= \int_M |\overline{\partial}^* \omega_0|^2 \, dV_{g_0} + \int_M T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \, dV_{g_0} \\
&\geq 0
\end{aligned}$$

and

$$\int_M (s_{\omega_0} - \widehat{s}_{\omega_0}) \omega^n \geq 0.$$

Since we have assumed $\mathrm{HSC}(\omega) > 0$, we have that $s_\omega + \widehat{s}_\omega > 0$ from Lemma 3.1, we get that by using (4.10),

$$\int_M s_{\omega_0}\omega_0^n = \frac{1}{2}\int_M (s_{\omega_0} + \widehat{s}_{\omega_0})\omega_0^n + \frac{1}{2}\int_M (s_{\omega_0} - \widehat{s}_{\omega_0})\omega_0^n \geq \frac{1}{2}\int_M f_0(s_\omega + \widehat{s}_\omega)\omega^n > 0,$$

where we have used the positivity of the smooth function f_0. □

From Proposition 1.26(ii), we conclude that we have $\kappa(M) = -\infty$, which means that Proposition 1.30 holds. By applying Lemma 1.12, we conclude that Theorem 1.31 holds. Since the following lemma tells us that we have $T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i} \geq 0$ on a real 4-dimensional almost Hermitian manifold, we conclude that Theorem 1.32 holds.

Lemma 4.6. *On a real* 4*-dimensional almost Hermitian manifold, we have that*

$$T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i} \geq 0. \tag{4.10}$$

The equality $T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i} = 0$ *holds if and only if* $T_{12}^{\overline{1}} = T_{12}^{\overline{2}}$.

Proof. We compute that

$$\begin{aligned}
&T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i} \\
&= g^{j\overline{k}} g^{i\overline{s}} g^{p\overline{q}} g_{p\overline{r}} g_{l\overline{s}} T_{ij}^{\overline{r}} T_{\overline{q}\overline{k}}^{l} = g^{j\overline{k}} \delta_{qr}\delta_{il} T_{ij}^{\overline{r}} T_{\overline{q}\overline{k}}^{l} = g^{j\overline{k}} (T_{1j}^{\overline{1}} T_{\overline{1}\overline{k}}^{1} + T_{1j}^{\overline{2}} T_{\overline{2}\overline{k}}^{1} + T_{2j}^{\overline{1}} T_{\overline{1}\overline{k}}^{2} + T_{2j}^{\overline{2}} T_{\overline{2}\overline{k}}^{2}) \\
&= T_{12}^{\overline{1}} T_{\overline{1}\overline{2}}^{1} + T_{12}^{\overline{2}} T_{\overline{2}\overline{1}}^{1} + T_{21}^{\overline{1}} T_{\overline{1}\overline{2}}^{2} + T_{21}^{\overline{2}} T_{\overline{2}\overline{1}}^{2} = T_{12}^{\overline{1}} T_{\overline{1}\overline{2}}^{1} - T_{12}^{\overline{2}} T_{\overline{1}\overline{2}}^{1} - T_{12}^{\overline{1}} T_{\overline{1}\overline{2}}^{2} + T_{12}^{\overline{2}} T_{\overline{1}\overline{2}}^{2} \\
&= (T_{12}^{\overline{1}} - T_{12}^{\overline{2}})(T_{\overline{1}\overline{2}}^{1} - T_{\overline{1}\overline{2}}^{2}) = (T_{12}^{\overline{1}} - T_{12}^{\overline{2}})\overline{(T_{12}^{\overline{1}} - T_{12}^{\overline{2}})} = |T_{12}^{\overline{1}} - T_{12}^{\overline{2}}|^2 \geq 0.
\end{aligned}$$
□

Remark 4.7. On a real 4-dimensional compact almost Hermitian manifold (M^4, J, ω), let us assume that there exists an almost Hermitian metric $\widetilde{\omega}$ on M^4 such that

$$\int_M s_{\widetilde{\omega}}\omega^2 \leq \int_M \widehat{s}_{\widetilde{\omega}}\omega^2.$$

Let $\widetilde{g}$ be the associated almost Hermitian metric with respect to $\widetilde{\omega}$ and let us define $dV_{\widetilde{g}} := \frac{\widetilde{\omega}^2}{2!}$. Then we obtain that

$$\begin{aligned}
0 &\geq \int_M (s_{\widetilde{\omega}} - \widehat{s}_{\widetilde{\omega}})\, dV_{\widetilde{g}} = \int_M \langle \overline{\partial}\overline{\partial}^* \widetilde{\omega}, \widetilde{\omega}\rangle\, dV_{\widetilde{g}} + \int_M T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i}\, dV_{\widetilde{g}} \\
&= \int_M |\overline{\partial}^* \widetilde{\omega}|^2\, dV_{\widetilde{g}} + \int_M T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i}\, dV_{\widetilde{g}} \\
\iff \quad 0 &\geq -\int_M |\overline{\partial}^* \widetilde{\omega}|^2\, dV_{\widetilde{g}} = \int_M T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i}\, dV_{\widetilde{g}}.
\end{aligned} \tag{4.11}$$

Combining $T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i} \geq 0$ on M^4 from Lemma 4.6 with (4.11), we obtain that

$$\int_M T_{ij}^{\overline{r}} T_{\overline{r}\overline{j}}^{i}\omega^2 = \int_M |T_{12}^{\overline{1}} - T_{12}^{\overline{2}}|^2\omega_0^2 = 0,$$

which tells that $T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} = 0$ (i.e., $T_{12}^{\bar{1}} = T_{12}^{\bar{2}}$). Then we get from (4.11), $\int_M |\overline{\partial}^* \widetilde{\omega}_0|^2 \omega^2 = 0$, which gives us that $\overline{\partial}^* \widetilde{\omega}_0 = 0$ on M^4. Hence, the metric $\widetilde{\omega}_0$ is almost Kähler on M^4, then we must have $T'' \equiv 0$ and the manifold must be Kähler from Lemma 1.12.

Proposition 4.8. *Let (M^4, J, ω) be a real 4-dimensional compact almost Hermitian manifold. Assume that there exists an almost Hermitian metric $\widetilde{\omega}$ on M^4 such that $\int_M s_{\widetilde{\omega}} \omega^2 \leq \int_M \widehat{s}_{\widetilde{\omega}} \omega^2$. Then the manifold is Kähler.*

Similarly, as in Proposition 4.5, assuming that there exists an almost Hermitian metric $\widetilde{\omega}$ such that $\int_M s_{\widetilde{\omega}} \widetilde{\omega}^n \leq \int_M \widehat{s}_{\widetilde{\omega}} \widetilde{\omega}^n$, by integrating the equation (4.9) for $\widetilde{\omega}$, then $\int_M T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} \widetilde{\omega}^n \leq 0$. Hence, the following statement holds.

Proposition 4.9. *Let (M^{2n}, J, ω) be a real $2n$-dimensional compact almost Kähler manifold with $n \geq 3$. Assume that there exists an almost Hermitian metric $\widetilde{\omega}$ such that $\int_M s_{\widetilde{\omega}} \widetilde{\omega}^n \leq \int_M \widehat{s}_{\widetilde{\omega}} \widetilde{\omega}^n$ on M. Then the manifold is Kähler.*

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Masaya Kawamura
School of Education, Sugiyama Jogakuen University and Graduate School, 17-3 Hoshigaoka-motomachi, Chikusa-ku, Nagoya, Aichi 464-8662, Japan
E-mail address: kawamura-m@sugiyama-u.ac.jp