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The A_α-spectral Radius and $[a, b]$-factors in Graphs

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Abstract. Let $A(G)$ and $D(G)$ be the adjacency matrix and the degree matrix of G, respectively. For any real $\alpha \in [0, 1]$, Nikiforov [12] defined the matrix $A_\alpha(G)$ as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

An $[a, b]$-factor of a graph G is a spanning subgraph H such that $a \leq d_H(v) \leq b$ for any $v \in V(G)$, where a and b are positive integers. In this paper, we give an upper bound of A_α-spectral radius of graphs with unique perfect matching, and then present A_α-spectral conditions for the existence of an $[a, b]$-factor in a graph. Our results extend the result of Fan et al. in [4] for the unique perfect matching and $[a, b]$-factor of graphs, and that of Zhao et al. in [16] for a $[1, b]$-odd factor of graphs.

1. Introduction

Throughout this paper, all graphs considered are simple connected and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $A(G)$ and $D(G)$ be the *adjacency matrix* and the *diagonal matrix* of vertex degrees of G, respectively. We write $d_G(v)$, i.e., $d(v)$, for the *degree* of the vertex $v \in V(G)$, $N_G(v)$ for the *neighbor set* of the vertex $v \in V(G)$, and $N_G[v]$ for $\{v\} \cup N_G(v)$. For any real $\alpha \in [0, 1]$, Nikiforov [12] defined the matrix $A_\alpha(G)$ as $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$. It is easy to see that $A_0(G) = A(G)$, $A_1(G) = D(G)$ and $2A_{1/2}(G) = Q(G)$, where $Q(G)$ is the *signless Laplacian matrix*. Moreover, $L(G) = (A_\alpha(G) - A_\beta(G))/(\alpha - \beta)$ if $\alpha \neq \beta$ for any $\alpha, \beta \in [0, 1]$, where $L(G)$ is the *Laplacian matrix*. The A_α-*spectral radius* of G is the largest eigenvalue of $A_\alpha(G)$, and denoted by $\rho_\alpha(G)$. The largest eigenvalue of $A(G)$, denoted by $\rho(G)$, is called the *spectral radius* of G. Obviously, $\rho_\alpha(G) = \rho(G)$ if $\alpha = 0$.

The *join* and *disjoint union* of graphs are denoted by the symbols ∇ and $\cup$, respectively. A *matching* M of G is a subset of $E(G)$ such that any two edges of M have no common vertices. Moreover, if M covers all vertex of G then it is said to be a *perfect matching* or a 1-*factor*. Suppose that G_1 is an empty graph with vertex set $U = \{u_1, u_2, \ldots, u_n\}$,

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and G_2 is a complete graph with vertex set $W = \{v_1, v_2, \ldots, v_n\}$. Let $G(2n, 1)$ be the graph of order $2n$ obtained from $G_1 \cup G_2$ by letting $N_{G_2}(u_i) = \{v_1, v_2, \ldots, v_i\}$ for $1 \leq i \leq n$. Clearly, $G(2n, 1)$ contains a unique perfect matching. More recently, Fan, Lin and Lu [4] determined the graph $G(2n, 1)$ attaining the maximum spectral radius among all graphs of order $2n$ with a unique perfect matching, and obtained the following result.

Theorem 1.1. [4, Theorem 1.1] *If G is a connected graph of order $2n$ with a unique perfect matching, then $\rho(G) \leq \rho(G(2n, 1))$, with equality if and only if $G \cong G(2n, 1)$.*

Let $A_\alpha(G)$ be the A_α*-matrix* of G, and $\rho_\alpha(G)$ be the A_α*-spectral radius* of G. Inspired by the result of Theorem 1.1, we extend this result by giving the graph attaining the maximum A_α-spectral radius among all graphs of order $2n$ with a unique perfect matching, and have the following theorem.

Theorem 1.2. *If G is a connected graph of order $2n$ with a unique perfect matching, then $\rho_\alpha(G) \leq \rho_\alpha(G(2n, 1))$, with equality if and only if $G \cong G(2n, 1)$.*

In 2021, Zhao, Huang and Wang [16] provided a lower bound for the A_α-spectral radius $\rho_\alpha(G)$ which guarantees the existence of a perfect matching in a connected graph G. Let

$$f(\alpha) = \begin{cases} 10 & \text{if } 0 \leq \alpha \leq 1/2, \\ 14 & \text{if } 1/2 < \alpha \leq 2/3, \\ 5/(1-\alpha) & \text{if } 2/3 < \alpha < 1. \end{cases}$$

Theorem 1.3. [16, Theorem 3] *Let $\alpha \in [0, 1)$, and let G be a connected graph of even order n with $n > f(\alpha)$. If $\rho_\alpha(G) \geq \rho_\alpha(K_1 \nabla (K_{n-3} \cup 2K_1))$, then G has a perfect matching unless $G = K_1 \nabla (K_{n-3} \cup 2K_1)$, where $\rho_\alpha(K_1 \nabla (K_{n-3} \cup 2K_1))$ is equal to the largest root of $x^3 - ((\alpha + 1)n + \alpha - 4)x^2 + (\alpha n^2 + (\alpha^2 - 2\alpha - 1)n - 2\alpha + 1)x - \alpha^2 n^2 + (5\alpha^2 - 3\alpha + 2)n - 10\alpha^2 + 15\alpha - 8 = 0$.*

A spanning subgraph H is an $[a, b]$-factor of a graph G if $a \leq d_H(v) \leq b$ for each $v \in V(G)$, where a and b are positive integers. Especially, when $a = b = 1$, a $[1, 1]$-factor of G is also called a perfect matching or a 1-factor of G. Moreover, a spanning subgraph H is a $[1, b]$-odd factor of a graph G if $d_H(v)$ is odd and $1 \leq d_H(v) \leq b$ for each $v \in V(G)$. In this paper, we extend the result of Theorem 1.3 by proving that the lower bound $\rho_\alpha(K_1 \nabla (K_{n-b-2} \cup (b + 1)K_1))$ can guarantee the existence of a $[1, b]$-odd factor where $\alpha \in [0, 1/2]$.

Theorem 1.4. *Let $\alpha \in [0, 1/2]$, and let G be a connected graph of even order n with $n > b + 2 + \alpha + \frac{2(b+1)(b+2-\alpha)^2}{b}$. If $\rho_\alpha(G) \geq \rho_\alpha(K_1 \nabla (K_{n-b-2} \cup (b + 1)K_1))$, then G has a $[1, b]$-odd factor unless $G = K_1 \nabla (K_{n-b-2} \cup (b+1)K_1)$, where $\rho_\alpha(K_1 \nabla (K_{n-b-2} \cup (b+1)K_1))$*

is equal to the largest root of $x^3-((\alpha+1)n-b+\alpha-3)x^2+(\alpha n^2+(\alpha^2-\alpha b-\alpha-1)n-2\alpha+1)x-\alpha^2n^2+(3\alpha^2+2\alpha^2b-\alpha-2\alpha b+b+1)n-\alpha^2b^2-5\alpha^2b-4\alpha^2+2\alpha b^2+8\alpha b+5\alpha-b^2-4b-3=0.$

Motivated by the theorem above, we pose a problem below.

Problem 1.5. Let $\alpha \in (1/2,1)$ and G be a connected graph of even order n. Investigate the lower bound of $\rho_\alpha(G)$ to guarantee the existence of a $[1,b]$-odd factor.

In the past decade, more researchers have presented different conditions for a graph to have an $[a,b]$-factor. Li and Cai [11] gave a degree condition that if $\delta(G) \geq a$, and for any two nonadjacent vertices $u,v \in V(G)$, $\max\{d_G(u), d_G(v)\} \geq \frac{an}{a+b}$, then G has an $[a,b]$-factor. Li [10] provided the neighborhood condition that G has an $[a,b]$-factor if $\delta(G) \geq (k-1)a$, $n \geq \frac{(a+b)(k(a+b)-2)}{b}$ and $|N_G(v_1)\cup N_G(v_2)\cup\cdots\cup N_G(v_k)| \geq \frac{an}{a+b}$ for any independent subset $\{v_1,v_2,\ldots,v_k\}$ of $V(G)$. Kouider and Lonc [8] gave sufficient conditions, which involve the minimum degree, the stability number and the connectivity of a graph. Chen [2] presented some sufficient conditions on the binding number and the minimum degree for a graph to have an $[a,b]$-factor. Cho and Park [3] gave counterexamples of Matsuda's conjecture and proposed the following conjecture about adjacent spectral lower bound for a graph to have an $[a,b]$-factor.

Conjecture 1.6. [3, Conjecture 4.4] *Let $a\cdot n$ be an even integer at least 2, where $n \geq a+1$. If G is a graph of order n with $\rho(G) > \rho(H_{n,a})$ where $H_{n,a} = K_{a-1}\nabla(K_1 \cup K_{n-a})$, then G contains an $[a,b]$-factor.*

Recently, Fan, Lin and Lu [4] confirmed Conjecture 1.6 for $n \geq 3a+b-1$ and gave the following theorem.

Theorem 1.7. [4, Theorem 1.3] *Let a, n be two positive integers such that $a\cdot n$ is even, and let $b \geq a \geq 1$. If G is a graph of order $n \geq 3a+b-1$ with $\rho(G) > \rho(H_{n,a})$, then G contains an $[a,b]$-factor.*

Later, Wei and Zhang [15] have completely proved that Conjecture 1.6 is true. Enlightened by the results above, we give an A_α-spectral condition to ensure that G has an $[a,b]$-factor. Moreover, it also extends the result of Theorem 1.7.

For convenience, suppose that a and b are two positive integers, and set $t = \lceil \frac{an}{a+b} \rceil - 1$. For $1 \leq a \leq 2$, let

$$f_1(\alpha) = 3a+b-1.$$

For $a \geq 3$, let

$$f_1(\alpha) = \begin{cases} \max\left\{3a+b-1, 2t+1+\frac{1+\alpha}{1-\alpha}\right\} & \text{if } 3/4 < \alpha < 1 \text{ and } b > a, \\ 3a+b-1 & \text{otherwise.} \end{cases}$$

In addition, we define

$$\mathscr{H} = \begin{cases} K_t \nabla (K_1 \cup K_{n-t-1}) & \text{if } 3/4 < \alpha < 1 \text{ and } b = a, \\ H_{n,a} & \text{otherwise.} \end{cases}$$

Theorem 1.8. *Let a, n be two positive integers such that $a \cdot n$ is even, and let $b \geq a \geq 1$. If G is a graph of order $n \geq f_1(\alpha)$ with $\rho_\alpha(G) > \rho_\alpha(\mathscr{H})$, then G contains an $[a,b]$-factor where $\alpha \in [0,1)$.*

2. Proof of Theorem 1.2

Firstly, we give some lemmas that will be used in the sequel.

Lemma 2.1. [9, Lemma 2.1], [13] *Let $\alpha \in [0,1)$ and G be a connected graph with $uv_i \in E(G)$ and $wv_i \notin E(G)$ for $i = 1, 2, \ldots, k$. Let $G' = G - \{uv_i\} + \{wv_i\}$ for $i = 1, 2, \ldots, k$ and $\mathbf{x}$ be a unit eigenvector of $A_\alpha(G)$ corresponding to $\rho_\alpha(G)$. If $x_w \geq x_u$, then $\rho_\alpha(G') > \rho_\alpha(G)$.*

An edge uv in graph G is said to be a *cut edge* if $\omega(G - uv) > \omega(G)$, where $\omega(G)$ denotes the number of the components of G.

Lemma 2.2. [7] *Let G be a connected graph with a unique perfect matching. Then G contains a cut edge uv that is an edge of the perfect matching of G.*

Lemma 2.3. [6, Lemma 8.7.2, p. 177] *If M_1 and M_2 are two nonnegative $n \times n$ matrices such that $M_1 - M_2$ is nonnegative, then*

$$\rho(M_1) \geq \rho(M_2),$$

where $\rho(M_i)$ is the spectral radius of M_i for $i = 1, 2$.

Lemma 2.4. [12, Proposition 14] *For $\alpha \in [0,1)$, let G be a graph, and $\mathbf{x}$ a nonnegative eigenvector to $\rho_\alpha(G)$:*

(i) *If G is connected, then $\mathbf{x}$ is positive and is unique up to scaling;*

(ii) *If G is disconnected and P is the set of vertices with positive entries in $\mathbf{x}$, then the subgraph induced by P is a union of components H of G with $\rho_\alpha(H) = \rho_\alpha(G)$;*

(iii) *If G is connected and μ is an eigenvalue of $A_\alpha(G)$ with a nonnegative eigenvector, then $\mu = \rho_\alpha(G)$;*

(iv) *If G is connected, and H is a proper subgraph of G, then $\rho_\alpha(H) < \rho_\alpha(G)$.*

Let $G[S]$ be the subgraph of G induced by S for any $S \subseteq V(G)$. If $d_G(u) \geq 2$ and $d_G(v) = 1$, then uv is called *pendant edge*, where $uv \in E(G)$. From Lemma 2.4, we noticed that $\mathbf{x}$ is positive if G is a connected graph, where $\mathbf{x}$ is a nonnegative eigenvector corresponding to $\rho_\alpha(G)$. In other words, if G is a connected graph, then one can set $\mathbf{x}$ as a positive unit eigenvector of $A_\alpha(G)$ corresponding to $\rho_\alpha(G)$. We now give a proof of Theorem 1.2.

Proof of Theorem 1.2. Assume that G is a connected graph of order $2n$, which has a unique perfect matching M. According to Lemma 2.2, there is a cut edge u_0v_0 in M. Then one can deduce that $G - u_0v_0$ is consisted of two odd components, and the edges of each component in M are unique. Let $\mathbf{x}^{(0)}$ be the positive unit eigenvector of $A_\alpha(G)$ corresponding to $\rho_\alpha(G)$. Without loss of generality, we suppose that $x^{(0)}_{u_0} \geq x^{(0)}_{v_0}$. Let

$$G_1 = G - \{v_0w : w \in N_G(v_0) \setminus \{u_0\}\} + \{u_0w : w \in V(G) \setminus N_G[u_0]\}.$$

Clearly, we can see that G_1 also has a unique perfect matching, say M_1. Let $H = G - \{v_0w : w \in N_G(v_0) \setminus \{u_0\}\} + \{u_0w : w \in N_G(v_0) \setminus \{u_0\}\}$. If $(N_G(v_0) \setminus \{u_0\}) \subseteq (N_G(u_0) \setminus \{v_0\})$, then $H = G$, which implies that $\rho_\alpha(H) \leq \rho_\alpha(G)$. Otherwise, there exists a vertex w such that $w \in N_G(v_0) \setminus \{u_0\}$ and $w \notin N_G(u_0) \setminus \{v_0\}$. It then follows from Lemma 2.1 that $\rho_\alpha(G) < \rho_\alpha(H)$. Consequently, $\rho_\alpha(G) \leq \rho_\alpha(H)$, with equality if and only if $G \cong H$. Meanwhile, by Lemma 2.3 it deduces that $\rho_\alpha(H) \leq \rho_\alpha(G_1)$, with equality if and only if $H \cong G_1$. Thus, one can obtain $\rho_\alpha(G) \leq \rho_\alpha(G_1)$, with equality if and only if $G \cong G_1$.

Let $S_1 = V(G_1) - \{u_0, v_0\}$. Note that u_0v_0 is a pendant edge of G_1 and $u_0v_0 \in M_1$. We have that the induced graph $G_1[S_1]$ also contains a unique perfect matching, i.e., $M_1 \setminus \{u_0v_0\}$. From the definition of $G_1[S_1]$, it is easy to see that each component of $G_1[S_1]$ has a unique perfect matching. Again by Lemma 2.2, there is a cut edge u_1v_1 in some component of $G_1[S_1]$ that is contained in $M_1 \setminus \{u_0v_0\}$. Let $\mathbf{x}^{(1)}$ be the positive unit eigenvector of $A_\alpha(G_1)$ corresponding to $\rho_\alpha(G_1)$. Assume that $x^{(1)}_{u_1} \geq x^{(1)}_{v_1}$. Let

$$G_2 = G_1 - \{v_1w : w \in N_{G_1[S_1]}(v_1) \setminus \{u_1\}\} + \{u_1w : w \in S_1 \setminus N_{G_1[S_1]}[u_1]\}.$$

Clearly, G_2 also has a unique perfect matching. Similar to that before, from Lemmas 2.1 and 2.3, we get $\rho_\alpha(G_1) \leq \rho_\alpha(G_2)$, with equality if and only if $G_1 \cong G_2$.

By repeating this process, one can construct a sequence of graphs $G_0, G_1, G_2, \ldots, G_{n-1}$, which have a unique perfect matching:

(i) $G_0 = G$;

(ii) for $i \in [0, n-2]$, let $S_i = V(G_i) - \{v_0, v_1, \ldots, v_{i-1}, u_0, u_1, \ldots, u_{i-1}\}$ and

$$G_{i+1} = G_i - \{v_iw : w \in N_{G_i[S_i]}(v_i) \setminus \{u_i\}\} + \{u_iw : w \in S_i \setminus N_{G_i[S_i]}[u_i]\},$$

where u_iv_i is a cut edge in some component of $G_i[S_i]$ that is contained in the unique perfect matching of $G_i[S_i]$ and $x^{(i)}_{v_i} \le x^{(i)}_{u_i}$, where $\mathbf{x}^{(i)}$ is the positive unit eigenvector of $A_\alpha(G_i)$ corresponding to $\rho_\alpha(G_i)$.

As mentioned above, we see that G_i has a unique perfect matching for each i, and $\rho_\alpha(G_i) \le \rho_\alpha(G_{i+1})$ with equality if and only if $G_i \cong G_{i+1} (0 \le i \le n-2)$. Note that $G_{n-1} \cong G(2n, 1)$. Hence the proof is completed. □

3. Proof of the Theorem 1.4

In 1985, Amahashi [1] gave a sufficient and necessary condition for the existence of an odd $[1, b]$-factor.

Lemma 3.1. [1, Theorem 2] *Let G be a graph and let b be a positive odd integer. Then G contains a $[1, b]$-odd factor if and only if for every subset $S \subseteq V(G)$,*

$$o(G - S) \le b|S|,$$

where $o(G - S)$ is the number of odd components in a graph $G - S$.

Let $\rho_\alpha(G) = \lambda_1(A_\alpha) \ge \lambda_2(A_\alpha) \ge \cdots \ge \lambda_n(A_\alpha)$ denotes all eigenvalues of $A_\alpha(G)$ for $\alpha \in [0, 1]$. Based on *Rayleigh's principle*, Nikiforov [12] obtained the following conclusion.

Lemma 3.2. [12, Proposition 2] *If $\alpha \in [0, 1]$ and G is a graph of order n, then*

$$\rho_\alpha(G) = \lambda_1(A_\alpha) = \max_{\|\mathbf{x}\|_2=1} \langle A_\alpha \mathbf{x}, \mathbf{x}\rangle \quad \text{and} \quad \lambda_n(A_\alpha) = \min_{\|\mathbf{x}\|_2=1} \langle A_\alpha \mathbf{x}, \mathbf{x}\rangle.$$

Moreover, if $\mathbf{x}$ is a unit n-vector, then $\rho_\alpha(G) = \lambda_1(A_\alpha) = \langle A_\alpha \mathbf{x}, \mathbf{x}\rangle$ if and only if $\mathbf{x}$ is an eigenvector to $\rho_\alpha(G)$, and $\lambda_n(A_\alpha) = \langle A_\alpha \mathbf{x}, \mathbf{x}\rangle$ if and only if $\mathbf{x}$ is an eigenvector to $\lambda_n(A_\alpha)$.

Lemma 3.3. *Let $\alpha \in [0, 1)$ and $n = \sum_{i=1}^{t} n_i + s$. If $n_1 \ge n_2 \ge \cdots \ge n_t \ge p$ and $n_1 < n - s - p(t - 1)$, then*

$$\rho_\alpha(K_s \nabla (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t})) < \rho_\alpha(K_s \nabla (K_{n-s-p(t-1)} \cup (t - 1)K_p)).$$

Proof. Let $G = K_s \nabla (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t})$ and $\mathbf{x}$ be a positive unit eigenvector of $A_\alpha(G)$ corresponding to $\rho_\alpha(G)$. By symmetry, one can suppose that $x_v = x_i$ for all $v \in V(K_{n_i})$, where $1 \le i \le t$, and $x_u = y_1$ for all $u \in V(K_s)$. Then it follows from $A_\alpha(G)\mathbf{x} = \rho_\alpha(G)\mathbf{x}$ that $(\rho_\alpha(G) - ((n_1 - 1) + \alpha s))x_1 = (1 - \alpha)sy_1 > 0$, which gives that $\rho_\alpha(G) > (n_1 - 1) + \alpha s$. Again, for $2 \le j \le t$, one can see that

$$(\rho_\alpha(G) - (n_j - 1) - \alpha s)(x_1 - x_j) = (n_1 - n_j)x_1 \ge 0.$$

Since $\rho_\alpha(G) > n_1 - 1 + \alpha s \geq n_j - 1 + \alpha s$, we have $x_1 \geq x_j$ for $2 \leq j \leq t$. Let $G' = K_s \nabla (K_{n-s-p(t-1)} \cup (t-1)K_p)$. By Lemma 3.2 and numbering the vertices of G' properly, we can get

$$\begin{aligned}
&\rho_\alpha(G') - \rho_\alpha(G) \\
&\geq x^T (A_\alpha(G') - A_\alpha(G))x \quad \text{(by Lemma 3.2)} \\
&= (1-\alpha)\left(\sum_{i=2}^{t} (n_i - p)x_i(n_1 x_1 - p x_i) + \sum_{i=2}^{t} (n_i - p)x_i \left(n_1 x_1 + \sum_{j=2}^{t} (n_j - p)x_j - n_i x_i \right) \right) \\
&\quad + \alpha \left(\sum_{i=2}^{t} (n_i - p)(n_1 x_1^2 - p x_i^2) + \sum_{i=2}^{t} (n_i - p)(n - p(t-1) - n_i - s)x_i^2 \right) > 0
\end{aligned}$$

and so, the result follows. □

Let H be a $[1,b]$-odd factor of a graph G. Then by the definition of $[1,b]$-odd factor, for each $v \in V(G)$, $1 \leq d_H(v) \leq b-1 \leq b$ if b is an even number. Thus, one can also call that $[1,b']$-odd factor is a $[1,b]$-odd factor of G, where $b' = b-1$ is an odd. Therefore, we always take b as an odd number to consider $[1,b]$-odd factor. Now we give a proof of Theorem 1.4.

Proof of Theorem 1.4. We here discuss two cases in the following.

If $b = 1$, then $n > b + 2 + \alpha + \frac{2(b+1)(b+2-\alpha)^2}{b} = 3 + \alpha + 4(3-\alpha)^2 > 10$, moreover, $\rho_\alpha(G) \geq \rho_\alpha(K_1 \nabla (K_{n-b-2} \cup (b+1)K_1)) = \rho_\alpha(K_1 \nabla (K_{n-3} \cup 2K_1))$. Thus, it follows from Theorem 1.3 that G has a $[1,1]$-factor unless $G = K_1 \nabla (K_{n-3} \cup 2K_1)$, it therefore means that if $n > b + 2 + \alpha + \frac{2(b+1)(b+2-\alpha)^2}{b}$ and $\rho_\alpha(G) \geq \rho_\alpha(K_1 \nabla (K_{n-b-2} \cup (b+1)K_1))$, then G also has a $[1,b]$-factor unless $G = K_1 \nabla (K_{n-b-2} \cup (b+1)K_1)$.

If $b > 1$, we assume, by a contradiction, that G contains no $[1,b]$-odd factor. Then by Lemma 3.1, there exists some nonempty subset S of $V(G)$ such that $q = o(G-S) > b|S|$. Let $|S| = s$. We assert that q and bs have the same parity. If s is an odd, then $n-s$ and bs are odd numbers since n is even and b is odd. As $n-s$ is odd, the number of odd components in $G-S$ must be odd, i.e., q is also odd. If not, q is even, then the number of vertices of all odd components in $G-S$ is also even. And together with the number of vertices of all even components in $G-S$, we have $n-s$ is even, a contradiction. So, q and bs are odd numbers. Similarly, one can prove that q and bs are even numbers if s is an even. Thus, q and bs have the same parity. Hence $q \geq bs+2$. To promote the proof, we first prove the following claims.

Claim 3.4. G is a spanning subgraph of $G_1 = K_s \nabla (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})$ for some positive odd integers $n_1 \geq n_2 \geq \cdots \geq n_q$ with $\sum_{i=1}^{q} n_i = n - s$.

Proof. Note that $G - S$ is consisted of $q \geq 1$ odd components and $k \geq 0$ (say) even components. We write X_i $(1 \leq i \leq q)$ for the odd components, Y_j $(0 \leq j \leq k-1)$ for the even components in $G - S$, where $|V(X_1)| \geq |V(X_2)| \geq \cdots \geq |V(X_q)|$. To obtain some positive odd integers $n_1 \geq n_2 \geq \cdots \geq n_q$ such that $\sum_{i=1}^{q} n_i = n - s$, we consider $X_i \nabla Y_j$ for some $1 \leq i \leq q$ and $0 \leq j \leq k-1$. Clearly, $|V(X_i \nabla Y_j)|$ is an odd number. Without loss of generality, let us join all even components of $G - S$ to X_1, i.e., $X_1 \nabla (Y_0 \cup Y_1 \cdots \cup Y_{k-1})$. We can see that $|V(X_1 \nabla (Y_0 \cup Y_1 \cdots \cup Y_{k-1}))|$ is also an odd number and $X_1 \nabla (Y_0 \cup Y_1 \cdots \cup Y_{k-1})$ must be a spanning subgraph of K_{n_1} for some odd integer $n_1 = |V(X_1 \nabla (Y_0 \cup Y_1 \cdots \cup Y_{k-1}))|$. Meanwhile, X_i $(2 \leq i \leq q)$ must be a spanning subgraph of K_{n_i} for some odd integers n_i $(2 \leq i \leq q)$, respectively, where $n_i = |V(X_i)|$. Recall that $|S| = s$ and $G[S]$ is a spanning subgraph of K_s. Thus, G is a spanning subgraph of $G_1 = K_s \nabla (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})$ for some positive odd integers $n_1 \geq n_2 \geq \cdots \geq n_q$ with $\sum_{i=1}^{q} n_i = n - s$. $\square$

In addition, it deduces from Lemma 2.4(iv) that $\rho_\alpha(G) \leq \rho_\alpha(G_1)$, where the equality holds if and only if $G \cong G_1$.

Claim 3.5. For $\alpha \in [0, 1)$, we have

$$\rho_\alpha(K_s \nabla (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})) \leq \rho_\alpha(K_s \nabla (K_{n-s-q+1} \cup (q-1)K_1)),$$

where the equality holds if and only if $(n_1, n_2, \ldots, n_q) = (n - s - q + 1, 1, \ldots, 1)$.

Proof. If $(n_1, n_2, \ldots, n_q) = (n - s - q + 1, 1, \ldots, 1)$, then $K_s \nabla (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q}) = K_s \nabla (K_{n-s-q+1} \cup (q-1)K_1)$. Hence $\rho_\alpha(K_s \nabla (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})) = \rho_\alpha(K_s \nabla (K_{n-s-q+1} \cup (q-1)K_1))$.

If $(n_1, n_2, \ldots, n_q) \neq (n-s-q+1, 1, \ldots, 1)$, it follows from Lemma 3.3 that $\rho_\alpha(K_s \nabla (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})) < \rho_\alpha(K_s \nabla (K_{n-s-q+1} \cup (q-1)K_1))$. So, this proves Claim 3.5. $\square$

Claim 3.6. For $\alpha \in [0, 1)$, we have

$$\rho_\alpha(K_s \nabla (K_{n-s-q+1} \cup (q-1)K_1)) \leq \rho_\alpha(K_s \nabla (K_{n-s-bs-1} \cup (bs+1)K_1)),$$

where the equality holds if and only if $q = bs + 2$.

Proof. If $q = bs + 2$, then $K_s \nabla (K_{n-s-q+1} \cup (q-1)K_1) = K_s \nabla (K_{n-s-bs-1} \cup (bs+1)K_1)$. So, we have $\rho_\alpha(K_s \nabla (K_{n-s-q+1} \cup (q-1)K_1)) = \rho_\alpha(K_s \nabla (K_{n-s-bs-1} \cup (bs+1)K_1))$.

If $q \geq bs + 4$, then $K_s \nabla (K_{n-s-q+1} \cup (q-1)K_1)$ is a subgraph of $K_s \nabla (K_{n-s-bs-1} \cup (bs+1)K_1)$. Thus, by Lemma 2.3 it deduces that $\rho_\alpha(K_s \nabla (K_{n-s-q+1} \cup (q-1)K_1)) \leq \rho_\alpha(K_s \nabla (K_{n-s-bs-1} \cup (bs+1)K_1))$. Now, we prove the inequation is strict, that is,

$$\rho_\alpha(K_s \nabla (K_{n-s-q+1} \cup (q-1)K_1)) < \rho_\alpha(K_s \nabla (K_{n-s-bs-1} \cup (bs+1)K_1)).$$

Let $G_2 = K_s\nabla(K_{n-s-q+1} \cup (q-1)K_1)$ and $G_3 = K_s\nabla(K_{n-s-bs-1} \cup (bs+1)K_1)$. Clearly, $V(G_3)$ can be partitioned as $V(G_3) = V(K_s) \cup V(K_{n-s-bs-1}) \cup V((bs+1)K_1)$, where $V(K_s) = \{u_1, u_2, \ldots, u_s\}$, $V((bs+1)K_1) = \{v_1, v_2, \ldots, v_{bs+1}\}$ and $V(K_{n-s-bs-1}) = \{w_1, w_2, \ldots, w_{n-s-q+1}, w_{n-s-q+2}, \ldots, w_{n-s-bs-1}\}$. In addition, we write $E_1 = \{w_iw_j \mid 1 \le i \le n-s-q+1, n-s-q+2 \le j \le n-s-bs-1\} \cup \{w_iw_k \mid n-s-q+2 \le i \le n-s-bs-2, i+1 \le k \le n-s-bs-1\}$. Obviously, $G_2 \cong G_3 - E_1$.

Let $\mathbf{x}$ (resp. $\mathbf{y}$) be the positive unit eigenvectors of $A_\alpha(G_3)$ (resp. $A_\alpha(G_2)$) corresponding to $\rho_\alpha(G_3)$ (resp. $\rho_\alpha(G_2)$). By symmetry, $\mathbf{x}$ takes the same value on the vertices of $V(K_s)$, $V(K_{n-s-bs-1})$ and $V((bs+1)K_1)$, respectively, say x_1, x_2 and x_3. Similarly, $\mathbf{y}$ takes the same value on the vertices of $V(K_s)$, $V(K_{n-s-q+1})$ and $V((q-1)K_1)$, respectively, say y_1, y_2 and y_3. Then, it follows from $A_\alpha(G_2)\mathbf{y} = \rho_\alpha(G_2)\mathbf{y}$ and $A_\alpha(G_3)\mathbf{x} = \rho_\alpha(G_3)\mathbf{x}$ that

$$\begin{aligned}
&\mathbf{x}^T(\rho_\alpha(G_3) - \rho_\alpha(G_2))\mathbf{y} \\
&= \mathbf{x}^T(A_\alpha(G_3) - A_\alpha(G_2))\mathbf{y} \\
&= \alpha\Bigg(\sum_{i=n-s-q+2}^{n-s-bs-1}(n-bs-s-2)x_{w_i}y_{w_i} + \sum_{i=1}^{n-s-q+1}(q-bs-2)x_{w_i}y_{w_i}\Bigg) \\
&\quad + (1-\alpha)\Bigg(\sum_{i=1}^{n-s-q+1}\sum_{j=n-s-q+2}^{n-s-bs-1}(x_{w_i}y_{w_j} + y_{w_i}x_{w_j}) + \sum_{i=n-s-q+2}^{n-s-bs-2}\sum_{k=i+1}^{n-s-bs-1}x_{w_i}y_{w_k}\Bigg) \\
&= \alpha((q-bs-2)(n-bs-s-2)x_2y_3 + (n-s-q+1)(q-bs-2)x_2y_2) \\
&\quad + (1-\alpha)((q-bs-2)(n-s-q+1)(x_2y_2 + y_3x_2) + (q-bs-2)(q-bs-3)x_2y_3) \\
&> 0 \quad (\text{since } q \ge bs+4).
\end{aligned}$$

Thus, $\rho_\alpha(K_s\nabla(K_{n-s-q+1} \cup (q-1)K_1)) < \rho_\alpha(K_s\nabla(K_{n-s-bs-1} \cup (bs+1)K_1))$. □

Claim 3.7. For $\alpha \in [0, 1/2]$, if $n > b+2+\alpha+\frac{2(b+1)(b+2-\alpha)^2}{b}$ and $b > 1$, then we have

$$\rho_\alpha(K_s\nabla(K_{n-s-bs-1} \cup (bs+1)K_1)) \le \rho_\alpha(K_1\nabla(K_{n-b-2} \cup (b+1)K_1))$$

with equality holding if and only if $s = 1$.

Proof. If $s = 1$, then $K_s\nabla(K_{n-s-bs-1} \cup (bs+1)K_1) = K_1\nabla(K_{n-b-2} \cup (b+1)K_1)$. So we have $\rho_\alpha(K_s\nabla(K_{n-s-bs-1} \cup (bs+1)K_1)) = \rho_\alpha(K_1\nabla(K_{n-b-2} \cup (b+1)K_1))$.

If $s \ge 2$, we should verify that $\rho_\alpha(K_s\nabla(K_{n-s-bs-1} \cup (bs+1)K_1)) < \rho_\alpha(K_1\nabla(K_{n-b-2} \cup (b+1)K_1))$. Let $G_4 = K_1\nabla(K_{n-b-2} \cup (b+1)K_1)$. Obviously, $G_4 \cong G_3 - \{u_iv_j \mid 2 \le i \le s, 1 \le j \le b+1\} + \{v_iv_j \mid b+2 \le i, j \le bs+1, i \ne j\} + \{w_iv_j \mid 1 \le i \le n-s-bs-1, b+2 \le j \le bs+1\}$.

Let $\mathbf{z}$ be the positive unit eigenvector of $A_\alpha(G_4)$ corresponding to $\rho_\alpha(G_4)$. By symmetry, $\mathbf{z}$ takes the same value on the vertices of $V(K_1)$, $V(K_{n-b-2})$ and $(b+1)V(K_1)$, respectively, say z_1, z_2 and z_3. Recall that $\mathbf{x}$ is the positive unit eigenvector of $A_\alpha(G_3)$ corresponding to $\rho_\alpha(G_3)$ and takes the same value on the vertices of $V(K_s)$, $V(K_{n-s-bs-1})$ and $V((bs+1)K_1)$, respectively, say x_1, x_2 and x_3. Then, from $A_\alpha(G_4)\mathbf{z} = \rho_\alpha(G_4)\mathbf{z}$ and $A_\alpha(G_3)\mathbf{x} = \rho_\alpha(G_3)\mathbf{x}$ it follows that

$$\rho_\alpha(G_4)z_2 = \alpha(n-b-2)z_2 + (1-\alpha)(z_1 + (n-b-3)z_2), \tag{3.1}$$

$$\rho_\alpha(G_4)z_3 = \alpha z_3 + (1-\alpha)z_1, \tag{3.2}$$

$$\rho_\alpha(G_3)x_2 = \alpha(n-bs-2)x_2 + (1-\alpha)(sx_1 + (n-bs-s-2)x_2), \tag{3.3}$$

$$\rho_\alpha(G_3)x_3 = \alpha s x_3 + (1-\alpha)sx_1. \tag{3.4}$$

From (3.1) and (3.2) we have

$$z_3 = \frac{\rho_\alpha(G_4) - \alpha - (n-b-3)}{\rho_\alpha(G_4) - \alpha} z_2. \tag{3.5}$$

From (3.3) and (3.4) we have

$$x_2 = \frac{(1-\alpha)s}{\rho_\alpha(G_3) - \alpha s - (n-bs-s-2)} x_1 \tag{3.6}$$

and

$$x_3 = \frac{(1-\alpha)s}{\rho_\alpha(G_3) - \alpha s} x_1. \tag{3.7}$$

Note that $K_1 \cup K_{n-b-2} \cup (b+1)K_1$ is a spanning subgraph of $K_1 \nabla (K_{n-b-2} \cup (b+1)K_1)$, meanwhile, both G_3 and G_4 are the proper subgraphs of K_n. It is easy to see that $\rho_\alpha(G_3) < n-1$ and $n-b-2 < \rho_\alpha(G_4) < n-1$. Together with $A_\alpha(G_3)\mathbf{x} = \rho_\alpha(G_3)\mathbf{x}$ and $A_\alpha(G_4)\mathbf{z} = \rho_\alpha(G_4)\mathbf{z}$ we get

$$\begin{aligned}
&\mathbf{z}^T(\rho_\alpha(G_4) - \rho_\alpha(G_3))\mathbf{x} \\
&= \mathbf{z}^T(A_\alpha(G_4) - A_\alpha(G_3))\mathbf{x} \\
&= \alpha\big(-(b+1)(s-1)z_2x_1 + b(s-1)(n-bs-s-1)z_2x_2 - (b+1)(s-1)z_3x_3 \\
&\qquad + b(s-1)(n-b-s-2)z_2x_3\big) \\
&\quad + (1-\alpha)\big(-(b+1)(s-1)z_3x_1 + b(s-1)(n-bs-s-1)z_2x_2 \\
&\qquad - (b+1)(s-1)z_2x_3 + b(s-1)(n-b-s-2)z_2x_3\big) \\
&= (s-1)\alpha\Bigg(-(b+1)z_2x_1 + b(n-bs-s-1)z_2\frac{(1-\alpha)s}{\rho_\alpha(G_3) - \alpha s - (n-bs-s-2)}x_1 \\
&\qquad - (b+1)\frac{\rho_\alpha(G_4) - \alpha - (n-b-3)}{\rho_\alpha(G_4) - \alpha}z_2\frac{(1-\alpha)s}{\rho_\alpha(G_3) - \alpha s}x_1 \\
&\qquad + b(n-b-s-2)z_2\frac{(1-\alpha)s}{\rho_\alpha(G_3) - \alpha s}x_1\Bigg) \\
&\quad + (s-1)(1-\alpha)\Bigg(-(b+1)\frac{\rho_\alpha(G_4) - \alpha - (n-b-3)}{\rho_\alpha(G_4) - \alpha}z_2x_1
\end{aligned}$$

$$
\begin{aligned}
&+ b(n-bs-s-1)z_2\frac{(1-\alpha)s}{\rho_\alpha(G_3)-\alpha s-(n-bs-s-2)}x_1-(b+1)z_2\frac{(1-\alpha)s}{\rho_\alpha(G_3)-\alpha s}x_1\\
&+ b(n-b-s-2)z_2\frac{(1-\alpha)s}{\rho_\alpha(G_3)-\alpha s}x_1\Big) \quad \text{(from (3.5), (3.6) and (3.7))}\\
=&\ \frac{(s-1)z_2x_1}{(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))}\\
&\times \alpha\Big(-(b+1)(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\\
&+ b(n-bs-s-1)(1-\alpha)s(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)\\
&+ b(n-b-s-2)(1-\alpha)s(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\\
&-(b+1)(\rho_\alpha(G_4)-\alpha-(n-b-3))(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))(1-\alpha)s\Big)\\
&+\frac{(s-1)z_2x_1}{(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))}(1-\alpha)\\
&\quad\times\Big(-(b+1)(\rho_\alpha(G_4)-\alpha-(n-b-3))(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\\
&\quad+ b(n-bs-s-1)(1-\alpha)s(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)\\
&\quad+ b(n-b-s-2)(1-\alpha)s(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\\
&\quad-(b+1)(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))(1-\alpha)s\Big)\\
=&\ \frac{(s-1)z_2x_1}{(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))}\\
&\times\Big(-(b+1)(\rho_\alpha(G_4)-\alpha-(n-b-3))(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\\
&+\alpha(b+1)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))(-(n-b-3))\\
&+ b(n-bs-s-1)(1-\alpha)s(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)\\
&-(b+1)(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))(1-\alpha)s\\
&+\alpha(b+1)(n-b-3)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))(1-\alpha)s\\
&+ b(n-b-s-2)(1-\alpha)s(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\Big)\\
=&\ \frac{(s-1)z_2x_1}{(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))}\\
&\times\Big(\alpha(b+1)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))(n-b-3)\big((1-\alpha)s-(\rho_\alpha(G_3)-\alpha s)\big)\\
&+ b(n-bs-s-1)(1-\alpha)s(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)\\
&-(b+1)(1-\alpha)s(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\\
&+ b(n-b-s-2)(1-\alpha)s(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\\
&-(b+1)(\rho_\alpha(G_4)-\alpha-(n-b-3))(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\Big)\\
=&\ \frac{(s-1)z_2x_1}{(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))}\\
&\times\Big(-\alpha(b+1)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))(n-b-3)(\rho_\alpha(G_3)-s)\\
&+ b(n-bs-s-1)(1-\alpha)s(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (b(n-b-s-2)-(b+1))(1-\alpha)s(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2)) \\
&\quad - (b+1)(\rho_\alpha(G_4)-\alpha-(n-b-3))(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\Big) \\
&= \frac{(s-1)z_2x_1}{(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))} \\
&\quad \times \Big((\rho_\alpha(G_3)-\alpha s-(n-bs-s-2)) \\
&\quad \times \big((1-\alpha)s(\rho_\alpha(G_4)-\alpha)(b(n-b-s-2)-(b+1))-\alpha(b+1)(n-b-3)(\rho_\alpha(G_3)-s)\big) \\
&\quad + (\rho_\alpha(G_3)-\alpha s)\big((1-\alpha)bs(n-bs-s-1)(\rho_\alpha(G_4)-\alpha) \\
&\qquad - (b+1)(\rho_\alpha(G_4)-\alpha-(n-b-3))(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\big)\Big) \\
&> \frac{(s-1)z_2x_1}{(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))} \\
&\quad \times \Bigg((\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\frac{n-b-3}{2} \\
&\quad \times \big(s(b(n-b-s-2)-(b+1))-(b+1)(\rho_\alpha(G_3)-s)\big) \\
&\quad + (\rho_\alpha(G_3)-\alpha s)\Big(\frac{1}{2}bs(n-bs-s-1)(\rho_\alpha(G_4)-\alpha) \\
&\qquad - (b+1)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))(\rho_\alpha(G_4)-\alpha-(n-b-3))\Big)\Bigg) \\
&\qquad\quad (\text{since } \alpha\in[0,1/2] \text{ and } \rho_\alpha(G_4)>n-b-2) \\
&> \frac{(s-1)z_2x_1}{(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))} \\
&\quad \times \Bigg((\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\frac{n-b-3}{2} \\
&\quad \times \big(2(b(n-b-s-2)-(b+1))-(b+1)(n-1-s)\big) \\
&\quad + (\rho_\alpha(G_3)-\alpha s)\Big(\frac{1}{2}bs(n-b-2-\alpha)-(b+1)(bs+2s-\alpha s)(b+2-\alpha)\Big)\Bigg) \\
&\qquad\quad (\text{since } s\ge 2, \rho_\alpha(G_3)<n-1 \text{ and } n-b-2<\rho_\alpha(G_4)<n-1) \\
&= \frac{(s-1)z_2x_1}{(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))} \\
&\quad \times \Bigg((\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\frac{n-b-3}{2} \\
&\quad \times \big((b-1)n-(2b(s+b+3)+2-(s+1)(b+1))\big) \\
&\quad + s(\rho_\alpha(G_3)-\alpha s)\Big(\frac{1}{2}b(n-b-2-\alpha)-(b+1)(b+2-\alpha)(b+2-\alpha)\Big)\Bigg) \\
&> \frac{(s-1)z_2x_1}{(\rho_\alpha(G_4)-\alpha)(\rho_\alpha(G_3)-\alpha s)(\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))} \\
&\quad \times \Bigg((\rho_\alpha(G_3)-\alpha s-(n-bs-s-2))\frac{n-b-3}{2} \\
&\quad \times \big((b-1)n-(2b(s+b+3)+2-2b(s+1))\big) \\
&\quad + s(\rho_\alpha(G_3)-\alpha s)\Big(\frac{1}{2}bn-\frac{1}{2}b(b+2+\alpha)-(b+1)(b+2-\alpha)(b+2-\alpha)\Big)\Bigg) \quad (\text{since } b>1)
\end{aligned}
$$

$$> 0 \quad \left(\text{since } n > b+2+\alpha+\frac{2(b+1)(b+2-\alpha)^2}{b}\right). \qquad \square$$

Claim 3.8. For any $b \geq 1$, we have $K_1\nabla(K_{n-b-2}\cup(b+1)K_1)$ contains no $[1,b]$-odd factor.

Proof. Let $V_1 = V(K_1)$, $V_2 = V(K_{n-b-2})$ and $V_3 = V((b+1)K_1)$. Taking $S = V_1$ we have $o(G-S) = b+2 > b|S| = b$. Thus, by Lemma 3.1 it follows that $K_1\nabla(K_{n-b-2}\cup(b+1)K_1)$ contains no $[1,b]$-odd factor. $\square$

Combining Claims 3.5, 3.6, 3.7 and 3.8, the proof is therefore completed. $\square$

4. Proof of Theorem 1.8

In this section, we firstly present some preliminaries, and then give a proof of Theorem 1.8.

Lemma 4.1. [11, Theorem 5] *Let G be a graph of order $n \geq 2a+b+\frac{a^2-a}{b}$ with minimum degree $\delta(G) \geq a$, and a, b be two integers such that $1 \leq a < b$. If*

$$\max\{d_G(u), d_G(w)\} \geq \frac{an}{a+b}$$

for any two nonadjacent vertices u and w of G, then G contains an $[a,b]$-factor.

A spanning subgraph H is an *$[a,b]$-factor* of a graph G, if $a \leq d_H(v) \leq b$ for each $v \in V(G)$, where a and b are positive integers. Especially, if $a = b = k$, then $[a,b]$-factor is also called a k-factor.

Lemma 4.2. [14] *Suppose $k \geq 3$. Let G be a connected graph of order $n \geq 4k-3$ with minimum degree $\delta(G)$ where $k\cdot n$ is even and $\delta(G) \geq k$. If*

$$\max\{d_G(u), d_G(w)\} \geq \frac{n}{2}$$

for any two nonadjacent vertices u and w of G, then G contains an k-factor.

Lemma 4.3. *Let G be a connected graph of order n and let u, w be two nonadjacent vertices of G. If $1 \leq \max\{d_G(u), d_G(w)\} \leq t$, then $\rho_\alpha(G) \leq \rho_\alpha(K_t\nabla(2K_1\cup K_{n-t-2}))$, with equality if and only if $G \cong K_t\nabla(2K_1\cup K_{n-t-2})$.*

Proof. Let $\mathbf{x}$ be a positive unit eigenvector of $A_\alpha(G)$ corresponding to $\rho_\alpha(G)$. By numbering the vertices in $V(G)\setminus\{u,w\}$ appropriately, we may assume that $V(G)\setminus\{u,w\} = \{v_1, v_2, \ldots, v_{n-2}\}$ with $x_{v_1} \geq x_{v_2} \geq \cdots \geq x_{v_{n-2}}$. Let

$$G' = G - \{uv \mid v \in N_G(u)\} - \{wv \mid v \in N_G(w)\} + \{uv_i, wv_i \mid 1 \leq i \leq t\}.$$

We note that u and w are two nonadjacent vertices of G such that $1 \leq \max\{d_G(u), d_G(w)\}$ $\leq t$, which implies that $t \leq n-2$. Thus, by Lemma 2.1 it follows that $\rho_\alpha(G) \leq \rho_\alpha(G')$, with equality holds if and only if $G \cong G'$. On the other hand, since G' is a spanning graph of $K_t\nabla(2K_1 \cup K_{n-t-2})$, from Lemma 2.4(iv) we have $\rho_\alpha(G') \le \rho_\alpha(K_t\nabla(2K_1 \cup K_{n-t-2}))$, the equality holds if and only if $G' \cong K_t\nabla(2K_1 \cup K_{n-t-2})$. Therefore, combining with above one can get that $\rho_\alpha(G) \le \rho_\alpha(K_t\nabla(2K_1 \cup K_{n-t-2}))$, with equality if and only if $G \cong G' \cong K_t\nabla(2K_1 \cup K_{n-t-2})$, i.e., $G \cong K_t\nabla(2K_1 \cup K_{n-t-2})$. □

Lemma 4.4. [12, Proposition 4] *Let $1 \ge \alpha > \beta \ge 0$. If G is a graph of order n with $A_\alpha(G) = A_\alpha$ and $A_\beta(G) = A_\beta$, then*

$$\lambda_k(A_\alpha) - \lambda_k(A_\beta) \ge 0$$

for any $1 \le k \le n$. If G is connected, then the inequality is strict, unless $k = 1$ and G is regular.

As usual, we write $K_{n-1} + v$ for $K_{n-1} \cup v$, and $K_{n-1} + e$ for the complete graph of order $n-1$ with a pendent edge.

Lemma 4.5. [5, Theorem 2] *Let G be a graph of order n and spectral radius $\rho(G)$. If*

$$\rho(G) \ge n-2,$$

then G contains a Hamiltonian path unless $G = K_{n-1} + v \cong H_{n,1}$. If the inequality is strict, then G contains a Hamiltonian cycle unless $G = K_{n-1} + e \cong H_{n,2}$.

Now we give a proof of Theorem 1.8 in the following.

Proof of Theorem 1.8. Let G be a graph satisfying the assumption of Theorem 1.8. We assert that G is connected. If not, we may assume that $G_1, G_2, \ldots, G_l$ ($l \ge 2$) are the components of G. Then $\rho_\alpha(G) = \max\{\rho_\alpha(G_1), \rho_\alpha(G_2), \ldots, \rho_\alpha(G_l)\} \le \rho_\alpha(K_{n-1}) = n-2$, a contradiction. Moreover, we declare that $\delta(G) \ge a$. If $1 \le \delta(G) \le a-1$, then together with $\delta(H_{n,a}) = a$ and the structure of $H_{n,a}$ one can see that G is a spanning subgraph of $H_{n,a}$, where $a \ge 2$. Hence $\rho_\alpha(G) \le \rho_\alpha(H_{n,a})$, it is a contradiction.

Case 1*:* $1 \le a \le 2$. Let $0 \le \alpha < 1$. Then by Lemma 4.4, $\rho_\alpha(G) = \lambda_1(A_\alpha) \ge \lambda_1(A_0) = \rho(G)$. Note that $\rho_\alpha(H_{n,2}) \ge \rho_\alpha(H_{n,1}) = n-2$. So, one can deduce that $\rho_\alpha(G) > \rho_\alpha(H_{n,1}) \ge \rho(H_{n,1}) = n-2$ for $a = 1$, and $\rho_\alpha(G) > \rho_\alpha(H_{n,2}) \ge \rho(H_{n,2}) \ge n-2$ for $a = 2$. On the other hand, from Lemma 4.5 we know that G contains a Hamiltonian path for $a = 1$ and a Hamiltonian cycle for $a = 2$. Thus, if $\rho_\alpha(G) > \rho_\alpha(H_{n,1})$, then G contains a 1-factor, and if $\rho_\alpha(G) > \rho_\alpha(H_{n,2})$, then G contains a 2-factor.

Case 2*:* $a \ge 3$. Assume by a contradiction, that G is a graph of order $n \ge f_1(\alpha)$ which contains no $[a,b]$-factor. Since $3a+b-1-(2a+b+\frac{a^2-a}{b}) = a-\left(1+\frac{a^2-a}{b}\right) = \frac{(a-1)(b-a)}{b} \ge 0$ and $3a+b-1-(4k-3) = 2 \ge 0$, we have $n \ge 3a+b-1 \ge 2a+b+\frac{a^2-a}{b}$ and $n \ge 3a+b-1 \ge 4k-3$. So, by Lemmas 4.1 and 4.2, there are two nonadjacent vertices

u and w such that $\max\{d_G(u), d_G(w)\} \leq \lceil \frac{an}{a+b} \rceil - 1 \leq \lceil \frac{n}{2} \rceil - 1$. Let $t = \lceil \frac{an}{a+b} \rceil - 1$. Then $t \geq \max\{d_G(u), d_G(w)\} \geq \delta(G) \geq a \geq 3$ and $n \geq 2t+1$ (since $t = \lceil \frac{an}{a+b} \rceil - 1 \leq \lceil \frac{n}{2} \rceil - 1 < (\frac{n}{2}+1)-1$). Thus, one can deduce that $\rho_\alpha(G) \leq \rho_\alpha(K_t \nabla (2K_1 \cup K_{n-t-2}))$ from Lemma 4.3.

In order to prove $\rho_\alpha(K_t \nabla (2K_1 \cup K_{n-t-2})) < \rho_\alpha(H_{n,a})$, we now discuss two subcases as follows.

Subcase 2.1*: $0 \leq \alpha \leq 3/4$ and $b \geq a$ or $3/4 < \alpha < 1$ and $b > a$.*

We first give a claim, which can be used in subsequent proof.

Claim 4.6. If $t \geq a \geq 3$ and $n \geq f_1(\alpha)$, then $(n-t-2)(n-2-\alpha(a-1))(1-\alpha)t - (t-a+1)(2\alpha+(1-\alpha)(t+2))(\alpha+(1-\alpha)a) > 0$.

Proof. If $0 \leq \alpha \leq 3/4$ and $b \geq a$, then $(n-t-2)-(t-a+1) = n-2t-1+a-2 \geq 1$ and

$$\begin{aligned} &n-2-\alpha(a-1)-2\alpha-(1-\alpha)(t+2) \\ &= n-t-4+\alpha(t-a)+\alpha \geq 2t+1-t-4+\alpha(t-a)+\alpha \geq \alpha \end{aligned}$$

by $n \geq 2t+1$ and $t \geq a \geq 3$. Thus, we have

$$\begin{aligned} &(n-t-2)(n-2-\alpha(a-1))(1-\alpha)t - (t-a+1)(2\alpha+(1-\alpha)(t+2))(\alpha+(1-\alpha)a) \\ &\geq (t-a+1+1)(\alpha+2\alpha+(1-\alpha)(t+2))(1-\alpha)t \\ &\quad - (t-a+1)(2\alpha+(1-\alpha)(t+2))(\alpha+(1-\alpha)a) \\ &= (t-a+1+1)\alpha(1-\alpha)t + (t-a+1+1)(2\alpha+(1-\alpha)(t+2))(1-\alpha)t \\ &\quad - (t-a+1)(2\alpha+(1-\alpha)(t+2))\alpha - (t-a+1)(2\alpha+(1-\alpha)(t+2))(1-\alpha)a \\ &\geq (t-a+1+1)\alpha(1-\alpha)t + (2\alpha+(1-\alpha)(t+2))(1-\alpha)t \\ &\quad - (t-a+1)(2\alpha+(1-\alpha)(t+2))\alpha. \end{aligned}$$

Set $h(\alpha) = (t-a+1+1)\alpha(1-\alpha)t + (2\alpha+(1-\alpha)(t+2))(1-\alpha)t - (t-a+1)(2\alpha+(1-\alpha)(t+2))\alpha$, i.e., $h(\alpha) = (t^2-t)\alpha^2 + (-2t^2-3t+2a-2)\alpha + t^2+2t$. Through a simple calculation, one can see that $h(\alpha)$ is opening up, and its symmetric axis is

$$\begin{aligned} \alpha &= \frac{2t^2+t+2(t-a+1)}{2(t^2-t)} \geq \frac{2t^2+t+2}{2(t^2-t)} \quad \text{(since } t \geq a) \\ &= \frac{2t^2-2t+3t+2}{2(t^2-t)} = 1 + \frac{3t+2}{2(t^2-t)} > 1. \end{aligned}$$

Hence, $h(\alpha)$ is monotonically decreasing in $[0,1)$.

Note that

$$\begin{aligned} h\left(\frac{3}{4}\right) &= (t^2-t)\left(\frac{3}{4}\right)^2 + (-2t^2-3t+2a-2)\left(\frac{3}{4}\right) + t^2+2t \\ &= \frac{1}{16}t^2 - \frac{13}{16}t + \frac{3}{2}(a-1) \geq \frac{1}{16}\left(t-\frac{13}{2}\right)^2 - \frac{1}{16}\frac{13^2}{4} + 3 \quad \text{(since } a \geq 3) \end{aligned}$$

$$> 0.$$

Thus, we can deduce that if $\alpha \in [0, 3/4]$, then $(n-t-2)(n-2-\alpha(a-1))(1-\alpha)t - (t-a+1)(2\alpha+(1-\alpha)(t+2))(\alpha+(1-\alpha)a) > 0$.

If $3/4 < \alpha < 1$ and $b > a$, then $(n-t-2)-(t-a+1) = n-2t-1+a-2 \geq 1+\frac{1+\alpha}{1-\alpha}$ and

$$\begin{aligned}
&n-2-\alpha(a-1)-2\alpha-(1-\alpha)(t+2)\\
&= n-t-4+\alpha(t-a)+\alpha \geq 2t+1+\frac{1+\alpha}{1-\alpha}-t-4+\alpha(t-a)+\alpha\\
&\geq \frac{1+\alpha}{1-\alpha}+\alpha > \alpha
\end{aligned}$$

by $n \geq 2t+1+\frac{1+\alpha}{1-\alpha}$ and $t \geq a \geq 3$. Thus, we have

$$\begin{aligned}
&(n-t-2)(n-2-\alpha(a-1))(1-\alpha)t-(t-a+1)(2\alpha+(1-\alpha)(t+2))(\alpha+(1-\alpha)a)\\
&\geq \left(t-a+1+1+\frac{1+\alpha}{1-\alpha}\right)(\alpha+2\alpha+(1-\alpha)(t+2))(1-\alpha)t\\
&\quad -(t-a+1)(2\alpha+(1-\alpha)(t+2))(\alpha+(1-\alpha)a)\\
&= \left(t-a+1+1+\frac{1+\alpha}{1-\alpha}\right)\alpha(1-\alpha)t\\
&\quad + \left(t-a+1+1+\frac{1+\alpha}{1-\alpha}\right)(2\alpha+(1-\alpha)(t+2))(1-\alpha)t\\
&\quad -(t-a+1)(2\alpha+(1-\alpha)(t+2))\alpha-(t-a+1)(2\alpha+(1-\alpha)(t+2))(1-\alpha)a\\
&\geq \left(t-a+1+1+\frac{1+\alpha}{1-\alpha}\right)\alpha(1-\alpha)t+\left(1+\frac{1+\alpha}{1-\alpha}\right)(2\alpha+(1-\alpha)(t+2))(1-\alpha)t\\
&\quad -(t-a+1)(2\alpha+(1-\alpha)(t+2))\alpha\\
&= (t-a+1)\alpha\big((1-\alpha)t-(2\alpha+(1-\alpha)(t+2))\big)+\left(1+\frac{1+\alpha}{1-\alpha}\right)\alpha(1-\alpha)t\\
&\quad + \left(1+\frac{1+\alpha}{1-\alpha}\right)(2\alpha+(1-\alpha)(t+2))(1-\alpha)t\\
&= -2\alpha(t-a+1)+\left(1+\frac{1+\alpha}{1-\alpha}\right)\alpha(1-\alpha)t+\left(1+\frac{1+\alpha}{1-\alpha}\right)(2\alpha+(1-\alpha)(t+2))(1-\alpha)t\\
&= -2\alpha(t-a+1)+2\alpha t+\left(1+\frac{1+\alpha}{1-\alpha}\right)(2\alpha+(1-\alpha)(t+2))(1-\alpha)t > 0.
\end{aligned}$$

Therefore, the claim holds. □

Let $G_1 \cong K_t\nabla(2K_1 \cup K_{n-t-2})$ and $G_2 \cong H_{n,a}$. Then $V(G_1)$ can be partitioned as $V(G_1) = V(2K_1) \cup V(K_t) \cup V(K_{n-t-2})$, where $V(2K_1) = \{u, w\}$, $V(K_t) = \{v_1, v_2, \ldots, v_t\}$ and $V(K_{n-t-2}) = \{v_{t+1}, \ldots, v_{n-2}\}$. Obviously, $G_2 \cong G_1 - \{uv_i \mid a \leq i \leq t\} + \{wv_j \mid t+1 \leq$

$j \leq n-2\}$. Let $\mathbf{x}$ (resp. $\mathbf{y}$) be the positive unit eigenvectors of $A_\alpha(G_1)$ (resp. $A_\alpha(G_2)$) corresponding to $\rho_\alpha(G_1)$ (resp. $\rho_\alpha(G_2)$). By symmetry, $\mathbf{x}$ takes the same components on the vertices of $V(2K_1)$, $V(K_t)$ and $V(K_{n-t-2})$ respectively, say x_1, x_2 and x_3. Similarly, $\mathbf{y}$ takes the same components on the vertices of $V(K_1)$, $V(K_{a-1})$ and $V(K_{n-a})$ respectively, say y_1, y_2 and y_3. Then, from $A_\alpha(G_1)\mathbf{x} = \rho_\alpha(G_1)\mathbf{x}$ and $A_\alpha(G_2)\mathbf{y} = \rho_\alpha(G_2)\mathbf{y}$, one can obtain that

$$\rho_\alpha(G_1)x_3 = \alpha(n-3)x_3 + (1-\alpha)(tx_2 + (n-t-3)x_3), \tag{4.1}$$

$$\rho_\alpha(G_2)y_1 = \alpha(a-1)y_1 + (1-\alpha)(a-1)y_2, \tag{4.2}$$

$$\rho_\alpha(G_2)y_3 = \alpha(n-2)y_3 + (1-\alpha)((a-1)y_2 + (n-a-1)y_3). \tag{4.3}$$

From (4.1) we have

$$x_2 = \frac{\rho_\alpha(G_1) - \alpha(n-3) - (1-\alpha)(n-t-3)}{(1-\alpha)t} x_3. \tag{4.4}$$

From (4.2) and (4.3) we get

$$(\rho_\alpha(G_2) - \alpha(a-1))y_1 = (\rho_\alpha(G_2) - \alpha(n-2) - (1-\alpha)(n-a-1))y_3. \tag{4.5}$$

Combining $\rho_\alpha(G_2) \geq n-2 = \alpha(n-2) + (1-\alpha)(n-2)$, $n \geq 3a+b-1$ and $b \geq a \geq 3$, it is easy to see that

$$\begin{aligned} \rho_\alpha(G_2) - \alpha(a-1) &\geq \alpha(n-2) + (1-\alpha)(n-2) - \alpha(a-1) \\ &= \alpha(n-a-1) + (1-\alpha)(n-2) \\ &\geq \alpha(2a+b-2) + (1-\alpha)(3a+b-3) > 0, \end{aligned}$$

and

$$\begin{aligned} &\rho_\alpha(G_2) - \alpha(n-2) - (1-\alpha)(n-a-1) \\ &\geq \alpha(n-2) + (1-\alpha)(n-2) - \alpha(n-2) - (1-\alpha)(n-a-1) = (1-\alpha)(a-1) > 0. \end{aligned}$$

Moreover,

$$\begin{aligned} &\rho_\alpha(G_2) - \alpha(a-1) - (\rho_\alpha(G_2) - \alpha(n-2) - (1-\alpha)(n-a-1)) \\ &= \alpha(n-2-a+1) + (1-\alpha)(n-a-1) = n-a-1 \geq 2a+b-2 > 0, \end{aligned}$$

that is, $\rho_\alpha(G_2) - \alpha(a-1) > \rho_\alpha(G_2) - \alpha(n-2) - (1-\alpha)(n-a-1)$.

Hence, it follows from (4.5) that $y_1 < y_3$ and

$$y_3 = \frac{\rho_\alpha(G_2) - \alpha(a-1)}{\rho_\alpha(G_2) - \alpha(n-2) - (1-\alpha)(n-a-1)} y_1. \tag{4.6}$$

Together with $A_\alpha(G_1)\mathbf{x} = \rho_\alpha(G_1)\mathbf{x}$ and $A_\alpha(G_2)\mathbf{y} = \rho_\alpha(G_2)\mathbf{y}$, we have

$$\begin{aligned}
&\mathbf{y}^T(\rho_\alpha(G_2) - \rho_\alpha(G_1))\mathbf{x} \\
&= \mathbf{y}^T(A_\alpha(G_2) - A_\alpha(G_1))\mathbf{x} \\
&= \alpha\big((n-t-2)(x_1y_3 + x_3y_3) - (t-a+1)(x_1y_1 + x_2y_3)\big) \\
&\quad + (1-\alpha)\bigg(\sum_{j=t+1}^{n-2}(x_w y_{v_j} + x_{v_j} y_w) - \sum_{i=a}^{t}(x_u y_{v_i} + x_{v_i} y_u)\bigg) \\
&= \alpha\big((n-t-2)(x_1y_3 + x_3y_3) - (t-a+1)(x_1y_1 + x_2y_3)\big) \\
&\quad + (1-\alpha)\big((n-t-2)(x_1y_3 + x_3y_3) - (t-a+1)(x_1y_3 + x_2y_1)\big) \\
&= \alpha\big((n-t-2)x_1y_3 - (t-a+1)x_1y_1 + (n-t-2)x_3y_3 - (t-a+1)x_2y_3\big) \\
&\quad + (1-\alpha)\big(((n-t-2) - (t-a+1))x_1y_3 + (n-t-2)x_3y_3 - (t-a+1)x_2y_1\big) \\
&> \alpha\big(((n-t-2) - (t-a+1))x_1y_3 + (n-t-2)x_3y_3 - (t-a+1)x_2y_3\big) \\
&\quad + (1-\alpha)\big(((n-t-2) - (t-a+1))x_1y_3 + (n-t-2)x_3y_3 - (t-a+1)x_2y_1\big) \quad (\text{since } y_1 < y_3) \\
&\geq \alpha((n-t-2)x_3y_3 - (t-a+1)x_2y_3) \\
&\quad + (1-\alpha)((n-t-2)x_3y_3 - (t-a+1)x_2y_1) \quad (\text{since } n \geq 2t+1) \\
&> (n-t-2)x_3y_3 - (t-a+1)x_2y_1 \quad (\text{since } y_1 < y_3) \\
&= x_3y_1\bigg((n-t-2)\frac{\rho_\alpha(G_2) - \alpha(a-1)}{\rho_\alpha(G_2) - \alpha(n-2) - (1-\alpha)(n-a-1)} \\
&\qquad - (t-a+1)\frac{\rho_\alpha(G_1) - \alpha(n-3) - (1-\alpha)(n-t-3)}{(1-\alpha)t}\bigg) \quad (\text{from (4.4) and (4.6)}) \\
&> x_3y_1\bigg(\frac{(n-t-2)(n-2-\alpha(a-1))}{\alpha + (1-\alpha)a} - \frac{(t-a+1)(2\alpha + (1-\alpha)(t+2))}{(1-\alpha)t}\bigg) \\
&\qquad (\text{since } \rho_\alpha(G_1) < n-1 \text{ and } n-2 \leq \rho_\alpha(G_2) < n-1) \\
&= x_3y_1\frac{(n-t-2)(n-2-\alpha(a-1))(1-\alpha)t - (t-a+1)(2\alpha + (1-\alpha)(t+2))(\alpha + (1-\alpha)a)}{(\alpha + (1-\alpha)a)(1-\alpha)t} \\
&> 0 \quad (\text{by Claim 4.6}).
\end{aligned}$$

Thus, we have $\rho_\alpha(G_1) < \rho_\alpha(G_2)$, i.e., $\rho_\alpha(K_t\nabla(2K_1 \cup K_{n-t-2})) < \rho_\alpha(H_{n,a})$. Together with $\rho_\alpha(G) \leq \rho_\alpha(K_t\nabla(2K_1 \cup K_{n-t-2}))$, one can obtain $\rho_\alpha(G) < \rho_\alpha(H_{n,a})$, a contradiction.

Subcase 2.2: $3/4 < \alpha < 1$ *and* $b = a$.

Note that $\rho_\alpha(G) \leq \rho_\alpha(K_t\nabla(2K_1 \cup K_{n-t-2}))$ and $K_t\nabla(2K_1 \cup K_{n-t-2})$ is a spanning graph of $K_t\nabla(K_1 \cup K_{n-t-1})$. According to Lemma 2.3, we get $\rho_\alpha(G) \leq \rho_\alpha(K_t\nabla(2K_1 \cup K_{n-t-2})) \leq \rho_\alpha(K_t\nabla(K_1 \cup K_{n-t-1}))$, also a contradiction.

Therefore, the proof is completed. □

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