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The Zero (Total) Forcing Number and Covering Number of Trees

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Abstract. Let $F(G)$, $F_t(G)$, $\beta(G)$, and $\beta'(G)$ be the zero forcing number, the total forcing number, the vertex covering number and the edge covering number of a graph G, respectively. In this paper, we first completely characterize all trees T with $F(T) = (\Delta - 2)\beta(T) + 1$, solving a problem proposed by Brimkov et al. in 2023. Next, we study the relationship between the zero (or total) forcing number of a tree and its edge covering number, and show that $F(T) \leq \beta'(T) - 1$ and $F_t(T) \leq \beta'(T)$ for any tree T of order $n \geq 3$. Moreover, we also characterize all trees T with $F(T) = \beta'(T) - 1$ and $F(T) = \beta'(T) - 2$, respectively.

1. Introduction

The graphs in this paper are undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ (or $d(v)$ and $N(v)$ for short) be the degree and the set of neighbors of v, respectively. Clearly, $d_G(v) = |N_G(v)|$. The maximum degree of G is denoted by $\Delta(G)$ (or Δ for short). For $v \in V(G)$ (resp., $e \in E(G)$), let $G - v$ (resp., $G - e$) be the graph obtained from G by deleting the vertex v (resp., the edge e). For a subset $S \subseteq V(G)$, the induced subgraph of G by S, denoted by $G[S]$, is the graph with vertex set S, in which two vertices are adjacent if and only if they are adjacent in G. A vertex (resp., edge) covering of a graph G is a set of vertices (resp., edges) of G such that every edge (resp., vertex) of G is incident with at least one vertex (resp., edge) of the set. The minimum cardinality of a vertex (resp., edge) covering of G is called the *vertex* (resp., *edge*) *covering number*, denoted by $\beta(G)$ (or $\beta'(G)$). As usual, the star and the path of order n are denoted by $K_{1,n-1}$, and P_n, respectively.

For a graph G, its vertices are colored with two different colors (white and black). Let $S \subseteq V(G)$ be the set of black vertices in G. If $u \in S$ and v is the only white neighbor of u, then u forces v to turn into black (color change rule). The set S is said to be a *zero forcing set* of G if by iteratively applying the color change rule such that all vertices of G become black. We also call such S an *F-set of G*. The *zero forcing number* of G is

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the minimum cardinality of an F-set of G, denoted by $F(G)$. Moreover, an F-set S of G is called a *total forcing set* of G if $G[S]$ contains no isolated vertex, we also call such S a TF-set of G. The *total forcing number* of G is the minimum cardinality of a TF-set of graph G, denoted by $F_t(G)$. The zero (or total) forcing number of G was introduced in [1, 6] and has been extensively studied in recent years, largely due to its connection to inverse eigenvalue problems for graphs and its applications to other problems. Up to now, there have been lots of research work on bounding the zero (or total) forcing number of a graph in terms of its various parameters [2, 3, 7–14].

Let $\mathcal{T}_n$ be the set of trees of order n. In this paper, we study the relationship between the zero (or total) forcing number of a tree T and its vertex (edge) covering number. First, we characterize all trees $T \in \mathcal{T}_n$ with $F(T) = (\Delta-2)\beta(T)+1$, solving a problem proposed by Brimkov [5]. Next, we prove that for any $T \in \mathcal{T}_n$ with $n \geq 3$, $F(T) \leq \beta'(T) - 1$ and $F_t(T) \leq \beta'(T)$. Moreover, we also characterize all trees with $F(T) = \beta'(T) - 1$ and $F(T) = \beta'(T) - 2$, respectively.

2. Zero forcing number and vertex covering number of a tree

Brimkov et al. [5] explored the following relationship between $F(G)$ and $\beta(G)$ for a connected graph G with $\Delta(G) \geq 3$.

Theorem 2.1. [5] *For any connected graph G with maximum degree $\Delta \geq 3$, we have $F(G) \leq (\Delta-2)\beta(G)+1$.*

In the same paper, they proposed a problem of characterizing all trees $T \in \mathcal{T}_n$ with $F(T) = (\Delta(T)-2)\beta(T)+1$. In this section, we solve this problem. Before then, we need the following definitions and lemmas.

Lemma 2.2. [9] *Let P_n and $K_{1,n-1}$ be the path and the star of order n, respectively. Then*

(1) *$F(P_n) = 1$ for $n \geq 2$ and $F(K_{1,n-1}) = n-2$ for $n \geq 3$;*

(2) *$F_t(P_n) = 2$ for $n \geq 2$ and $F_t(K_{1,n-1}) = n-1$ for $n \geq 3$.*

Lemma 2.3. *For any $T \in \mathcal{T}_n$ with $n \geq 3$, we have $F(T) \leq (\Delta-2)\beta(T)+1$, where Δ is the maximum degree of T.*

Proof. For $T \in \mathcal{T}_n$ with $n \geq 3$, if $\Delta = 2$, then $T \cong P_n$. Hence Lemma 2.2 implies that $F(T) = 1$, the result follows. If $\Delta \geq 3$, then the result follows from Theorem 2.1. □

Lemma 2.4. [13] *Let G be a graph obtained from a graph H and a star $K_{1,n}$ with $n \geq 2$, by adding an edge to join a vertex of H and the central vertex of $K_{1,n}$. Then $F(G) = F(H) + n - 1$.*

Suppose $G = K_{1,s}$ and $H = K_{1,t}$. The *double star* $S_{t,s}$ is obtained from G and H by adding an edge to join the central vertices of two stars. Clearly $|S_{t,s}| = t + s + 2$.

Lemma 2.5. *Suppose* $1 \leq t \leq s$. *Then*

$$F(S_{t,s}) = \begin{cases} s & \text{if } t = 1, \\ s + t - 2 & \text{if } t \geq 2. \end{cases}$$

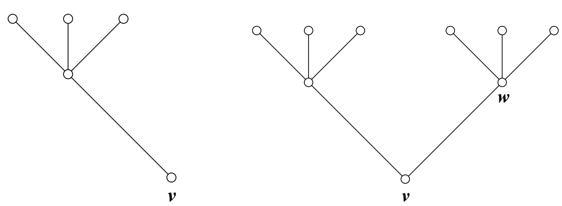
Proof. If $s = 1$, then $S_{1,1} = P_4$. Hence Lemma 2.2 implies that $F(S_{1,1}) = 1 = s$. If $s \geq 2$, then by Lemma 2.4, we have

$$F(S_{t,s}) = F(K_{1,t}) + s - 1 = \begin{cases} s & \text{if } t = 1, \\ s + t - 2 & \text{if } t \geq 2, \end{cases}$$

as desired. □

A pendant vertex (or leaf) in a graph G is a vertex with degree 1 and the edge incident with it is a pendant edge. We call a vertex is a *strong* (or *weak*) *support vertex* of G if it has at least two leaf neighbors (or only one leaf neighbor).

We now introduce a general operation called *k-leaf support vertex addition* on G, abbreviated k-LSVA. For a graph G with maximum degree Δ, we define k-LSVA on G to be the process of attaching to a vertex $v \in V(G)$ with $d_G(v) \leq k - 1$ by a new vertex w, and then attaching k leaves to w. Figure 2.1 is an example.



(a) G (b) G'

Figure 2.1: The star $G = K_{1,4}$ and the graph G' obtained by performing a 3-LSVA on $K_{1,4}$.

We use the standard notation $[\boldsymbol{h}] = \{1, 2, \ldots, h\}$. For $k \geq 3$, let $\mathcal{T}_n^+$ be the set of trees of order n obtained by starting with $K_{1,k}$ and applying as many $(k-1)$-LSVA as wanted. In other words, $\mathcal{T}_n^+$ is the family of all trees T with maximum degree $\Delta \geq 3$ whose vertex set $V(T)$ can be partitioned into sets $(V_1, \ldots, V_h)$ such that

(1) $T_i = G[V_i]$ for $i \in [\boldsymbol{h}]$;

(2) $T_1 \cong K_{1,\Delta}$ and $T_i \cong K_{1,\Delta-1}$ for $i \in [\boldsymbol{h}] \setminus \{1\}$;

(3) for $i \in [\boldsymbol{h}]$, the central vertex v_i of the star T_i has degree Δ in the tree T;

(4) $\{v_1, \ldots, v_h\}$ is an independent set in T.

Thus, if $T \in \mathcal{T}_n^+$, then $n = h\Delta + 1$ for some $h \geq 1$. In addition, we call the trees $T_1, \ldots, T_h$ the *basic trees* of T. Obviously, $\{v_1, \ldots, v_h\}$ is a minimum vertex covering of T.

Lemma 2.6. *For any $T \in \mathcal{T}_n^+$ with maximum degree $\Delta \geq 3$, we have $F(T) = (\Delta - 2)\beta(T) + 1$.*

Proof. Assuming T is the tree obtained by applying $h-1$ times $(\Delta-1)$-LSVA starting from $K_{1,\Delta}$, where $\Delta \geq 3$. Let $T_1, T_2, \ldots, T_h$ be the basic trees of T and v_i be the central vertex of T_i for $i \in [\boldsymbol{h}]$. Clearly $\beta(T) = h$ since $\{v_1, v_2, \ldots, v_h\}$ is a minimum vertex covering of T. Then Lemmas 2.4 and 2.2 imply that

$$F(T) = F(K_{1,\Delta}) + (h-1)F(K_{1,\Delta-1}) = \Delta - 1 + (h-1)(\Delta-2) = (\Delta-2)h + 1,$$

as desired. □

Lemma 2.7. *For $T \in \mathcal{T}_n^+$, let T' be a tree obtained by adding an edge connecting a leaf of a basic tree in T and a vertex of P_2. Then we have $F(T') = F(T)$.*

Proof. The proof is similar to that in Lemma 2.6. Obviously, $\Delta(T') = \Delta(T)$. Without lost of generality, we divide T' into h basic trees $T'_1, \ldots, T'_h$, where $T'_i = K_{1,\Delta-1}$, $i \in [\boldsymbol{h-1}]$, and T'_h is a tree obtained by adding an edge connecting a leaf of $K_{1,\Delta}$ and a vertex of P_2. Since any $\Delta - 1$ leaves in T'_h form a minimum F-set of T'_h, $F(T'_h) = \Delta - 1$. By Lemmas 2.4, 2.2 and 2.6, we have

$$F(T') = F(T'_h) + (h-1)F(K_{1,\Delta-1}) = \Delta - 1 + (h-1)(\Delta-2) = (\Delta-2)h + 1 = F(T),$$

as desired. □

Let $\alpha(G)$ be the independence number of a graph G. Recall that $\beta(G)$ and $\beta'(G)$ are the vertex covering number and the edge covering number of G, respectively. Then $\beta(G) + \alpha(G) = n$ (see [4, Corollary 7.1]) and $\alpha(G) \leq \beta'(G)$ for any connected graph G of order n. Moreover, $\alpha(G) = \beta'(G)$ when G is a bipartite graph (see [4, Theorem 7.3]). Hence we have the following result for trees.

Lemma 2.8. *For any $T \in \mathcal{T}_n$ with $n \geq 2$, we have $\beta(T) = n - \beta'(T)$.*

For $u, v \in V(G)$, we use $P(u,v)$ to denote the shortest path from u to v. The *distance* between u and v is the length of a shortest path $P(u,v)$ in G. And the *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum distance among every pair of distinct vertices of G.

A rooted tree T distinguishes one vertex r called the *root* (see Figure 2.2). For each vertex $v \neq r$ of T, the parent of v is the neighbor of v on the unique $P(v, r)$, while a child of v is any other neighbor of v.

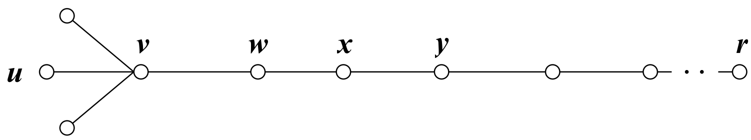


Figure 2.2: Example of root tree T with the root r.

Theorem 2.9. *For any $T \in \mathcal{T}_n$ with $n \geq 3$, we have $F(T) = (\Delta - 2)\beta(T) + 1$ if and only if $T \in \mathcal{T}_n^+ \cup \{P_n\}$.*

Proof. The sufficient part follows from Lemmas 2.2 and 2.6. We shall show the necessary part by mathematical induction on n.

If $n = 3$, then $T = P_3$. Hence we are done.

We assume that, for any tree T' of order n', $3 \leq n' < n$, $F(T') = (\Delta' - 2)\beta(T') + 1$ implies $T' \in \mathcal{T}_{n'}^+ \cup \{P_{n'}\}$, where $\Delta' = \Delta(T')$.

Now, suppose $T \in \mathcal{T}_n$ and $F(T) = (\Delta - 2)\beta(T) + 1$, where $n \geq 4$. If $\Delta = 2$, then $T = P_n$. We are done. If $\Delta \geq 3$ and $\operatorname{diam}(T) = 2$, then $T = K_{1,\Delta-1} \in \mathcal{T}_n^+$. We are done too. Thus, we only need to deal with $\Delta \geq 3$ and $\operatorname{diam}(T) \geq 3$.

If $\operatorname{diam}(T) = 3$, then $T \cong S_{t,s}$, here $S_{t,s}$ is a double star, where $1 \leq t \leq s$ and $s \geq 2$. Let x and y be the (only) vertices in $S_{r,s}$ of degree greater than 1. Clearly $\beta(S_{t,s}) = 2$ since $\{x, y\}$ is a minimum vertex covering of $S_{t,s}$. Then

$$(\Delta - 2)\beta(S_{t,s}) + 1 = 2(s + 1 - 2) + 1 = 2s - 1.$$

From Lemma 2.4, we can see that $F(S_{t,s}) < 2s - 1$. Thus $\operatorname{diam}(T) = 3$ is not a case. Thus $\operatorname{diam}(T) \geq 4$.

Let $u, r \in V(T)$ such that $\operatorname{diam}(T) = d(u, r)$. Clearly, u and r are two leaves in T. Let r be the root of T and $P(u, r) = uvwxy \cdots r$. Note that $y = r$ when $\operatorname{diam}(T) = 4$ and $y \neq r$ when $\operatorname{diam}(T) > 4$.

Let $d_T(v) = t$. Clearly, $2 \leq t \leq \Delta$. Let T_v be the subtree of T which is induced by the vertex v and its children. Let $T' = T - V(T_v)$, $n' = |T'|$ and S' be a minimum F-set of T'. Clearly T' is a tree. Since w, x, y are distinct vertices of T', $n' \geq 3$.

Now, let us compute $\beta(T')$. Let A be a minimum edge covering of T. Since every edge covering contains all pendant edges of T, then $E(T_v) \subseteq A$. If $vw \notin A$, then $A \setminus E(T_v)$ is an edge covering of T'. Since the minimum property of A and $E(T_v) \subseteq A$, then $A \setminus E(T_v)$ is a minimum edge covering of T'. If $vw \in A$, then A has no other edges incident with w.

Otherwise, assuming there is an edge incident with w, if necessary, the edge is wx. Then, $A \setminus \{vw\}$ is a smaller edge covering of T, a contradiction. Let $A' = (A \setminus \{vw\}) \cup \{wx\}$. Then $|A'| = |A|$ and A' is also a minimum edge covering. Since the minimum property of A' and $E(T_v) \subseteq A'$, then $A' \setminus E(T_v)$ is a minimum edge covering of T'. In conclusion, $\beta'(T') = \beta'(T) - (t-1) = \beta'(T) - t + 1$ and $n' = n - d_T(v) = n - t$. Then Lemma 2.8 implies that

$$\beta(T') = n' - \beta'(T') = n - \beta'(T) - 1 = \beta(T) - 1.$$

Recall that $2 \le t \le \Delta$ and $\Delta \ge 3$. Let Δ' be the maximum degree of T'.

(A) Suppose $t = 2$. Let S' be an F-set of T' such that $F(T') = |S'|$. Then $S' \cup \{v\}$ is an F-set of T. Hence $F(T') + 1 \ge F(T)$. By the assumption and Lemma 2.3 we have

$$\begin{aligned}(\Delta'-2)\beta(T') + 1 \ge F(T') \ge F(T) - 1 = (\Delta-2)\beta(T) \ge (\Delta'-2)(\beta(T')+1)\\ = (\Delta'-2)\beta(T') + \Delta' - 2 \ge (\Delta'-2)\beta(T') + 1.\end{aligned}$$

Thus $F(T') = (\Delta'-2)\beta(T') + 1$, $\Delta' = \Delta$ and $F(T') + 1 = F(T)$.

By induction hypothesis, $T' = P_{n'}$ or $T' \in \mathcal{T}^+_{n'}$. Since $\Delta' = \Delta \ge 3$, $T' \in \mathcal{T}^+_{n'}$.

Since $\Delta = \Delta'$, w is a leaf of a basis tree. And T is the tree obtained by adding an edge connecting a leaf of a basic tree in T' and a vertex of P_2. Then by Lemma 2.7, we have $F(T) = F(T')$ which contradicts $F(T') + 1 = F(T)$.

(B) Suppose $t \ge 3$. From Lemma 2.4 we have

$$\begin{aligned}F(T') &= F(T) - (t-2) = (\Delta-2)\beta(T) + 1 - t + 2\\ &= (\Delta-2)(\beta(T')+1) + 1 - t + 2 \qquad\qquad (\text{since } \Delta \ge \Delta' \text{ and } \Delta \ge t)\\ &= (\Delta-2)\beta(T') + \Delta + 1 - t \ge (\Delta'-2)\beta(T') + 1.\end{aligned}$$

Together with Lemma 2.3 we have $F(T') = (\Delta'-2)\beta(T') + 1$ and $\Delta = \Delta' = t \ge 3$. By induction hypothesis, $T' = P_{n'}$ or $T' \in \mathcal{T}^+_{n'}$. Since $\Delta' = \Delta \ge 3$, $T' \in \mathcal{T}^+_{n'}$.

Let $T_1, T_2, \ldots, T_h$ be the basic trees of T', where $T_1 = K_{1,\Delta'}$ and if $h \ge 2$, then $T_i = K_{1,\Delta'-1}$ for $i \in [\boldsymbol{h}] \setminus \{1\}$. Let v_i be the central vertex of T_i. Then $d_{T'}(v_i) = \Delta'$ for $i \in [\boldsymbol{h}]$. Hence $\{v_1, \ldots, v_h\}$ is an independent set and a minimum vertex covering of T'. Furthermore, $\beta(T) = h + 1 \ge 2$.

Since $\Delta = \Delta'$, w is a leaf of a basic tree of T'. Let $v = v_{h+1}$ and $T_{h+1} = K_{1,\Delta-1}$. Then T is obtained from T' by applying once $(\Delta-1)$-LSVA process. That is $T \in \mathcal{T}^+_n$.

The proof is complete. $\square$

3. Zero (Total) forcing number and edge covering number of a tree

In this section, we study the relationship between the zero (total) forcing number of a tree and its edge covering number. Before then, we introduce some definitions and lemmas as follows.

The contraction of an edge $e = uv \in E(G)$ is the graph obtained from G by replacing the vertices u and v by a new vertex and joining this new vertex to all vertices that are adjacent to u or v in G. For any $T \in \mathcal{T}_n$ with $n \geq 2$, the trimmed tree of T, denoted by $\mathrm{trim}(T)$, is the tree obtained from T by iteratively contracting edges with one of its incident vertices of degree exactly 2 and with the other incident vertex of degree at most 2 until no such edge remains. For instance, $\mathrm{trim}(P_n) = P_2$ for $n \geq 2$. While if $T \neq P_n$, then every edge in $\mathrm{trim}(T)$ is incident with a vertex of degree at least 3.

Lemma 3.1. [8] *For any $T \in \mathcal{T}_n$ with $n \geq 2$, we have*

(1) $F(T) = F(\mathrm{trim}(T))$;

(2) $F_t(T) = F_t(\mathrm{trim}(T))$;

(3) *both trees T and $\mathrm{trim}(T)$ have the same number of leaves.*

Lemma 3.2. *Let G be a graph obtained from a graph H and a star $K_{1,n}$ with $n \geq 2$, by adding an edge to join a vertex of H and the central vertex of $K_{1,n}$. Then $F_t(G) \leq F_t(H) + n$.*

Proof. Let S_1 be a minimum TF-set of H and S_2 be a set containing the central vertex and $n-1$ leaves of $K_{1,n}$. Then Lemma 2.2 implies that S_2 is a minimum TF-set of $K_{1,n}$. Hence, $S_1 \cup S_2$ is a TF-set of G. So $F_t(G) \leq |S_1| + |S_2| = F_t(H) + n$, as desired. □

In particular, when H is a tree, in view of Lemma 2.8 and the discussion in Theorem 2.9, we then have the following result.

Lemma 3.3. *Let T be a tree obtained from a tree T' and a star $K_{1,n}$ with $n \geq 2$, by adding an edge to join a vertex of T' and the central vertex of $K_{1,n}$. Then $\beta'(T) = \beta'(T') + n$.*

Lemma 3.4. *Let G be a connected graph of order at least 3 and $e = uv \in E(G)$. If H is the graph obtained from G by contracting e, then $\beta'(H) \leq \beta'(G)$.*

Proof. Let x be the resulting new vertex in H after contracting e. Since the order of G is at least 3, without loss of generality, we assume $d(u) \geq 2$ and let w be another neighbor of u rather than v. Thus $xw \in E(H)$.

Let A be a minimum edge covering of G. If $uv \in A$, then let $A' = (A \setminus \{uv\}) \cup \{xw\}$. If $uv \notin A$, then let $A' = A$.

Clearly A' is an edge covering of H and $|A'| \leq |A|$ (since $vw \in E(G)$ may be in A and it is the same edge $xw \in E(H)$). Thus $\beta'(H) \leq \beta'(G)$. □

Corollary 3.5. *For any $T \in \mathcal{T}_n$ with $n \geq 2$, we have $\beta'(\mathrm{trim}(T)) \leq \beta'(T)$.*

Lemma 3.6. [9] *If G is an isolate-free graph, then every vertex v of G with at least two leaf neighbors is contained in every* TF-set*, and all except possibly one leaf neighbor of v is contained in every* TF-set*.*

Theorem 3.7. *For any $T \in \mathcal{T}_n$ with $n \geq 3$, we have $F_t(T) \leq \beta'(T)$.*

Proof. We shall prove this theorem by mathematical induction on n.

If $n = 3$, then $T = P_3$. Thus the result follows from Lemma 2.2 since $\beta'(P_3) = 2$.

Assume that $F(T') \leq \beta'(T')$ holds for any T' of order n', where $3 \leq n' < n$.

Now let T be a tree of order $n \geq 4$. If $T = P_n$, then Lemma 2.2 implies that $F_t(P_n) = 2$. Thus we have $F_t(P_n) \leq \beta'(P_n)$ as $\beta'(P_n) \geq 2$. In what follows, we assume that $T \neq P_n$. We now consider the following two cases.

(a) $T = \mathrm{trim}(T)$. The tree T is obtained from a tree T' and a star $K_{1,k}$ with $k \geq 2$, by adding an edge to join a vertex of T' and the central vertex of $K_{1,k}$. By induction hypothesis and Lemmas 3.2 and 3.3, we then have

$$F_t(T) \leq F_t(T') + k \leq \beta'(T') + k = \beta'(T),$$

as desired.

(b) $T \neq \mathrm{trim}(T)$. Lemma 3.1 and Corollary 3.5 imply that

$$F_t(T) = F_t(\mathrm{trim}(T)) \leq \beta'(\mathrm{trim}(T)) \leq \beta'(T),$$

as desired. This completes the proof. □

A tree T is called a *spider with k legs*, where $k \geq 2$, if $\Delta(T) = k$ and T contains only one vertex of degree k. This vertex is called the *core* of the spider. Let $T(n_1, \ldots, n_k)$ be the spider of k legs shown in Figure 3.1, where v is its core and $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$. We shall adopt a spider has only 2 legs. In this case, the spider $T(n_1, n_2)$ is a path of length $n_1 + n_2$.

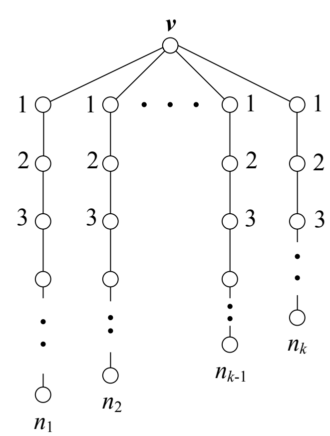


Figure 3.1: The tree $T(n_1, \ldots, n_k)$.

Note that $\mathrm{trim}(T(n_1, \ldots, n_k)) = K_{1,k}$. By Lemmas 2.2 and 3.1 we have

Lemma 3.8. *If T is a spider with $k \geq 2$ legs, then $F(T) = k - 1$.*

Combining this with Lemma 2.4, we then have the following result.

Lemma 3.9. *Let G be a graph obtained from a graph H and a spider $T(n_1, \ldots, n_k)$, by adding an edge to join a vertex of H and the core of $T(n_1, \ldots, n_k)$, $k \geq 2$. Then $F(G) = F(H) + k - 1$.*

Lemma 3.10. [8] *For any $T \in \mathcal{T}_n$ with $n \geq 2$, we have $F_t(T) \geq F(T) + 1$.*

Lemma 3.11. *For any $T \in \mathcal{T}_n$ with $n \geq 3$, we have $F(T) \leq \beta'(T) - 1$.*

Proof. The result follows form Theorem 3.7 and Lemma 3.10. □

Lemma 3.12. *For an isolate-free graph G with $k \geq 1$ strong support vertices, we have $F_t(G) \geq F(G) + k$.*

Proof. Let $v_1, \ldots, v_k$ be strong support vertices of G. Assume that S is a minimum TF-set of G with $|S| = F_t(G)$. By Lemma 3.6, we have $\{v_1, \ldots, v_k\} \subseteq S$ and there is a leaf neighbor, say x_i, of v_i such that $x_i \in S$ for each $i \in [\boldsymbol{k}]$. Let $S' = S \setminus \{v_1, \ldots, v_k\}$. We claim that S' is an F-set of G. Indeed, for the forcing process of S', firstly, x_1 forces v_1 to color. Next, x_i gradually forces v_i to color for $[\boldsymbol{k}] \setminus \{1\}$. Finally, the set of colored vertices is S. Hence, S' is an F-set of G since S is a TF-set of G. That is $F(G) \leq |S'| = |S| - k = F_t(G) - k$. It follows that $F_t(G) \geq F(G) + k$. □

A subpath $P = vu_1u_2 \cdots u_l$ of a graph G is referred to a *pendent path* if $d_G(v) \geq 3$, $d_G(u_1) = \cdots = d_G(u_{l-1}) = 2$, $d_G(u_l) = 1$, and l is the length of the pendant path. We use $p(v)$ to denote the number of pendant paths which attached to $v \in V(G)$. If $p(v) \geq 2$, we call v a *strong major vertex*; if $p(v) = 1$, we call v a *weak major vertex*.

Lemma 3.13. *For $T \in \mathcal{T}_n$ with $n \geq 4$, if T has $k \geq 1$ strong major vertices, then $F(T) \leq \beta'(T) - k$.*

Proof. Let T be a tree with $k \geq 1$ strong major vertices. Note that every strong major vertex in T is a strong support vertex in $\mathrm{trim}(T)$. Then $\mathrm{trim}(T)$ has k strong support vertices.

$$\begin{aligned}
F(T) &= F(\mathrm{trim}(T)) && \text{(by Lemma 3.1)}\\
&\leq F_t(\mathrm{trim}(T)) - k && \text{(by Lemma 3.12)}\\
&\leq \beta'(\mathrm{trim}(T)) - k && \text{(by Theorem 3.7)}\\
&\leq \beta'(T) - k. && \text{(by Corollary 3.5)}
\end{aligned}$$

This completes the proof. □

Before proving Theorems 3.16 and 3.17, we introduce the following types of spiders. Let

(1) $\mathcal{G}_1$ be the set of spiders $T(n_1,\dots,n_k)$ for some $k \geq 2$ with $n_1 \leq 2$ and $n_k = 1$;

(2) $\mathcal{G}_2$ be the set of spiders $T(n_1,\dots,n_k)$ for some $k \geq 2$ with $n_1 = n_2 = \cdots = n_k = 2$;

(3) $\mathcal{G}_3$ be the set of trees $T(n_1,\dots,n_k)$ spiders for some $k \geq 2$ with $n_1 = 3$ and $n_2 \leq 2$;

(4) $\mathcal{G}_4$ be the set of trees $T(n_1,\dots,n_k)$ spiders for some $k \geq 3$, $n_1 = 4$, $n_2 \leq 2$ and $n_k = 1$.

Remark 3.14. Clearly, for a spider with 2 legs, $\{P_3, P_4\} \subset \mathcal{G}_1$, $P_5 \in \mathcal{G}_2$ and $\{P_5, P_6\} \subset \mathcal{G}_3$.

Lemma 3.15. *For $k \geq 2$,*

$$\beta'(T(n_1,\dots,n_k)) = \begin{cases} \sum_{i=1}^{k} \lceil n_i/2 \rceil + 1 & \text{if all } n_i\text{'s are even,} \\ \sum_{i=1}^{k} \lceil n_i/2 \rceil & \text{otherwise.} \end{cases}$$

Proof. Let v be the core of the spider $T = T(n_1,\dots,n_k)$ and let $R_i = vx_1^i \cdots x_{n_i}^i$ be the pendant path of length n_i. Let A be a minimum edge covering of T and $A_i = A \cap E(R_i)$, $1 \leq i \leq k$.

Suppose n_i is even. The pendant edge $x_{n_i-1}^i x_{n_i}^i \in A_i$. Since all vertices of $R_i - v$ are covered by A_i, $x_{n_i-3}^i x_{n_i-2}^i, \dots, x_2^i x_1^i \in A_i$ gradually. Thus $|A_i| = n_i/2$ or $n_i/2 + 1$ when $vx_1^i \notin A_i$ or $vx_1^i \in A_i$, respectively.

Suppose n_i is odd. Similarly, $x_{n_i-1}^i x_{n_i}^i, \dots, x_3^i x_2^i \in A_i$. Since x_1^i is also covered by A_i, $vx_1^i \in A_i$. Thus $|A_i| = (n_i+1)/2 = \lceil n_i/2 \rceil$.

Suppose there is an odd n_j. By the proof above, $vx_1^j \in A$. By the minimality, those A_i's do not contain vx_1^i for all even n_i's. Hence $\beta'(T(n_1,\dots,n_k)) = \sum_{i=1}^{k} \lceil n_i/2 \rceil$.

Suppose all n_i's are even. Since v must be covered, by the minimality only one of A_i contains vx_1^i. Hence $\beta'(T(n_1,\dots,n_k)) = \sum_{i=1}^{k} \lceil n_i/2 \rceil + 1$. □

Let G and H be two disjoint connected graphs with $v \in V(G)$ and $u \in V(H)$. Define the graph $G(v) \circ H(u)$ is obtained from $G \cup H$ by identifying v with u. For example, let v be a leaf of P_2 and u be a leaf of P_3, then $G(v) \circ H(u) = P_4$. Let G_1, G_2 and G_3 be three mutually disjoint connected graphs, and let $x_1 \in V(G_1)$, $x_2 \in V(G_2)$, $y_1, y_2 \in V(G_3)$, where $y_1 \neq y_2$. A connected graph G obtained from $G_1 \cup G_2 \cup G_3$ by identifying x_1 with y_1 and identifying x_2 with y_2 is denoted by $G_1(x_1 \circ y_1) \bigcup_{G_3} G_2(x_2 \circ y_2)$.

We define

$$\mathcal{Q}_1 = \left\{ T_1(v_1 \circ x_1) \bigcup_{K_{1,3}} T_2(v_2 \circ x_2) \;\middle|\; \begin{array}{c} T_i \in \mathcal{G}_1 \text{ with core } v_i, i = 1, 2, \\ x_1, x_2 \text{ are two different leaves of } K_{1,3} \end{array} \right\},$$

$$\mathcal{Q}_2 = \left\{ T_1(v_1 \circ x_1) \bigcup_{P_2} T_2(v_2 \circ x_2) \;\middle|\; T_i \in \mathcal{G}_1 \text{ with core } v_i, i = 1,2, P_2 = x_1x_2 \right\},$$

$$\mathcal{Q}_3 = \left\{ T_1(v_1 \circ x_1) \bigcup_{P_3} T_2(v_2 \circ x_2) \;\middle|\; T_i \in \mathcal{G}_1 \text{ with core } v_i, i = 1,2, P_3 = x_1yx_2 \right\},$$

$$\mathcal{Q}_4 = \left\{ T_1(v_1 \circ x_1) \bigcup_{P_4} T_2(v_2 \circ x_2) \;\middle|\; T_i \in \mathcal{G}_1 \text{ with core } v_i, i = 1,2, P_4 = x_1yzx_2 \right\},$$

$$\mathcal{Q}_5 = \left\{ T_1(v_1 \circ x_1) \bigcup_{P_3} T_2(v_2 \circ x_2) \;\middle|\; T_1 \in \mathcal{G}_1, T_2 \in \mathcal{G}_2 \text{ with core } v_i, i = 1,2, P_3 = x_1yx_2 \right\}.$$

Theorem 3.16. *For any $T \in \mathcal{T}_n$ with $n \geq 2$, we have $F(T) = \beta'(T) - 1$ if and only if $T \in \mathcal{G}_1$.*

Proof. Suppose $T \in \mathcal{T}_n$ with $F(T) = \beta'(T) - 1$. If $T = P_n$, then by Lemma 2.2, we check that only $T = P_3$ or $T = P_4$ satisfies that $F(T) = \beta'(T) - 1$, as desired.

If $T \neq P_n$, let $l \geq 1$ be the number of the strong major vertices in T. Since $F(T) = \beta'(T) - 1$, Lemma 3.13 implies that $l = 1$. Then T is a spider. Let $T = T(n_1, \ldots, n_k)$ with $k \geq 3$ and v be the unique major vertex of T. Then Lemma 3.8 implies $F(T) = k - 1$. Thus $k = \beta'(T)$. By Lemma 3.15, there exists an odd n_j and all $\lceil n_i/2 \rceil = 1$. This implies that $n_i \leq 2$ and $n_j = 1$. By definition, $n_1 \leq 2$ and $n_k = 1$. Hence $T \in \mathcal{G}_1$.

The converse follows from Lemmas 3.8 and 3.15. This completes the proof. □

Note that if T has exactly two strong major vertices v_1 and v_2 and some weak major vertices, then each weak major vertex should be in the path $P(v_1, v_2)$. Let $T(v_i)$ be the induced subgraph of vertices of all pendant paths attached to v_i, i.e., $T(v_i)$ is a spider for $i = 1, 2$.

Theorem 3.17. *For any $T \in \mathcal{T}_n$ with $n \geq 2$, we have $F(T) = \beta'(T) - 2$ if and only if T is an element in one of the following classes:*

$$\mathcal{G}_2,\ \mathcal{G}_3,\ \mathcal{G}_4,\ \mathcal{Q}_1,\ \mathcal{Q}_2,\ \mathcal{Q}_3,\ \mathcal{Q}_4,\ \mathcal{Q}_5.$$

Proof. For $T \in \mathcal{T}_n$ with $F(T) = \beta'(T) - 2$, if $T = P_n$, then by Lemma 2.2, one may check that only P_5 and P_6 satisfy that $F(P_5) = \beta'(P_5) - 2$ and $F(P_6) = \beta'(P_6) - 2$, as desired.

Now we consider $T \neq P_n$. Let $l \geq 1$ be the number of strong major vertices in T. By Lemma 3.13, we have $l \leq 2$, i.e., $l = 1$ or $l = 2$. We now consider the following two cases.

(A) Suppose $l = 1$. Then $T = T(n_1, \ldots, n_k)$ with $k \geq 3$. Let v be the major vertex (core) and let u be one of its neighbor. Let $\mathcal{P}_i$ be the pendant paths with length of n_i, where $1 \leq i \leq k$. Then $F(T) = k - 1$ by Lemma 3.8.

Suppose all n_i are even. By Lemma 3.15, $k-1 = \sum_{i=1}^{k}(n_i/2) - 1$. This implies that $n_i = 2$. So $T \in \mathcal{G}_2$.

Suppose there is an odd n_j. By Lemma 3.15, $k-1 = \sum_{i=1}^{k}\lceil n_i/2\rceil - 2$. This implies that $\lceil n_1/2\rceil = 2$ and $\lceil n_i/2\rceil = 1$ for $2 \le i \le k$. Thus $n_i \le 2$ for $2 \le i \le k$.

Suppose $n_1 = 3$. $T \in \mathcal{G}_3$. Suppose $n_1 = 4$. Since there is an odd n_j, $n_k = 1$. Hence $T \in \mathcal{G}_4$.

(B) Suppose $l = 2$. Let v_1 and v_2 be two strong major vertices of T, u_i be the neighbor of v_i in $P = P(v_1, v_2)$, $i = 1, 2$ and $w_1, w_2, \ldots, w_h$ be h weak major vertices on P, $h \ge 0$.

(B1) Suppose $h \ge 1$. Let $P(v_1, v_2) = v_1 u w \cdots v_2$ and $T - v_1 u = T_1 \cup T_2$, where $v_1 \in V(T_1)$. Here w may be v_2.

We let $T' = \mathrm{trim}(T)$. Then $T' - v_1 u = T'_1 \cup T'_2$. Moreover $\mathrm{trim}(T_i) = T'_i$, $i = 1, 2$.

Let S_i be an F-set of T'_i with minimum cardinality, $i = 1, 2$. Clearly $S_1 \cup S_2$ is an F-set of T'. Thus $F(T') \le F(T'_1) + F(T'_2)$.

Since T'_1 is a star graph, $\beta'(T'_1) = |E(T'_1)|$. Hence Lemma 2.2 implies $F(T'_1) = \beta'(T'_1) - 1$.

Let A be a minimum edge covering of T' and let

$$A' = \begin{cases} A \setminus E(T'_1) & \text{if } v_1 u \notin A, \\ (A \cup \{uw\}) \setminus (E(T'_1) \cup \{v_1 u\}) & \text{if } v_1 u \in A. \end{cases}$$

Then A' is a minimum edge covering of T'_2 since A is a minimum edge covering of T' and $E(T'_1) \subseteq A$. That is, $\beta'(T'_2) \le \beta'(T') - \beta'(T'_1)$ (since uw may be already in A). Hence, by assumption and Lemma 2.4 we have

$$\begin{aligned} \beta'(T) - 2 = F(T) = F(T') &\le F(T'_1) + F(T'_2) \\ &\le \beta'(T'_1) - 1 + \beta'(T'_2) - 1 \le \beta'(T') - 2 \le \beta'(T) - 2. \end{aligned}$$

Thus all inequalities become equalities. Hence $T'_1, T'_2 \in \mathcal{G}_1$. Since $T'_2 \in \mathcal{G}_1$, $h = 1$. Furthermore, $\beta'(T'_1) = k_1$ and $\beta'(T'_2) = k_2 + 1$.

Now T_1 is a spider of k_1 legs and T_2 is a spider of $k_2 + 1$ legs. By Lemma 3.15, it forces that $T_1, T_2 \in \mathcal{G}_1$.

Let us look at the weak major vertex w_1. Let R be the pendant path attached to w_1. Since w_1 is a vertex in one of a leg of T_2 and $T_2 \in \mathcal{G}_1$, the distance between v_2 and w_1 is 1 and the length of R is 1. Also since w_1 is a vertex of the path $P(v_1, v_2)$, $w_1 = u$. Thus $P(v_1, v_2) = v_1 w_1 v_2$. Thus, $T_2 = T(2, m_1, \ldots, m_{k_2})$ with $m_i \le 2$ for $i \ge 1$ and $m_{k_2} = 1$. Hence $T \in \mathcal{Q}_1$.

(B2) Suppose $h = 0$. Suppose $P(v_1, v_2) = v_1 v_2$. Let $T - v_1 v_2 = T_1 \cup T_2$. By the same proof of Case (B1), we get $T_1, T_2 \in \mathcal{G}_1$. Thus $T \in \mathcal{Q}_2$.

Suppose $P(v_1, v_2) = v_1 u \cdots v_2$. Let $T - v_1 u = T_1 \cup T_2$. By the same proof of Case (B1), we get $T_1, T_2 \in \mathcal{G}_1$. T_2 is a spider with a leg $P(v_1, v_2) - v_1$. So the length of $P(v_1, v_2)$ is less than 3.

Suppose the length of $P(v_1, v_2)$ is 3. Since $T_2 \in \mathcal{G}_1$, $T_2 = T(2, m_1, \ldots, m_{k_2})$ with $m_i \leq 2$ for $i \geq 1$ and $m_{k_2} = 1$. So $T(v_2) \in \mathcal{G}_1$. Hence $T \in \mathcal{Q}_4$.

Suppose the length of $P(v_1, v_2)$ is 2. Since $T_2 \in \mathcal{G}_1$, $T_2 = T(m_1, \ldots, m_{k_2}, 1)$ with $m_i \leq 2$ for $i \geq 1$. So $T(v_2) \in \mathcal{G}_1 \cup \mathcal{G}_2$. Hence $T \in \mathcal{Q}_3 \cup \mathcal{Q}_5$.

The converse follows from Lemmas 3.8 and 3.15. This completes the proof. $\square$

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