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Marginally Trapped Ruled Surfaces and Their Gauss Map in Minkowski Space

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Abstract. In 1991, Chen proposed a conjecture which is the relationship between biharmonic submanifolds and harmonic submanifolds in Euclidean space and quite a few related studies have supported it. Around the same time, it was proved that Chen's conjecture does not extend to submanifolds in Minkowski space. In this paper, as part of these researches, we investigate biharmonic marginally trapped ruled surfaces in Minkowski m-space and then construct some examples about them in which Chen's conjecture does not hold.

1. Introduction

In the middle of 1980s, Chen introduced the biharmonic submanifold in Euclidean space, which is the notion generated from the studies of the finite-type immersion of submanifolds in Euclidean space (see [5]): An isometric immersion x of a Riemannian submanifold M into a Euclidean space $\mathbb{E}^m$ is said to be *biharmonic* if it satisfies

$$\Delta^2 x = \mathbf{0},$$

where Δ and $\mathbf{0}$ denote the Laplace operator defined on M and zero vector, respectively. And he proved that there are no biharmonic surfaces in $\mathbb{E}^3$ except the minimal ones (see [5]) and there exist no biharmonic submanifolds of $\mathbb{E}^m$ which lie in a hypersphere of $\mathbb{E}^m$ (see [7]). In [14, 15], Dimitric showed that biharmonic curves in Euclidean space $\mathbb{E}^m$ are part of straight lines (that is, minimal), biharmonic submanifolds of finite-type in $\mathbb{E}^m$ are minimal, and pseudo umbilical submanifolds M of $\mathbb{E}^m$ with $\dim M \neq 4$ are minimal. Based on these results, in 1991 Chen proposed the following:

Chen's Conjecture. *Biharmonic submanifolds of Euclidean spaces are minimal.*

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Since then, Chen's Conjceture has been investigated by some mathematicians for some submanifolds of Euclidean space. Hasanis and Vlachos showed that a biharmonic hypersurface in $\mathbb{E}^4$ is harmonic (see [20]) and Fu showed that biharmonic hypersurfaces with 3 distinct principle curvatures in $\mathbb{E}^m$ with arbitrary m are harmonic (see [18, 19]). In [37], it was proved that this conjecture is true for G-invariant hypersurfaces in $\mathbb{E}^m$. Although it has been proven for some cases, Chen's conjecture has not been verified yet, in general. Interestingly, in [9, 10] Chen et al. showed that biharmonic surfaces in pseudo-Euclidean 3-spaces are minimal and that there exist proper biharmonic surfaces in 4-dimensional pseudo-Euclidean spaces $\mathbb{E}^4_s$ (with index $s = 1, 2, 3$). This means that his conjecture does not extend to submanifolds in pseudo-Euclidean space, which indicates that biharmonic submanifolds in pseudo-Euclidean space are worth studying (see [16, 17, 35, 36, 38]).

Meanwhile, since the concept of *trapped surfaces* was introduced by Penrose in [39], it has played an important role in general relativity, for example, the singularity theorems, the analysis of gravitational collapse, the cosmic censorship hypothesis, etc. In the theory of cosmic black holes, a *marginally trapped surface* was considered to separate the *trapped surfaces* from the untrapped ones and it is well known that the surface of a black hole is located by the marginally trapped surface. From the perspective of a differential geometry, a marginally trapped surface in pseudo-Euclidean space is a Riemannian surface whose mean curvature vector field is null at every point of the surface. In the last decade or so, many mathematicians have investigated marginally trapped surfaces in a specific pseudo-Euclidean space with some geometric conditions (see [1, 2, 6, 11, 12, 21, 22]).

In 1966, Takahashi proved an eigenvalue problem of immersion $x\colon M \to \mathbb{E}^m$ of a Riemannian manifold M into a Euclidean space $\mathbb{E}^m$, namely, if $\Delta x = \lambda x$, $\lambda \in \mathbb{R}\setminus\{0\}$ holds on M, then M is a minimal submanifold in Euclidean space or a minimal submanifold in a hypersphere of Euclidean space, where Δ is the Laplace operator defined on M (see [40]). It was the cornerstone for studying minimal submanifolds with an algebraic condition.

In particular, ruled surfaces and ruled submanifolds in Euclidean space or pseudo-Euclidean space, which are the typical and interesting objects in differential geometry, have been intensively studied and characterized in many researches (see [8, 13, 25, 30, 31, 34]). The Gauss map on submanifolds of Euclidean space or pseudo-Euclidean space gives some useful geometrical and topological properties on that submanifold. For that reasons, the Gauss map of ruled surfaces and ruled submanifolds in Euclidean space or pseudo-Euclidean space, which satisfies some geometric properties, has been treated as the main subject of study (see [3, 23, 24, 27–29, 32, 33]).

In this paper, we precisely give a parametrization of marginally trapped ruled surfaces in Minkowski m-space having harmonic Gauss map, which can be obtained naturally from previous our researches, and then we study biharmonic marginally trapped ruled surfaces in Minkowski m-space associated with Chen's Conjecture. Most of all, we will characterize proper biharmonic marginally trapped ruled surfaces in Minkowski space, including constructing some examples.

All of geometric objects under consideration are smooth and submanifolds are assumed to be connected unless otherwise stated.

2. Preliminaries

Let $\mathbb{E}^m_s$ be an m-dimensional pseudo-Euclidean space of signature $(m-s, s)$ with the standard scalar product $\langle\,\cdot\,,\cdot\,\rangle$. For $m \geq 2$, in particular, $\mathbb{E}^m_1$ is called a *Lorentz–Minkowski* m-space or simply *Minkowski* m-space, which is denoted by $\mathbb{L}^m$. A vector X of $\mathbb{L}^m$ is said to be *space-like* if $\langle X, X\rangle > 0$ or $X = \mathbf{0}$, *time-like* if $\langle X, X\rangle < 0$ and *null* (or *light-like*) if $\langle X, X\rangle = 0$ and $X \neq \mathbf{0}$, where $\mathbf{0}$ denotes zero vector. A time-like or null vector in $\mathbb{L}^m$ is said to be *causal*. A curve in $\mathbb{L}^m$ is said to be *space-like*, *time-like* or *null* if its tangent vector field is space-like, time-like or null, respectively.

Lemma 2.1. [32] *There are no causal vectors in $\mathbb{L}^m$ orthogonal to a time-like vector.*

Lemma 2.2. [32] *Two null vectors are orthogonal if and only if they are linearly dependent.*

Let $x\colon M \to \mathbb{E}^m_s$ be an isometric immersion of an n-dimensional pseudo-Riemannian manifold M into $\mathbb{E}^m_s$. From now on, a submanifold in $\mathbb{E}^m_s$ always means pseudo-Riemannian, that is, each tangent space of the submanifold is non-degenerate. Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^m_s$ and ∇ the induced connection on M. Then, the Gauss formula is obtained by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

for the vector fields X, Y tangent to M, where h is the second fundamental form of M in $\mathbb{E}^m_s$. The mean curvature vector field is defined by $H = \frac{1}{n}\operatorname{trace} h$. In other words, for every point $p \in M$, H is given by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i h(e_i, e_i),$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of T_pM and $\varepsilon_i = \langle e_i, e_i\rangle = \pm 1$. We say that a submanifold M in a pseudo-Riemannian manifold N is *marginally trapped* (or *pseudo-minimal*) if its mean curvature vector is null for every point of M.

Let $(x_1, x_2, \ldots, x_n)$ be a local standard coordinate system of M in $\mathbb{E}^m_s$. For the components g_{ij} of the pseudo-Riemannian metric on M induced from that of $\mathbb{E}^m_s$, we denote by (g^{ij}) (respectively, $\mathcal{G}$) the inverse matrix (respectively, the determinant) of the matrix (g_{ij}). The *Laplacian* (or *Laplace operator*) Δ on M is then defined by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial x_j} \right).$$

A pseudo-Riemannian submanifold M of an m-dimensional pseudo-Euclidean space $\mathbb{E}^m_s$ is said to be of *harmonic* if its position vector x satisfies

$$\Delta x = \mathbf{0}.$$

For an adapted local orthonormal frame $\{e_1, e_2, \ldots, e_m\}$ in $\mathbb{E}^m_s$ such that $e_1, e_2, \ldots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \ldots, e_m$ normal to M, the map $G\colon M \to G(n,m) \subset \mathbb{E}^N$ $(N = {}_mC_n)$ defined by $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$ is called the *Gauss map* of M that is a smooth map which carries a point p in M into an oriented n-plane passing through the origin in $\mathbb{E}^m_s$ obtained from the parallel translation of the tangent space of M at p in $\mathbb{E}^m_s$, where $G(n,m)$ is the Grassmann manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m_s$. An indefinite scalar product $\langle\langle \cdot , \cdot \rangle\rangle$ on $G(n,m) \subset \mathbb{E}^N$ is defined by

$$\langle\langle e_{i_1} \wedge \cdots \wedge e_{i_n}, e_{j_1} \wedge \cdots \wedge e_{j_n} \rangle\rangle = \det(\langle e_{i_l}, e_{j_k} \rangle),$$

where l, k run over the range $1, 2, \ldots, n$.

A non-degenerate $(r+1)$-dimensional submanifold M in $\mathbb{L}^m$ is called a *ruled submanifold* if M is foliated by r-dimensional totally geodesic submanifolds $E(s,r)$ of $\mathbb{L}^m$ along a regular curve $\alpha = \alpha(s)$ on M defined on an open interval I. Then, a ruled submanifold M in $\mathbb{L}^m$ can be parameterized by

$$x = x(s, t_1, t_2, \ldots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s), \quad s \in I,\ t_i \in I_i,$$

where I_i's are some open intervals for $i = 1, \ldots, r$. For every s, $E(s,r)$ is open in $\mathrm{Span}\{e_1(s), e_2(s), \ldots, e_r(s)\}$ that is the linear span of linearly independent vector fields $e_1(s), e_2(s), \ldots, e_r(s)$ along the curve α. Here, we assume $E(s,r)$ are either non-degenerate or degenerate for all s along α. We call $E(s,r)$ the *rulings* and α the *base curve* of the ruled submanifold M. The ruled submanifold M is said to be *cylindrical* if $E(s,r)$ is parallel along α, or *non-cylindrical* otherwise.

Remark 2.3. [26] (1) If the rulings of M are non-degenerate, then the base curve α can be chosen to be orthogonal to the rulings as follows: Let V be a unit vector field on M which is orthogonal to the rulings. Then α can be taken as an integral curve of V.

(2) If the rulings are degenerate, we can choose a null base curve which is transversal to the rulings: Let V be a null vector field on M which is not tangent to the rulings. An integral curve of V can be the base curve.

In [4], Barbosa et al. chose a frame along a base curve on a ruled submanifold in $\mathbb{E}^m$ satisfying a special property of a system of ordinary differential equations regarding a frame. Similarly, we have

Lemma 2.4. [25] *Let $V(s)$ be a smooth l-dimensional non-degenerate distribution in the Minkowski m-space $\mathbb{L}^m$ along a curve $\alpha = \alpha(s)$, where $l \geq 2$ and $m \geq 3$. Then, we can choose orthonormal vector fields $e_1(s), \ldots, e_{m-l}(s)$ along α which generate the orthogonal complement $V^{\perp}(s)$ satisfying $e_i'(s) \in V(s)$ for $1 \leq i \leq m-l$.*

3. The Gauss map of marginally trapped ruled surfaces in $\mathbb{L}^m$

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ with non-degenerate rulings. By Remark 2.3, the base curve $\alpha = \alpha(s)$ can be chosen to be orthogonal to the rulings. Without loss of generality, we may assume that α is a unit speed curve, that is, $\langle \alpha', \alpha' \rangle = 1$. From now on, the prime $'$ denotes d/ds unless otherwise stated. By Lemma 2.4, we may choose vector fields $e_1(s), e_2(s), \ldots, e_r(s)$ along α satisfying

$$\langle \alpha'(s), e_i(s) \rangle = 0, \quad \langle e_i(s), e_j(s) \rangle = \delta_{ij} \quad \text{and} \quad \langle e_i'(s), e_j(s) \rangle = 0 \tag{3.1}$$

for $i, j = 1, 2, \ldots, r$. Then, a parametrization of M is given by

$$x = x(s, t_1, t_2, \ldots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s). \tag{3.2}$$

We always assume that the parametrization (3.2) satisfies condition (3.1). Because of (3.1), the Gauss map G of M is naturally expressed as

$$G = \frac{1}{\|x_s\|} x_s \wedge x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_r},$$

or, equivalently

$$G = \frac{1}{\sqrt{q}} \left(\alpha' \wedge e_1 \wedge \cdots \wedge e_r + \sum_{i=1}^{r} t_i e_i' \wedge e_1 \wedge \cdots \wedge e_r \right),$$

where q is the function of $s, t_1, t_2, \ldots, t_r$ defined by $q = \langle x_s, x_s \rangle$.

In [27], the authors studied and characterized ruled submanifolds in $\mathbb{L}^m$ with harmonic Gauss map. In particular, for ruled submanifolds with non-degenerate rulings in $\mathbb{L}^m$, it is shown:

Theorem 3.1. [27] *Let M be an $(r+1)$-dimensional ruled submanifold with non-degenerate rulings in the Minkowski m-space $\mathbb{L}^m$. Then, M has harmonic Gauss map if and only if*

M is part of either an $(r+1)$-plane or a ruled submanifold up to cylinders over a certain submanifold with the parametrization given by

$$x(s,t_1,t_2,\dots,t_r) = f(s)\mathbf{N} + s\mathbf{E} + \sum_{j=1}^{r} t_j(p_j(s)\mathbf{N} + \mathbf{F}_j)$$

for some polynomials f and p_j in s with $\deg f \le 2$, $\deg p_j \le 1$ and some constant vector fields $\mathbf{N}$, $\mathbf{E}$, $\mathbf{F}_j$ with $\langle \mathbf{E},\mathbf{E}\rangle = 1$, $\langle \mathbf{N},\mathbf{N}\rangle = \langle \mathbf{N},\mathbf{E}\rangle = \langle \mathbf{N},\mathbf{F}_j\rangle = \langle \mathbf{E},\mathbf{F}_j\rangle = 0$ and $\langle \mathbf{F}_j,\mathbf{F}_i\rangle = \delta_{ji}$ for $i,j = 1,2,\dots,r$. In particular, if $\deg p_j = 0$ for all j, then M is cylindrical and otherwise M is non-cylindrical.

Remark 3.2. We should point out that some detailed properties for the polynomials f and p_j in s were mistakenly dropped for $j = 1,\dots,r$ in Theorem 3.6 of [27].

Thus, Theorem 3.1 implies that if M is non-planar, then the mean curvature vector field H of M vanishes if $\deg f < 2$ and is a null constant vector field if $\deg f = 2$. Therefore, we have

Theorem 3.3. *Let M be an $(r+1)$-dimensional non-planar ruled submanifold with non-degenerate rulings in the Minkowski m-space $\mathbb{L}^m$. If M has harmonic Gauss map, then M is minimal or marginally trapped.*

4. Biharmonic marginally trapped ruled surfaces in $\mathbb{L}^m$

In this section, we consider biharmonic marginally trapped ruled surfaces in $\mathbb{L}^m$. Let M be a biharmonic marginally trapped ruled surface in $\mathbb{L}^m$ parameterized by

$$x = x(s,t) = \alpha(s) + t\beta(s) \tag{4.1}$$

satisfying

$$\langle \alpha',\alpha'\rangle = \langle \beta,\beta\rangle = 1 \quad \text{and} \quad \langle \alpha',\beta\rangle = 0.$$

Then, $\Delta^2 x = \mathbf{0}$. By definition, the Laplacian Δ of M is expressed as

$$\Delta = \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q}\frac{\partial q}{\partial t}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2}.$$

Since $\Delta x = -2H$, the mean curvature vector field H on M is given by

$$\begin{aligned} H = \frac{1}{2q^2}\Big\{&(\alpha'' + u\beta) + (2u\alpha'' + \beta'' - u'\alpha' + 2u^2\beta + w\beta)t \\ &+ \Big(w\alpha'' + 2u\beta'' - \frac{w'}{2}\alpha' - u'\beta' + 3uw\beta\Big)t^2 + \Big(w\beta'' - \frac{w'}{2}\beta' + w^2\beta\Big)t^3\Big\} \end{aligned} \tag{4.2}$$

which is null at each point of M, where $u(s) = \langle \alpha',\beta'\rangle$ and $w(s) = \langle \beta',\beta'\rangle$.

Lemma 4.1. *If M is a biharmonic marginally trapped ruled surface in $\mathbb{L}^m$ parameterized by (4.1), then the functions $u = u(s)$ and $w = w(s)$ are constant on M, i.e.,*

$$\frac{\partial q}{\partial s} = 0 \quad \textit{on } M.$$

Proof. From (4.2), equation $\langle H, H\rangle = 0$ gives a polynomial in t of degree 6 with functions of s as coefficients, which is vanishing everywhere on M. Then, by considering the coefficient functions of this polynomial, we can get

$$\begin{gathered}\langle \alpha'', \alpha''\rangle = u^2, \quad \langle \alpha'', \beta''\rangle = uw, \quad \langle \beta'', \beta''\rangle = (u')^2 + w^2, \\ u'w' = 2u(u')^2 \quad \text{and} \quad (w')^2 = 4(u')^2 w.\end{gathered} \tag{4.3}$$

Equation '$u'w' = 2u(u')^2$' of (4.3) yields

$$u'(w' - 2uu') = 0.$$

We consider the open set $U = \{s \in \operatorname{dom}\alpha \mid u' \neq 0\}$ and suppose that U is non-empty. Here $\operatorname{dom}\alpha$ means the domain of α. Then, $w' = 2uu'$ on U. With the help of '$(w')^2 = 4(u')^2 w$' of (4.3), it follows that

$$w = u^2 \quad \text{on } U$$

and hence Δx is given by

$$\begin{aligned}\Delta x = -\frac{1}{(1+ut)^4}\big\{&(\alpha'' + u\beta) + (2u\alpha'' + \beta'' - u'\alpha' + 3u^2\beta)t \\ &+ (u^2\alpha'' + 2u\beta'' - uu'\alpha' - u'\beta' + 3u^3\beta)t^2 + (u^2\beta'' - uu'\beta' + u^4\beta)t^3\big\}.\end{aligned}$$

By a straightforward computation, we get

$$\Delta^2 x = \frac{1}{(1+ut)^7} Q(s,t),$$

where $Q(s,t)$ is a polynomial in t of degree 4 with functions of s as coefficients, given by

$$\begin{aligned}&Q(s,t) \\ &= (u^3\beta + 5uu'\alpha' + 4u^2\alpha'' - 2u\beta'' + u''\beta + \alpha^{(4)}) \\ &\quad + \big\{(u^4 - 3(u')^2)\beta - (u^3 + 9u^2u')\alpha' + 7uu'\beta' - 4u''\alpha'' + 2u^2\beta'' - 6u'\alpha''' + \beta^{(4)}\big\}t \\ &\quad + \big\{3u(u')^2\beta + (9u^3u' + 10u'u'' - 2uu''' + 3u^4)\alpha' - (9u^2u' + u''')\beta' \\ &\qquad + (15(u')^2 + 4uu'')\alpha'' - 4u''\beta'' + 6uu'\alpha''' - 6u'\beta'''\big\}t^2 \\ &\quad + \big\{6u^2(u')^2\beta + (18u^4u' - 15(u')^3 + 10uu'u'' - u^2u''' + 3u^5)\alpha' \\ &\qquad - (18u^3u' - 10u'u'' + 2uu''')\beta' + (15u(u')^2 + 8u^2u'')\alpha'' \\ &\qquad + (15(u')^2 - 8uu'')\beta'' + 12u^2u'\alpha''' - 12uu'\beta'''\big\}t^3 \\ &\quad + \big\{u^7\beta - (2u^4u' + 15(u')^3 - 10uu'u'' + u^5)\beta' + (2u^5 + 15u(u')^2 - 4u^2u'')\beta'' \\ &\qquad - 6u^2u'\beta''' + u^3\beta^{(4)}\big\}t^4.\end{aligned} \tag{4.4}$$

Since M is biharmonic, $Q(s,t) = \mathbf{0}$. Thus, the constant terms of (4.4) with respect to t tell us

$$\alpha^{(4)} = -(u^3 + u'')\beta - 5uu'\alpha' - 4u^2\alpha'' + 2u\beta''$$

which implies that

$$\langle \alpha^{(4)}, \alpha' \rangle = -5uu' + 2u\langle \beta'', \alpha' \rangle,$$

or, equivalently

$$\langle \beta'', \alpha' \rangle = u' \tag{4.5}$$

because of $\langle \alpha^{(4)}, \alpha' \rangle = -3uu'$ with the help of $\langle \alpha', \alpha' \rangle = 1$ and $\langle \alpha'', \alpha'' \rangle = u^2$. In (4.4), we multiply the coefficients of the terms containing t^2 with $-2u$ and then add the equation obtained in such a way to the coefficients of the terms containing t^3. Then, we have

$$(-15(u')^3 - 10uu'u'' + 3u^2u''' - 3u^5)\alpha' + 10u'u''\beta' - 15u(u')^2\alpha'' + 15(u')^2\beta'' = \mathbf{0} \tag{4.6}$$

which allows us to have

$$-15(u')^3 + 3u^2u''' - 3u^5 + 15(u')^2\langle \beta'', \alpha' \rangle = 0$$

by taking the scalar product with α'. Putting (4.5) into the above equation gives

$$u''' = u^3. \tag{4.7}$$

The coefficients of the terms containing t of (4.4) indicates

$$\beta^{(4)} = (3(u')^2 - u^4)\beta + (u^3 + 9u^2u')\alpha' - 7uu'\beta' + 4u''\alpha'' - 2u^2\beta'' + 6u'\alpha'''. \tag{4.8}$$

Substituting (4.8) into the coefficients of the terms containing t^4 of (4.4), we get

$$\begin{aligned} &3u^3(u')^2\beta + (u^6 + 9u^5u')\alpha' + (10uu'u'' - 9u^4u' - 15(u')^3 - u^5)\beta' \\ &+ 4u^3u''\alpha'' + (15u(u')^2 - 4u^2u'')\beta'' + 6u^3u'\alpha''' - 6u^2u'\beta''' = \mathbf{0}. \end{aligned} \tag{4.9}$$

Multiplying the coefficients of the terms containing t^2 of (4.4) with $-u^2$ and then comparing the equation obtained in such a way and (4.9), we get

$$10u^2u'u''\alpha' + (15(u')^3 - 10uu'u'')\beta' + 15u^2(u')^2\alpha'' - 15u(u')^2\beta'' = \mathbf{0}. \tag{4.10}$$

Combining (4.6) and (4.10), we can obtain

$$\beta' = u\alpha' \quad \text{on } U \tag{4.11}$$

because of $u' \neq 0$, and then

$$\beta'' = u'\alpha' + u\alpha'' \quad \text{and} \quad \beta''' = u''\alpha' + 2u'\alpha'' + u\alpha'''. \tag{4.12}$$

With the help of (4.7), (4.11) and (4.12), the coefficients of the terms containing t^2 of (4.4) implies that

$$\alpha'' = -u\beta \quad \text{on } U \tag{4.13}$$

because of $u' \neq 0$. Using (4.11) and (4.13), we have $H = \mathbf{0}$ of (4.2) on U, which contradicts to the character of H. Therefore, we see that $U = \emptyset$, which means that u' is identically zero on M. From $(w')^2 = 4(u')^2 w$ of (4.3), it is obvious that $w' = 0$. Clearly, $\frac{\partial q}{\partial s} = 0$ on M. □

By Lemma 4.1, Δx and $\Delta^2 x$ are reduced to

$$\begin{aligned} \Delta x = -\frac{1}{q^2}\big\{&(\alpha'' + u\beta) + (2u\alpha'' + \beta'' + (2u^2 + w)\beta)t \\ &+ (w\alpha'' + 2u\beta'' + 3uw\beta)t^2 + (w\beta'' + w^2\beta)t^3\big\} \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \Delta^2 x = \frac{1}{q^3}\big\{&(6u^3 - 5uw)\beta + (6u^2 - 2w)\alpha'' - 2u\beta'' + \alpha^{(4)} \\ &+ ((8u^2 w - 5w^2)\beta + 8uw\alpha'' + (2u^2 - 4w)\beta'' + 2u\alpha^{(4)} + \beta^{(4)})t \\ &+ (3uw^2\beta + 4w^2\alpha'' + 2uw\beta'' + w\alpha^{(4)} + 2u\beta^{(4)})t^2 \\ &+ (w^3\beta + 2w^2\beta'' + w\beta^{(4)})t^3\big\}, \end{aligned}$$

respectively. Here, we denote the numerator of $\Delta^2 x$ by $R(s,t)$, which is a polynomial in t of degree 3 with functions of s as coefficients. Since $\Delta^2 x = \mathbf{0}$, the constant terms and the coefficients of the terms containing t of $R(s,t)$ induce

$$\alpha^{(4)} = (5uw - 6u^3)\beta + (2w - 6u^2)\alpha'' + 2u\beta'' \tag{4.15}$$

and

$$\beta^{(4)} = (12u^4 + 5w^2 - 18u^2 w)\beta + (12u^3 - 12uw)\alpha'' + (4w - 6u^2)\beta'', \tag{4.16}$$

respectively. Using (4.15) and (4.16), the coefficients of the terms containing t^2 and t^3 of $R(s,t)$ are rewritten as

$$2u(u^2 - w)\beta'' = (4u^5 - 7u^3 w + 3uw^2)\beta + (4u^4 - 5u^2 w + w^2)\alpha'' \tag{4.17}$$

and

$$w(u^2 - w)\beta'' = (2u^4 w - 3u^2 w^2 + w^3)\beta + (2u^3 w - 2uw^2)\alpha'', \tag{4.18}$$

respectively. We multiply (4.17) with u and (4.18) with $2u$, respectively, and compare the two equations obtained in such a way. Then, it follows that

$$w^2(u^2 - w)(\alpha'' + u\beta) = \mathbf{0}.$$

If the constant $w^2(u^2 - w) \neq 0$, we have $\alpha'' = -u\beta$ and then (4.18) gives $\beta'' = -w\beta$. Thus, we have $\Delta x = \mathbf{0}$ of (4.14), which means that $H = \mathbf{0}$, a contradiction. Therefore, we see that $w^2(u^2 - w) = 0$.

Lemma 4.2. *If M is a biharmonic marginally trapped ruled surface in $\mathbb{L}^m$ parameterized by* (4.1), *then $w = u^2$ on M.*

Proof. Since u and w are constant satisfying $w^2(u^2 - w) = 0$, we have either $w = u^2$ or $w \neq u^2$ on M. We suppose that $w \neq u^2$. Then, $w = 0$ on M and thus $u \neq 0$. In this case, (4.15) and (4.16) are reduced to

$$\alpha^{(4)} = -6u^3\beta - 6u^2\alpha'' + 2u\beta'' \quad \text{and} \quad \beta^{(4)} = 12u^4\beta + 12u^3\alpha'' - 6u^2\beta'',$$

respectively, which imply that

$$\beta^{(4)} = -2u\alpha^{(4)} - 2u^2\beta''. \tag{4.19}$$

Equation (4.17) is simplified as

$$\beta'' = 2u^2\beta + 2u\alpha'' \quad \text{and therefore} \quad \beta^{(4)} = 2u^2\beta'' + 2u\alpha^{(4)} \tag{4.20}$$

because of $q_s = 0$. Comparing (4.19) and (4.20) gives

$$\beta^{(4)} = \mathbf{0} \quad \text{and} \quad \alpha^{(4)} = -u\beta''. \tag{4.21}$$

Thus, the derivatives of the vector field β can be put as

$$\beta'''(s) = \mathbf{D}, \quad \beta''(s) = \mathbf{D}s + \mathbf{C} \quad \text{and} \quad \beta'(s) = \frac{1}{2}\mathbf{D}s^2 + \mathbf{C}s + \mathbf{F} \tag{4.22}$$

for some constant vector fields $\mathbf{D}$, $\mathbf{C}$ and $\mathbf{F}$. Since $\langle \beta'', \beta'' \rangle = (u')^2 + w^2$ of (4.3), we have $\langle \beta'', \beta'' \rangle = 0$ for all s, which implies

$$\langle \mathbf{C}, \mathbf{C} \rangle = \langle \mathbf{C}, \mathbf{D} \rangle = \langle \mathbf{D}, \mathbf{D} \rangle = 0. \tag{4.23}$$

From $w = \langle \beta', \beta' \rangle = 0$, we get $\langle \beta'', \beta' \rangle = 0$ for all s and therefore we see that

$$\langle \mathbf{D}, \mathbf{F} \rangle = \langle \mathbf{C}, \mathbf{F} \rangle = 0. \tag{4.24}$$

With the help of (4.23) and (4.24), $w = 0$ guarantees

$$\langle \mathbf{F}, \mathbf{F} \rangle = 0. \tag{4.25}$$

On the other hand, we note that $\langle \beta'', \beta'' \rangle = 0$. If the interior W_o of the set $W = \{s \in \operatorname{dom}\alpha \mid \beta'' = \mathbf{0}\}$ is non-empty, then (4.20) indicates $\alpha'' = -u\beta$ on W_o, which implies that $\Delta x = \mathbf{0}$ of (4.14), a contradiction. Therefore, the vector field β'' is non-vanishing for all s, i.e., β'' is null for all s. Thus, at least one of $\mathbf{D}$ and $\mathbf{C}$ is a null constant vector field. With the aid of (4.23), (4.24) and (4.25), we see that $\mathbf{D} \wedge \mathbf{C} = \mathbf{D} \wedge \mathbf{F} = \mathbf{C} \wedge \mathbf{F} = \mathbf{0}$. As a result, we can put as

$$\beta' = g_1(s)\mathbf{N} \quad \text{and} \quad \beta'' = g_2(s)\mathbf{N} \tag{4.26}$$

for some polynomials g_1 and g_2 in s satisfying $g_1'(s) = g_2(s)$ and constant null vector field $\mathbf{N}$ with $\mathbf{N} \wedge \mathbf{D} = \mathbf{N} \wedge \mathbf{C} = \mathbf{N} \wedge \mathbf{F} = \mathbf{0}$.

From equations $\beta'' = 2u^2\beta + 2u\alpha''$ of (4.20) and $\beta''(s) = \mathbf{D}s + \mathbf{C}$ of (4.22), the vector field α'' is given by

$$\alpha''(s) = -u\beta(s) + \frac{1}{2u}\mathbf{D}s + \frac{1}{2u}\mathbf{C}. \tag{4.27}$$

Since $\alpha^{(4)} = -u\beta''$ of (4.21) and $\beta''(s) = \mathbf{D}s + \mathbf{C}$ of (4.22), we have

$$\alpha'''(s) = -\frac{1}{2}u\mathbf{D}s^2 - u\mathbf{C}s + \mathbf{n}_1 \tag{4.28}$$

for some constant vector $\mathbf{n}_1$. Consequently, we get

$$\beta' = \frac{1}{2u^2}\mathbf{D} + \frac{1}{2}\mathbf{D}s^2 + \mathbf{C}s - \frac{1}{u}\mathbf{n}_1$$

which implies

$$\mathbf{n}_1 = \frac{1}{2u}\mathbf{D} - u\mathbf{F}$$

in comparison to (4.22). Since $\mathbf{N} \wedge \mathbf{D} = \mathbf{N} \wedge \mathbf{F} = \mathbf{0}$, it is obvious that $\mathbf{n}_1 \wedge \mathbf{N} = \mathbf{0}$. Therefore, (4.28) gives

$$\alpha'''(s) = h(s)\mathbf{N}$$

and hence

$$\alpha''(s) = \left(\int h(s)\, ds\right)\mathbf{N} + \mathbf{n}_2 \tag{4.29}$$

for some function h and constant vector $\mathbf{n}_2$. Considering (4.26) and (4.27), we can get $\langle \beta', \alpha'' \rangle = 0$ and hence $\langle \mathbf{N}, \mathbf{n}_2 \rangle = 0$. Since $\langle \alpha'', \alpha'' \rangle = u^2$ of (4.3), (4.29) tells us $\langle \mathbf{n}_2, \mathbf{n}_2 \rangle = u^2$.

We note that $\langle \beta', \beta \rangle = 0$ and $\langle \beta, \beta \rangle = 1$. From (4.26), we can put as

$$\beta(s) = \left(\int g_1(s)\, ds\right)\mathbf{N} + \mathbf{n}_3$$

which implies that $\langle \mathbf{N}, \mathbf{n}_3\rangle = 0$ and $\langle \mathbf{n}_3, \mathbf{n}_3\rangle = 1$. Together with (4.26) and (4.29), equation $\beta'' = 2u^2\beta + 2u\alpha''$ of (4.20) can be rewritten as

$$g_2(s)\mathbf{N} = 2u^2\left(\int g_1(s)\,ds\mathbf{N} + \mathbf{n}_3\right) + 2u\left(\int h(s)\,ds\mathbf{N} + \mathbf{n}_2\right),$$

or,

$$\left(g_2(s) - 2u^2\int g_1(s)\,ds - 2u\int h(s)\,ds\right)\mathbf{N} = 2u^2\mathbf{n}_3 + 2u\mathbf{n}_2. \tag{4.30}$$

Taking the scalar product to (4.30) with itself gives

$$0 = 4u^4\langle \mathbf{n}_3, \mathbf{n}_3\rangle + 8u^3\langle \mathbf{n}_3, \mathbf{n}_2\rangle + 4u^2\langle \mathbf{n}_2, \mathbf{n}_2\rangle,$$

that is,

$$\langle \mathbf{n}_3, \mathbf{n}_2\rangle = -u. \tag{4.31}$$

If we put $\Gamma(s) = \int h(s)\,ds$ of (4.29), then

$$\alpha'(s) = \left(\int \Gamma(s)\,ds\right)\mathbf{N} + \mathbf{n}_2 s + \mathbf{n}_4$$

for some constant vector $\mathbf{n}_4$. Here, $\langle \mathbf{N}, \mathbf{n}_4\rangle = 0$ because of $\langle \beta'', \alpha'\rangle = 0$. Thus,

$$0 = \langle \alpha', \beta\rangle = \left\langle \int \Gamma(s)\,ds\mathbf{N} + \mathbf{n}_2 s + \mathbf{n}_4, \int g(s)\,ds\mathbf{N} + \mathbf{n}_3\right\rangle = \langle \mathbf{n}_2, \mathbf{n}_3\rangle s + \langle \mathbf{n}_4, \mathbf{n}_3\rangle$$

which implies that $\langle \mathbf{n}_2, \mathbf{n}_3\rangle = 0$, i.e., $u = 0$ of (4.31), a contradiction. Therefore, we have $w = u^2$ on M. □

By Lemma 4.2, we divide the problem into two cases which are $u = 0$ or $u \neq 0$.

Case 1. Suppose that $u = w = 0$. By (4.3), (4.15) and (4.16), it gives

$$\langle \alpha'', \alpha''\rangle = \langle \alpha'', \beta''\rangle = \langle \beta'', \beta''\rangle = 0 \quad \text{and} \quad \alpha^{(4)} = \beta^{(4)} = \mathbf{0}.$$

Therefore, we can put as

$$\alpha''' = \mathbf{A}, \quad \alpha'' = \mathbf{A}s + \mathbf{A}_1, \quad \beta''' = \mathbf{B}, \quad \beta'' = \mathbf{B}s + \mathbf{B}_1$$

for some constant vector fields $\mathbf{A}$, $\mathbf{A}_1$, $\mathbf{B}$ and $\mathbf{B}_1$ satisfying

$$\begin{aligned}\langle \mathbf{A}, \mathbf{A}\rangle &= \langle \mathbf{A}, \mathbf{A}_1\rangle = \langle \mathbf{A}, \mathbf{B}\rangle = \langle \mathbf{A}, \mathbf{B}_1\rangle = \langle \mathbf{A}_1, \mathbf{A}_1\rangle \\ &= \langle \mathbf{A}_1, \mathbf{B}\rangle = \langle \mathbf{A}_1, \mathbf{B}_1\rangle = \langle \mathbf{B}, \mathbf{B}\rangle = \langle \mathbf{B}, \mathbf{B}_1\rangle = \langle \mathbf{B}_1, \mathbf{B}_1\rangle = 0.\end{aligned}$$

Without loss of generality, we may assume that

$$\alpha'' = (a_1 s + a_2)\mathbf{N} \quad \text{and} \quad \beta'' = (b_1 s + b_2)\mathbf{N}$$

for some null constant vector field $\mathbf{N}$ with $\mathbf{N} \wedge \mathbf{A} = \mathbf{0}$ and constants $a_i, b_i \in \mathbb{R}$, $i = 1, 2$. In this case, the mean curvature vector field H is given by $H(s,t) = -\frac{1}{2}(\alpha''(s) + t\beta''(s))$, which is null for all s and t, so at least one of a_1 and a_2 is non-zero. And, we have

$$\alpha' = \left(\frac{1}{2}a_1 s^2 + a_2 s + a_3\right)\mathbf{N} + \mathbf{F}_1 \quad \text{and} \quad \beta' = \left(\frac{1}{2}b_1 s^2 + b_2 s + b_3\right)\mathbf{N} + \mathbf{F}_2$$

for some constants a_3, b_3 and constant vector fields $\mathbf{F}_1$, $\mathbf{F}_2$. Since α is a space-like curve, we see that

$$\langle \mathbf{F}_1, \mathbf{F}_1 \rangle = 1 \quad \text{and} \quad \langle \mathbf{F}_1, \mathbf{N} \rangle = 0.$$

From $w = \langle \beta', \beta' \rangle = 0$, β' is vanishing or null for s. Therefore, we can put as

$$\beta' = \left(\frac{1}{2}b_1 s^2 + b_2 s + b_3\right)\mathbf{N} \quad \text{and} \quad \beta = \left(\frac{1}{6}b_1 s^3 + \frac{1}{2}b_2 s^2 + b_3 s\right)\mathbf{N} + \mathbf{F}$$

for some constant vector field $\mathbf{F}$. Since $\langle \beta, \beta \rangle = 1$ for all s, we have

$$\langle \mathbf{F}, \mathbf{F} \rangle = 1 \quad \text{and} \quad \langle \mathbf{F}, \mathbf{N} \rangle = 0.$$

Furthermore, we get $\langle \mathbf{F}, \mathbf{F}_1 \rangle = 0$ because of $\langle \alpha', \beta \rangle = 0$.

Therefore, we can parameterize M by

$$x(s,t) = \alpha(s) + t\beta(s) = (p_1(s) + tp_2(s))\mathbf{N} + \mathbf{F}_1 s + \mathbf{D} + t\mathbf{F}, \tag{4.32}$$

where p_1 and p_2 are polynomials in s with $2 \leq \deg p_1 \leq 3$ and $\deg p_2 \leq 3$, respectively, and $\mathbf{N}$, $\mathbf{F}_1$, $\mathbf{D}$, $\mathbf{F}$ are constant vector fields such that

$$\langle \mathbf{N}, \mathbf{N} \rangle = 0, \quad \langle \mathbf{F}_1, \mathbf{F}_1 \rangle = \langle \mathbf{F}, \mathbf{F} \rangle = 0 \quad \text{and} \quad \langle \mathbf{N}, \mathbf{F}_1 \rangle = \langle \mathbf{N}, \mathbf{F} \rangle = \langle \mathbf{F}_1, \mathbf{F} \rangle = 0.$$

From this, M is cylindrical if $\deg p_2 = 0$ and otherwise, M is non-cylindrical.

Case 2. If $uw \neq 0$, Δx and $\Delta^2 x$ are given by

$$\Delta x = -\frac{1}{(1+ut)^2}\left\{(\alpha'' + u\beta) + t(\beta'' + u^2\beta)\right\}$$

and

$$\Delta^2 x = \frac{1}{(1+ut)^4}\left\{(\alpha^{(4)} - 2u\beta'' + 4u^2\alpha'' + u^3\beta) + t(\beta^{(4)} + 2u^2\beta'' + u^4\beta)\right\},$$

respectively. Since $\Delta^2 x = \mathbf{0}$, we have

$$\alpha^{(4)} = -u^3\beta - 4u^2\alpha'' + 2u\beta'' \quad \text{and} \quad \beta^{(4)} = -u^4\beta - 2u^2\beta''. \tag{4.33}$$

Equation $\beta^{(4)} = -u^4\beta - 2u^2\beta''$ of (4.33) can be rewritten as

$$(\beta'' + u^2\beta)'' = -u^2(\beta'' + u^2\beta)$$

which indicates that

$$\beta'' + u^2\beta = \cos(us)\mathbf{b}_1 + \sin(us)\mathbf{b}_2$$

for some constant vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, from which,

$$\beta(s) = \cos(us)\mathbf{a}_1 + \sin(us)\mathbf{a}_2 + s\cos(us)\mathbf{a}_3 + s\sin(us)\mathbf{a}_4$$

for some constant vectors $\mathbf{a}_i$, $i = 1, \ldots, 4$. Since $\langle \beta, \beta \rangle = 1$ for all s, we can obtain

$$\begin{aligned} &(\langle \mathbf{a}_1, \mathbf{a}_1 \rangle - 1)\cos^2(us) + 2\langle \mathbf{a}_1, \mathbf{a}_2 \rangle \cos(us)\sin(us) + (\langle \mathbf{a}_2, \mathbf{a}_2 \rangle - 1)\sin^2(us) \\ &+ 2\langle \mathbf{a}_1, \mathbf{a}_3 \rangle s\cos^2(us) + 2(\langle \mathbf{a}_1, \mathbf{a}_4 \rangle + \langle \mathbf{a}_2, \mathbf{a}_3 \rangle)s\cos(us)\sin(us) + 2\langle \mathbf{a}_2, \mathbf{a}_4 \rangle s\sin^2(us) \\ &+ \langle \mathbf{a}_3, \mathbf{a}_3 \rangle s^2\cos^2(us) + 2\langle \mathbf{a}_3, \mathbf{a}_4 \rangle s^2\cos(us)\sin(us) + \langle \mathbf{a}_4, \mathbf{a}_4 \rangle s^2\sin^2(us) = 0. \end{aligned}$$

With the help of the Wronskian, we see that all the constant coefficients of the above vanish. Thus, β can be put

$$\beta(s) = \cos(us)\mathbf{a}_1 + \sin(us)\mathbf{a}_2 + s(\cos(us) + c\sin(us))\mathbf{N} \tag{4.34}$$

for some constant c and a constant null vector $\mathbf{N}$ with $\langle \mathbf{a}_1, \mathbf{N} \rangle = \langle \mathbf{a}_2, \mathbf{N} \rangle = 0$. Here, unit space-like vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal.

On the other hand, combining two equations of (4.33) yields $\beta^{(4)} - u\alpha^{(4)} = -4u^2(\beta'' - u\alpha'')$, i.e.,

$$(\beta'' - u\alpha'')'' = -4u^2(\beta'' - u\alpha'')$$

which gives

$$\beta'' - u\alpha'' = \cos(2us)\mathbf{m}_1 + \sin(2us)\mathbf{m}_2$$

for some constant vectors $\mathbf{m}_1$ and $\mathbf{m}_2$. We note that $\langle \beta'' - u\alpha'', \beta'' - u\alpha'' \rangle = 0$ by virtue of (4.3). Without loss of generality, we may assume that

$$\beta'' - u\alpha'' = \big(\widetilde{c_1}\cos(2us) + \widetilde{c_2}\sin(2us)\big)\widetilde{\mathbf{N}}$$

for some constants $\widetilde{c_1}$, $\widetilde{c_2}$ and a constant vector $\widetilde{\mathbf{N}}$, which is null or zero. Since $\langle \beta' - u\alpha', \beta' - u\alpha' \rangle = 0$ and $\langle \beta' - u\alpha', \beta'' - u\alpha'' \rangle = 0$, it follows that

$$\beta' - u\alpha' = \left(\frac{\widetilde{c_1}}{2u}\sin(2us) - \frac{\widetilde{c_2}}{2u}\cos(2us) + \widetilde{c_3}\right)\widetilde{\mathbf{N}} \tag{4.35}$$

for some constant $\widetilde{c_3}$. Taking the scalar product to both sides of (4.35) with β, we can get $\langle \widetilde{\mathbf{N}}, \beta \rangle = 0$. Together with (4.34) and equation $\langle \widetilde{\mathbf{N}}, \beta \rangle = 0$, we have

$$\langle \widetilde{\mathbf{N}}, \mathbf{N} \rangle = 0, \quad \text{i.e.,} \quad \widetilde{\mathbf{N}} \wedge \mathbf{N} = \mathbf{0} \tag{4.36}$$

by the linear independence of the functions '$\cos(us)$', '$\sin(us)$', '$s\cos(us)$' and '$s\sin(us)$' for all s. With the help of (4.34), (4.35) and (4.36), a parametrization of M is then given by

$$
\begin{aligned}
x(s,t) &= \alpha(s) + t\beta(s) \\
&= \left(\frac{1}{u} + t\right)\beta + (c_1\sin(2us) + c_2\sin(2us) + c_3 s)\mathbf{N} + \mathbf{D} \\
&= \left(\frac{1}{u} + t\right)\big\{\cos(us)\mathbf{a}_1 + \sin(us)\mathbf{a}_2 + s(\cos(us) + c\sin(us))\mathbf{N}\big\} \\
&\quad + (c_1\sin(2us) + c_2\sin(2us) + c_3 s)\mathbf{N} + \mathbf{D}
\end{aligned}
\tag{4.37}
$$

for some constants c, c_1, c_2, c_3 and constant vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{N}$, $\mathbf{D}$ satisfying

$$\langle \mathbf{a}_1, \mathbf{a}_1\rangle = \langle \mathbf{a}_2, \mathbf{a}_2\rangle = 1, \quad \langle \mathbf{a}_1, \mathbf{a}_2\rangle = \langle \mathbf{a}_1, \mathbf{N}\rangle = \langle \mathbf{a}_2, \mathbf{N}\rangle = \langle \mathbf{N}, \mathbf{N}\rangle = 0.$$

Conversely, by a direct computation, we can see easily that ruled surfaces parameterized by (4.32) or (4.37) are marginally trapped and $\Delta^2 x = \mathbf{0}$. Consequently, we have

Theorem 4.3. *A marginally trapped ruled surface M in $\mathbb{L}^m$ is biharmonic if and only if M is parameterized by either* (4.32) *or* (4.37)*.*

Example 4.4. We consider the following vectors in $\mathbb{L}^4$:

$$\mathbf{N} = \big(2, 1, 1, \sqrt{2}\big), \quad \mathbf{F}_1 = \big(1, 0, 0\sqrt{2}\big) \quad \text{and} \quad \mathbf{F} = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right).$$

Using these, we can construct a ruled surface M in $\mathbb{L}^4$ given by

$$
\begin{aligned}
x(s,t) &= (s^3 + ts^2)\big(2, 1, 1, \sqrt{2}\big) + s\big(1, 0, 0\sqrt{2}\big) + t\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \\
&= \big(2s^3 + s, s^3, s^3, \sqrt{2}s^3 + \sqrt{2}s\big) + t\left(2s^2, s^2 + \frac{1}{\sqrt{2}}, s^2 - \frac{1}{\sqrt{2}}, \sqrt{2}s^2\right).
\end{aligned}
$$

By computation, we have

$$\Delta x = (6s + 2t)\big(2, 1, 1, \sqrt{2}\big) \quad \text{and hence} \quad \Delta^2 x = \mathbf{0}$$

which means that M is biharmonic marginally trapped.

Example 4.5. Let u be a non-zero real number. Define a vector field $\beta(s)$ by

$$
\begin{aligned}
\beta(s) &= \cos(us)\mathbf{a}_1 + \sin(us)\mathbf{a}_2 + s(\cos(us) + 2\sin(us))\mathbf{N} \\
&= \big(s(\cos(us) + 2\sin(us)), s(\cos(us) + 2\sin(us)), \cos(us), \sin(us), 0\big),
\end{aligned}
$$

where $\mathbf{a}_1 = (0,0,1,0,0)$, $\mathbf{a}_2 = (0,0,0,1,0)$ and $\mathbf{N} = (1,1,0,0,0)$. Define a curve α by

$$\alpha(s) = \frac{1}{u}\beta + (\cos(2us) - \sin(2us) + 2s)\mathbf{N} + (1,2,3,4,5).$$

Then, we can obtain a biharmonic marginally trapped ruled surface in $\mathbb{L}^5$ given by

$$\begin{aligned} x(s,t) &= \alpha(s) + t\beta(s) \\ &= \left(\frac{1}{u} + t\right)\beta(s) + (\cos(2us) - \sin(2us) + 2s)\mathbf{N} + (1,2,3,4,5). \end{aligned}$$

In fact, it satisfies

$$\Delta x = -\frac{1}{(1+ut)^2}\big\{2(1+ut)(-\sin(us) + 2\cos(us)) - 4u^2(\cos(2us) - \sin(2us))\big\}\mathbf{N}$$

and

$$\Delta^2 x = \mathbf{0}.$$

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References

[1] H. Anciaux and N. Cipriani, *Codimension two marginally trapped submanifolds in Robertson–Walker spacetimes*, J. Geom. Phys. **88** (2015), 105–112.

[2] H. Anciaux and Y. Godoy, *Marginally trapped submanifolds in Lorentzian space forms and in the Lorentzian product of a space form by the real line*, J. Math. Phys. **56** (2015), no. 2, 023502, 12 pp.

[3] C. Baikoussis, *Ruled submanifolds with finite type Gauss map*, J. Geom. **49** (1994), no. 1-2, 42–45.

[4] J. M. Barbosa, M. Dajczer and L. P. Jorge, *Minimal ruled submanifolds in spaces of constant curvature*, Indiana Univ. Math. J. **33** (1984), no. 4, 531–547.

[5] B.-Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math. **17** (1991), no. 2, 169–188.

[6] ______, *Black holes, marginally trapped surfaces and quasi-minimal surfaces*, Tamkang J. Math. **40** (2009), no. 4, 313–341.

[7] ______, *Total Mean Curvature and Submanifolds of Finite Type*, Second edition, Series in Pure Mathematics **27**, World Scientific, Singapore, 2015.

[8] B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Ruled surfaces of finite type*, Bull. Austral. Math. Soc. **42** (1990), no. 3, 447–453.

[9] B.-Y. Chen and S. Ishikawa, *Biharmonic surfaces in pseudo-Euclidean spaces*, Mem. Fac. Sci. Kyushu Univ. Ser. A **45** (1991), no. 2, 323–347.

[10] ———, *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*, Kyushu J. Math. **52** (1998), no. 1, 167–185.

[11] B.-Y. Chen and J. Van der Veken, *Marginally trapped surfaces in Lorentzian space forms with positive relative nullity*, Classical Quantum Gravity **24** (2007), no. 3, 551–563.

[12] P. T. Chruściel, G. J. Galloway and D. Pollack, *Mathematical general relativity: a sampler*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), no. 4, 567–638.

[13] F. Dillen, *Ruled submanifolds of finite type*, Proc. Amer. Math. Soc. **114** (1992), no. 3, 795–798.

[14] I. M. Dimitrić, *Quadric Representation and Submanifolds of Finite Type*, Ph.D. Thesis, Michigan State University, 1989.

[15] ———, *Submanifolds of E^n with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sinica **20** (1992), no. 1, 53–65.

[16] Y. Dong and Y.-L. Ou, *Biharmonic submanifolds of pseudo-Riemannian manifolds*, J. Geom. Phys. **112** (2017), 252–262.

[17] L. Du and J. Zhang, *Biharmonic submanifolds with parallel normalized mean curvature vector field in pseudo-Riemannian space forms*, Bull. Malays. Math. Sci. Soc. **42** (2019), no. 4, 1469–1484.

[18] Y. Fu, *Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean 5-space*, J. Geom. Phys. **75** (2014), 113–119.

[19] ———, *Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean space*, Tohoku Math. J. (2) **67** (2015), no. 3, 465–479.

[20] T. Hasanis and T. Vlachos, *Hypersurfaces in E^4 with harmonic mean curvature vector field*, Math. Nachr. **172** (1995), 145–169.

[21] S. Heasen and M. Ortega, *Boost invariant marginally trapped surfaces in Minkowski 4-space*, Classical Quantum Gravity **24** (2007), no. 22, 5441–5452.

[22] ———, *Marginally trapped surfaces in Minkowski 4-space invariant under a rotation subgroup of the Lorentz group*, Gen. Relativity Gravitation **41** (2009), no. 8, 1819–1834.

[23] S. M. Jung, D.-S. Kim and Y. H. Kim, *Minimal ruled submanifolds associated with Gauss map*, Taiwanese J. Math. **22** (2018), no. 3, 567–605.

[24] S. M. Jung, D.-S. Kim, Y. H. Kim and D. W. Yoon, *Gauss maps of ruled submanifolds and applications I*, J. Korean Math. Soc. **53** (2016), no. 6, 1309–1330.

[25] D.-S. Kim and Y. H. Kim, *Minimal ruled submanifolds in Minkowski space* $\mathbb{L}^m$, J. Geom. Phys. **62** (2012), no. 9, 1893–1902.

[26] ______, *Some classification results on finite-type ruled submanifolds in a Lorentz–Minkowski space*, Taiwanese J. Math. **16** (2012), no. 4, 1475–1488.

[27] D.-S. Kim, Y. H. Kim and S. M. Jung, *Ruled submanifolds with harmonic Gauss map*, Taiwanese J. Math. **18** (2014), no. 1, 53–76.

[28] ______, *Some classifications of ruled submanifolds in Minkowski space and their Gauss map*, Taiwanese J. Math. **18** (2014), no. 4, 1021–1040.

[29] D.-S. Kim, Y. H. Kim and D. W. Yoon, *Extended B-scrolls and their Gauss maps*, Indian J. Pure Appl. Math. **33** (2002), no. 7, 1031–1040.

[30] ______, *Characterization of generalized B-scrolls and cylinders over finite type curves*, Indian J. Pure Appl. Math. **34** (2003), no. 11, 1523–1532.

[31] ______, *Finite type ruled surfaces in Lorentz–Minkowski space*, Taiwanese J. Math. **11** (2007), no. 1, 1–13.

[32] Y. H. Kim and D. W. Yoon, *Ruled surfaces with finite type Gauss map in Minkowski spaces*, Soochow J. Math. **26** (2000), no. 1, 85–96.

[33] ______, *Ruled surfaces with pointwise 1-type Gauss map*, J. Geom. Phys. **34** (2000), no. 3-4, 191–205.

[34] ______, *Classification of ruled surfaces in Minkowski 3-spaces*, J. Geom. Phys. **49** (2004), no. 1, 89–100.

[35] N. Koiso and H. Urakawa, *Biharmonic submanifolds in a Riemannian manifold*, Osaka J. Math. **55** (2018), no. 2, 325–346.

[36] J. Liu, L. Du and J. Zhang, *Minimality on biharmonic space-like submanifolds in pseudo-Riemannian space forms*, J. Geom. Phys. **92** (2015), 69–77.

[37] S. Montaldo, C. Oniciuc and A. Ratto, *On cohomogeneity one biharmonic hypersurfaces into the Euclidean space*, J. Geom. Phys. **106** (2016), 305–313.

[38] Y.-L. Ou and L. Tang, *On the generalized Chen's conjecture on biharmonic submanifolds*, Michigan Math. J. **61** (2012), no. 3, 531–542.

[39] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** (1965), 57–59.

[40] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.

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