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Multiplicity of Normalized Solutions for Schrödinger Equation with Mixed Nonlinearity

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Abstract. In this paper, we explore the multiplicity of normalized solutions for Schrödinger equation with mixed nonlinearities

$$\begin{cases} -\Delta u + V(\epsilon x)u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, dx = c, \end{cases}$$

where $\mu > 0$, $c > 0$, $2 < q < 2 + 4/N < p < 2N/(N-2)$, $N \geq 3$, $\epsilon > 0$ is a parameter and $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier. The potential V is a bounded and continuous nonnegative function, satisfying some suitable global conditions. By employing the minimization techniques and the truncated argument, we obtain that the number of normalized solutions is not less than the number of global minimum points of V when the parameter ϵ is sufficiently small.

1. Introduction and main results

This paper concentrates on investigating the multiplicity of standing waves for nonlinear Schrödinger equation with combined power nonlinearities

$$i\phi_t + \Delta\phi + V(x)\phi + \mu|\phi|^{q-2}\phi + |\phi|^{p-2}\phi = 0 \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\phi\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $\mu > 0$, and $2 < q < p < 2N/(N-2)$, while V is a potential function. Over the last decade, significant interest has grown around the nonlinear Schrödinger equations with combined nonlinearities, primarily ignited by the influential work of Tao et al. [36]. Their pioneering research has triggered extensive exploration and studied in this domain. To identify stationary states, we adopt the ansatz $\phi(t,x) = e^{i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$, and $u\colon \mathbb{R}^N \to \mathbb{C}$ is a time-independent function. Through straightforward calculation, we determine that u satisfies the following equation

$$-\Delta u + V(x)u = \lambda u + \mu|u|^{q-2}u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

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Currently, there exist two significantly distinct viewpoints regarding the frequency λ in equation (1.1). The first perspective treats λ as a constant, a scenario known as the *fixed frequency problem*. In this context, the variational methods can be applied to identify the critical points of the energy functional corresponding to equation (1.1) or other topological techniques. The *fixed frequency problem* has been the subject of extensive research over recent decades, see [14, 16, 17, 32, 37].

The alternative perspective is to treat λ as an unknown variable in equation (1.1). In this view, it's reasonable to set the mass value in a way that λ can be interpreted as a Lagrange multiplier. Then, equation (1.1) with mass constraint is called *fixed mass problem*. Currently, physicists express great interest in solutions that adhere to the normalized condition:

$$\int_{\mathbb{R}^3} |u|^2 \, dx = c$$

for a priori given c, since the mass admits a clear physical meaning. From a physical perspective, the normalized condition may signify the count of particles in each component of Bose–Einstein condensates or the power supply in the nonlinear optics domain. Furthermore, such solutions can provide deeper insights into dynamic attributes such as orbital stability or instability, and they can describe attractive Bose–Einstein condensates [15, 19, 41, 42]. These solutions are typically referred to as prescribed mass solutions or normalized solutions in mathematics. A natural approach to obtaining normalized solutions for equation (1.1) is to identify the critical points of the associated energy functional under the mass constraint. For the Schrödinger equation, we categorize cases into three types: mass subcritical for $2 < q < 2 + 4/N$, mass critical for $q = 2 + 4/N$, and mass supercritical for $2 + 4/N < q < 2N/(N-2)$. Naturally, the techniques for handling these cases differ, and the results on the existence of normalized solutions for these cases can be found in references such as [23, 31, 33–35] and related references.

For the more general Laplacian operator, we introduce some (p, q)-Laplacian equations here. Baldelli and Yang [8] investigated the existence of normalized solutions to a class of $(2, q)$-Laplacian equations

$$-\Delta u - \Delta_q u = \lambda u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

under the constraint $\int_{\mathbb{R}^N} |u|^2 \, dx = c$, where $\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u)$ is the q-Laplacian of u. The authors tackled novel challenges presented by the quasi-linear term and considered the different behaviors of the equation for $q < 2$ and $q > 2$. Subsequently, Baldelli et al. [7] utilized variational methods and explored the existence of solutions for a wide range of quasi-linear problems, including those involving the Born–Infeld operator. For *fixed frequency problem* of (p, q)-Laplacian equations, we refer the readers to [4–6, 18].

In [13], Ding and Zhong established the existence of a solution $(u,\lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$, posed by

$$\begin{cases} -\Delta u + V(x)u + \lambda u = g(u) & \text{in } \mathbb{R}^N,\ N \geq 3, \\ u \geq 0, \quad \int_{\mathbb{R}^N} |u|^2\,dx = c, \end{cases} \tag{1.2}$$

where $V(x) \leq 0$ satisfies some regularity conditions and g satisfying suitable conditions. Ikoma and Miyamoto [21] investigated the existence and nonexistence of normalized solutions for equation (1.2). They also investigated distinctions between the cases when $V(x) = 0$ and $V(x) \not\equiv 0$ using a scheme introduced by Shibata in [31]. Bartsch et al. [9] studied the existence of a solution for $g(u) = |u|^{p-2}u$, $2 + 4/N < p < 2N/(N-2)$ and assumed that $V(x) \geq 0$, $V(x) \to 0$ as $|x| \to \infty$. Very recently, in [2], Alves and Ji studied the existence a solution for $g(u) = |u|^{p-2}u$, $p \in (2, 2 + 4/N)$, the potential V satisfies different types of potentials. When $V(x) = 0$ and g with mass critical growth close to 0, Bieganowski and Mederski [11] proved the existence of a normalized ground state solution for equation (1.2) by using the minimization method of functional on the linear combination of Nehari and Pohozaev constraints. Subsequently, Liu and Zhao [30] obtained some similar results by weakening the assumptions on g.

Numerous authors have extensively investigated the existence of infinitely many normalized solutions for the nonlinear Schrödinger equations with combined power nonlinearities. They employed genus theory and deformation arguments in their studies [3, 10, 12, 22, 25–27]. Recently, without using of the genus theory, Alves [1] explored the existence of multiple solutions for a problem in the form of

$$\begin{cases} -\Delta u = \lambda u + h(\epsilon x) f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0, \quad \int_{\mathbb{R}^N} |u|^2\,dx = c, \end{cases}$$

where $c > 0$, $\epsilon > 0$, and $\lambda \in \mathbb{R}$ as unknown parameters. Here, $h\colon \mathbb{R}^N \to [0,+\infty)$ is a continuous function, and f exhibits mass subcritical growth. Alves demonstrated that, for sufficiently small ϵ, the number of normalized solutions is at least equal to the number of global maximum points of h. Yang et al. [40] extended the findings from [1] to fractional Schrödinger equations. They also introduced new insights into the existence of normalized ground states for non-autonomous elliptic equations. Li et al. [28] investigated the multiplicity of normalized solutions for the Schrödinger equation with mixed nonlinearities, as expressed by

$$\begin{cases} -\Delta u = \lambda u + h(\epsilon x)|u|^{q-2}u + \eta |u|^{p-2}u & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2\,dx = c, \end{cases}$$

where $\eta > 0$, $q < 2 + 4/N < p \leq 2N/(N-2)$, and $\lambda \in \mathbb{R}$ as unknown parameters.

The study established that a similar result to [1]. Additionally, the research included an analysis of the orbital stability of the obtained solutions.

Building upon the work of [1, 28], this paper studies the multiplicity of normalized solutions for Schrödinger equation with mixed nonlinearities

$$
\begin{cases} -\Delta u + V(\epsilon x)u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2\,dx = c, \end{cases} \tag{1.3}
$$

where $\mu > 0$, $c > 0$, $2 < q < 2 + 4/N < p < 2N/(N-2)$, $N \geq 3$, $\epsilon > 0$ is a parameter and $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier. The potential V is a bounded and nonnegative continuous function satisfying the following conditions:

(V_1) $V \in L^\infty(\mathbb{R}^N)$, $V(x) \geq 0$ for all $x \in \mathbb{R}^N$.

(V_2) $V_\infty = \lim_{|x|\to+\infty} V(x) > V_0 := \min_{x\in\mathbb{R}^N} V(x) = 0$.

(V_3) $V^{-1}(\{0\}) = \{a_1, a_2, \ldots, a_l\}$ with $a_1 = 0$ and $a_j \neq a_s$ if $j \neq s$.

Theorem 1.1. *Assume that* (V_1)–(V_3) *hold. Then there exist* $\widetilde{\epsilon}$, V_* *and* $\overline{c} > 0$, *such that problem* (1.3) *admits at least* l *couples* $(u_j, \lambda_j) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ *of weak solutions for* $\|V\|_\infty < V_*$, $\epsilon \in (0, \widetilde{\epsilon})$ *and* $c \in (0, \overline{c}]$ *with* $\int_{\mathbb{R}^N} |u_j|^2\,dx = c$, $\lambda_j < 0$ *for* $j = 1, 2, \ldots, l$.

A solution u to the problem (1.3) corresponds to a critical point of the functional

$$
I_\epsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\epsilon x)|u|^2\,dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dx,
$$

restricted to the sphere

$$
S_c := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2\,dx = c \right\}.
$$

In this paper, our primary focus lies on the mass supercritical case, which arises due to the presence of the term $|u|^{p-2}u$ where $2 + 4/N < p \leq 2N/(N-2)$. Consequently, we observe that the functional I_ϵ becomes unbounded from below on S_c for any $c > 0$. This is contrast to the mass subcritical case, where the constrained functional I_ϵ remains bounded from below and exhibits coercive behavior. Furthermore, it should be noted that an arbitrary Palais Smale sequence does not necessarily exhibit boundedness in the space $H^1(\mathbb{R}^N)$. To address the first difficulty, we utilize the truncation technique as demonstrated in [3, 28], in which they studied a modified functional that is both bounded from below and coercive (see Lemma 3.1). The second major challenge lies in establishing the compactness of the Palais Smale sequence. This is thoroughly examined in Lemmas 4.4 and 4.5, where the crucial step is to prove that the Lagrange multiplier λ is negative, which heavily depends on the properties outlined in conditions (V_1)–(V_3).

The remaining sections of this paper are structured as follows. Section 2 provides some technical results. Section 3 investigates the characteristics of the truncated functional. Section 4 will show the proof of Theorem 1.1.

Notations. In this paper, unless otherwise specified, the following notations are employed: $\|\cdot\|$ represents the standard norm in $H^1(\mathbb{R}^N)$. $\|\cdot\|_r$ signifies the standard norm in $L^r(\mathbb{R}^N)$ for $r \in [1,+\infty]$. $H^{-1}(\mathbb{R}^N)$ refers to the dual space of $H^1(\mathbb{R}^N)$. $B_r(u)$ defines an open ball centered at u with radius $r > 0$. Constants, such as $C, C_1, C_2, \ldots$, denote positive values whose value is not relevant. $o_n(1)$ represents a real sequence with $o_n(1) \to 0$ as $n \to +\infty$. Symbols like $:=$ and $=:$ are used for definitions. The symbols $\rightharpoonup$ and $\to$ indicate weak and strong convergence, respectively, in the relevant function spaces.

2. Preliminary results

In this paper, we shall usually use the following Gagliardo–Nirenberg type result. The proof can be found in [34, 38].

Lemma 2.1. *For $t \in (2, 2N/(N-2))$, then there exists a constant $C_t := \big(\frac{t}{2\|W_t\|_2^{t-2}}\big)^{1/t} > 0$ such that*

$$\|u\|_t \leq C_t \|\nabla u\|_2^{\gamma_t} \|u\|_2^{(1-\gamma_t)} \quad \text{for any } u \in H^1(\mathbb{R}^N),$$

where $\gamma_t := \frac{N(t-2)}{2t}$ and W_t is the unique positive solution of $-\Delta W + \big(\frac{1}{\gamma_t} - 1\big)W = \frac{2}{t\gamma_t}|W|^{t-2}W$.

Set $\alpha_1 := q\gamma_q - 2$, $\alpha_2 := p\gamma_p - 2$, where γ_t and C_t are given by Lemma 2.1. For any $c > 0$, we consider the function

$$f(c,r) := \frac{1}{2} - \frac{\mu}{q} C_q^q c^{\frac{q(1-\gamma_q)}{2}} r^{\alpha_1} - \frac{1}{p} C_p^p c^{\frac{p(1-\gamma_p)}{2}} r^{\alpha_2}, \quad r > 0.$$

Moreover, if $c \in (0,\infty)$ is fixed, we regard that $f_c(r) := f(c,r)$. Note that $t\gamma_t < 2$ for $2 < t < 2 + 4/N$, $t\gamma_t = 2$ for $t = 2 + 4/N$ and $t\gamma_t > 2$ for $2 + 4/N < t \leq 2N/(N-2)$. Since $2 < q < 2 + 4/N < p < 2N/(N-2)$, similar to that of [24, Lemma 2.1], we know that $f_c(r) \to -\infty$ as $r \to 0^+$ and $f_c(r) \to -\infty$ as $r \to \infty$. There exists $c_* > 0$, the function $f_c(r)$ has a unique global maximum and the maximum value satisfies

$$\max_{r>0} f_c(r) \begin{cases} > 0 & \text{if } c < c_*, \\ = 0 & \text{if } c = c_*, \\ < 0 & \text{if } c > c_*. \end{cases} \tag{2.1}$$

Moreover, for any $c > 0$, with the similar setting, we define that

$$g(c,r) := \frac{1}{2} r^2 - \frac{\mu}{q} C_q^q c^{\frac{q(1-\gamma_q)}{2}} r^{q\gamma_q} - \frac{1}{p} C_p^p c^{\frac{p(1-\gamma_p)}{2}} r^{p\gamma_p}, \quad r > 0.$$

Now, we investigate the properties of $g_c(r) := g(c,r) = r^2 f_c(r)$. For $c < c_*$, $\lim_{r\to 0^+} g_c(r) = 0^-$ and $\lim_{r\to+\infty} g_c(r) = -\infty$. By (2.1), $g_c(r)$ attains its positive global maximum if $c < c_*$. In the following sections, we always assume $c \le \overline{c}$, where $\overline{c} < c_*$. Then, there exist

(2.2) $\quad 0 < R_0 < R_1 < +\infty$ dependent of c such that $g_c(R_0) = g_c(R_1) = 0$.

Moreover, $g_c(r) < 0$ in the intervals $(0,R_0)$, $(R_1,+\infty)$ and $g_c(r) > 0$ in the interval (R_0,R_1). Setting $\tau\colon \mathbb{R}^+ \to [0,1]$ as being a non-increasing and C^∞ function that satisfies

$$\tau(x) = \begin{cases} 1 & \text{if } 0 \le x \le R_0, \\ 0 & \text{if } x \ge R_1. \end{cases}$$

By Lemma 2.1 and (V_1), for any $u \in S_c$, we have

$$\begin{aligned} I_\epsilon(u) &\ge \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{p}\|u\|_p^p \\ &\ge \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}C_q^q c^{\frac{q(1-\gamma_q)}{2}}\|\nabla u\|_2^{q\gamma_q} - \frac{1}{p}C_p^p c^{\frac{p(1-\gamma_p)}{2}}\|\nabla u\|_2^{p\gamma_p} \\ &= g(c,\|\nabla u\|_2). \end{aligned}$$

Similarly, we consider the truncated functional

$$I_{\epsilon,T}(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\epsilon x)|u|^2\,dx - \frac{\mu}{q}\|u\|_q^q - \frac{\tau(\|\nabla u\|_2)}{p}\|u\|_p^p.$$

By Lemma 2.1 and (V_1), we get

$$\begin{aligned} I_{\epsilon,T}(u) &\ge \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}C_q^q c^{\frac{q(1-\gamma_q)}{2}}\|\nabla u\|_2^{q\gamma_q} - \frac{\tau(\|\nabla u\|_2)}{p}C_p^p c^{\frac{p(1-\gamma_p)}{2}}\|\nabla u\|_2^{p\gamma_p} \\ &:= g_T(c,\|\nabla u\|_2), \end{aligned}$$

where $g_T(c,r) := \frac{1}{2}r^2 - \frac{\mu}{q}C_q^q c^{\frac{q(1-\gamma_q)}{2}} r^{q\gamma_q} - \frac{\tau(r)}{p}C_p^p c^{\frac{p(1-\gamma_p)}{2}} r^{p\gamma_p}$, $r > 0$. It is easy to see that $g_{T,c}(r) := g_T(c,r)$ has the following properties

$$\begin{cases} g_{T,c}(r) \equiv g_c(r) \text{ for all } r \in (0,R_0], \\ g_{T,c}(r) \text{ is positive and strictly increasing in } (R_0,+\infty). \end{cases}$$

Correspondingly, for any $\omega \in [0,\|V\|_\infty]$, we denote by $I_\omega, I_{\omega,T}\colon H^1(\mathbb{R}^N) \to \mathbb{R}$ the following functionals

$$I_\omega(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{\omega}{2}\|u\|_2^2 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{p}\|u\|_p^p$$

and

$$I_{\omega,T}(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{\omega}{2}\|u\|_2^2 - \frac{\mu}{q}\|u\|_q^q - \frac{\tau(\|\nabla u\|_2)}{p}\|u\|_p^p.$$

3. The autonomous problem with truncated

In this section, we study the properties of the functional $I_{\omega,T}$ restricted on S_c.

Lemma 3.1. *The functional $I_{\omega,T}$ is bounded from below in S_c and coercive.*

Proof. Following the properties of $g_{T,c}$, for any $u \in S_c$,

$$I_{\omega,T}(u) \geq g_{T,c}(\|\nabla u\|_2) \geq \inf_{r>0} g_{T,c}(r) > -\infty.$$

Moreover, $I_{\omega,T}(u) \to \infty$ as $\|\nabla u\|_2 \to \infty$. □

Have this in mind, we can define that

$$\Upsilon_{\omega,T,c} := \inf_{u \in S_c} I_{\omega,T}(u). \tag{3.1}$$

The following lemma presents an important property of $\Upsilon_{\omega,T,c}$.

Lemma 3.2. *For $c \leq \overline{c}$, there exists $V_* > 0$ such that $\Upsilon_{\omega,T,c} < 0$ if $\omega < V_*$.*

Proof. Fixed $u \in S_c$, we set

$$s \star u := e^{\frac{Ns}{2}} u(e^s x) \quad \text{for all } s \in \mathbb{R}.$$

A direct computation gives

$$\|s \star u\|_2^2 = c, \quad \text{and} \quad \|s \star u\|_t^t = e^{t\gamma_t s}\|u\|_t^t \ \text{ for } 2 < t < 2N/(N-2),$$

which lead to

$$I_{\omega,T}(s \star u) \leq \frac{e^{2s}}{2}\|\nabla u\|_2^2 + \frac{\omega \overline{c}}{2} - \frac{\mu e^{q\gamma_q s}}{q}\|u\|_q^q.$$

Since $q < 2 + 4/N$, there exists $s < 0$ such that

$$\frac{e^{2s}}{2}\|\nabla u\|_2^2 - \frac{\mu e^{q\gamma_q s}}{q}\|u\|_q^q := A_s < 0.$$

Therefore, setting $\omega < V_* := \frac{-A_s}{\overline{c}}$, it follows that

$$I_{\omega,T}(s \star u) < A_s - \frac{A_s}{2} = \frac{A_s}{2} < 0,$$

which implies that $\Upsilon_{\omega,T,c} < 0$. □

Now we always assume that $\omega < V_*$ holds. The proof of the following lemma is standard and a similar proof can been seen in [28, Lemma 3.3].

Lemma 3.3. (i) $I_{\omega,T} \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$.

(ii) *If $I_{\omega,T} \le 0$ then $\|\nabla u\|_2 < R_0$, and $I_{\omega,T}(v) = I_\omega(v)$ for all v in a small neighborhood of u in $H^1(\mathbb{R}^N)$.*

We recall

$$g_T(c,r) = \frac{1}{2}r^2 - \frac{\mu}{q}C_q^q c^{\frac{q(1-\gamma_q)}{2}} r^{q\gamma_q} - \frac{\tau(r)}{p}C_p^p c^{\frac{p(1-\gamma_p)}{2}} r^{p\gamma_p}, \quad r>0,$$

and define $f_T(c,r) := \frac{1}{2} - \frac{\mu}{q}C_q^q c^{\frac{q(1-\gamma_q)}{2}} r^{\alpha_1} - \frac{\tau(r)}{p}C_p^p c^{\frac{p(1-\gamma_p)}{2}} r^{\alpha_2}$. We consider $f_{T,c}(r)$, which is defined on $(0,\infty)$ by $r \mapsto f_T(c,r)$.

Lemma 3.4. *Let $(c_2, r_2) \in (0,\infty)\times(0,\infty)$ satisfy $f(c_2,r_2) \ge 0$. Then for any $c_1 \in (0, c_2]$, we have that*

$$f_T(c_1,r_1) \ge 0 \quad \textit{if } r_1 \in \left[\sqrt{\frac{c_1}{c_2}}r_2, r_2\right].$$

Proof. Since $c \to f_T(\,\cdot\,, r)$ is a non-increasing function, we clearly have that

$$f_T(c_1,r_2) \ge f_T(c_2,r_2) \ge f(c_2,r_2) \ge 0.$$

Through direct calculations, we can get that

$$\begin{aligned}
f_T\left(c_1, \sqrt{\frac{c_1}{c_2}}r_2\right) &\ge f\left(c_1, \sqrt{\frac{c_1}{c_2}}r_2\right) \\
&= \frac{1}{2} - \frac{\mu}{q}C_q^q \left(\frac{c_1}{c_2}\right)^{\alpha_3} c_2^{\frac{q(1-\gamma_q)}{2}} r_2^{\alpha_1} - \frac{1}{p}C_p^p \left(\frac{c_1}{c_2}\right)^{\alpha_4} c_2^{\frac{p(1-\gamma_p)}{2}} r_2^{\alpha_2} \\
&\ge \frac{1}{2} - \frac{\mu}{q}C_q^q c_2^{\frac{q(1-\gamma_q)}{2}} r_2^{\alpha_1} - \frac{1}{p}C_p^p c_2^{\frac{p(1-\gamma_p)}{2}} r_2^{\alpha_2} = f(c_2,r_2) \ge 0,
\end{aligned}$$

where $\alpha_3 := \frac{q(1-\gamma_q)+\alpha_1}{2}$, $\alpha_4 := \frac{p(1-\gamma_p)+\alpha_2}{2} > 0$. Then, this means that

$$f_T\left(c_1, \sqrt{\frac{c_1}{c_2}}r_2\right) \ge 0 \quad \text{and} \quad f_T(c_1,r_2) \ge 0.$$

By the definition of τ, we know that $f_T(c_1,r_1) \ge 0$ for $r_1 \in \left[\sqrt{\frac{c_1}{c_2}}r_2, r_2\right]$. The proof is completed. □

Lemma 3.5. *For any $u \in S_c$, we have that*

$$I_{\omega,T}(u) \ge \|\nabla u\|_2^2 f_T(c, \|\nabla u\|_2).$$

Proof. Applying the Gagliardo–Nirenberg inequality, we obtain that, for any $u \in S_c$,

$$\begin{aligned}
I_{\omega,T}(u) &\ge \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}C_q^q c^{\frac{q(1-\gamma_q)}{2}} \|\nabla u\|_2^{q\gamma_q} - \frac{\tau(\|\nabla u\|_2)}{p}C_p^p c^{\frac{p(1-\gamma_p)}{2}} \|\nabla u\|_2^{p\gamma_p} \\
&= \|\nabla u\|_2^2 \left[\frac{1}{2} - \frac{\mu}{q}C_q^q c^{\frac{q(1-\gamma_q)}{2}} \|\nabla u\|_2^{\alpha_1} - \frac{\tau(\|\nabla u\|_2)}{p}C_p^p c^{\frac{p(1-\gamma_p)}{2}} \|\nabla u\|_2^{\alpha_2}\right] \\
&= \|\nabla u\|_2^2 f_T(c, \|\nabla u\|_2).
\end{aligned}$$

The lemma is proved. □

The proof process of the following lemma is standard and for detailed information, see [20, Lemma 2.3], we omit the proof. We recall the definition of $\Upsilon_{\omega,T,c}$ in (3.1).

Lemma 3.6. *$\Upsilon_{\omega,T,c}$ is continuous with regard to $c \in (0, \overline{c}]$.*

For any $\overline{c} < c_*$, by setting of (2.2) and $g(\overline{c}, r) = r^2 f(\overline{c}, r)$, it follows that $f(\overline{c}, R_0) = 0$. Moreover, $c \to f(\,\cdot\,, r)$ is a non-increasing function, it follows that $f(c, R_0) \geq 0$ for all $c \in (0, \overline{c}]$.

Lemma 3.7. *$\frac{c_1}{c_2}\Upsilon_{\omega,T,c_2} < \Upsilon_{\omega,T,c_1} < 0$ where $0 < c_1 < c_2 \leq \overline{c}$.*

Proof. Set $\xi = \sqrt{c_2/c_1}$ then $\xi > 1$. Let $\{u_n\} \subset S_{c_1}$ be a minimizing sequence with respect to Υ_{ω,T,c_1}, that is, $I_{\omega,T}(u_n) \to \Upsilon_{\omega,T,c_1} < 0$ as $n \to +\infty$ by Lemma 3.2. Consequently, there exists n_0 such that

$$I_{\omega,T}(u_n) < 0 \quad \text{for } n \geq n_0. \tag{3.2}$$

In view of Lemma 3.4 and $f(c_2, R_0) \geq 0$, $f_T(c_1, r) \geq 0$ for any $r \in \big[\sqrt{\frac{c_1}{c_2}}R_0, R_0\big]$. Hence, we can deduce from Lemma 3.5 and (3.2) that

$$\|\nabla u_n\|_2 < \sqrt{\frac{c_1}{c_2}}R_0 \quad \text{for } n \geq n_0. \tag{3.3}$$

Setting $v_n = \xi u_n$, then $v_n \in S_{c_2}$. By (3.3), one has $\|\nabla v_n\|_2 = \xi\|\nabla u_n\|_2 < R_0$. Therefore, $\tau(\|\nabla u_n\|_2) = \tau(\|\nabla v_n\|_2) = 1$. Through direct calculations, we find that

$$\begin{aligned}
\Upsilon_{\omega,T,c_2} &\leq I_{\omega,T}(v_n) \\
&= \xi^2 I_{\omega,T}(u_n) + \frac{(\xi^2 - \xi^q)}{q}\mu\|u_n\|_q^q + \frac{\tau(\|\nabla u_n\|_2)\xi^2 - \tau(\|\nabla v_n\|_2)\xi^p}{p}\|u_n\|_p^p \\
&= \xi^2 I_{\omega,T}(u_n) + \frac{(\xi^2 - \xi^q)}{q}\mu\|u_n\|_q^q + \frac{(\xi^2 - \xi^p)}{p}\|u_n\|_p^p.
\end{aligned}$$

For any $t \in (2, 2N/(N-2))$, there exist positive constants C and n_0 such that $\|u_n\|_t^t \geq C$ for all $n \geq n_0$. If not, there exists $t_1 \in (2, 2N/(N-2))$ such that $\|u_n\|_{t_1}^{t_1} \to 0$ as $n \to +\infty$, then by the Vanishing Lemma [29, Lemma I.1], $\|u_n\|_p^p \to 0$ and $\|u_n\|_q^q \to 0$ as $n \to +\infty$. Now, recalling that

$$0 > \Upsilon_{\omega,T,c_1} = I_{\omega,T}(u_n) \geq -\frac{\mu}{q}\|u_n\|_q^q - \frac{\tau(\|\nabla u\|_2)}{p}\|u_n\|_p^p \to 0 \quad \text{as } n \to \infty,$$

which is a contradiction and the result holds. We obtain that for $n \in \mathbb{N}$ large

$$\Upsilon_{\omega,T,c_2} \leq \xi^2 I_{\omega,T}(u_n) + \frac{(\xi^2 - \xi^q)C}{q}.$$

By Lemma 3.6 and let $n \to +\infty$, it follows that $\Upsilon_{\omega,T,c_2} < \xi^2\Upsilon_{\omega,T,c_1}$, which implies that

$$\frac{c_1}{c_2}\Upsilon_{\omega,T,c_2} < \Upsilon_{\omega,T,c_1}.$$

This completes the proof of this lemma. □

Lemma 3.8. *Let $\{u_n\} \subset S_c$ be a minimizing sequence with respect to $\Upsilon_{\omega,T,c}$. Then, for some subsequence, either*

(i) *$\{u_n\}$ is strongly convergent, or*

(ii) *There exists $\{y_n\} \subset \mathbb{R}^N$ with $|y_n| \to \infty$ such that the sequence $\overline{u}_n(x) = u_n(x + y_n)$ is strongly convergent to a function $\overline{u} \in S_c$ with $I_{\omega,T}(\overline{u}) = \Upsilon_{\omega,T,c}$.*

Proof. Observing that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ by Lemmas 3.1 and 3.2. There exists $u \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ for some subsequence. Now, let's explore the following three possibilities.

Case 1*:* If $u \not\equiv 0$ and $\|u\|_2^2 = b < c$. It follows from the Fatou Lemma and the Brézis–Lieb Lemma [39, Lemma 3.5] that $\|\nabla u\|_2^2 \le \liminf_{n\to+\infty} \|\nabla u_n\|_2^2$ and

$$\|u_n\|_2^2 = \|v_n\|_2^2 + \|u\|_2^2 + o_n(1), \quad \|\nabla u_n\|_2^2 = \|\nabla u\|_2^2 + \|\nabla v_n\|_2^2 + o_n(1).$$

If we set $v_n := u_n - u$, $d_n = \|v_n\|_2^2$, one has that $\|v_n\|_2^2 \to d$, where $c = d + b$. Noting that $d_n \in (0, c)$ for n large enough, and by the fact that τ is continuous, non-increasing and Lemma 3.7, we obtain that

$$\begin{aligned}
\Upsilon_{\omega,T,c} + o_n(1) &= I_{\omega,T}(u_n) \\
&= \frac{1}{2}\|\nabla v_n\|_2^2 + \frac{1}{2}\omega\|v_n\|_2^2 - \frac{\mu}{q}\|v_n\|_q^q - \frac{\tau(\|\nabla u_n\|_2)}{p}\|v_n\|_p^p \\
&\quad + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\omega\|u\|_2^2 - \frac{\mu}{q}\|u\|_q^q - \frac{\tau(\|\nabla u_n\|_2)}{p}\|u\|_p^p + o_n(1) \\
&\ge I_{\omega,T}(v_n) + I_{\omega,T}(u) + o_n(1) \\
&\ge \Upsilon_{\omega,T,d_n} + \Upsilon_{\omega,T,b} + o_n(1) \\
&\ge \frac{d_n}{c}\Upsilon_{\omega,T,c} + \Upsilon_{\omega,T,b} + o_n(1).
\end{aligned}$$

By Lemma 3.7, letting $n \to +\infty$, we obtain that

$$\Upsilon_{\omega,T,c} \ge \frac{d}{c}\Upsilon_{\omega,T,c} + \Upsilon_{\omega,T,b} > \frac{d}{c}\Upsilon_{\omega,T,c} + \frac{b}{c}\Upsilon_{\omega,T,c} = \Upsilon_{\omega,T,c},$$

which is a contradiction.

Case 2*:* If $\|u\|_2^2 = c$, then $u_n \to u$ in $L^2(\mathbb{R}^N)$. Moreover, $u_n \to u$ in $L^t(\mathbb{R}^N)$ for all $t \in (2, 2N/(N-2))$. Then

$$\begin{aligned}
\Upsilon_{\omega,T,c} &= \lim_{n\to+\infty} I_{\omega,T}(u_n) \\
&= \lim_{n\to+\infty} \left(\frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{2}\omega\|u_n\|_2^2 - \frac{\mu}{q}\|u_n\|_q^q - \frac{1}{p}\tau(\|\nabla u_n\|_2)\|u_n\|_p^p\right) \\
&\ge I_{\omega,T}(u).
\end{aligned}$$

As $u \in S_c$, we conclude that $I_{\omega,T}(u) = \Upsilon_{\omega,T,c}$, then $u_n \to u$ in $H^1(\mathbb{R}^N)$, which implies that (i) occurs.

Case 3*:* If $u \equiv 0$, then $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, we assert the existence of R', $k_1 > 0$, and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that for all n,

$$\int_{B_{R'}(y_n)} |u_n|^2 \, dx \geq k_1. \tag{3.4}$$

Otherwise, it would imply that $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (2, 2N/(N-2))$ by the Vanishing Lemma. As a result, we have $I_{\omega,T}(u_n) \geq \frac{1}{2}\|\nabla u_n\|_2^2 + o_n(1)$. However, this contradicts the fact that $I_{\omega,T}(u_n) \to \Upsilon_{\omega,T,c} < 0$. Hence, in all cases, (3.4) holds and $|y_n| \to +\infty$ obviously. Consequently, by defining $\overline{u}_n(x) = u_n(x + y_n)$, it is evident that $\{\overline{u}_n\} \subset S_c$ and it is also a minimizing sequence with respect to $\Upsilon_{\omega,T,c}$. Furthermore, there exists $\overline{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\overline{u}_n \rightharpoonup \overline{u}$ in $H^1(\mathbb{R}^N)$. Following the approach used in the first two proof progress, we conclude that $\overline{u}_n \to \overline{u}$ in $H^1(\mathbb{R}^N)$, confirming the occurrence of (ii) and thereby establishing the lemma. □

Lemma 3.9. *$\Upsilon_{\omega,T,c}$ is attained.*

Proof. By Lemmas 3.1 and 3.8, there exists a bounded minimizing sequence $\{u_n\} \subset S_c$ and $u_n \to u$ in $H^1(\mathbb{R}^N)$ with respect to $\Upsilon_{\omega,T,c} = I_{\omega,T}(u) < 0$. Then $\{u_n\}$ is also a minimizing sequence for $I_\omega(u)$ and $\Upsilon_{\omega,T,c}(u) = I_\omega(u)$ by Lemma 3.3. □

A direct result of Lemma 3.9 is the subsequent corollary.

Corollary 3.10. *If $\omega_1 < \omega_2 < V_*$. Then $\Upsilon_{\omega_1,T,c} < \Upsilon_{\omega_2,T,c}$.*

Proof. Let $u \in S_c$ satisfy $I_{\omega_2,T}(u) = \Upsilon_{\omega_2,T,c}$. Then, $\Upsilon_{\omega_1,T,c} \leq I_{\omega_1,T}(u) < I_{\omega_2,T}(u) = \Upsilon_{\omega_2,T,c}$. □

4. Proof of Theorem 1.1

In this section, some properties of the following functional $I_{\epsilon,T} \colon H^1(\mathbb{R}^N) \to \mathbb{R}$ are given:

$$I_{\epsilon,T}(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\epsilon x)|u|^2 \, dx - \frac{\mu}{q}\|u\|_q^q - \frac{\tau(\|\nabla u\|_2)}{p}\|u\|_p^p.$$

More precisely, we study the following the minimum value:

$$\Upsilon_{\epsilon,T,c} := \inf_{u \in S_c} I_{\epsilon,T}(u),$$

where $\Upsilon_{\epsilon,T,c}$ is well defined by the properties of $g_{T,c}(r)$. We shall denote by $I_{0,T}, I_{\infty,T} \colon H^1(\mathbb{R}^N) \to \mathbb{R}$ the following functionals:

$$I_{0,T}(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}\|u\|_q^q - \frac{\tau(\|\nabla u\|_2)}{p}\|u\|_p^p$$

and

$$I_{\infty,T}(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty |u|^2\,dx - \frac{\mu}{q}\|u\|_q^q - \frac{\tau(\|\nabla u\|_2)}{p}\|u\|_p^p.$$

By (V_1)–(V_3), $V_\infty < V_*$, and Lemma 3.9 in Section 3, the minimum value $\Upsilon_{0,T,c}$ and $\Upsilon_{\infty,T,c}$ defined by

$$\Upsilon_{0,T,c} := \inf_{u\in S_c} I_{0,T}(u), \quad \Upsilon_{\infty,T,c} := \inf_{u\in S_c} I_{\infty,T}(u),$$

respectively, are attained. There exist $u_0, u_\infty \in S_c$ such that $I_{0,T}(u_0) = \Upsilon_{0,T,c}$ and $I_{\infty,T}(u_\infty) = \Upsilon_{\infty,T,c}$. Furthermore, by Corollary 3.10 and $V_0 < V_\infty$, we know $\Upsilon_{0,T,c} < \Upsilon_{\infty,T,c} < 0$.

Lemma 4.1. $\limsup_{\epsilon\to 0^+} \Upsilon_{\epsilon,T,c} \le \Upsilon_{0,T,c}$.

Proof. By Lemma 3.9, let $u_0 \in S_c$ with $I_{0,T}(u_0) = \Upsilon_{0,T,c}$. Then

$$\Upsilon_{\epsilon,T,c} \le I_{\epsilon,T}(u_0) = \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\epsilon x)|u_0|^2\,dx - \frac{\mu}{q}\|u_0\|_q^q - \frac{\tau(\|\nabla u_0\|_2)}{p}\|u_0\|_p^p.$$

Letting $\epsilon \to 0^+$, we obtain, by the Lebesgue dominated convergence theorem,

$$\limsup_{\epsilon\to 0^+} \Upsilon_{\epsilon,T,c} \le \limsup_{\epsilon\to 0^+} I_{\epsilon,T}(u_0) = I_{0,T}(u_0) = \Upsilon_{0,T,c},$$

which completes the proof of this lemma. □

From Lemma 4.1 and $\Upsilon_{0,T,c} < \Upsilon_{\infty,T,c}$, there exists $\epsilon_0 > 0$ such that

$$\Upsilon_{\epsilon,T,c} < \Upsilon_{\infty,T,c} \quad \text{for all } \epsilon \in (0,\epsilon_0).$$

Similar to the proof of Lemma 3.3, we have the following result, whose proof is omitted.

Lemma 4.2. (i) $I_{\epsilon,T} \in C^1(H^1(\mathbb{R}^N),\mathbb{R})$.

(ii) *If $I_{\epsilon,T}(u) \le 0$ then $\|\nabla u\|_2 < R_0$ and $I_{\epsilon,T}(v) = I_\epsilon(v)$ for all v in a small neighborhood of u in $H^1(\mathbb{R}^N)$.*

Let $\{u_n\} \subset S_c$ be a minimizing sequence of $I_{\epsilon,T}(u_n)$ with respect to any $m < \Upsilon_{\infty,T,c} < 0$. Similar to the proofs of Lemmas 3.1 and 3.2, $\{\|\nabla u_n\|_2\}$ is bounded. Hence, there exists $u \in H^1(\mathbb{R}^N)$ and a subsequence of $\{u_n\}$, still denoted by itself, such that

$$u_n \rightharpoonup u_\varepsilon \ \text{ in } H^1(\mathbb{R}^N) \quad \text{and} \quad u_n(x) \to u_\varepsilon(x) \ \text{ a.e. in } \mathbb{R}^N.$$

Lemma 4.3. *The weak limit u_ε of $\{u_n\}$ is nontrivial.*

Proof. Assume by contradiction that $u_\varepsilon = 0$. Then

$$m + o_n(1) = I_{\epsilon,T}(u_n) = I_{\infty,T}(u_n) + \frac{1}{2}\int_{\mathbb{R}^N} (V(\epsilon x) - V_\infty)|u_n|^2\,dx.$$

By (V_1)–(V_3), for any given $\zeta > 0$, there exists $R > 0$ such that

$$V(x) \geq V_\infty - \zeta \quad \text{for all } |x| \geq R.$$

Hence,

$$m + o_n(1) = I_{\epsilon,T}(u_n) \geq I_{\infty,T}(u_n) + \frac{1}{2}\int_{B_{R/\epsilon}(0)} (V(\epsilon x) - V_\infty)|u_n|^2\,dx - \frac{\zeta}{2}\int_{B^c_{R/\epsilon}(0)} |u_n|^2\,dx.$$

Recalling that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $u_n \to 0$ in $L^2(B_{R/\epsilon}(0))$, it follows that

$$m + o_n(1) \geq I_{\infty,T}(u_n) - \zeta C \geq \Upsilon_{\infty,T,c} - \zeta C. \tag{4.1}$$

Since $\zeta > 0$ is arbitrary, we infer that $m \geq \Upsilon_{\infty,T,c}$, which is a contradiction. Thus, the weak limit u of $\{u_n\}$ is nontrivial. □

Lemma 4.4. *Let $\{u_n\}$ be a $(PS)_m$ sequence of $I_{\epsilon,T}$ restricted to S_c with $m < \Upsilon_{\infty,T,c}$ and u_ϵ is the weak limit of $\{u_n\}$ in $H^1(\mathbb{R}^N)$. If $u_n \nrightarrow u_\epsilon$ in $H^1(\mathbb{R}^N)$, there exists $\beta > 0$ independent of ϵ such that*

$$\liminf_{n\to+\infty} \|u_n - u_\epsilon\|_2 \geq \beta.$$

Proof. By Lemma 4.2 and $m < \Upsilon_{\infty,T,c} < 0$, this implies that $\|\nabla u_n\|_2 < R_0$ for n sufficiently large. Consequently, the sequence $\{u_n\}$ also qualifies as a $(PS)_m$ sequence of I_ϵ constrained to S_c, that is,

$$I_\epsilon(u_n) \to m \quad \text{and} \quad \big\|I_\epsilon|'_{S_c}(u_n)\big\|_{H^{-1}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to +\infty.$$

Introducing the functional $\Psi\colon H^1(\mathbb{R}^N) \to \mathbb{R}$ defined as $\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2\,dx$, we observe that $S_c = \Psi^{-1}(c/2)$. By referencing Willem [39, Proposition 5.12], we deduce the existence of a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\big\|I'_\epsilon(u_n) - \lambda_n \Psi'(u_n)\big\|_{H^{-1}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to +\infty. \tag{4.2}$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, then $u_n \rightharpoonup u_\epsilon$ and we let $v_n := u_n - u_\epsilon$. It follows that $\{\lambda_n\}$ is also bounded, for some subsequence, there exists λ_ϵ such that $\lambda_n \to \lambda_\epsilon$ as $n \to +\infty$. The combination of this with (4.2) results in

$$I'_\epsilon(u_\epsilon) - \lambda_\epsilon \Psi'(u_\epsilon) = 0 \ \text{ in } H^{-1}(\mathbb{R}^N), \quad \big\|I'_\epsilon(v_n) - \lambda_\epsilon \Psi'(v_n)\big\|_{H^{-1}(\mathbb{R}^N)} \to 0 \ \text{ as } n \to +\infty. \tag{4.3}$$

Through straightforward calculations, we get that

$$0 > \Upsilon_{\infty,T,c} > \lim_{n\to+\infty} J_\epsilon(u_n) = \lim_{n\to+\infty} \left(I_\epsilon(u_n) - \frac{1}{2} I'_\epsilon(u_n)u_n + \frac{\lambda_n}{2}\|u_n\|_2^2 + o_n(1) \right) \geq \frac{1}{2}\lambda_\epsilon c,$$

implying that

$$\lambda_\epsilon \leq \frac{2\Upsilon_{\infty,T,c}}{c} < 0 \quad \text{for all } \epsilon \in (0,\epsilon_0). \tag{4.4}$$

By (4.3), the above analysis yields that

$$\|\nabla v_n\|_2^2 + \int_{\mathbb{R}^N} V(\epsilon x)|v_n|^2\,dx - \lambda_\epsilon \|v_n\|_2^2 = \mu\|v_n\|_q^q + \|v_n\|_p^p + o_n(1),$$

which combined with (4.4) give that

$$\|\nabla v_n\|_2^2 + \int_{\mathbb{R}^N} V(\epsilon x)|v_n|^2\,dx - \frac{2\Upsilon_{\infty,T,c}}{c}\|v_n\|_2^2 \leq \mu\|v_n\|_q^q + \|v_n\|_p^p + o_n(1). \tag{4.5}$$

By (4.5) and the Sobolev inequality, we deduce that

$$C_1\|v_n\|^2 \leq \mu\|v_n\|_q^q + \|v_n\|_p^p + o_n(1) \leq \mu C_2\|v_n\|^q + C_3\|v_n\|^p + o_n(1).$$

Since $v_n \nrightarrow 0$ in $H^1(\mathbb{R}^N)$, there exists C_4 independent of ϵ such that $\|v_n\| \geq C_4$. Moreover, there holds,

$$\liminf_{n\to+\infty} (\mu\|v_n\|^q + \|v_n\|^p) \geq C_5 \tag{4.6}$$

for some $C_5 > 0$. By (4.6) and the Gagliardo–Nirenberg inequality, there exists $\beta > 0$ independent of $\epsilon \in (0,\epsilon_0)$ such that

$$\liminf_{n\to+\infty} \|v_n\|_2 \geq \beta.$$

The proof is complete. □

From now on, we fix $0 < \rho_0 < \min\left\{\Upsilon_{\infty,T,c} - \Upsilon_{0,T,c}, \frac{\beta^2}{c}(\Upsilon_{\infty,T,c} - \Upsilon_{0,T,c})\right\}$.

Lemma 4.5. *$I_{\epsilon,T}$ satisfies the $(PS)_m$ condition restricted to S_c if $m < \Upsilon_{0,T,c} + \rho_0$.*

Proof. Let $\{u_n\} \subset S_c$ be a $(PS)_m$ sequence of $I_{\epsilon,T}$ restricted to S_c. Noting that $m < \Upsilon_{\infty,T,c} < 0$, by Lemma 4.2, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Let $u_n \rightharpoonup u_\epsilon$ in $H^1(\mathbb{R}^N)$ and $u_\epsilon \not\equiv 0$, see Lemma 4.3. A straightforward computation gives that $v_n := u_n - u_\epsilon$ is a $(PS)_{m'}$ sequence of $I_{\epsilon,T}$ restricted to S_c and $m' < m$. If $v_n \nrightarrow 0$ in $H^1(\mathbb{R}^N)$, by Lemma 4.4, $\liminf_{n\to+\infty} \|v_n\|_2 \geq \beta$.

Setting $b = \|u_\epsilon\|_2^2$, $d_n = \|v_n\|_2^2$ and supposing that $\|v_n\|_2^2 \to d$, then we get $d \geq \beta^2 > 0$ and $c = b + d$. By the fact that $v_n \rightharpoonup 0$ with a similar proof of (4.1), we can obtain that $I_{\epsilon,T}(v_n) \geq \Upsilon_{\infty,T,d_n} + o(1)$. From $d_n \in (0,c)$ for n large enough, we have

$$m + o_n(1) = I_{\epsilon,T}(u_n) \geq I_{\epsilon,T}(v_n) + I_{\epsilon,T}(u_\epsilon) \geq \Upsilon_{\infty,T,d_n} + \Upsilon_{0,T,b} + o_n(1). \tag{4.7}$$

From the proof of Lemma 3.7 and (4.7), it follows that

$$\Upsilon_{0,T,c} + \rho_0 \geq m + o_n(1) \geq \frac{d_n}{c}\Upsilon_{\infty,T,c} + \frac{b}{c}\Upsilon_{0,T,c}.$$

Letting $n \to +\infty$, we obtain that

$$\rho_0 \geq \frac{d}{c}(\Upsilon_{\infty,T,c} - \Upsilon_{0,T,c}) \geq \frac{\beta^2}{c}(\Upsilon_{\infty,T,c} - \Upsilon_{0,T,c}),$$

which contradicts $\rho_0 < \frac{\beta^2}{c}(\Upsilon_{\infty,T,c} - \Upsilon_{0,T,c})$. Thus, we must have $u_n \to u_\epsilon$ in $H^1(\mathbb{R}^N)$. □

4.1. Multiplicity result

In this section, we manage to prove our multiplicity result and borrow some arguments from [1]. In what follows, we fix $\widetilde{\rho}, \widetilde{r} > 0$ satisfying

- $\overline{B_{\widetilde{\rho}}(a_i)} \cap \overline{B_{\widetilde{\rho}}(a_j)} = \emptyset$, for i, j and a_i, a_j are defined in (V_3).
- $\bigcup_{i=1}^{l} B_{\widetilde{\rho}}(a_i) \subset B_{\widetilde{r}}(0)$.
- $K_{\widetilde{\rho}/2} = \bigcup_{i=1}^{l} \overline{B_{\widetilde{\rho}/2}(a_i)}$.

We set the function $Q_\epsilon \colon H^1(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ by

$$Q_\epsilon(u) := \frac{\int_{\mathbb{R}^N} \chi(\epsilon x)|u|^2\,dx}{\int_{\mathbb{R}^N} |u|^2\,dx},$$

where $\chi \colon \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\chi(x) := \begin{cases} x & \text{if } |x| \leq \widetilde{r}, \\ \widetilde{r}\frac{x}{|x|} & \text{if } |x| > \widetilde{r}. \end{cases}$$

The following two lemmas will be instrumental in generating (PS) sequences for $I_{\epsilon,T}$ within the constraints of S_c.

Lemma 4.6. *There exist $\epsilon_1 \in (0,\epsilon_0]$ and $\rho_1 \in (0,\rho_0]$ such that if $\epsilon \in (0,\epsilon_1)$, $u \in S_c$ and $I_{\epsilon,T}(u) \leq \Upsilon_{0,T,c} + \rho_1$, then*

$$Q_\epsilon(u) \in K_{\widetilde{\rho}/2}.$$

Proof. If not, there exist sequences $\rho_n \to 0$, $\epsilon_n \to 0$ and $\{u_n\} \subset S_c$ such that

$$I_{\epsilon_n,T}(u_n) \leq \Upsilon_{0,T,c} + \rho_n, \quad Q_{\epsilon_n}(u_n) \notin K_{\widetilde{\rho}/2}. \tag{4.8}$$

Consequently, we have

$$\Upsilon_{0,T,c} \leq I_{0,T}(u_n) \leq I_{\epsilon_n,T}(u_n) \leq \Upsilon_{0,T,c} + \rho_n,$$

thus, $\{u_n\} \subset S_c$ and $I_{0,T}(u_n) \to \Upsilon_{0,T,c}$. According to Lemma 3.8, we have two cases:

(i) $u_n \to u$ in $H^1(\mathbb{R}^N)$ for some $u \in S_c$, or

(ii) There exists $\{y_n\} \subset \mathbb{R}^N$ with $|y_n| \to +\infty$ such that $v_n(x) = u_n(x+y_n)$ converges to some $v \in S_c$ in $H^1(\mathbb{R}^N)$.

Analysis of (i): Applying the Lebesgue dominated convergence theorem, it follows that

$$Q_{\epsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \chi(\epsilon_n x)|u_n|^2\,dx}{\int_{\mathbb{R}^N} |u_n|^2\,dx} \to \frac{\int_{\mathbb{R}^N} \chi(0)|u|^2\,dx}{\int_{\mathbb{R}^N} |u|^2\,dx} = 0 \in K_{\widetilde{\rho}/2},$$

which contradicts to $Q_{\epsilon_n}(u_n) \notin K_{\widetilde{\rho}/2}$.

Analysis of (ii): Now, we will examine two cases, (I) $|\epsilon_n y_n| \to +\infty$ and (II) $\epsilon_n y_n \to y$ for some $y \in \mathbb{R}^N$.

If (I) holds, as the limit $v_n \to v$ in $H^1(\mathbb{R}^N)$, we obtain

$$\begin{aligned} I_{\epsilon_n,T}(u_n) &= \frac{1}{2}\|\nabla v_n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\epsilon_n x + \epsilon_n y_n)|v_n|^2\,dx - \frac{\mu}{q}\|v_n\|_q^q - \frac{\tau(\|\nabla v_n\|_2)}{p}\|v_n\|_p^p \\ &\to I_{\infty,T}(v). \end{aligned} \tag{4.9}$$

Since $I_{\epsilon_n,T}(u_n) \le \Upsilon_{0,T,c} + \rho_n$, we conclude that $\Upsilon_{\infty,T,c} \le I_{\infty,T}(v) \le \Upsilon_{0,T,c}$, which contradicts $\Upsilon_{\infty,T,c} > \Upsilon_{0,T,c}$.

If (II) holds, similar to (4.9), $I_{\epsilon_n,T}(u_n) \to I_{y,T}(v)$, which combined with $I_{\epsilon_n,T}(u_n) \le \Upsilon_{0,T,c} + \rho_n$ imply that $\Upsilon_{y,T,c} \le I_{y,T}(v) \le \Upsilon_{0,T,c}$. According to Corollary 3.10, it follows that $V(y) = V_0$ and $y = a_i$ for some $i = 1, 2, \dots, l$. Consequently,

$$Q_{\epsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \chi(\epsilon_n x + \epsilon_n y_n)|v_n|^2\,dx}{\int_{\mathbb{R}^N} |v_n|^2\,dx} \to \frac{\int_{\mathbb{R}^N} \chi(y)|v|^2\,dx}{\int_{\mathbb{R}^N} |v|^2\,dx} = a_i \in K_{\widetilde{\rho}/2},$$

which implies that $Q_{\epsilon_n}(u_n) \in K_{\widetilde{\rho}/2}$ for n large enough. This contradicts to (4.8) and completes the proof. $\square$

From now on, we will use the following notations:

- $\theta_\epsilon^i := \{u \in S_c : |Q_\epsilon(u) - a_i| < \widetilde{\rho}\}$, $\partial\theta_\epsilon^i := \{u \in S_c : |Q_\epsilon(u) - a_i| = \widetilde{\rho}\}$.
- $\beta_\epsilon^i := \inf_{u\in\theta_\epsilon^i} I_{\epsilon,T}(u)$, $\widetilde{\beta}_\epsilon^i := \inf_{u\in\partial\theta_\epsilon^i} I_{\epsilon,T}(u)$.

Lemma 4.7. *There exists $\epsilon_2 \in (0, \epsilon_1]$ such that*

$$\beta_\epsilon^i < \Upsilon_{0,T,c} + \frac{\rho_1}{2} \quad \textit{and} \quad \beta_\epsilon^i < \widetilde{\beta}_\epsilon^i \quad \textit{for any } \epsilon \in (0, \epsilon_2). \tag{4.10}$$

Proof. In what follows, let $u \in S_c$ satisfy $I_{0,T}(u) = \Upsilon_{0,T,c}$. For $1 \le i \le l$, we define the function $\widehat{u}_\epsilon^i \colon \mathbb{R}^N \to \mathbb{R}$ as

$$\widehat{u}_\epsilon^i(\cdot) := u\left(\cdot - \frac{a_i}{\epsilon}\right).$$

Therefore, $\widehat{u}_\epsilon^i \in S_c$ for all $\epsilon > 0$ and $1 \le i \le l$. Through a straightforward change of variable, it can be shown that

$$I_{\epsilon,T}(\widehat{u}_\epsilon^i) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\epsilon x + a_i)|u|^2\,dx - \frac{\mu}{q}\|u\|_q^q - \frac{\tau(\|\nabla u\|_2)}{p}\|u\|_p^p,$$

and

$$\lim_{\epsilon\to 0^+} I_{\epsilon,T}(\widehat{u}_\epsilon^i) = I_{a_i,T}(u) = I_{0,T}(u) = \Upsilon_{0,T,c}. \tag{4.11}$$

Note that, as $\epsilon \to 0^+$, $Q_\epsilon(\widehat{u}_\epsilon^i) \to a_i$, this implies that $\widehat{u}_\epsilon^i \in \theta_\epsilon^i$ when ϵ is sufficiently small. According to (4.11), there exists $\epsilon_2 \in (0, \epsilon_1]$ such that

$$\beta_\epsilon^i < \Upsilon_{0,T,c} + \frac{\rho_1}{2} \quad \text{for any } \epsilon \in (0, \epsilon_2),$$

indicating the first inequality in (4.10).

For any $v \in \partial\theta_\epsilon^i$, we can conclude that $Q_\epsilon(v) \notin K_{\widetilde{\rho}/2}$. Thus, by Lemma 4.6, one has

$$I_{\epsilon,T}(v) > \Upsilon_{0,T,c} + \rho_1 \quad \text{for all } v \in \partial\theta_\epsilon^i \text{ and } \epsilon \in (0, \epsilon_2).$$

This implies that $\widetilde{\beta}_\epsilon^i = \inf_{v\in\partial\theta_\epsilon^i} I_{\epsilon,T}(v) \ge \Upsilon_{0,T,c} + \rho_1$ for all $\epsilon \in (0, \epsilon_2)$. Consequently,

$$\beta_\epsilon^i < \widetilde{\beta}_\epsilon^i \quad \text{for all } \epsilon \in (0, \epsilon_2).$$

This completes the proof. □

Proof of Theorem 1.1. Let $\epsilon \in (0, \widetilde{\epsilon})$ where $\widetilde{\epsilon} := \epsilon_2$ as determined in Lemma 4.7. For each $i \in \{1, 2, \ldots, l\}$, we can apply the Ekeland's variational principle to find a sequence $\{u_n^i\} \subset \theta_\epsilon^i$ satisfying

$$I_{\epsilon,T}(u_n^i) \to \beta_\epsilon^i \quad \text{and} \quad \big\|I_{\epsilon,T}|'_{S_c}(u_n^i)\big\|_{H^{-1}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to +\infty.$$

In other words, $\{u_n^i\}$ is a $(PS)_{\beta_\epsilon^i}$ sequence for $I_{\epsilon,T}$ when restricted on S_c. Because $\beta_\epsilon^i < \Upsilon_{0,T,c} + \rho_0$, Lemma 4.5 guarantees the existence of u^i with $u_n^i \to u^i$ in $H^1(\mathbb{R}^N)$. Therefore,

$$u^i \in \theta_\epsilon^i, \quad I_{\epsilon,T}(u^i) = \beta_\epsilon^i \quad \text{and} \quad I_{\epsilon,T}|'_{S_c}(u^i) = 0.$$

For

$$Q_\epsilon(u^i) \in \overline{B_{\widetilde{\rho}}(a_i)}, \quad Q_\epsilon(u^j) \in \overline{B_{\widetilde{\rho}}(a_j)} \quad \text{and} \quad \overline{B_{\widetilde{\rho}}(a_i)} \cap \overline{B_{\widetilde{\rho}}(a_j)} = \emptyset \quad \text{for } i \ne j,$$

then $u^i \not\equiv u^j$ for $i \ne j$, where $1 \le i, j \le l$. This argument shows that $I_{\epsilon,T}$ possesses at least l nontrivial critical points for any $\epsilon \in (0, \widetilde{\epsilon})$.

Using Lemma 4.2 and the fact that $I_{\epsilon,T}(u^i) < 0$ for any $i = 1, 2, \ldots, l$, it becomes evident that u^i are in fact the critical points of I_ϵ restricted on S_c with $I_\epsilon(u^i) = \beta_\epsilon^i < 0$ and $I'_\epsilon(u^i)u^i = \lambda_i c$. Then, we deduce that

$$\frac{1}{2}\lambda_i c = I_\epsilon(u^i) + \left(\frac{1}{q} - \frac{1}{2}\right)\mu\|u^i\|_q^q + \left(\frac{1}{p} - \frac{1}{2}\right)\|u^i\|_p^p.$$

This implies that $\lambda_i < 0$ for $i = 1, 2, \ldots, l$. The proof of Theorem 1.1 is completed. □

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