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The Primitive Ideal Space of the Partial-isometric Crossed Product by Automorphic Actions of the Semigroup $\mathbb{N}^2$

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Abstract. Let $(A, \mathbb{N}^2, \alpha)$ be a dynamical system consisting of a C^*-algebra A and an action α of $\mathbb{N}^2$ on A by automorphisms. Let $A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2$ be the partial-isometric crossed product of the system. We apply the fact that it is a full corner of a crossed product by the group $\mathbb{Z}^2$ in order to give a complete description of its primitive ideal space.

1. Introduction

The C^*-algebras associated with semigroup dynamical systems have been extensively studied in recent years. Recall that a semigroup dynamical system is a trio (A, P, α) consisting of a C^*-algebra A and an action α of a (unital) semigroup P on A by endomorphisms. The C^*-algebra B corresponding to the system (A, P, α) that we study in the present work is the one in which the endomorphisms α_s, where $s \in P$, are implemented by partial-isometries. It is universal for covariant representations of the system which means that its (nondegenerate) representations are in bijective correspondence with covariant representations of the system. However, this construction requires some conditions on the system as well as the semigroup P. In [5], P is considered to be the positive cone of a group G such that (G, P) is quasi-lattice ordered in the sense of Nica [9], and the algebra B is called the *Nica–Toeplitz crossed product* of the system (A, P, α). This algebra is then studied in [7] for the positive cones of totally ordered abelian groups and called the *partial-isometric crossed product* of the system (A, P, α). Note that the algebra B is denoted by $A \times_\alpha^{\mathrm{piso}} P$. Finally, following these efforts, the study of the semigroup crossed product $A \times_\alpha^{\mathrm{piso}} P$ is extended to (left) LCM semigroups in [14]. Next, to understand about the algebra $A \times_\alpha^{\mathrm{piso}} P$ more, we would like to investigate its ideal structure. From [13], if P is the positive cone of an abelian lattice-ordered group G, then $A \times_\alpha^{\mathrm{piso}} P$ is a full corner in a classical crossed product by the group G. Therefore, this corner realization provides a way through our investigations. This is due to the fact that if a C^*-algebra $\mathcal{B}$ is a full corner in a C^*-algebra $\mathcal{A}$, then they are Morita equivalent, and hence, their primitive

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ideal spaces are homeomorphic (see [10]). Moreover, under some certain conditions, the description of the primitive ideal space of group crossed products is available in earlier works such as [4,11]. We recall that the primitive ideal space of a C^*-algebra $\mathcal{A}$ is denoted by $\operatorname{Prim}\mathcal{A}$. To study the theory of the partial-isometric crossed products, readers may refer to [5,7,14] as a preliminary background. Further studies in this regard are available in [1–3,6,12,13,15].

Now, in the present work, we consider the dynamical system $(A, \mathbb{N}^2, \alpha)$ in which $\mathbb{N}^2$ denotes the positive cone of the (abelian lattice-ordered) group $\mathbb{Z}^2$, A is a C^*-algebra, which is not necessarily unital, and α is an action of $\mathbb{N}^2$ on A by automorphisms such that $\alpha_0 = \mathrm{id}$. Our goal is to describe the primitive ideal space of the partial-isometric crossed product $A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2$ of the system and its hull-kernel (Jacobson) topology completely. To do so, since $A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2$ is a full corner in a group crossed product $(B_{\mathbb{Z}^2} \otimes A) \rtimes \mathbb{Z}^2$ (see [13, §5]), it suffices to describe $\operatorname{Prim}((B_{\mathbb{Z}^2} \otimes A) \rtimes \mathbb{Z}^2)$, for which, we then apply the works on the ideal structure of crossed products by groups available in [4,11]. So, we need to consider the following two conditions:

(i) when A is separable and abelian;

(ii) when A is separable and $\mathbb{Z}^2$ acts on $\operatorname{Prim} A$ freely.

Under the first condition, we apply [11, Theorem 8.39] to see that $\operatorname{Prim}((B_{\mathbb{Z}^2} \otimes A) \rtimes \mathbb{Z}^2)$ is homeomorphic to a quotient of the product space

$$\Delta(B_{\mathbb{Z}^2}) \times \Delta(A) \times \widehat{\mathbb{Z}^2} = \Delta(B_{\mathbb{Z}}) \times \Delta(B_{\mathbb{Z}}) \times \Delta(A) \times \mathbb{T}^2, \tag{1.1}$$

where $\Delta(B_{\mathbb{Z}})$ and $\Delta(A)$ are the spectrums of the (abelian) C^*-algebras $B_{\mathbb{Z}}$ and A, respectively. Then, the quotient space is identified by the disjoint union

$$\Delta(A) \sqcup \operatorname{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z}) \sqcup \operatorname{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z}) \sqcup \operatorname{Prim}(A \rtimes_{\alpha} \mathbb{Z}^2) \tag{1.2}$$

through parameterizing the equivalent classes, where $\dot{\alpha}$ and $\ddot{\alpha}$ are two automorphisms corresponding to two generators of the group $\mathbb{Z}^2$. Finally, the open sets in (1.2) are precisely identified by using the fact the quotient map of (1.1) onto (1.2) is open (see [11, Remark 8.40]). Under the second condition, we apply [4, Corollary 7.35] to see that $\operatorname{Prim}((B_{\mathbb{Z}^2} \otimes A) \rtimes \mathbb{Z}^2)$ is homeomorphic to a quotient of the product space

$$\operatorname{Prim}(B_{\mathbb{Z}^2} \otimes A) = \operatorname{Prim} B_{\mathbb{Z}^2} \times \operatorname{Prim} A = \operatorname{Prim} B_{\mathbb{Z}} \times \operatorname{Prim} B_{\mathbb{Z}} \times \operatorname{Prim} A.$$

This quotient space is called the *quasi-orbit space*, which, by a similar discussion to the first condition, will be described along with its quotient topology precisely. Note that [15, Corollary 3.13] already indicates that the primitive ideals of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ are coming from the four sets

$$\operatorname{Prim} A, \quad \operatorname{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z}), \quad \operatorname{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z}) \quad \text{and} \quad \operatorname{Prim}(A \rtimes_{\alpha} \mathbb{Z}^2).$$

So, all these ideals will also be identified in the present work under the conditions (i) and (ii) mentioned earlier. We would like to mention that the present work is therefore a generalization of the effort in [6] based on the results of [2].

We begin with a preliminary section containing a quick recall on some results from [13–15], and a brief discussion on the primitive ideal space of crossed products by groups from [4,11]. In Section 3, we identify the primitive ideals of the algebra $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ derived from $\operatorname{Prim} A$. In Sections 4 and 5, by applying the realization of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ as a full corner of a crossed product by the group $\mathbb{Z}^2$, $\operatorname{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2)$ is completely described under some certain conditions. Moreover, we identify all primitive ideals of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$, and provide necessary and sufficient conditions under which $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ is GCR (type I or postliminal). In Section 6, the final section, we see that $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ is primitive precisely when A is primitive.

2. Preliminaries

2.1. The algebra $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ as a full corner

First of all, since $\mathbb{Z}^2$ is an additive group, the notation "+" is used for its action in the present work.

Suppose that $(A, \mathbb{N}^2, \alpha)$ is a dynamical system consisting of a C^*-algebra A and an action α of $\mathbb{N}^2$ on A by automorphisms. Let π be a nondegenerate representation of A on a Hilbert space H. If the maps

$$\widetilde{\pi}\colon A \to B(\ell^2(\mathbb{N}^2) \otimes H) \quad \text{and} \quad V\colon \mathbb{N}^2 \to B(\ell^2(\mathbb{N}^2) \otimes H)$$

are defined by

$$(\widetilde{\pi}(a)f)(s) = \pi(\alpha_s(a))f(s) \quad \text{and} \quad (V_t f)(s) = f(s+t)$$

for all $f \in \ell^2(\mathbb{N}^2) \otimes H \simeq \ell^2(\mathbb{N}^2, H)$ and $s, t \in \mathbb{N}^2$, then the pair $(\widetilde{\pi}, V)$ is a covariant partial-isometric representation of $(A, \mathbb{N}^2, \alpha)$ on $\ell^2(\mathbb{N}^2, H)$. Moreover, if π is faithful, so is $\widetilde{\pi}$ (see [14, Example 4.3]).

Let $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ be the partial-isometric crossed product of the system $(A, \mathbb{N}^2, \alpha)$. Recall from [13, §2] (see also [15, Remark 3.11]) that we have the short exact sequence

$$0 \longrightarrow \ker q \longrightarrow A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2 \xrightarrow{\;q\;} A \rtimes_{\alpha} \mathbb{Z}^2 \longrightarrow 0 \tag{2.1}$$

of C^*-algebras, and by [15, Corollary 3.13], the algebra $\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A$ of compact operators is contained in $\ker q$ as an (essential) ideal such that

$$\ker q/[\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A] \simeq \big[\mathcal{K}(\ell^2(\mathbb{N}))\otimes (A\rtimes_{\dot{\alpha}}\mathbb{Z})\big]\oplus\big[\mathcal{K}(\ell^2(\mathbb{N}))\otimes (A\rtimes_{\ddot{\alpha}}\mathbb{Z})\big],$$

where $\dot{\alpha}$ and $\ddot{\alpha}$ are two automorphic actions corresponding to two generators of the group $\mathbb{Z}^2$. These results in [15] are obtained by applying the fact that the algebra $A\times_{\alpha}^{\mathrm{piso}}\mathbb{N}^2$ is a full corner in a crossed product by the group $\mathbb{Z}^2$, which is provided in [13]. More precisely, let $B_{\mathbb{Z}^2}$ be the C^*-subalgebra of $\ell^\infty(\mathbb{Z}^2)$ generated by the characteristic functions $\{1_x\in\ell^\infty(\mathbb{Z}^2): x\in\mathbb{Z}^2\}$, such that

$$1_x(y)=\begin{cases}1 & \text{if } x\le y,\\ 0 & \text{otherwise.}\end{cases}$$

Then, there is an action τ of $\mathbb{Z}^2$ on $B_{\mathbb{Z}^2}$ given by translation. So, the system $(A,\mathbb{N}^2,\alpha)$ gives rise to the group dynamical system $(B_{\mathbb{Z}^2}\otimes A,\mathbb{Z}^2,\tau\otimes\alpha^{-1})$. Also, if $B_{\mathbb{Z}^2,\infty}$ is the C^*-subalgebra of $B_{\mathbb{Z}^2}$ generated by the elements $\{1_x-1_y : x\le y\in\mathbb{Z}^2\}$, then it is a τ-invariant (essential) ideal of $B_{\mathbb{Z}^2}$. Now, by [13, Corollary 5.3], $A\times_{\alpha}^{\mathrm{piso}}\mathbb{N}^2$ and the ideal $\ker q$ sit in the group crossed products $(B_{\mathbb{Z}^2}\otimes A)\times_{\tau\otimes\alpha^{-1}}\mathbb{Z}^2$ and $(B_{\mathbb{Z}^2,\infty}\otimes A)\times_{\tau\otimes\alpha^{-1}}\mathbb{Z}^2$, respectively, as full corners. Thus, the information on $A\times_{\alpha}^{\mathrm{piso}}\mathbb{N}^2$ in [15] are indeed imported from the group crossed product $(B_{\mathbb{Z}^2}\otimes A)\times_{\tau\otimes\alpha^{-1}}\mathbb{Z}^2$.

2.2. The primitive ideal space of crossed products by groups

Suppose that G is an abelian countable discrete group which acts on a second countable locally compact Hausdorff space X. So, the pair (G,X) is a second countable locally compact transformation group, which gives rise to the separable group dynamical system $(C_0(X),G,\mathrm{lt})$ with G abelian. If $C_0(X)\rtimes_{\mathrm{lt}}G$ is the group crossed product of the system, then its primitive ideals are known by [11, Theorem 8.21], and a complete description of the topology of $\mathrm{Prim}(C_0(X)\rtimes_{\mathrm{lt}}G)$ is available in [11, Theorem 8.39]. In brief, for every $x\in X$, let

$$\varepsilon_x\colon C_0(X)\to\mathbb{C}$$

be the evaluation map at x, and the sets

$$G\cdot x:=\{t\cdot x : t\in G\}\quad\text{and}\quad G_x:=\{t\in G : t\cdot x=x\}$$

the *G-orbit* and the *stability group* of x, respectively. Now, there is an equivalence relation on the product space $X\times\widehat{G}$ such that $(x,\gamma)\sim(y,\mu)$ if

$$\overline{G\cdot x}=\overline{G\cdot y}\ \text{ (which implies that } G_x=G_y)\quad\text{and}\quad \gamma|_{G_x}=\mu|_{G_x}.$$

Let $X\times\widehat{G}/\!\sim$ be the quotient space equipped with the quotient topology. We have

Theorem 2.1. [11, Theorem 8.39] *Let (G, X) be a second countable locally compact transformation group with G abelian. Then, the map*

$$\Phi\colon X \times \widehat{G} \to \mathrm{Prim}(C_0(X) \rtimes_{\mathrm{lt}} G)$$

defined by

$$\Phi(x, \gamma) := \ker\left(\mathrm{Ind}_{G_x}^{G}(\varepsilon_x \rtimes \gamma|_{G_x})\right)$$

is a continuous and open surjection which factors through a homeomorphism of $X \times \widehat{G}/\!\sim$ onto $\mathrm{Prim}(C_0(X) \rtimes_{\mathrm{lt}} G)$.

To see more details, interested readers are referred to [11].

Next, recall that if G is a (discrete) group with the unit element e which acts on a topological space Z, then the action of G on Z is called *free* (or we say G acts on Z *freely*) if all stability groups are just the trivial subgroup $\{e\}$. Also, there is an equivalence relation $\sim$ on Z such that

$$z_1 \sim z_2 \quad \Longleftrightarrow \quad \overline{G \cdot z_1} = \overline{G \cdot z_2}$$

for all $z_1, z_2 \in Z$. If $\mathcal{O}(Z)$ denotes the set of all equivalence classes, then it is called the *quasi-orbit space* when equipped with the quotient topology, which is always a T_0-topological space. The equivalence class of each $z \in Z$ is called the *quasi-orbit* of z and denoted by $\mathcal{O}(z)$. As an example, if (A, G, α) is a group dynamical system, then we can talk about the quasi-orbit space $\mathcal{O}(\mathrm{Prim}\, A)$. This is due to the fact that the system defines an action of G on $\mathrm{Prim}\, A$ by

$$t \cdot P := \alpha_t(P) = \{\alpha_t(a) : a \in P\}$$

for all $t \in G$ and $P \in \mathrm{Prim}\, A$.

Let (A, G, α) be a group dynamical system and π a nondegenerate representation of A on a Hilbert space H such that $\ker \pi = J$. Recall that there is a covariant representation $(\widetilde{\pi}, U)$ of (A, G, α) on the Hilbert space $\ell^2(G, H) \simeq \ell^2(G) \otimes H$ defined by

$$(\widetilde{\pi}(a)f)(s) = \pi(\alpha_{s^{-1}}(a))f(s) \quad \text{and} \quad (U_t f)(s) = f(t^{-1}s)$$

for every $a \in A$, $f \in \ell^2(G, H)$, and $s, t \in G$. The corresponding (nondegenerate) representation $\widetilde{\pi} \rtimes U$ of the crossed product $A \rtimes_\alpha G$ of the system is denoted by $\mathrm{Ind}\, \pi$, and $\ker(\mathrm{Ind}\, \pi)$ by $\mathrm{Ind}\, J = \mathrm{Ind}(\ker \pi)$. Now, if in the system (A, G, α), A is separable and G is an abelian discrete countable group, which acts on $\mathrm{Prim}\, A$ freely, then each primitive idea of $A \rtimes_\alpha G$ is of the form $\mathrm{Ind}\, P$ induced by a primitive ideal P of A. More precisely,

Theorem 2.2. [4, Corollary 7.35] *Let (A, G, α) be a dynamical system in which A is separable and G is an amenable discrete countable group. If G acts on* $\operatorname{Prim} A$ *freely, then the map*

$$\begin{aligned} \mathcal{O}(\operatorname{Prim} A) \quad &\to \quad \operatorname{Prim}(A \rtimes_\alpha G) \\ \mathcal{O}(P) \quad &\mapsto \quad \operatorname{Ind} P = \ker(\operatorname{Ind} \pi) \end{aligned}$$

is a homeomorphism, where π is an irreducible representation of A with $\ker \pi = P$*. In particular, $A \rtimes_\alpha G$ is simple if and only if every G-orbit is dense in* $\operatorname{Prim} A$*, and $A \rtimes_\alpha G$ is primitive if and only if there exists a dense G-orbit in* $\operatorname{Prim} A$*.*

Recall that we say a C^*-algebra is *simple* if it does not have any nontrivial ideal. It is called *primitive* if it has a faithful nonzero irreducible representation, in other words, the zero ideal is a primitive ideal of it.

3. Primitive ideals of $A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2$ derived from $\operatorname{Prim} A$

Let $(A, \mathbb{N}^2, \alpha)$ be a dynamical system consisting of a C^*-algebra A and an action α of $\mathbb{N}^2$ on A by automorphisms. Suppose that $A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2$ is the partial-isometric crossed product of the system. Since it is a full corner in the group crossed product $(B_{\mathbb{Z}^2} \otimes A) \rtimes_{\tau \otimes \alpha^{-1}} \mathbb{Z}^2$ by [13, Corollary 5.3], in order to describe $\operatorname{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2)$, it is enough to describe $\operatorname{Prim}((B_{\mathbb{Z}^2} \otimes A) \rtimes_{\tau \otimes \alpha^{-1}} \mathbb{Z}^2)$ and its topology. First of all, since the algebra $\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A$ of compact operators sits in $A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2$ as an essential ideal (see [15, Corollary 3.13]),

$$\operatorname{Prim}(\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A) \simeq \operatorname{Prim} A$$

sits in $\operatorname{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2)$ as an open dense subset. More precisely, there is a homeomorphism of $\operatorname{Prim} A$ onto the open dense subset

$$U := \left\{\mathcal{I} \in \operatorname{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2) : \mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A \not\subset \mathcal{I}\right\}$$

of $\operatorname{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2)$. So, we first identify the elements of U, namely, the primitive ideals of $A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2$ derived from $\operatorname{Prim} A$. Then, in order to identify other primitive ideals of $A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2$ derived from the other three sets

$$\operatorname{Prim}(A \rtimes_{\dot\alpha} \mathbb{Z}), \quad \operatorname{Prim}(A \rtimes_{\ddot\alpha} \mathbb{Z}) \quad \text{and} \quad \operatorname{Prim}(A \rtimes_\alpha \mathbb{Z}^2) \quad (\text{see §2 or §1}),$$

and describe the topology of

$$\operatorname{Prim}((B_{\mathbb{Z}^2} \otimes A) \rtimes_{\tau \otimes \alpha^{-1}} \mathbb{Z}^2) \simeq \operatorname{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2),$$

we will consider some conditions on the system $(A, \mathbb{N}^2, \alpha)$ so that we can apply the results of [4, 11].

Now, the following proposition identifies the elements of U.

Proposition 3.1. *Let $\pi\colon A \to B(H)$ be a nonzero irreducible representation of A such that $P = \ker\pi$. If $(\widetilde{\pi}, V)$ is the pair defined in* [14, Example 4.3] *(see* §2*), then the corresponding representation $\widetilde{\pi}\times V$ of $(A\times_\alpha^{\mathrm{piso}}\mathbb{N}^2, i_A, i_{\mathbb{N}^2})$ is irreducible on $\ell^2(\mathbb{N}^2)\otimes H$ which lives on $\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A$.*

Proof. We first show that the restriction of $\widetilde{\pi}\times V$ to the (essential) ideal $\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A \simeq \mathcal{K}(\ell^2(\mathbb{N}^2)\otimes A)$ is the representation $\mathrm{id}\otimes\pi$. Let $\{e_{(m,n)} : (m,n)\in\mathbb{N}^2\}$ be the usual orthonormal basis of $\ell^2(\mathbb{N}^2)$, and $\xi_{(m,n)}^{(x,y)}(a)$ denote the element

$$i_{\mathbb{N}^2}(m,n)^* i_A(a)\big[1 - i_{\mathbb{N}^2}(1,0)^* i_{\mathbb{N}^2}(1,0)\big] i_{\mathbb{N}^2}(x,y)$$

of the algebra $A\times_\alpha^{\mathrm{piso}}\mathbb{N}^2$. Recall that, by [15, Lemma 3.8], the elements of the form

$$\xi_{(m,n)}^{(x,y)}(a) - \xi_{(m,n+1)}^{(x,y+1)}(\alpha_{(0,1)}(a))$$

span an (essential) ideal L of $A\times_\alpha^{\mathrm{piso}}\mathbb{N}^2$ which is isomorphic to the algebra $\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A$ of compact operators via an isomorphism φ, such that

$$\varphi\big((e_{(m,n)}\otimes\overline{e_{(x,y)}})\otimes ab^*\big) = \xi_{(m,n)}^{(x,y)}(ab^*) - \xi_{(m,n+1)}^{(x,y+1)}(\alpha_{(0,1)}(ab^*))$$

for all $a,b\in A$ and $(m,n),(x,y)\in\mathbb{N}^2$ (see [15, Theorem 3.10]), where $(e_{(m,n)}\otimes\overline{e_{(x,y)}})$ is the rank-one operator on $\ell^2(\mathbb{N}^2)$ defined by $g\mapsto\langle g|e_{(x,y)}\rangle e_{(m,n)}$. Now, by calculation on spanning elements, we have

$$\begin{aligned}
&(\widetilde{\pi}\times V)\left(\xi_{(m,n)}^{(x,y)}(ab^*) - \xi_{(m,n+1)}^{(x,y+1)}(\alpha_{(0,1)}(ab^*))\right)(e_{(r,s)}\otimes h)\\
&= [(\widetilde{\pi}\times V)\circ\varphi]\big((e_{(m,n)}\otimes\overline{e_{(x,y)}})\otimes ab^*\big)(e_{(r,s)}\otimes h)\\
&= (\mathrm{id}\otimes\pi)\big((e_{(m,n)}\otimes\overline{e_{(x,y)}})\otimes ab^*\big)(e_{(r,s)}\otimes h)
\end{aligned}\tag{3.1}$$

for all $a,b\in A$, $(m,n),(x,y),(r,s)\in\mathbb{N}^2$, and $h\in H$. Thus, $(\widetilde{\pi}\times V)|_{\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A} = \mathrm{id}\otimes\pi$, from which, since the representation π is nonzero, it follows that $\widetilde{\pi}\times V$ lives on the ideal $\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A$.

At last, to see that the representation $\widetilde{\pi}\times V$ is irreducible, let f be a nonzero vector in $\ell^2(\mathbb{N}^2)\otimes H\simeq\ell^2(\mathbb{N}^2, H)$. So, there is $(x,y)\in\mathbb{N}^2$ such that $f(x,y)\neq 0$, and hence, since the representation π is irreducible, the nonzero vector $f(x,y)$ of H is a cyclic vector for π. Thus, it follows that

$$\ell^2(\mathbb{N}^2)\otimes H = \overline{\mathrm{span}}\big\{e_{(m,n)}\otimes(\pi(a)f(x,y)) : (m,n)\in\mathbb{N}^2, a\in A\big\}.$$

However, we have

$$e_{(m,n)}\otimes(\pi(a)f(x,y)) = (\widetilde{\pi}\times V)\left(\xi_{(m,n)}^{(x,y)}(a) - \xi_{(m,n+1)}^{(x,y+1)}(\alpha_{(0,1)}(a))\right)f \quad \text{(see (3.1))},$$

which implies that

$$\ell^2(\mathbb{N}^2) \otimes H = \overline{\operatorname{span}}\big\{(\widetilde{\pi} \times V)(\xi)f : \xi \in A \times_\alpha^{\text{piso}} \mathbb{N}^2\big\}.$$

So, every nonzero vector f of $\ell^2(\mathbb{N}^2) \otimes H$ is a cyclic vector for $\widetilde{\pi} \times V$, and therefore, $\widetilde{\pi} \times V$ is irreducible. This completes the proof. $\square$

Remark 3.2. Therefore, by Proposition 3.1, each element of U is the kernel of an irreducible representation $\widetilde{\pi} \times V$ induced by a primitive ideal $P = \ker \pi$ of A. So, we denote $\ker(\widetilde{\pi} \times V)$ by $\operatorname{ind} P$, and hence,

$$U = \{\operatorname{ind} P : P \in \operatorname{Prim} A\},$$

which is homeomorphic to $\operatorname{Prim} A$ via the homeomorphism $P \mapsto \operatorname{ind} P$. This homeomorphism is obtained by the composition of homeomorphisms

$$\operatorname{ind} P \in U \mapsto \operatorname{ind} P \cap (\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A) \in \operatorname{Prim}(\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A),$$

where

$$\begin{aligned}\operatorname{ind} P \cap (\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A) &= \ker((\widetilde{\pi} \times V)|_{\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A}) \\ &= \ker(\operatorname{id} \otimes \pi) = \mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes P,\end{aligned}$$

and

$$P \in \operatorname{Prim} A \mapsto \mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes P \in \operatorname{Prim}(\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A) \quad \text{(the Rieffel homeomorphism).}$$

Remark 3.3. One can immediately see that $A \times_\alpha^{\text{piso}} \mathbb{N}^2$ is not simple as it contains the algebra $\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A$ as a proper nonzero ideal and $A \neq 0$.

4. The topology of $\operatorname{Prim}(A \times_\alpha^{\text{piso}} \mathbb{N}^2)$ when A is abelian and separable

First, recall that $\widehat{\mathbb{Z}}$, the dual of the group $\mathbb{Z}$, is isomorphic to $\mathbb{T}$ via the map $z \in \mathbb{T} \mapsto \gamma_z \in \widehat{\mathbb{Z}}$, such that $\gamma_z(n) = z^n$ for all $n \in \mathbb{Z}$. Therefore, $\widehat{\mathbb{Z}^2} \simeq \mathbb{T}^2$ via the isomorphism $(z_1, z_2) \in \mathbb{T}^2 \mapsto \gamma_{(z_1,z_2)} \in \widehat{\mathbb{Z}^2}$, such that

$$\gamma_{(z_1,z_2)}(m,n) = z_1^m z_2^n = \gamma_{z_1}(m)\gamma_{z_2}(n).$$

Now, if in the system $(A, \mathbb{N}^2, \alpha)$, A is abelian and separable, then $(B_{\mathbb{Z}^2} \otimes A) \rtimes_{\tau\otimes\alpha^{-1}} \mathbb{Z}^2$ is isomorphic to the crossed product $C_0(X) \rtimes_{\text{lt}} \mathbb{Z}^2$ associated with the second countable locally compact transformation group $(\mathbb{Z}^2, X)$, where $X = \Delta(B_{\mathbb{Z}^2} \otimes A)$ is the spectrum of the (abelian) algebra $B_{\mathbb{Z}^2} \otimes A$. Thus, by Theorem 2.1,

$$\operatorname{Prim}((B_{\mathbb{Z}^2} \otimes A) \rtimes_{\tau\otimes\alpha^{-1}} \mathbb{Z}^2) \simeq \operatorname{Prim}(A \times_\alpha^{\text{piso}} \mathbb{N}^2)$$

is homeomorphic to the quotient space $X \times \mathbb{T}^2/\sim$. In order to get a precise description of $X \times \mathbb{T}^2/\sim$, firstly, since

$$B_{\mathbb{Z}^2} = B_{(\mathbb{Z}\times\mathbb{Z})} \simeq B_{\mathbb{Z}} \otimes B_{\mathbb{Z}} \ \text{(this isomorphisms intertwines the actions } \tau \text{ and } (\mathrm{lt}\otimes\mathrm{lt})\text{)},$$

it follows by [10, Theorem B.45] (or [10, Theorem B.37]) that

$$X \simeq \Delta(B_{\mathbb{Z}}) \times \Delta(B_{\mathbb{Z}}) \times \Delta(A).$$

Moreover, by [6, Lemma 3.3], $\Delta(B_{\mathbb{Z}})$ is homeomorphic to the open dense subset $\mathbb{Z}\cup\{\infty\}$ of the two-point compactification $\{-\infty\}\cup\mathbb{Z}\cup\{\infty\}$ of $\mathbb{Z}$. Therefore, we actually need to describe

$$\big[(\mathbb{Z}\cup\{\infty\}) \times (\mathbb{Z}\cup\{\infty\}) \times \Delta(A)\big] \times \mathbb{T}^2/\sim . \tag{4.1}$$

To do so, we first need to see that how the group $\mathbb{Z}^2$ acts on the product space

$$X \simeq (\mathbb{Z}\cup\{\infty\}) \times (\mathbb{Z}\cup\{\infty\}) \times \Delta(A). \tag{4.2}$$

For every $(m,n)\in\mathbb{Z}^2$, $\phi\in\Delta(A)$, and $r,s\in\mathbb{Z}$, we have

$$(m,n)\cdot((r,s),\phi) = ((r+m,s+n),\phi).$$

This is due to the fact that $((r,s),\phi)$ is an element of the spectrum of the (essential) ideal

$$C_0(\mathbb{Z}^2)\otimes A \simeq C_0(\mathbb{Z})\otimes C_0(\mathbb{Z})\otimes A$$

of the algebra $B_{\mathbb{Z}^2}\otimes A$, which is invariant under the action $\tau\otimes\alpha^{-1}$. Therefore, the crossed product $(C_0(\mathbb{Z}^2)\otimes A)\rtimes_{\tau\otimes\alpha^{-1}}\mathbb{Z}^2$ sits in the algebra $(B_{\mathbb{Z}^2}\otimes A)\rtimes_{\tau\otimes\alpha^{-1}}\mathbb{Z}^2$ as an (essential) ideal (see [15, Theorem 3.7]). Furthermore, by [11, Lemma 7.4], we have

$$(C_0(\mathbb{Z}^2)\otimes A)\rtimes_{\tau\otimes\alpha^{-1}}\mathbb{Z}^2 \simeq (C_0(\mathbb{Z}^2)\otimes A)\rtimes_{\tau\otimes\mathrm{id}}\mathbb{Z}^2 \simeq \mathcal{K}(\ell^2(\mathbb{Z}^2))\otimes A.$$

Next, for every $(m,n)\in\mathbb{Z}^2$ and $\phi\in\Delta(A)$,

$$(m,n)\cdot((\infty,\infty),\phi) = ((\infty+m,\infty+n),\phi\circ\alpha_{(m,n)}) = ((\infty,\infty),\phi\circ\alpha_{(m,n)}).$$

Finally, to see that how $\mathbb{Z}^2$ acts on the elements of the forms $((r,\infty),\phi)$ and $((\infty,r),\phi)$, where $r\in\mathbb{Z}$ and $\phi\in\Delta(A)$, first note that the action α of $\mathbb{Z}^2$ induces two actions $\dot\alpha$ and $\ddot\alpha$ (corresponding to two generators of the group $\mathbb{Z}^2$) of (the subgroup) $\mathbb{Z}$ on A by automorphisms, such that

$$\dot\alpha_n := \alpha_{(n,0)} \quad \text{and} \quad \ddot\alpha_n := \alpha_{(0,n)}$$

for every $n \in \mathbb{Z}$. Thus, there are two group crossed products $A \rtimes_{\dot{\alpha}} \mathbb{Z}$ and $A \rtimes_{\ddot{\alpha}} \mathbb{Z}$, correspondingly. The primitive ideal spaces of them are quotients of the space $\Delta(A) \times \mathbb{T}$ which we denote them by $\Delta(A) \times \mathbb{T}/\sim^{(1)}$ and $\Delta(A) \times \mathbb{T}/\sim^{(2)}$, respectively. Now, $((r, \infty), \phi)$ is an element of the spectrum of the (essential) ideal

$$(C_0(\mathbb{Z}) \otimes B_{\mathbb{Z}}) \otimes A$$

of $B_{\mathbb{Z}^2} \otimes A \simeq (B_{\mathbb{Z}} \otimes B_{\mathbb{Z}}) \otimes A$, which is indeed invariant under the action $\tau \otimes \alpha^{-1}$. Moreover, there is an isomorphism

$$\psi \colon (C_0(\mathbb{Z}) \otimes B_{\mathbb{Z}}) \otimes A \to C_0(\mathbb{Z}) \otimes (B_{\mathbb{Z}} \otimes A)$$

such that

$$((1_x - 1_{x+1}) \otimes 1_y) \otimes a \mapsto (1_x - 1_{x+1}) \otimes (1_y \otimes \dot{\alpha}_x(a))$$

for all $x, y \in \mathbb{Z}$ and $a \in A$. By inspection on spanning elements, one can see that for every $(m, n) \in \mathbb{Z}^2$ the following diagram commutes:

$$\begin{array}{ccc} (C_0(\mathbb{Z}) \otimes B_{\mathbb{Z}}) \otimes A & \xrightarrow{\psi} & C_0(\mathbb{Z}) \otimes (B_{\mathbb{Z}} \otimes A) \\ {\scriptstyle \tau_{(m,n)} \otimes \alpha_{(-m,-n)}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{lt}_m \otimes (\mathrm{lt}_n \otimes \ddot{\alpha}_{-n})} \\ (C_0(\mathbb{Z}) \otimes B_{\mathbb{Z}}) \otimes A & \xrightarrow{\psi} & C_0(\mathbb{Z}) \otimes (B_{\mathbb{Z}} \otimes A). \end{array}$$

Therefore, it follows that

$$(m, n) \cdot ((r, \infty), \phi) = ((r + m, \infty + n), \phi \circ \ddot{\alpha}_n) = ((r + m, \infty), \phi \circ \alpha_{(0,n)})$$

for every $(m, n) \in \mathbb{Z}^2$, $r \in \mathbb{Z}$, and $\phi \in \Delta(A)$. Note that also, by [11, Lemma 2.65], the isomorphism ψ induces an isomorphism of the (essential) ideal

$$((C_0(\mathbb{Z}) \otimes B_{\mathbb{Z}}) \otimes A) \rtimes_{\tau \otimes \alpha^{-1}} \mathbb{Z}^2$$

of $(B_{\mathbb{Z}^2} \otimes A) \rtimes_{\tau \otimes \alpha^{-1}} \mathbb{Z}^2$ onto the algebra

$$\begin{aligned} &(C_0(\mathbb{Z}) \otimes (B_{\mathbb{Z}} \otimes A)) \rtimes_{\mathrm{lt} \otimes (\mathrm{lt} \otimes \ddot{\alpha}^{-1})} (\mathbb{Z} \times \mathbb{Z}) \\ &\simeq [C_0(\mathbb{Z}) \rtimes_{\mathrm{lt}} \mathbb{Z}] \otimes \big[(B_{\mathbb{Z}} \otimes A) \rtimes_{\mathrm{lt} \otimes \ddot{\alpha}^{-1}} \mathbb{Z}\big] \\ &\simeq \mathcal{K}(\ell^2(\mathbb{Z})) \otimes \big[(B_{\mathbb{Z}} \otimes A) \rtimes_{\mathrm{lt} \otimes \ddot{\alpha}^{-1}} \mathbb{Z}\big] \end{aligned}$$

of compact operators (see also [15, Theorem 3.7] and [15, Remark 3.11]). A similar discussion shows that

$$(m, n) \cdot ((\infty, r), \phi) = ((\infty + m, r + n), \phi \circ \dot{\alpha}_m) = ((\infty, r + n), \phi \circ \alpha_{(m,0)})$$

for every $(m, n) \in \mathbb{Z}^2$, $r \in \mathbb{Z}$, and $\phi \in \Delta(A)$.

Lemma 4.1. *The quotient space* (4.1)*, as a set, is identified by the disjoint union of four sets*

$$\Delta(A) \sqcup \mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z}) \sqcup \mathrm{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z}) \sqcup \mathrm{Prim}(A \rtimes_{\alpha} \mathbb{Z}^2), \tag{4.3}$$

where $\mathrm{Prim}(A \rtimes_{\alpha} \mathbb{Z}^2)$ *is a quotient of the product space* $\Delta(A) \times \mathbb{T}^2$*.*

Proof. For every $r, s \in \mathbb{Z}$ and $\phi \in \Delta(A)$, the stability group of the element $((r,s),\phi)$ of (4.2) is the trivial subgroup $\{(0,0)\}$, and its $\mathbb{Z}^2$-orbit is $\mathbb{Z}^2 \times \{\phi\} = \mathbb{Z} \times \mathbb{Z} \times \{\phi\}$. So, the element $((r,s),\phi,(z_1,z_2))$ of the product space

$$X \times \mathbb{T}^2 \simeq \big[(\mathbb{Z} \cup \{\infty\}) \times (\mathbb{Z} \cup \{\infty\}) \times \Delta(A)\big] \times \mathbb{T}^2 \tag{4.4}$$

can only be equivalent to an element $((m,n),\psi,(w_1,w_2))$ of the same type, and since $\Delta(A)$ is Hausdorff, we have

$$\begin{aligned}
& ((r,s),\phi,(z_1,z_2)) \sim ((m,n),\psi,(w_1,w_2)) \\
\Longleftrightarrow\ & \overline{\mathbb{Z}^2 \cdot ((r,s),\phi)} = \overline{\mathbb{Z}^2 \cdot ((m,n),\psi)} \\
\Longleftrightarrow\ & \overline{\mathbb{Z}^2 \times \{\phi\}} = \overline{\mathbb{Z}^2 \times \{\psi\}} \\
\Longleftrightarrow\ & \overline{\mathbb{Z}^2} \times \overline{\{\phi\}} = \overline{\mathbb{Z}^2} \times \overline{\{\psi\}} \\
\Longleftrightarrow\ & \overline{\mathbb{Z}^2} \times \{\phi\} = \overline{\mathbb{Z}^2} \times \{\psi\}.
\end{aligned}$$

It thus follows that

$$((r,s),\phi,(z_1,z_2)) \sim ((m,n),\psi,(w_1,w_2)) \quad \Longleftrightarrow \quad \phi = \psi.$$

Therefore, all elements $((r,s),\phi,(z_1,z_2))$ in which $\phi \in \Delta(A)$ is fixed and (r,s) and (z_1,z_2) are running in $\mathbb{Z}^2$ and $\mathbb{T}^2$, respectively, are in the same equivalence class in (4.1), which can be parameterized by $\phi \in \Delta(A)$.

Next, the stability group of the element $((\infty,\infty),\phi)$ of (4.2) equals the stability group $\mathbb{Z}^2_\phi$ of ϕ when $\mathbb{Z}^2$ acts on $\Delta(A)$ via the action α (corresponding to the crossed product $A \rtimes_\alpha \mathbb{Z}^2$). Its $\mathbb{Z}^2$-orbit is $\{(\infty,\infty)\} \times (\mathbb{Z}^2 \cdot \phi)$, where $\mathbb{Z}^2 \cdot \phi$ is the $\mathbb{Z}^2$-orbit of ϕ. Moreover, the element $((\infty,\infty),\phi,(z_1,z_2))$ of (4.4) can only be equivalent to an element $((\infty,\infty),\psi,(w_1,w_2))$ of the same type, and we have

$$\begin{aligned}
& ((\infty,\infty),\phi,(z_1,z_2)) \sim ((\infty,\infty),\psi,(w_1,w_2)) \\
\Longleftrightarrow\ & \overline{\mathbb{Z}^2 \cdot ((\infty,\infty),\phi)} = \overline{\mathbb{Z}^2 \cdot ((\infty,\infty),\psi)} \ \text{ and } \ \gamma_{(z_1,z_2)}|_{\mathbb{Z}^2_\phi} = \gamma_{(w_1,w_2)}|_{\mathbb{Z}^2_\phi} \\
\Longleftrightarrow\ & \overline{\{(\infty,\infty)\} \times (\mathbb{Z}^2 \cdot \phi)} = \overline{\{(\infty,\infty)\} \times (\mathbb{Z}^2 \cdot \psi)} \ \text{ and } \ \gamma_{(z_1,z_2)}|_{\mathbb{Z}^2_\phi} = \gamma_{(w_1,w_2)}|_{\mathbb{Z}^2_\phi} \\
\Longleftrightarrow\ & \overline{\{(\infty,\infty)\}} \times \overline{\mathbb{Z}^2 \cdot \phi} = \overline{\{(\infty,\infty)\}} \times \overline{\mathbb{Z}^2 \cdot \psi} \ \text{ and } \ \gamma_{(z_1,z_2)}|_{\mathbb{Z}^2_\phi} = \gamma_{(w_1,w_2)}|_{\mathbb{Z}^2_\phi} \\
\Longleftrightarrow\ & \{(\infty,\infty)\} \times \overline{\mathbb{Z}^2 \cdot \phi} = \{(\infty,\infty)\} \times \overline{\mathbb{Z}^2 \cdot \psi} \ \text{ and } \ \gamma_{(z_1,z_2)}|_{\mathbb{Z}^2_\phi} = \gamma_{(w_1,w_2)}|_{\mathbb{Z}^2_\phi}.
\end{aligned}$$

Therefore,

$$\begin{aligned} &((\infty,\infty),\phi,(z_1,z_2)) \sim ((\infty,\infty),\psi,(w_1,w_2)) \\ \Longleftrightarrow\quad & \overline{\mathbb{Z}^2\cdot\phi} = \overline{\mathbb{Z}^2\cdot\psi} \ \text{ and } \ \gamma_{(z_1,z_2)}|_{\mathbb{Z}^2_\phi} = \gamma_{(w_1,w_2)}|_{\mathbb{Z}^2_\phi}, \end{aligned}$$

which implies that $((\infty,\infty),\phi,(z_1,z_2)) \sim ((\infty,\infty),\psi,(w_1,w_2))$ precisely when the pairs $(\phi,(z_1,z_2))$ and $(\psi,(w_1,w_2))$ are in the same equivalence class in the quotient space $\Delta(A)\times\mathbb{T}^2/\!\sim = \operatorname{Prim}(A\rtimes_\alpha\mathbb{Z}^2)$. Consequently, we can parameterize the equivalence class of each element $((\infty,\infty),\phi,(z_1,z_2))$ in (4.1) by the class of the pair $(\phi,(z_1,z_2))$ in $\operatorname{Prim}(A\rtimes_\alpha\mathbb{Z}^2)$.

At last, it is left to discuss on the parametrization of the equivalence classes of the elements of the forms $((\infty,r),\phi,(z_1,z_2))$ and $((r,\infty),\phi,(z_1,z_2))$ in (4.1). We only do this for $((\infty,r),\phi,(z_1,z_2))$ as the parametrization of the other one follows similarly. Firstly, let $\dot{\mathbb{Z}}_\phi$ and $\dot{\mathbb{Z}}\cdot\phi$ denote the stability group and the $\mathbb{Z}$-orbit of ϕ, respectively, when the group $\mathbb{Z}$ acts on $\Delta(A)$ via the action $\dot\alpha$ (corresponding to the crossed product $A\rtimes_{\dot\alpha}\mathbb{Z}$). Then, the stability group of the element $((\infty,r),\phi)$ of (4.2) is $\dot{\mathbb{Z}}_\phi\times\{0\}$, which is isomorphic to $\dot{\mathbb{Z}}_\phi$, and its $\mathbb{Z}^2$-orbit is

$$\big(\{\infty\}\times\mathbb{Z}\big)\times\dot{\mathbb{Z}}\cdot\phi.$$

It therefore follows that the element $((\infty,r),\phi,(z_1,z_2))$ can only be equivalent to an element $((\infty,s),\psi,(w_1,w_2))$ of the same type. Moreover,

$$\begin{aligned} &((\infty,r),\phi,(z_1,z_2)) \sim ((\infty,s),\psi,(w_1,w_2)) \\ \Longleftrightarrow\quad & \overline{\mathbb{Z}^2\cdot((\infty,r),\phi)} = \overline{\mathbb{Z}^2\cdot((\infty,s),\psi)} \ \text{ and } \ \gamma_{(z_1,z_2)}|_{(\dot{\mathbb{Z}}_\phi\times\{0\})} = \gamma_{(w_1,w_2)}|_{(\dot{\mathbb{Z}}_\phi\times\{0\})} \\ \Longleftrightarrow\quad & \overline{\big(\{\infty\}\times\mathbb{Z}\big)\times\dot{\mathbb{Z}}\cdot\phi} = \overline{\big(\{\infty\}\times\mathbb{Z}\big)\times\dot{\mathbb{Z}}\cdot\psi} \ \text{ and } \ \gamma_{z_1}|_{\dot{\mathbb{Z}}_\phi} = \gamma_{w_1}|_{\dot{\mathbb{Z}}_\phi} \\ \Longleftrightarrow\quad & \overline{\{\infty\}\times\mathbb{Z}}\times\overline{\dot{\mathbb{Z}}\cdot\phi} = \overline{\{\infty\}\times\mathbb{Z}}\times\overline{\dot{\mathbb{Z}}\cdot\psi} \ \text{ and } \ \gamma_{z_1}|_{\dot{\mathbb{Z}}_\phi} = \gamma_{w_1}|_{\dot{\mathbb{Z}}_\phi} \\ \Longleftrightarrow\quad & \overline{\dot{\mathbb{Z}}\cdot\phi} = \overline{\dot{\mathbb{Z}}\cdot\psi} \ \text{ and } \ \gamma_{z_1}|_{\dot{\mathbb{Z}}_\phi} = \gamma_{w_1}|_{\dot{\mathbb{Z}}_\phi}. \end{aligned}$$

This implies that $((\infty,r),\phi,(z_1,z_2)) \sim ((\infty,s),\psi,(w_1,w_2))$ if and only if the pairs (ϕ,z_1) and (ψ,w_1) are in the same equivalence class in the quotient space $\Delta(A)\times\mathbb{T}/\sim^{(1)} = \operatorname{Prim}(A\rtimes_{\dot\alpha}\mathbb{Z})$. Thus, the equivalence class of each element $((\infty,r),\phi,(z_1,z_2))$ in (4.1) can be parameterized by the class of the pair (ϕ,z_1) in $\operatorname{Prim}(A\rtimes_{\dot\alpha}\mathbb{Z})$. Note that a similar discussion shows that each element $((r,\infty),\phi,(z_1,z_2))$ can only be equivalent to an element of the same type, and its equivalence class in (4.1) is parameterized by the class of the pair (ϕ,z_2) in $\Delta(A)\times\mathbb{T}/\sim^{(2)} = \operatorname{Prim}(A\rtimes_{\ddot\alpha}\mathbb{Z})$. This completes the proof. □

We are now ready to describe the topology of $\operatorname{Prim}(A\times_\alpha^{\mathrm{piso}}\mathbb{N}^2)$ precisely.

Theorem 4.2. *Let $(A,\mathbb{N}^2,\alpha)$ be a dynamical system consisting of a separable abelian C^*-algebra A and an action α of $\mathbb{N}^2$ on A by automorphisms. Then,* $\operatorname{Prim}(A\times_\alpha^{\mathrm{piso}}\mathbb{N}^2)$ *is* *homeomorphic to the disjoint union* (4.3) *equipped with the quotient topology in which the open sets are in the following four forms:*

(a) $O \subset \Delta(A)$*, where* O *is open in* $\Delta(A)$*;*

(b) $O \cup W_1$*, where* O *is a nonempty open subset of* $\Delta(A)$ *and* W_1 *is an open set in* $\mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$*;*

(c) $O \cup W_2$*, where* O *is a nonempty open subset of* $\Delta(A)$ *and* W_2 *is an open set in* $\mathrm{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z})$*; and*

(d) $O \cup W_1 \cup W_2 \cup W$*, where* O*,* W_1*, and* W_2 *are nonempty open subsets of* $\Delta(A)$*,* $\mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$*, and* $\mathrm{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z})$*, respectively, and* W *is an open set in* $\mathrm{Prim}(A \rtimes_{\alpha} \mathbb{Z}^2)$*.*

Proof. Assume that $\widetilde{q}$ is the quotient map of the product space

$$\big[(\mathbb{Z} \cup \{\infty\}) \times (\mathbb{Z} \cup \{\infty\}) \times \Delta(A)\big] \times (\mathbb{T} \times \mathbb{T}) \tag{4.5}$$

onto (4.3). Let $q_1 \colon \Delta(A) \times \mathbb{T} \to \mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$, $q_2 \colon \Delta(A) \times \mathbb{T} \to \mathrm{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z})$, and $q \colon \Delta(A) \times \mathbb{T}^2 \to \mathrm{Prim}(A \rtimes_{\alpha} \mathbb{Z}^2)$ be the quotient maps. Recall that these quotient maps are all open (see [11, Remark 8.40]). Let $\widetilde{\mathfrak{B}}$ be the set of all elements

$$U_1 \times U_2 \times O \times (V_1 \times V_2),$$

where each U_i is either $\{n_i\}$ or $J_{n_i} = \{n_i, n_i + 1, n_i + 2, \ldots\} \cup \{\infty\}$ for some $n_i \in \mathbb{Z}$ (see [6, Lemma 3.3]), O is an open set in $\Delta(A)$, and each V_i is an open subset of $\mathbb{T}$. Obviously, $\widetilde{\mathfrak{B}}$ is a basis for the topology of the product space (4.5). Therefore, the forward image of the elements of $\widetilde{\mathfrak{B}}$ by $\widetilde{q}$ forms a basis for the quotient topology of (4.3) which we denote it by $\mathfrak{B}$. But, first note that, since each U_i has two forms, the elements of $\widetilde{\mathfrak{B}}$ have totally four forms as follows:

(i) $\{n_1\} \times \{n_2\} \times O \times (V_1 \times V_2)$;

(ii) $J_{n_1} \times \{n_2\} \times O \times (V_1 \times V_2)$;

(iii) $\{n_1\} \times J_{n_2} \times O \times (V_1 \times V_2)$; and

(iv) $J_{n_1} \times J_{n_2} \times O \times (V_1 \times V_2)$.

Therefore, accordingly, $\mathfrak{B}$ is the union of the following four sets (see Lemma 4.1):

$$\mathfrak{B}_1 := \{O \subset \Delta(A) : O \text{ is open in } \Delta(A)\},$$
$$\mathfrak{B}_2 := \{O \cup q_1(O \times V_1) : O \text{ is a nonempty open subset of } \Delta(A), \text{ and } V_1 \text{ is open in } \mathbb{T}\},$$
$$\mathfrak{B}_3 := \{O \cup q_2(O \times V_2) : O \text{ is a nonempty open subset of } \Delta(A), \text{ and } V_2 \text{ is open in } \mathbb{T}\},$$

and

$$\begin{aligned}\mathfrak{B}_4 := \{O \cup q_1(O \times V_1) \cup q_2(O \times V_2) \cup q(O \times (V_1 \times V_2)) :\\ O \text{ is a nonempty open subset of } \Delta(A),\\ \text{and each } V_i \text{ is a nonempty open subset of } \mathbb{T}\}.\end{aligned}$$

So, the rest follows from the facts that the open sets $q_1(O\times V_1)$, $q_2(O\times V_2)$, and $q(O\times(V_1\times V_2))$ form bases for the topological spaces $\mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$, $\mathrm{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z})$, $\mathrm{Prim}(A \rtimes_{\alpha} \mathbb{Z}^2)$, respectively. □

Remark 4.3. Recall that the primitive ideals of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ derived from $\mathrm{Prim}(A \rtimes_{\alpha} \mathbb{Z}^2)$ form a closed subset of $\mathrm{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2)$ (see (2.1)), which is

$$F := \{\mathcal{J} \in \mathrm{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2) : \ker q \subset \mathcal{J}\}.$$

Under the conditions of Theorem 4.2, these ideals are actually the kernels of the irreducible representations $\big(\mathrm{Ind}_{\mathbb{Z}^2_\phi}^{\mathbb{Z}^2}(\phi \rtimes \gamma_{(z,w)}|_{\mathbb{Z}^2_\phi})\big) \circ q$ corresponding to the elements (equivalence classes) $[(\phi,(z,w))]$ of $\Delta(A) \times \mathbb{T}^2/\sim = \mathrm{Prim}(A \rtimes_\alpha \mathbb{Z}^2)$. We denote $\ker\big([\mathrm{Ind}_{\mathbb{Z}^2_\phi}^{\mathbb{Z}^2}(\phi \rtimes \gamma_{(z,w)}|_{\mathbb{Z}^2_\phi})] \circ q\big)$ by $\mathcal{J}_{[(\phi,(z,w))]}$, and therefore,

$$F = \{\mathcal{J}_{[(\phi,(z,w))]} : \phi \in \Delta(A), (z,w) \in \mathbb{T}^2\}.$$

Also, the homeomorphism of $\mathrm{Prim}(A \rtimes_\alpha \mathbb{Z}^2)$ onto F is given by the map

$$[(\phi,(z,w))] \mapsto \mathcal{J}_{[(\phi,(z,w))]}.$$

Next, we want to identify the primitive ideals of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ derived from $\mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$ and $\mathrm{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z})$, respectively, under the conditions of Theorem 4.2. Consider the semigroup dynamical system $(A, \mathbb{N}, \dot{\alpha})$, corresponding to which, there is the following short exact sequence

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \longrightarrow A \times_{\dot{\alpha}}^{\mathrm{piso}} \mathbb{N} \xrightarrow{\dot{q}} A \rtimes_{\dot{\alpha}} \mathbb{Z} \longrightarrow 0 \tag{4.6}$$

of C^*-algebras (see [2,12]). So, corresponding to each element (equivalent class) $[(\phi,z)]$ of $\Delta(A) \times \mathbb{T}/\sim^{(1)} \simeq \mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$, the composition

$$\big(\mathrm{Ind}_{\dot{\mathbb{Z}}_\phi}^{\mathbb{Z}}(\phi \rtimes \gamma_z|_{\dot{\mathbb{Z}}_\phi})\big) \circ \dot{q} \tag{4.7}$$

gives a nonzero irreducible representation $\dot{\pi}\colon A \times_{\dot{\alpha}}^{\mathrm{piso}} \mathbb{N} \to B(H)$ of $(A \times_{\dot{\alpha}}^{\mathrm{piso}} \mathbb{N}, j_A, v)$ on a Hilbert space H, where $\dot{\mathbb{Z}}_\phi$ denotes the stability group of ϕ when the group $\mathbb{Z}$ acts on $\Delta(A)$ via the action $\dot{\alpha}$. It follows that $\mathcal{K}(\ell^2(\mathbb{N})) \otimes (\ker \dot{\pi})$ is a primitive ideal of the algebra

$\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\dot{\alpha}}^{\mathrm{piso}} \mathbb{N})$, which by [15, Corollary 3.12] sits in $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ as an (essential) ideal $\mathcal{I}_1$ (more precisely, $\mathcal{I}_1$ is the ideal $\mathcal{I}_\gamma$ in [15]). Also, $\mathcal{I}_1$ contains the algebra

$$\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A \simeq \mathcal{K}(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})) \otimes A \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes [\mathcal{K}(\ell^2(\mathbb{N})) \otimes A]$$

of compact operators as an (essential) ideal (see again [15]). Recall that the map $T\colon \mathbb{N} \to B(\ell^2(\mathbb{N}))$ defined by $T_n(e_m) = e_{m+n}$ on the usual orthonormal basis $\{e_m : m \in \mathbb{N}\}$ of $\ell^2(\mathbb{N})$ is a representation of $\mathbb{N}$ by isometries, such that

$$\mathcal{K}(\ell^2(\mathbb{N})) = \overline{\mathrm{span}}\{T_m(1 - TT^*)T_n^* : m, n \in \mathbb{N}\}, \quad T := T_1.$$

Indeed, for every $m, n \in \mathbb{N}$, $T_m(1 - TT^*)T_n^*$ is a rank-one operator on $\ell^2(\mathbb{N})$ such that $f \mapsto \langle f | e_n \rangle e_m$. Now, the following lemma identifies the primitive ideals of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ coming from $\mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$.

Lemma 4.4. *Define the maps*

$$\dot{\rho}\colon A \to B(\ell^2(\mathbb{N}) \otimes H) \quad \textit{and} \quad \dot{W}\colon \mathbb{N}^2 \to B(\ell^2(\mathbb{N}) \otimes H)$$

by

$$(\dot{\rho}(a)f)(n) = (\dot{\pi} \circ j_A)(\ddot{\alpha}_n(a))f(n) \quad \textit{and} \quad \dot{W}_{(m,n)} = T_n^* \otimes \overline{\dot{\pi}}(v_m),$$

respectively, for all $a \in A$, $f \in \ell^2(\mathbb{N}) \otimes H$, and $m, n \in \mathbb{N}$. Then, the pair $(\dot{\rho}, \dot{W})$ is a covariant partial-isometric representation of the system $(A, \mathbb{N}^2, \alpha)$ on the Hilbert space $\ell^2(\mathbb{N}) \otimes H \simeq \ell^2(\mathbb{N}, H)$, such that the corresponding (nondegenerate) representation $\dot{\rho} \times \dot{W}$ of $(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2, i)$ is irreducible on $\ell^2(\mathbb{N}) \otimes H$, which lives on the ideal $\mathcal{I}_1 \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\dot{\alpha}}^{\mathrm{piso}} \mathbb{N})$ but vanishes on $\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A$.

Proof. Firstly, some routine calculations (on spanning elements) show that the pair $(\dot{\rho}, \dot{W})$ is indeed a covariant partial-isometric representation of $(A, \mathbb{N}^2, \alpha)$ on $\ell^2(\mathbb{N}) \otimes H$ which we skip them here.

Next, to see that the corresponding representation $\dot{\rho} \times \dot{W}\colon (A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2, i) \to B(\ell^2(\mathbb{N}) \otimes H)$ is irreducible on $\ell^2(\mathbb{N}) \otimes H$, we show that every nonzero vector $f \in \ell^2(\mathbb{N}) \otimes H$ is a cyclic vector for $\dot{\rho} \times \dot{W}$. Since $f \neq 0$, there is $y \in \mathbb{N}$ such that $f(y)$ is a nonzero vector in H. Therefore, since the representation $\dot{\pi}$ is irreducible, $f(y)$ is a cyclic vector for $\dot{\pi}$, and hence, the elements

$$\big\{e_n \otimes [\dot{\pi}(v_m^* j_A(a) v_x) f(y)] : a \in A, n, m, x \in \mathbb{N}\big\}$$

span the Hilbert space $\ell^2(\mathbb{N}) \otimes H$ (recall that $A \times_{\dot{\alpha}}^{\mathrm{piso}} \mathbb{N} = \overline{\mathrm{span}}\big\{v_m^* j_A(a) v_x : a \in A, m, x \in \mathbb{N}\big\}$). We show the each spanning element $e_n \otimes [\dot{\pi}(v_m^* j_A(a) v_x) f(y)]$ belongs to

$$\overline{\mathrm{span}}\big\{(\dot{\rho} \times \dot{W})(\eta) f : \eta \in A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2\big\},$$

which implies that f is cyclic for $\dot{\rho} \times \dot{W}$. Take the element

$$\eta^{(x,y)}_{(m,n)}(a) := i_{\mathbb{N}^2}(m,n)^* i_A(a)\big[1 - i_{\mathbb{N}^2}(0,1)^* i_{\mathbb{N}^2}(0,1)\big] i_{\mathbb{N}^2}(x,y) \tag{4.8}$$

of $A \times_{\alpha}^{\text{piso}} \mathbb{N}^2$. See in [15, Lemma 3.8] that, in fact, the elements of the form (4.8) span the (essential) ideal $\mathcal{I}_1$ of $A \times_{\alpha}^{\text{piso}} \mathbb{N}^2$. Now, one can calculate to see that

$$\begin{aligned}
(\dot{\rho} \times \dot{W})\left(\eta^{(x,y)}_{(m,n)}(a)\right) f &= \big(T_n \otimes \overline{\pi}(v_m^*)\big)\dot{\rho}(a)\big[1 - (T \otimes 1)(T^* \otimes 1)\big]\big(T_y^* \otimes \overline{\pi}(v_x)\big) f \\
&= \big(T_n \otimes \overline{\pi}(v_m^*)\big)\dot{\rho}(a)\big[(1 - TT^*) \otimes 1\big]\big(T_y^* \otimes \overline{\pi}(v_x)\big) f \\
&= \big(T_n \otimes \overline{\pi}(v_m^*)\big)\dot{\rho}(a)\big[(1 - TT^*)T_y^* \otimes \overline{\pi}(v_x)\big] f \\
&= e_n \otimes \big[\dot{\pi}(v_m^* j_A(a) v_x) f(y)\big]
\end{aligned} \tag{4.9}$$

for all $a \in A$ and $n, m, x \in \mathbb{N}$. It thus follows that f is a cyclic vector for $\dot{\rho} \times \dot{W}$.

To see that the restriction of $\dot{\rho} \times \dot{W}$ to the ideal $\mathcal{I}_1 \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\dot{\alpha}}^{\text{piso}} \mathbb{N})$ is nonzero, we first show that the restriction $(\dot{\rho} \times \dot{W})|_{\mathcal{I}_1}$ is the representation

$$\mathrm{id} \otimes \dot{\pi} \colon \mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\dot{\alpha}}^{\text{piso}} \mathbb{N}) \to B(\ell^2(\mathbb{N}) \otimes H),$$

such that $(\mathrm{id} \otimes \dot{\pi})(S \otimes \xi) = S \otimes \dot{\pi}(\xi)$ for all $S \in \mathcal{K}(\ell^2(\mathbb{N}))$ and $\xi \in A \times_{\dot{\alpha}}^{\text{piso}} \mathbb{N}$. It is enough to see this on spanning elements, and therefore, we calculate

$$\begin{aligned}
&(\mathrm{id} \otimes \dot{\pi})\big([T_n(1 - TT^*)T_y^*] \otimes [v_m^* j_A(a) v_x]\big)(e_r \otimes h) \\
&= \big(T_n(1 - TT^*)T_y^* \otimes \dot{\pi}(v_m^* j_A(a) v_x)\big)(e_r \otimes h) \\
&= [T_n(1 - TT^*)T_y^* e_r] \otimes \big[\dot{\pi}(v_m^* j_A(a) v_x) h\big] \in \ell^2(\mathbb{N}) \otimes H,
\end{aligned} \tag{4.10}$$

which is equal to $e_n \otimes [\dot{\pi}(v_m^* j_A(a) v_x) h]$ if $r = y$, otherwise, zero for all $m, n, x, y, r \in \mathbb{N}$, $a \in A$, and $h \in H$. On the other hand, see in [15, Proposition 3.9] that the isomorphism $\mathcal{I}_1 \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\dot{\alpha}}^{\text{piso}} \mathbb{N})$, which we denote by Ψ_1 here, takes each spanning element $[T_n(1 - TT^*)T_y^*] \otimes [v_m^* j_A(a) v_x]$ of the algebra $\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\dot{\alpha}}^{\text{piso}} \mathbb{N})$ to the spanning element $\eta^{(x,y)}_{(m,n)}(a)$ of the ideal $\mathcal{I}_1$ (see (4.8)). Now, by a similar calculation to (4.9), we have

$$\begin{aligned}
&(\dot{\rho} \times \dot{W})|_{\mathcal{I}_1}\left(\Psi_1\big([T_n(1 - TT^*)T_y^*] \otimes [v_m^* j_A(a) v_x]\big)\right)(e_r \otimes h) \\
&= (\dot{\rho} \times \dot{W})|_{\mathcal{I}_1}\left(\eta^{(x,y)}_{(m,n)}(a)\right)(e_r \otimes h) \\
&= e_n \otimes \big[\dot{\pi}(v_m^* j_A(a) v_x) h\big]
\end{aligned} \tag{4.11}$$

if $r = y$, otherwise, zero for all $m, n, x, y, r \in \mathbb{N}$, $a \in A$, and $h \in H$. Thus, it follows by comparing (4.10) and (4.11) that we indeed have

$$(\dot{\rho} \times \dot{W})|_{\mathcal{I}_1 \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\dot{\alpha}}^{\text{piso}} \mathbb{N})} = \mathrm{id} \otimes \dot{\pi}. \tag{4.12}$$

Consequently, since the representations id and $\dot{\pi}$ are nonzero, it follows from (4.12) that the restriction of $\dot{\rho} \times \dot{W}$ to the ideal $\mathcal{I}_1$ must be nonzero.

Finally, since

$$(\dot{\rho} \times \dot{W})|_{\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A} = (\mathrm{id} \otimes \dot{\pi})|_{\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A}$$

and $\ker \dot{\pi}$ contains the algebra $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ as an ideal (see (4.6) and the definition of $\dot{\pi}$ in (4.7)), it follows that

$$\ker(\mathrm{id} \otimes \dot{\pi}) = \mathcal{K}(\ell^2(\mathbb{N})) \otimes \ker \dot{\pi} \supset \mathcal{K}(\ell^2(\mathbb{N})) \otimes \big(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A\big) \simeq \mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A,$$

and therefore, the representation $\dot{\rho}\times\dot{W}$ vanishes on the ideal $\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A$. This completes the proof. $\square$

Remark 4.5. It therefore follows by Lemma 4.4 that, under the conditions of Theorem 4.2, each primitive ideal of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ coming from $\mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$ is the kernel of an irreducible representation $\dot{\rho} \times \dot{W}$ corresponding to the pair $(\dot{\rho}, \dot{W})$ induced by an element $[(\phi, z)]$ of $\Delta(A) \times \mathbb{T}/\sim^{(1)} \simeq \mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$. Let $\dot{\mathcal{J}}_{[(\phi,z)]}$ denote $\ker(\dot{\rho} \times \dot{W})$. So, the map

$$\dot{\mathcal{J}}_{[(\phi,z)]} \mapsto \dot{\mathcal{J}}_{[(\phi,z)]} \cap \mathcal{I}_1 = \ker((\dot{\rho} \times \dot{W})|_{\mathcal{I}_1}) = \ker(\mathrm{id} \otimes \dot{\pi}) = \mathcal{K}(\ell^2(\mathbb{N})) \otimes \ker \dot{\pi}$$

is a bijection between the subset of $\mathrm{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2)$ consisting of the primitive ideals $\dot{\mathcal{J}}_{[(\phi,z)]}$ and the closed subspace

$$F_1 := \big\{P \in \mathrm{Prim}(\mathcal{I}_1) : \mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A \subset P\big\}$$

of $\mathrm{Prim}(\mathcal{I}_1) \simeq \mathrm{Prim}(A \times_{\dot{\alpha}}^{\mathrm{piso}} \mathbb{N})$. Moreover, F_1 is homeomorphic to $\mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$ by the composition of the following homeomorphisms

$$\mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z}) \longrightarrow \big\{I \in \mathrm{Prim}(A \times_{\dot{\alpha}}^{\mathrm{piso}} \mathbb{N}) : \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \subset I\big\} \xrightarrow{\text{the Rieffel homeomorphism}} F_1,$$

such that

$$[(\phi, z)] \mapsto \ker\big([\mathrm{Ind}_{\dot{\mathbb{Z}}_\phi}^{\mathbb{Z}} (\phi \rtimes \gamma_z|_{\dot{\mathbb{Z}}_\phi})] \circ \dot{q}\big) = \ker \dot{\pi} \mapsto \mathcal{K}(\ell^2(\mathbb{N})) \otimes \ker \dot{\pi}.$$

Therefore, the map $[(\phi, z)] \mapsto \dot{\mathcal{J}}_{[(\phi,z)]}$ embeds the set $\mathrm{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$ in $\mathrm{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2)$ as a subset.

Similarly, the semigroup dynamical system $(A, \mathbb{N}, \ddot{\alpha})$ gives rise to the following short exact sequence

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \longrightarrow A \times_{\ddot{\alpha}}^{\mathrm{piso}} \mathbb{N} \xrightarrow{\ddot{q}} A \rtimes_{\ddot{\alpha}} \mathbb{Z} \longrightarrow 0. \tag{4.13}$$

Therefore, corresponding to each element $[(\phi, w)]$ of $\Delta(A) \times \mathbb{T}/\sim^{(2)} \simeq \mathrm{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z})$, the composition

$$\big(\mathrm{Ind}_{\ddot{\mathbb{Z}}_\phi}^{\mathbb{Z}} (\phi \rtimes \gamma_w|_{\ddot{\mathbb{Z}}_\phi})\big) \circ \ddot{q}$$

defines a nonzero irreducible representation $\ddot{\pi}$ of $(A \times_{\ddot{\alpha}}^{\text{piso}} \mathbb{N}, k_A, u)$ on a Hilbert space H, where $\ddot{\mathbb{Z}}_\phi$ denotes the stability group of ϕ when the group $\mathbb{Z}$ acts on $\Delta(A)$ via the action $\ddot{\alpha}$. Hence, $\mathcal{K}(\ell^2(\mathbb{N})) \otimes (\ker \ddot{\pi})$ is a primitive ideal of the algebra $\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\ddot{\alpha}}^{\text{piso}} \mathbb{N})$, which again by [15, Corollary 3.12], sits in $A \times_{\alpha}^{\text{piso}} \mathbb{N}^2$ as an (essential) ideal $\mathcal{I}_2$ (note that, $\mathcal{I}_2$ is actually the ideal $\mathcal{I}_\delta$ in [15]). Now, we have

Lemma 4.6. *Define the maps*

$$\ddot{\rho}\colon A \to B(\ell^2(\mathbb{N}) \otimes H) \quad \textit{and} \quad \ddot{W}\colon \mathbb{N}^2 \to B(\ell^2(\mathbb{N}) \otimes H)$$

by

$$(\ddot{\rho}(a)f)(m) = (\ddot{\pi} \circ k_A)(\dot{\alpha}_m(a))f(m) \quad \textit{and} \quad \ddot{W}_{(m,n)} = T_m^* \otimes \overline{\ddot{\pi}}(u_n)$$

respectively, for all $a \in A$, $f \in \ell^2(\mathbb{N}) \otimes H$, and $m, n \in \mathbb{N}$. Then, the pair $(\ddot{\rho}, \ddot{W})$ is a covariant partial-isometric representation of the system $(A, \mathbb{N}^2, \alpha)$ on the Hilbert space $\ell^2(\mathbb{N}) \otimes H$, such that the corresponding (nondegenerate) representation $\ddot{\rho} \times \ddot{W}$ of $(A \times_{\alpha}^{\text{piso}} \mathbb{N}^2, i)$ is irreducible on $\ell^2(\mathbb{N}) \otimes H$, which lives on the ideal $\mathcal{I}_2 \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\ddot{\alpha}}^{\text{piso}} \mathbb{N})$ but vanishes on $\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A$.

Proof. We skip the proof as it is similar to the proof of Lemma 4.4. $\square$

Remark 4.7. Thus, by Lemma 4.6, under the conditions of Theorem 4.2, the primitive ideals of $A \times_{\alpha}^{\text{piso}} \mathbb{N}^2$ derived from $\mathrm{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z})$ are the kernels of the irreducible representations $\ddot{\rho} \times \ddot{W}$ induced by elements $[(\phi, w)]$ of $\Delta(A) \times \mathbb{T}/\sim^{(2)} \simeq \mathrm{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z})$. So, we denote these ideals by $\ddot{\mathcal{J}}_{[(\phi,w)]}$, and hence, the map $[(\phi, w)] \in \mathrm{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z}) \mapsto \ddot{\mathcal{J}}_{[(\phi,w)]} \in \mathrm{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N}^2)$ is an embedding (of sets).

In addition, as a refinement of Lemma 4.1, we would like to mention that the maps

$$P \mapsto \operatorname{ind} P, \quad [(\phi, z)] \mapsto \dot{\mathcal{J}}_{[(\phi,z)]}, \quad [(\phi, w)] \mapsto \ddot{\mathcal{J}}_{[(\phi,w)]}, \quad \text{and} \quad [(\phi, (z, w))] \mapsto \mathcal{J}_{[(\phi,(z,w))]}$$

combine to give a bijective correspondence of the disjoint union (4.3) onto $\mathrm{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N}^2)$.

Proposition 4.8. *Let $(A, \mathbb{N}^2, \alpha)$ be a dynamical system consisting of a separable abelian C^*-algebra A and an action α of $\mathbb{N}^2$ on A by automorphisms. Then $A \times_{\alpha}^{\text{piso}} \mathbb{N}^2$ is GCR if and only if the orbit space $\mathbb{Z}^2 \setminus \Delta(A)$ is T_0.*

Proof. Recall that, by [8, Theorem 5.6.2], $A \times_{\alpha}^{\text{piso}} \mathbb{N}^2$ is GCR if and only if

$$\ker q \quad \text{and} \quad A \rtimes_\alpha \mathbb{Z}^2 \simeq C_0(\Delta(A)) \rtimes_{\text{lt}} \mathbb{Z}^2$$

are GCR (see (2.1)), and by [11, Theorem 8.43], $A \rtimes_\alpha \mathbb{Z}^2$ is GCR if and only if the orbit space $\mathbb{Z}^2 \setminus \Delta(A)$ is T_0. So, it is enough to see that if $\mathbb{Z}^2 \setminus \Delta(A)$ is a T_0 space, then the ideal $\ker q$ is GCR. Suppose that $\mathbb{Z}^2 \setminus \Delta(A)$ is T_0. To see that the algebra $\ker q$ is GCR, since, by [15, Corollary 3.13], we have

$$\ker q/[\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A] \simeq \big[\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \rtimes_{\dot{\alpha}} \mathbb{Z})\big] \oplus \big[\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \rtimes_{\ddot{\alpha}} \mathbb{Z})\big],$$

and A is abelian, again, by [8, Theorem 5.6.2], it is enough show that the algebras

$$A \rtimes_{\dot{\alpha}} \mathbb{Z} \quad \text{and} \quad A \rtimes_{\ddot{\alpha}} \mathbb{Z}$$

are GCR. Let $\dot{\mathbb{Z}} \setminus \Delta(A)$ and $\ddot{\mathbb{Z}} \setminus \Delta(A)$ denote the orbit spaces corresponding to the actions $\dot{\alpha}$ and $\ddot{\alpha}$ of $\mathbb{Z}$ on A, respectively. Suppose that $\sigma\colon \Delta(A) \to \mathbb{Z}^2 \setminus \Delta(A)$ and $\sigma_1\colon \Delta(A) \to \dot{\mathbb{Z}} \setminus \Delta(A)$ are the orbit maps. One can see that the map $\Psi_1\colon \dot{\mathbb{Z}} \setminus \Delta(A) \to \mathbb{Z}^2 \setminus \Delta(A)$ defined by

$$\dot{\mathbb{Z}} \cdot \phi \mapsto \mathbb{Z}^2 \cdot \phi$$

is bijective, where $\dot{\mathbb{Z}} \cdot \phi$ denotes the $\mathbb{Z}$-orbit of $\phi \in \Delta(A)$ corresponding to the action $\dot{\alpha}$. Moreover, we clearly have $\Psi_1 \circ \sigma_1 = \sigma$, by applying which, it follows the map Ψ_1 is actually a homeomorphism. Therefore, the orbit space $\dot{\mathbb{Z}} \setminus \Delta(A)$ must also be T_0. A similar argument shows that $\ddot{\mathbb{Z}} \setminus \Delta(A)$ is T_0, too, and hence, again by [11, Theorem 8.43], the (group) crossed products $A \rtimes_{\dot{\alpha}} \mathbb{Z}$ and $A \rtimes_{\ddot{\alpha}} \mathbb{Z}$ are GCR. This completes the proof. □

Proposition 4.9. *Let $(A, \mathbb{N}^2, \alpha)$ be a dynamical system consisting of a separable abelian C^*-algebra A and an action α of $\mathbb{N}^2$ on A by automorphisms. Then $A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2$ is not CCR.*

Proof. Since $A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2$ and

$$(B_{\mathbb{Z}^2} \otimes A) \rtimes_{\tau \otimes \alpha^{-1}} \mathbb{Z}^2 \simeq C_0(X) \rtimes_{\mathrm{lt}} \mathbb{Z}^2$$

are Morita equivalent, it is enough to see that $C_0(X) \rtimes_{\mathrm{lt}} \mathbb{Z}^2$ is not CCR (see [11, Proposition I.43]). Since for the element $((m,n),\phi) \in X$, where $m,n \in \mathbb{Z}$ and $\phi \in \Delta(A)$ (see (4.2)), we have

$$\overline{\mathbb{Z}^2 \cdot ((m,n),\phi)} = \overline{\mathbb{Z}^2 \times \{\phi\}} = \overline{\mathbb{Z}^2} \times \overline{\{\phi\}} = \big(\overline{\mathbb{Z}} \times \overline{\mathbb{Z}}\big) \times \{\phi\} = \big[(\mathbb{Z} \cup \{\infty\}) \times (\mathbb{Z} \cup \{\infty\})\big] \times \{\phi\},$$

it follows that the $\mathbb{Z}^2$-orbit of $((m,n),\phi)$ is not closed in X. Thus, by [11, Theorem 8.44], $C_0(X) \rtimes_{\mathrm{lt}} \mathbb{Z}^2$ is not CCR. □

5. The topology of $\mathrm{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N}^2)$ when A is separable and $\mathbb{Z}^2$ acts on $\mathrm{Prim}\,A$ freely

Assume that in the system $(A, \mathbb{N}^2, \alpha)$, the C^*-algebra A is separable, and the action of $\mathbb{Z}^2$ on $\mathrm{Prim}\,A$ is free. Now, consider the group dynamical system $(B_{\mathbb{Z}^2} \otimes A, \mathbb{Z}^2, \tau \otimes \alpha^{-1})$

in which the algebra $(B_{\mathbb{Z}^2} \otimes A)$ is certainly separable and $\mathbb{Z}^2$ is an abelian (discrete) countable group. To describe the action of $\mathbb{Z}^2$ on $\operatorname{Prim}(B_{\mathbb{Z}^2} \otimes A)$, first note that, by [10, Theorem B.45], $\operatorname{Prim}(B_{\mathbb{Z}^2} \otimes A) = \operatorname{Prim}(B_{\mathbb{Z}} \otimes B_{\mathbb{Z}} \otimes A)$ is homeomorphic to

$$\operatorname{Prim} B_{\mathbb{Z}} \times \operatorname{Prim} B_{\mathbb{Z}} \times \operatorname{Prim} A \simeq (\mathbb{Z} \cup \{\infty\}) \times (\mathbb{Z} \cup \{\infty\}) \times \operatorname{Prim} A. \tag{5.1}$$

Then, by a similar discussion to the one given at the beginning of §4, one can see that $\mathbb{Z}^2$ acts on the product space (5.1) as follows:

$$(m,n) \cdot ((r,s),P) = ((r+m, s+n), P), \quad (m,n) \cdot ((\infty,\infty),P) = ((\infty,\infty), \alpha_{(-m,-n)}(P)),$$
$$(m,n) \cdot ((\infty,s),P) = ((\infty, s+n), \alpha_{(-m,0)}(P)) = ((\infty, s+n), \dot{\alpha}_{-m}(P)),$$

and

$$(m,n) \cdot ((r,\infty),P) = ((r+m,\infty), \alpha_{(0,-n)}(P)) = ((r+m,\infty), \ddot{\alpha}_{-n}(P))$$

for all $(m,n) \in \mathbb{Z}^2$, $P \in \operatorname{Prim} A$, and $r, s \in \mathbb{Z}$. So, it is not difficult to see that, in fact, $\mathbb{Z}^2$ also acts on $\operatorname{Prim}(B_{\mathbb{Z}^2} \otimes A)$ freely, and therefore, by Theorem 2.2,

$$\operatorname{Prim}((B_{\mathbb{Z}^2} \otimes A) \rtimes_{\tau \otimes \alpha^{-1}} \mathbb{Z}^2) \simeq \operatorname{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N}^2)$$

is homeomorphic to the quasi-orbit space

$$\mathcal{O}\big(\operatorname{Prim}(B_{\mathbb{Z}^2} \otimes A)\big) = \mathcal{O}\big((\mathbb{Z} \cup \{\infty\}) \times (\mathbb{Z} \cup \{\infty\}) \times \operatorname{Prim} A\big). \tag{5.2}$$

But again, a similar discussion to Lemma 4.1 shows that the quasi-orbit space (5.2), as a set, corresponds to the disjoint union

$$\operatorname{Prim} A \sqcup \mathcal{O}_1(\operatorname{Prim} A) \sqcup \mathcal{O}_2(\operatorname{Prim} A) \sqcup \mathcal{O}(\operatorname{Prim} A), \tag{5.3}$$

where $\mathcal{O}_1(\operatorname{Prim} A)$, $\mathcal{O}_2(\operatorname{Prim} A)$, and $\mathcal{O}(\operatorname{Prim} A)$ are the quasi-orbit spaces homeomorphic to $\operatorname{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z})$, $\operatorname{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z})$, and $\operatorname{Prim}(A \rtimes_{\alpha} \mathbb{Z}^2)$, respectively. Now, we have

Theorem 5.1. *Let $(A, \mathbb{N}^2, \alpha)$ be a dynamical system consisting of a separable C^*-algebra A and an action α of $\mathbb{N}^2$ on A by automorphisms. Suppose that the action of $\mathbb{Z}^2$ on $\operatorname{Prim} A$ is free. Then, $\operatorname{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N}^2)$ is homeomorphic to the disjoint union (5.3) equipped with the quotient topology in which the open sets are in the following four forms:*

(a) *$O \subset \operatorname{Prim} A$, where O is open in $\operatorname{Prim} A$;*

(b) *$O \cup W_1$, where O is a nonempty open subset of $\operatorname{Prim} A$ and W_1 is an open set in $\mathcal{O}_1(\operatorname{Prim} A)$;*

(c) *$O \cup W_2$, where O is a nonempty open subset of $\operatorname{Prim} A$ and W_2 is an open set in $\mathcal{O}_2(\operatorname{Prim} A)$; and*

(d) $O \cup W_1 \cup W_2 \cup W$, *where* O, W_1, *and* W_2 *are nonempty open subsets of* $\operatorname{Prim} A$, $\mathcal{O}_1(\operatorname{Prim} A)$, *and* $\mathcal{O}_2(\operatorname{Prim} A)$, *respectively, and* W *is an open set in* $\mathcal{O}(\operatorname{Prim} A)$.

Proof. We skip the proof as it follows by a similar argument to the proof of Theorem 4.2, using the fact that, for any (group) dynamical system (B, G, β), the quasi-orbit map $\sigma\colon \operatorname{Prim} B \to \mathcal{O}(\operatorname{Prim} B)$ is continuous and open (see [11, Lemma 6.12]). □

Remark 5.2. Under the conditions of Theorem 5.1, the primitive ideals of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ coming from $\operatorname{Prim}(A \rtimes_{\alpha} \mathbb{Z}^2) \simeq \mathcal{O}(\operatorname{Prim} A)$ are indeed the kernels of the irreducible representations $(\operatorname{Ind} \pi) \circ q = (\widetilde{\pi} \rtimes U) \circ q$ of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$, where π is an irreducible representation of A such that $\ker \pi = P$ (see §2). Moreover, $(\operatorname{Ind} \pi) \circ q$ is actually the associated representation $\widetilde{\pi} \times^{\mathrm{piso}} U$ of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ corresponding to the covariant partial-isometric representation $(\widetilde{\pi}, U)$ of $(A, \mathbb{N}^2, \alpha)$. Therefore, each element of the closed subspace

$$F = \{\mathcal{J} \in \operatorname{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2) : \ker q \subset \mathcal{J}\}$$

of $\operatorname{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2)$ is the kernel of an irreducible representation $\widetilde{\pi} \times^{\mathrm{piso}} U$ corresponding to the quasi-orbit $\mathcal{O}(P)$, which we denote by $\mathcal{J}_{\mathcal{O}(P)}$. Hence, the map $\mathcal{O}(P) \to \mathcal{J}_{\mathcal{O}(P)}$ is a homeomorphism of $\mathcal{O}(\operatorname{Prim} A) \simeq \operatorname{Prim}(A \rtimes_{\alpha} \mathbb{Z}^2)$ onto the closed subspace F.

Also, note that, the primitive ideals of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ derived from $\operatorname{Prim}(A \rtimes_{\dot{\alpha}} \mathbb{Z}) \simeq \mathcal{O}_1(\operatorname{Prim} A)$ and $\operatorname{Prim}(A \rtimes_{\ddot{\alpha}} \mathbb{Z}) \simeq \mathcal{O}_2(\operatorname{Prim} A)$ can similarly be identified by looking at Lemmas 4.4 and 4.6, respectively. To be more precise, for each quasi-orbit $\mathcal{O}_1(P) \in \mathcal{O}_1(\operatorname{Prim} A)$ and $\mathcal{O}_2(P) \in \mathcal{O}_2(\operatorname{Prim} A)$, there is an irreducible representation π of A such that $P = \ker \pi$. Now, if $\operatorname{Ind}_1 \pi$ and $\operatorname{Ind}_2 \pi$ denote the induced representations of the crossed products $A \rtimes_{\dot{\alpha}} \mathbb{Z}$ and $A \rtimes_{\ddot{\alpha}} \mathbb{Z}$, respectively, then the compositions

$$\dot{\pi} := (\operatorname{Ind}_1 \pi) \circ \dot{q} \quad \text{and} \quad \ddot{\pi} := (\operatorname{Ind}_2 \pi) \circ \ddot{q}$$

are nonzero irreducible representations of the algebras $A \times_{\dot{\alpha}}^{\mathrm{piso}} \mathbb{N}$ and $A \times_{\ddot{\alpha}}^{\mathrm{piso}} \mathbb{N}$, respectively (see (4.6) and (4.13)). Then, the rest follows from Lemmas 4.4 and 4.6. We denote the primitive ideal of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}^2$ corresponding to $\mathcal{O}_1(P) \in \mathcal{O}_1(\operatorname{Prim} A)$ by $\mathcal{J}_{\mathcal{O}_1(P)}$, and similarly, the one corresponding to $\mathcal{O}_2(P) \in \mathcal{O}_2(\operatorname{Prim} A)$ by $\mathcal{J}_{\mathcal{O}_2(P)}$.

Remark 5.3. Recall that the primitive ideal space of any C^*-algebra is a locally compact space, and if a C^*-algebra is separable, then its primitive ideal space is second countable. A (not necessarily Hausdorff) locally compact space X is called *almost Hausdorff* if each locally compact subspace V contains a relatively open nonempty Hausdorff subset (see [11, Definition 6.1]). If a C^*-algebra is GCR, then its primitive ideal space is almost Hausdorrff with respect to the hull-kernel (Jacobson) topology (see [11, pages 171, 172]). Therefore, if A is a separable GCR C^*-algebra, then the spectrum $\widehat{A}$ of A is a second countable almost Hausdorff locally compact space as it is homeomorphic to $\operatorname{Prim} A$. Now, suppose that (A, G, α) is a group dynamical system in which the algebra A is separable and G is an abelian discrete countable group. If G acts on $\widehat{A}$ freely, then it follows from [16] that $A \rtimes_\alpha G$ is GCR if and only if A is GCR and every G-orbit in $\widehat{A}$ is discrete. However, every G-orbit in $\widehat{A}$ is discrete if and only if, for every $[\pi] \in \widehat{A}$, the map $G \to G \cdot [\pi]$ defined by

$$s \mapsto s \cdot [\pi] := [\pi \circ \alpha_{s^{-1}}]$$

is a homeomorphism, which by [11, Theorem 6.2 (Mackey-Glimm Dichotomy)], is equivalent to saying that the orbit space $G \setminus \widehat{A}$ is T_0. Therefore, if (the abelian group) G in the separable system (A, G, α) acts on $\widehat{A}$ freely, then $A \rtimes_\alpha G$ is GCR if and only if A is GCR and the orbit space $G \setminus \widehat{A}$ is T_0.

Proposition 5.4. *Let $(A, \mathbb{N}^2, \alpha)$ be a dynamical system consisting of a separable C^*-algebra A and an action α of $\mathbb{N}^2$ on A by automorphisms. Suppose that $\mathbb{Z}^2$ acts on $\widehat{A}$ freely. Then, $A \times_\alpha^{\text{piso}} \mathbb{N}^2$ is GCR if and only if A is GCR and the orbit space $\mathbb{Z}^2 \setminus \widehat{A}$ is T_0.*

Proof. We skip the proof as it follows by a similar discussion to the proof of Proposition 4.8 and Remark 5.3. □

6. Primitivity of $A \times_\alpha^{\text{piso}} \mathbb{N}^2$

Finally, we have

Theorem 6.1. *Let $(A, \mathbb{N}^2, \alpha)$ be a dynamical system consisting of a (nonzero) C^*-algebra A and an action α of $\mathbb{N}^2$ on A by automorphisms. Then, $A \times_\alpha^{\text{piso}} \mathbb{N}^2$ is primitive if and only if A is primitive.*

Proof. If $A \times_\alpha^{\text{piso}} \mathbb{N}^2$ is primitive, then it has a faithful nonzero irreducible representation ρ, and hence, $\ker \rho$, which is the zero ideal, is a primitive ideal of $A \times_\alpha^{\text{piso}} \mathbb{N}^2$. But, this ideal can only be derived from $\operatorname{Prim} A$ as all primitive ideals of $A \times_\alpha^{\text{piso}} \mathbb{N}^2$ except the ones derived from $\operatorname{Prim} A$ contain the algebra $\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A$. Therefore, it follows from Proposition 3.1 that the representation ρ is the representation $\widetilde{\pi} \times V$ corresponding to a pair $(\widetilde{\pi}, V)$ induced by a (nonzero) irreducible representation π of A. It thus follows that the restriction of $\rho = \widetilde{\pi} \times V$ to the ideal $\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A$ of $A \times_\alpha^{\text{piso}} \mathbb{N}^2$, which is the representation $\operatorname{id} \otimes \pi$, is a (nonzero) faithful irreducible representation of $\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A$. So, the algebra $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ is primitive, which implies that A must be primitive (in fact, $\ker \pi = \{0\}$).

Conversely, if A is primitive, then it has a faithful nonzero irreducible representation π. Let $\widetilde{\pi} \times V$ be the (nonzero) irreducible representation of $A \times_\alpha^{\text{piso}} \mathbb{N}^2$ corresponding to the pair $(\widetilde{\pi}, V)$ induced by the representation π (see again Proposition 3.1). Now, the restriction $(\widetilde{\pi} \times V)|_{\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A} = \operatorname{id} \otimes \pi$ is faithful as the representations id and π are (see [10, Corollary B.11]). Therefore, since $\mathcal{K}(\ell^2(\mathbb{N}^2))\otimes A$ is an essential ideal of $A\times_\alpha^{\mathrm{piso}}\mathbb{N}^2$ (see [15, Corollary 3.13]), it follows that the representation $\widetilde{\pi}\times V$ must be faithful. So, the algebra $A\times_\alpha^{\mathrm{piso}}\mathbb{N}^2$ is primitive. □

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References

[1] S. Adji and A. Hosseini, *The partial-isometric crossed products of c_0 by the forward and the backward shifts*, Bull. Malays. Math. Sci. Soc. (2) **33** (2010), no. 3, 487–498.

[2] S. Adji and S. Zahmatkesh, *The partial-isometric crossed products by semigroups of endomorphisms as full corners*, J. Aust. Math. Soc. **96** (2014), no. 2, 145–166.

[3] ______, *The composition series of ideals of the partial-isometric crossed product by semigroup of endomorphisms*, J. Korean Math. Soc. **52** (2015), no. 4, 869–889.

[4] S. Echterhoff, *Crossed products and the Mackey–Rieffel–Green machine*, arXiv:1006.4975.

[5] N. J. Fowler, *Discrete product systems of Hilbert bimodules*, Pacific J. Math. **204** (2002), no. 2, 335–375.

[6] W. Lewkeeratiyutkul and S. Zahmatkesh, *The primitive ideal space of the partial-isometric crossed product of a system by a single automorphism*, Rocky Mountain J. Math. **47** (2017), no. 8, 2699–2722.

[7] J. Lindiarni and I. Raeburn, *Partial-isometric crossed products by semigroups of endomorphisms*, J. Operator Theory **52** (2004), no. 1, 61–87.

[8] G. J. Murphy, *C^*-algebras and Operator Theory*, Academic Press, Boston, MA, 1990.

[9] A. Nica, *C^*-algebras generated by isometries and Wiener–Hopf operators*, J. Operator Theory **27** (1992), no. 1, 17–52.

[10] I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-trace C^*-algebras*, Math. Surveys Monogr. **60**, American Mathematical Society, Providence, RI, 1998.

[11] D. P. Williams, *Crossed Products of C^*-algebras*, Math. Surveys Monogr. **134**, American Mathematical Society, Providence, RI, 2007.

[12] S. Zahmatkesh, *The partial-isometric crossed products by semigroups of endomorphisms are Morita equivalent to crossed products by groups*, New Zealand J. Math. **47** (2017), 121–139.

[13] ———, *The Nica–Toeplitz algebras of dynamical systems over abelian lattice-ordered groups as full corners*, Houston J. Math. **47** (2021), no. 1, 115–149.

[14] ———, *Partial-isometric crossed products of dynamical systems by left LCM semigroups*, New York J. Math. **29** (2023), no. 3, 51–106.

[15] ———, *The composition series of ideals of the partial-isometric crossed product by the semigroup* $\mathbb{N}^2$, Adv. Oper. Theory **9** (2024), Paper No. 2, 32 pp.

[16] G. Zeller-Meier, *Produits croisés d'une C^*-algèbre par un groupe d'automorphismes*, J. Math. Pures Appl. (9) **47** (1968), 101–239.

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