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On the Exterior Problem for Parabolic Hessian Quotient Equations

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Abstract. We prove the existence of ancient solutions of the exterior problem for parabolic Hessian quotient equations $-u_t S_{k,l}(D^2u) = 1$ with prescribed asymptotic behavior at infinity. We construct a subsolution to it and use Perron method to finish the proof.

1. Introduction

Let

$$\mathbb{R}^{n+1}_- = \{(x,t) \mid x \in \mathbb{R}^n, t \leq 0\}.$$

Denote

$$S_{k,l}(D^2u) = \frac{\sigma_k(\lambda(D^2u))}{\sigma_l(\lambda(D^2u))},$$

where $\lambda = \lambda(D^2u) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ are the eigenvalues of D^2u, the Hessian matrix on x,

$$\sigma_k(\lambda) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

In this paper, we consider the solvability of the exterior problem of the parabolic Hessian quotient equation

$$-u_t S_{k,l}(D^2u) = 1 \quad \text{in } \mathbb{R}^{n+1}_- \setminus \overline{D}, \tag{1.1}$$

$$u = \varphi(x,t) \quad \text{on } \partial_p D, \tag{1.2}$$

where $0 \leq l < k \leq n$, $n \geq 3$,

$$D = \{(x,t) \mid Q(x) < t \leq 0\}, \quad \partial_p D = \{(x,t) \mid Q(x) = t \leq 0\},$$

$Q(x)$ is a strictly convex second-order differentiable function such that D is bounded and not empty.

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The Hessian quotient equation is an extension of the Monge–Ampère equation. In [3], Caffarelli and Li first studied the asymptotic behavior near infinity and the exterior problem for the elliptic Monge–Ampère equation

$$\det D^2 u = 1.$$

In 2015, Bao, Li and Zhang [2] generalized the results in [3] to

$$\det D^2 u = f,$$

where f is a perturbation of 1 near infinity. In [8,11], Bao, Wang and Zhang obtained the asymptotic behavior of the parabolic Monge–Ampère equation.

In 2011, Dai [5] studied the existence of solutions of the exterior problem for the elliptic Hessian quotient equation

$$\begin{cases} S_{k,l}(D^2u) = 1 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ u(x) = \varphi(x) & \text{on } \partial\Omega, \\ \limsup_{|x|\to\infty} |x|^{k-l-2}\big|u(x) - \big(\frac{1}{2}a|x|^2 + c\big)\big| < \infty, \end{cases}$$

where Ω is a smooth, bounded and strictly convex domain, $a = (C_n^l/C_n^k)^{k-l}$, $0 \le l < k \le n$, $k - l \ge 3$, $c \in \mathbb{R}$. In [1], Bao, Li and Li constructed a generalized symmetric subsolution and proved the solvability of the exterior problem of the elliptic Hessian equation.

In [12], we considered the existence of solutions of the exterior problem of the parabolic Monge–Ampère equation. We raised this problem for the first time and constructed a subsolution to it. In this paper, we continue to extend this problem to the Hessian quotient equation and generalize the result in [5] to the parabolic case.

Our main theorem is

Theorem 1.1. *Let $n \ge 3$. For any $\varphi \in C^2(\overline{D})$, there exists $c^* \in \mathbb{R}$, depending on n, k, l, D and $\|\varphi\|_{C^2(\overline{D})}$, such that for any $c > c^*$, there exists a unique viscosity solution $u_c \in C^0(\mathbb{R}^{n+1}_- \setminus D)$ of* (1.1), (1.2) *and*

$$\limsup_{|x|^2 - t\to+\infty} (|x|^2 - t)^{\frac{k-l-2}{2}} \left| u_c(x,t) - \left(-t + \frac{1}{2}a|x|^2 + c\right)\right| < \infty,$$

where $a = (C_n^l/C_n^k)^{k-l}$, $0 \le l < k \le n$, $k - l \ge 3$, $c \in \mathbb{R}$.

The paper is arranged as follows. In Section 2, we state some notations and lemmas. In Section 3, we construct subsolutions of the problem. Finally in Section 4, we use Perron method to finish the proof.

2. Notations and lemmas

We begin with some notations. Given a bounded set $\Omega \subset \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$, we denote

$$\Omega(t) = \{x : (x,t) \in \Omega\}.$$

Let $t_0 = \inf\{t : \Omega(t) \neq \emptyset\}$. The parabolic boundary of the bounded domain Ω is defined by

$$\partial_p \Omega = (\overline{\Omega}(t_0) \times \{t_0\}) \cup \left(\bigcup_{t \in \mathbb{R}} (\partial\Omega(t) \times \{t\}) \right),$$

where $\overline{\Omega}$ denotes the closure of Ω and $\partial\Omega(t)$ denotes the boundary of $\Omega(t)$. We say that the set $\Omega \subset \mathbb{R}^{n+1}$ is a bowl-shaped domain if $\Omega(t)$ is convex for each t and $\Omega(t_1) \subset \Omega(t_2)$ for $t_1 \leq t_2$.

Let

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, j = 1, 2, \ldots, k\}.$$

We say that a function $u \in C^2(\Omega)$ is admissible (or k-convex) if $\lambda(D^2 u) \in \overline{\Gamma}_k$ in Ω.

We say a function $u \in C^{k,j}(\Omega)$ which means that u is k-th continuous differentiable with spatial variables $x \in \mathbb{R}^n$ and j-th continuous differentiable with time variable t for $(x,t) \in \Omega$. A function u is called locally parabolically k-convex if u is locally k-convex in x and nonincreasing in t. We say Ω is an open set in the parabolic sense if $\Omega = \overline{\Omega} \setminus \partial_p \Omega$.

The following is the definition of viscosity solutions.

Definition 2.1. Let u be an upper-semicontinuous (USC for short) (resp. lower-semicontinuous (LSC for short)) function in Ω. Then u is called a viscosity subsolution (supersolution) of

$$-u_t S_{k,l}(D^2 u) = 1 \quad \text{in } \Omega, \tag{2.1}$$

if for any point $(\overline{x}, \overline{t}) \in \Omega$ and any function $h \in C^{2,1}(Q_r(\overline{x}, \overline{t}))$ satisfying

$$u(x,t) - h(x,t) \leq u(\overline{x}, \overline{t}) - h(\overline{x}, \overline{t}) \quad (u(x,t) - h(x,t) \geq u(\overline{x}, \overline{t}) - h(\overline{x}, \overline{t}))$$

for all $(x,t) \in Q_r(\overline{x}, \overline{t})$, where

$$Q_r(\overline{x}, \overline{t}) := \{(x,t) \mid |x - \overline{x}| < r, \overline{t} - r^2 < t \leq \overline{t}\} \subset \Omega,$$

we have

$$-h_t(\overline{x}, \overline{t}) S_{k,l}(D^2 h(\overline{x}, \overline{t})) \geq 1 \quad (-h_t(\overline{x}, \overline{t}) S_{k,l}(D^2 h(\overline{x}, \overline{t})) \leq 1).$$

For the supersolution, we also require that h is locally parabolically k-convex.

A function $u \in C^0(\Omega)$ is called a viscosity solution of (2.1), if it is both a viscosity subsolution and supersolution of (2.1).

Definition 2.2. A function u is called a viscosity subsolution (supersolution) of the problem (1.1), (1.2), if u is a viscosity subsolution (supersolution) of (1.1), and $u(x,t) \le \varphi(x,t)$ ($u(x,t) \ge \varphi(x,t)$) on $\partial_p D$.

A function $u \in C^0(\mathbb{R}^{n+1}_- \setminus D)$ is called a viscosity solution of (1.1), (1.2), if u is a viscosity solution of (1.1), and $u(x,t) = \varphi(x,t)$ on $\partial_p D$.

Definition 2.3. We call u a generalized parabolically symmetric function with respect to a if u is a function of

$$h = -t + \frac{1}{2}a|x|^2.$$

Lemma 2.4. *Let $\Omega_1 \subset \Omega_2$ be two open subsets in $\mathbb{R}^{n+1}$ in the parabolic sense. Suppose $u \in \mathrm{USC}(\Omega_2)$ and $v \in \mathrm{USC}(\overline{\Omega}_1)$ satisfy*

$$-u_t S_{k,l}(D^2 u) \ge 1 \quad \text{in } \Omega_2 \tag{2.2}$$

and

$$-v_t S_{k,l}(D^2 v) \ge 1 \quad \text{in } \Omega_1 \tag{2.3}$$

in the viscosity sense, respectively. Furthermore, assume

$$u \le v \ \text{ in } \Omega_1, \quad u = v \ \text{ on } \partial\Omega_1 \setminus (\partial\Omega_1 \cap \partial\Omega_2).$$

Let

$$w(x,t) = \begin{cases} v(x,t), & (x,t) \in \Omega_1, \\ u(x,t), & (x,t) \in \Omega_2 \setminus \Omega_1. \end{cases}$$

Then $w \in \mathrm{USC}(\Omega_2)$ satisfies

$$-w_t S_{k,l}(D^2 w) \ge 1 \quad \text{in } \Omega_2$$

in the viscosity sense.

Proof. Let $h \in C^{2,1}(\Omega_2)$ and $(\overline{x},\overline{t}) \in \Omega_2$ satisfying

$$w(x,t) - h(x,t) \le w(\overline{x},\overline{t}) - h(\overline{x},\overline{t}), \quad \forall\,(x,t) \in Q_r(\overline{x},\overline{t})$$

for some $Q_r(\overline{x},\overline{t}) \subset \Omega_2$.

If $(\overline{x},\overline{t}) \in \Omega_1$, then for some $Q_{r_1}(\overline{x},\overline{t}) \subset Q_r(\overline{x},\overline{t}) \cap \Omega_1$,

$$v(x,t) - h(x,t) = w(x,t) - h(x,t) \le w(\overline{x},\overline{t}) - h(\overline{x},\overline{t}) = v(\overline{x},\overline{t}) - h(\overline{x},\overline{t}), \ \ \forall\,(x,t) \in Q_{r_1}(\overline{x},\overline{t}).$$

By (2.3), we have

$$-h_t(\overline{x},\overline{t}) S_{k,l}(D^2 h(\overline{x},\overline{t})) \ge 1.$$

If $(\overline{x},\overline{t}) \in \Omega_2 \setminus \Omega_1$, then

$$u(x,t)-h(x,t) \le w(x,t)-h(x,t) \le w(\overline{x},\overline{t})-h(\overline{x},\overline{t}) = u(\overline{x},\overline{t})-h(\overline{x},\overline{t}), \quad \forall\,(x,t) \in Q_r(\overline{x},\overline{t}).$$

By (2.2), we have

$$-h_t(\overline{x},\overline{t})S_{k,l}(D^2h(\overline{x},\overline{t})) \ge 1. \qquad \square$$

Based on Jensen approximations [6], and referring to the parabolic analogue in [9], we can obtain our comparison principle below.

Lemma 2.5 (Comparison Principle)**.** *Let Ω be a bounded open set in $\mathbb{R}^{n+1}$ in the parabolic sense. Let $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ satisfy*

$$-u_tS_{k,l}(D^2u) \ge 1 \ \text{ in } \Omega \quad \text{and} \quad -v_tS_{k,l}(D^2v) \le 1 \ \text{ in } \Omega$$

in the viscosity sense respectively. Then we have

$$\sup_{\Omega}(u-v) \le \sup_{\partial_p\Omega}(u-v). \tag{2.4}$$

Proof. By replacing u by $\beta^{\frac{1}{k+1}}u$, we may assume that u satisfies

$$-u_tS_{k,l}(D^2u) \ge \beta \quad \text{in } \Omega$$

in the viscosity sense, where $\beta > 1$. Construct the sup- and inf-convolution u_ε^+ and v_ε^- of u and v, respectively, on $\overline{\Omega}$:

$$\begin{aligned} u_\varepsilon^+(x,t) &= \sup_{(y,s)\in\Omega}\left\{u(y,s) - \frac{|x-y|^2}{\varepsilon} - \frac{|t-s|^2}{\varepsilon}\right\} = u(y_\varepsilon^+,s_\varepsilon^+) - \frac{|x-y_\varepsilon^+|^2}{\varepsilon} - \frac{|t-s_\varepsilon^+|^2}{\varepsilon}, \\ v_\varepsilon^-(x,t) &= \inf_{(y,s)\in\Omega}\left\{v(y,s) + \frac{|x-y|^2}{\varepsilon} + \frac{|t-s|^2}{\varepsilon}\right\} = v(y_\varepsilon^-,s_\varepsilon^-) + \frac{|x-y_\varepsilon^-|^2}{\varepsilon} + \frac{|t-s_\varepsilon^-|^2}{\varepsilon}, \end{aligned}$$

where $\varepsilon > 0$ is an arbitrarily small parameter.

Since u is upper semi-continuous, we have that u is bounded above in Ω, and for any $(x,t) \in \Omega$,

$$|x-y_\varepsilon^+|^2 + |t-s_\varepsilon^+|^2 = \varepsilon(u(y_\varepsilon^+,s_\varepsilon^+) - u_\varepsilon^+(x,t)) \le \varepsilon\left(\sup_{\Omega} u - u(x,t)\right).$$

Therefore,

$$u(x,t) \le \liminf_{\varepsilon\to 0} u_\varepsilon^+(x,t) \le \limsup_{\varepsilon\to 0} u_\varepsilon^+(x,t) \le \limsup_{\varepsilon\to 0} u(y_\varepsilon^+,s_\varepsilon^+) \le u(x,t),$$

which implies that u_ε^+ converges to u in Ω. Similarly, we can also prove that v_ε^- converges to v in Ω. Then if (2.4) were false, there would exists a small constant $\varepsilon_0 > 0$, such that whenever $0 < \varepsilon \le \varepsilon_0$, the function $w_\varepsilon := u_\varepsilon^+ - v_\varepsilon^-$ attains its local maximum at some interior point $(x_\varepsilon, t_\varepsilon) \in \Omega$ in the parabolic sense.

Let

$$\Gamma_\varepsilon = \{(x,t) \in \Omega : \exists\, p \in \mathbb{R}^n \text{ s.t. } w_\varepsilon(x,t) + p\cdot(y-x) \ge w_\varepsilon(y,s), \forall\,(y,s) \in \Omega, s \le t\}$$

be the upper contact set of w_ε. Similar to the proof of Lemma A.3 in [4], we can prove that Γ_ε has positive measure. Indeed, by mollification, we may assume that w_ε is smooth. Note that w_ε attains its local maximum at $(x_\varepsilon, t_\varepsilon)$. Let $w_{\varepsilon,p}(x,t) := w_\varepsilon(x,t) - p\cdot x$. If δ is sufficiently small and $p \in B_\delta$, then every maximum of $w_{\varepsilon,p}$ lies in the interior of Ω. Since $Dw_\varepsilon - p = 0$ holds at maximum points of $w_{\varepsilon,p}$, we know that $Dw_\varepsilon(\Gamma_\varepsilon) \supset B_\delta$. Noting that w_ε is semi-convex, there exists $\lambda_\varepsilon > 0$ such that $-\lambda_\varepsilon I \le D^2 w_\varepsilon \le 0$ on Γ_ε. Thus,

$$|B_\delta|_{\mathcal{L}^n} \le |Dw_\varepsilon(\Gamma_\varepsilon)|_{\mathcal{L}^n} \le \int_{\Gamma_\varepsilon} |\det D^2 w_\varepsilon(x,t)|\, dxdt \le |\Gamma_\varepsilon|_{\mathcal{L}^{n+1}} \cdot \lambda_\varepsilon^n.$$

Then we obtain that $|\Gamma_\varepsilon|_{\mathcal{L}^{n+1}} > 0$.

Since $w_\varepsilon(x,t) \ge w(x,s)$ for all $(x,t) \in \Gamma_\varepsilon$ and $s \le t$, then $(w_\varepsilon)_t \ge 0$ a.e. in Γ_ε. Moreover,

$$w_\varepsilon(x,t) + p\cdot h \ge w_\varepsilon(x+h,t) \quad \text{and} \quad w_\varepsilon(x,t) - p\cdot h \ge w_\varepsilon(x-h,t)$$

for all $(x,t) \in \Gamma_\varepsilon$ and all sufficiently small vectors h. So

$$2w_\varepsilon(x,t) \ge w_\varepsilon(x+h,t) + w_\varepsilon(x-h,t).$$

Since

$$\frac{w_\varepsilon(x+h,t) + w_\varepsilon(x-h,t) - 2w_\varepsilon(x,t)}{|h|^2}$$

converges weakly to $h' D^2 w_\varepsilon(x,t) h$ as $|h| \to 0$, which follows that $D^2 w_\varepsilon \le 0$ a.e. in Γ_ε (see page 159 in [7]). Thus, we have

$$D^2 u_\varepsilon^+ \le D^2 v_\varepsilon^-, \quad (u_\varepsilon^+)_t \ge (v_\varepsilon^-)_t \quad \text{a.e. in } \Gamma_\varepsilon. \tag{2.5}$$

Since u_ε^+ is a semi-convex function, it is twice differentiable almost everywhere in Ω (see Lemma A.2 in [4]), and at such point (x,t), we have, for $(y,s), (\xi,\tau) \in \Omega$,

$$\begin{aligned}
&u(y,s) - \frac{|\xi - y|^2}{\varepsilon} - \frac{|\tau - s|^2}{\varepsilon} \\
&\le u_\varepsilon^+(\xi,\tau) \\
&= u_\varepsilon^+(x,t) + \partial_t u_\varepsilon^+(x,t)(\tau - t) + D_x u_\varepsilon^+(x,t)\cdot(\xi - x) \\
&\quad + \frac{1}{2}(\xi - x)' D_x^2 u_\varepsilon^+(x,t)(\xi - x) + o(|\xi - x|^2 + |\tau - t|)
\end{aligned}$$

$$
\begin{aligned}
&= u(y_\varepsilon^+, s_\varepsilon^+) - \frac{|x-y_\varepsilon^+|^2}{\varepsilon} - \frac{|t-s_\varepsilon^+|^2}{\varepsilon} \\
&\quad + \partial_t u_\varepsilon^+(x,t)(\tau - t) + D_x u_\varepsilon^+(x,t)\cdot(\xi - x) \\
&\quad + \frac{1}{2}(\xi - x)' D_x^2 u_\varepsilon^+(x,t)(\xi - x) + o(|\xi - x|^2 + |\tau - t|).
\end{aligned}
$$

Taking $\xi = y - y_\varepsilon^+ + x$, $\tau = s - s_\varepsilon^+ + t$, we obtain that

$$
\begin{aligned}
u(y,s) &\le u(y_\varepsilon^+, s_\varepsilon^+) + \partial_t u_\varepsilon^+(x,t)(s - s_\varepsilon^+) + D_x u_\varepsilon^+(x,t)\cdot(y - y_\varepsilon^+) \\
&\quad + \frac{1}{2}(y - y_\varepsilon^+)' D_x^2 u_\varepsilon^+(x,t)(y - y_\varepsilon^+) + o(|y - y_\varepsilon^+|^2 + |s - s_\varepsilon^+|).
\end{aligned} \tag{2.6}
$$

By the definition of viscosity subsolutions, we know that

$$
-(u_\varepsilon^+)_t S_k(D^2 u_\varepsilon^+)(x,t) \ge \beta \quad \text{and} \quad \lambda(D^2 u_\varepsilon^+(x,t)) \in \Gamma_k. \tag{2.7}
$$

Hence, from (2.5) and (2.7), for a.e. $(x,t) \in \Gamma_\varepsilon$,

$$
-(v_\varepsilon^-)_t S_k(D^2 v_\varepsilon^-)(x,t) \ge \beta \quad \text{and} \quad \lambda(D^2 v_\varepsilon^-(x,t)) \in \Gamma_k. \tag{2.8}
$$

On the other hand, similarly as (2.6), we can also obtain that

$$
\begin{aligned}
v(y,s) &\ge v(y_\varepsilon^-, s_\varepsilon^-) + \partial_t v_\varepsilon^-(x,t)(s - s_\varepsilon^-) + D_x v_\varepsilon^-(x,t)\cdot(y - y_\varepsilon^-) \\
&\quad + \frac{1}{2}(y - y_\varepsilon^-)' D_x^2 v_\varepsilon^-(x,t)(y - y_\varepsilon^-) + o(|y - y_\varepsilon^-|^2 + |s - s_\varepsilon^-|).
\end{aligned}
$$

Therefore, by the definition of viscosity supersolutions, we have

$$
-(v_\varepsilon^-)_t S_{k,l}(D_x^2 v_\varepsilon^-)(x,t) \le 1,
$$

which is a contradiction to (2.8). □

To introduce the Perron method for parabolic equations, we first define weak viscosity solutions which do not satisfy (semi) continuous properties.

Definition 2.6. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set in the parabolic sense. We say a function u is a weak viscosity subsolution of

$$
-u_t S_{k,l}(D^2 u) = 1 \quad \text{in } \Omega
$$

if the USC envelope of u, namely,

$$
u^*(x,t) = \lim_{r\to 0} \sup_{(y,s)\in B_r(x,t)} u(y,s)
$$

is finite and a viscosity subsolution, where

$$
B_r(x,t) := \{(y,s) \mid |x-y|^4 + |t-s|^2 < r^2\} \subset \Omega.
$$

Similarly, one uses LSC envelope $u_* = -(-u)^*$ for supersolutions. If u is a weak viscosity sub- and supersolution, we call u a weak viscosity solution.

We can also define weak viscosity solutions of the problem (1.1), (1.2) by giving the boundary condition like Definition 2.2.

From the result in [10], we have the two lemmas below.

Lemma 2.7. *Let Ω be an open set in $\mathbb{R}^{n+1}$ in the parabolic sense. Let $\mathcal{S}$ denote any nonempty set of weak viscosity subsolutions of*

$$-v_t S_{k,l}(D^2 v) = 1 \quad \text{in } \Omega. \tag{2.9}$$

Set

$$u(x,t) = \sup\{v(x,t) \mid v \in \mathcal{S}\} \quad \text{for } (x,t) \in \Omega.$$

Suppose $u^*(x,t) < \infty$ for $(x,t) \in \Omega$, then u is a weak viscosity subsolution of (2.9).

Lemma 2.8. *Let g be a weak viscosity supersolution of (2.9). Let*

$$S_g := \{v \mid v \text{ is a weak viscosity subsolution of (2.9) and } v \le g\}$$

and

$$u(x,t) := \sup\{v(x,t) \mid v \in S_g\}.$$

If S_g is not empty, then u is a weak viscosity solution of (2.9).

In [1], the authors derived a formula of $\sigma_k(\lambda(M))$ for matrices M of the form

$$M = (p_i \delta_{ij} - \beta q_i q_j)_{n\times n}, \tag{2.10}$$

where $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n)$ and $\beta \in \mathbb{R}$.

Proposition 2.9 (Bao–Li–Li)**.** *If M is an $n \times n$ matrix of the form (2.10) for $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n)$ and $\beta \in \mathbb{R}$, then we have*

$$\sigma_k(\lambda(M)) = \sigma_k(p) - \beta \sum_{i=1}^{n} q_i^2 \sigma_{k-1;i}(p),$$

where $\sigma_{k-1;i}(p) = \sigma_{k-1}(p)|_{p_i=0}$.

3. Construction of subsolutions

Denote

$$\mathcal{A}_{k,l} := \{A : A \text{ is a real } n \times n \text{ symmetric positive definite matrix and } S_{k,l}(A) = 1\}.$$

Lemma 3.1. *Let $n \geq 3$, $\varphi \in C^2(\overline{D})$ and $A \in \mathcal{A}_{k,l}$. Then there exists some positive constant c_0 depending only on n, $\|\varphi\|_{C^2(\overline{D})}$, D, A, such that for any $\overline{c} > c_0$, and $(\xi, \lambda) \in \partial_p D$, there exist C_0 depending only on n, $\|\varphi\|_{C^2(\overline{D})}$, D, A, $\overline{c}$, and $\overline{x}(\xi, \lambda) \in \mathbb{R}^n$ satisfying*

$$|\overline{x}(\xi, \lambda)| \leq C_0 \quad \text{and} \quad w_{\xi,\lambda}(x,t) < \varphi(x,t) \ \text{ on } \partial_p D \setminus \{(\xi, \lambda)\},$$

where

$$w_{\xi,\lambda}(x,t) = \varphi(\xi,\lambda) - \overline{c}(t-\lambda) + \frac{1}{2}(x-\overline{x})'A(x-\overline{x}) - \frac{1}{2}(\xi-\overline{x})'A(\xi-\overline{x}), \quad (x,t) \in \mathbb{R}^{n+1}_-.$$

Proof. Denote

$$I := \{x \in \mathbb{R}^n \mid Q(x) \leq 0\}.$$

Let $(\xi, \lambda) \in \partial_p D$. By the mean value theorem, for $x \in I$, there exist $\xi_1, \xi_2 \in I$ such that

$$Q(x) = Q(\xi) + DQ(\xi_1) \cdot (x - \xi),$$
$$Q(x) = Q(\xi) + DQ(\xi) \cdot (x - \xi) + \frac{1}{2}(x-\xi)'D^2Q(\xi_2)(x-\xi).$$

Let

$$M_1 = \max_{x \in I} |DQ(x)|,$$

and M_2 be the half of the minimum of the smallest eigenvalue of $D^2Q(x)$ over $x \in I$. Then we have

$$|Q(x) - Q(\xi)| \leq M_1|x - \xi|,$$
$$Q(x) \geq Q(\xi) + DQ(\xi) \cdot (x - \xi) + M_2|x-\xi|^2.$$

Again by the mean value theorem, for $(x,t) \in \partial_p D$,

$$\begin{aligned}
&\varphi(x,t) \\
&= \varphi(\xi,\lambda) + D_{x,t}\varphi(\xi,\lambda) \cdot ((x,t) - (\xi,\lambda)) + \frac{1}{2}((x,t)-(\xi,\lambda))'D^2_{x,t}\varphi(\overline{\xi},\overline{\lambda})((x,t)-(\xi,\lambda)) \\
&\geq \varphi(\xi,\lambda) + D_x\varphi(\xi,\lambda) \cdot (x-\xi) + \varphi_t(\xi,\lambda)(t-\lambda) - C\big(|x-\xi|^2 + (t-\lambda)^2\big) \\
&= \varphi(\xi,\lambda) + D_x\varphi(\xi,\lambda) \cdot (x-\xi) + \varphi_t(\xi,\lambda)(Q(x)-Q(\xi)) - C\big(|x-\xi|^2 + (Q(x)-Q(\xi))^2\big),
\end{aligned}$$

where $(\overline{\xi}, \overline{\lambda}) \in \overline{D}$, and $C = \frac{1}{2}\big(\max_{\overline{D}} |D^2_{x,t}\varphi| + \max_{\overline{D}} |\varphi_t|\big)$.

Define

$$w_{\xi,\lambda}(x,t) = \varphi(\xi,\lambda) - \overline{c}(t-\lambda) + \frac{1}{2}(x-\overline{x})'A(x-\overline{x}) - \frac{1}{2}(\xi-\overline{x})'A(\xi-\overline{x}), \ \ (x,t) \in \mathbb{R}^{n+1}_-,$$

where

$$\overline{x}(\xi,\lambda) = -A^{-1}D_x\varphi(\xi,\lambda) + \xi - (\overline{c} + \varphi_t(\xi,\lambda))A^{-1}DQ(\xi).$$

Then on $\partial_p D$,

$$
\begin{aligned}
w_{\xi,\lambda}(x,t) &= \varphi(\xi,\lambda) - \overline{c}(t-\lambda) + \frac{1}{2}(x'Ax - \xi' A\xi) - (x-\xi)'A\overline{x} \\
&= \varphi(\xi,\lambda) - \overline{c}(Q(x)-Q(\xi)) + \frac{1}{2}(x-\xi)'A(x-\xi) + D_x\varphi(\xi,\lambda)\cdot(x-\xi) \\
&\quad + (\overline{c} + \varphi_t(\xi,\lambda))DQ(\xi)\cdot(x-\xi).
\end{aligned}
$$

Thus for $\overline{c} \geq \max_{\overline{D}} |\varphi_t|$,

$$
\begin{aligned}
&(w_{\xi,\lambda} - \varphi)(x,t) \\
&\leq (-\overline{c} - \varphi_t(\xi,\lambda))(Q(x)-Q(\xi)) + \frac{1}{2}(x-\xi)'A(x-\xi) + C\big(|x-\xi|^2 + (Q(x)-Q(\xi))^2\big) \\
&\quad + (\overline{c} + \varphi_t(\xi,\lambda))DQ(\xi)\cdot(x-\xi) \\
&\leq (-\overline{c} - \varphi_t(\xi,\lambda))M_2|x-\xi|^2 + \frac{A_{\max}}{2}|x-\xi|^2 + C\big(|x-\xi|^2 + M_1^2|x-\xi|^2\big) \\
&= \left[(-\overline{c} - \varphi_t(\xi,\lambda))M_2 + \frac{A_{\max}}{2} + C(1+M_1^2)\right]|x-\xi|^2,
\end{aligned}
$$

where $A_{\max}$ is the upper bound of A. Set $c_0 = \frac{1}{M_2}\big(\frac{A_{\max}}{2} + C(1+M_1^2)\big) + 2C$, then for $\overline{c} > c_0$,

$$
(-\overline{c} - \varphi_t(\xi,\lambda))M_2 + \frac{A_{\max}}{2} + C(1+M_1^2) < 0,
$$

and

$$
(w_{\xi,\lambda} - \varphi)(x,t) < 0 \quad \text{on } \partial_p D \setminus \{(\xi,\lambda)\}. \qquad \square
$$

By Lemma 3.1, for $(\xi,\lambda) \in \partial_p D$, there exist $c_0 > 0$ and $\overline{x}(\xi,\lambda) \in \mathbb{R}^n$, $|\overline{x}(\xi,\lambda)| < \infty$ such that

$$
w_{\xi,\lambda}(x,t) < \varphi(x,t) \quad \text{on } \partial_p D \setminus \{(\xi,\lambda)\},
$$

where

$$
w_{\xi,\lambda}(x,t) = \varphi(\xi,\lambda) - \overline{c}(t-\lambda) + \frac{1}{2}a|x-\overline{x}|^2 - \frac{1}{2}a|x-\overline{x}|^2, \quad (x,t) \in \mathbb{R}^{n+1}_-,
$$

and $\overline{c} \geq \max\{1, c_0\}$. Then

$$
-(w_{\xi,\lambda})_t S_{k,l}(D^2 w_{\xi,\lambda}) = \overline{c} \geq 1, \quad (x,t) \in \mathbb{R}^{n+1}_-.
$$

Set

$$
w(x,t) = \max_{(\xi,\lambda)\in\partial_p D} w_{\xi,\lambda}(x,t), \quad (x,t) \in \mathbb{R}^{n+1}_-.
$$

Then w is a locally Lipschitz function in $\mathbb{R}^{n+1}_-$,

$$
w(x,t) = \varphi(x,t), \quad (x,t) \in \partial_p D, \tag{3.1}
$$

and by Lemma 2.7,

$$-w_t S_{k,l}(D^2 w) \geq 1, \quad (x,t) \in \mathbb{R}^{n+1}_- \tag{3.2}$$

in the viscosity sense.

Let $D_H = \left\{(x,t) \mid \frac{1}{2}a|x|^2 - H^2 < t \leq 0\right\}$. Without loss of generality, we can assume that $D_{H_1} \subset\subset D \subset\subset D_{H_2}$, where $H_2 > H_1$. For $b > 0$, $h = \sqrt{-t + \frac{1}{2}a|x|^2}$, define

$$u_-(x,t) = U(h) = 2\int_{H_2}^{h} (s^{k-l} + b)^{\frac{1}{k-l}}\, ds + \inf_{D_{H_2}} w, \quad (x,t) \in \mathbb{R}^{n+1}_-,$$

and

$$\overline{u}(x,t) = -t + \frac{1}{2}a|x|^2 + c, \quad (x,t) \in \mathbb{R}^{n+1}_-.$$

We see that

$$u_-(x,t) \leq 2\int_{H_2}^{H_2} (s^{k-l} + b)^{\frac{1}{k-l}}\, ds + \inf_{D_{H_2}} w \leq w(x,t) \quad \text{on } \partial_p D. \tag{3.3}$$

Choose $H_3 = H_2 + 1$ and sufficiently large b, c such that the following three inequalities hold at the same time

$$u_-(x,t) = 2\int_{H_2}^{H_3} (s^{k-l} + b)^{\frac{1}{k-l}}\, ds + \inf_{D_{H_2}} w \geq w(x,t) \quad \text{on } \partial_p D_{H_3},$$

$$\overline{u}(x,t) = -t + \frac{1}{2}a|x|^2 + c \geq w(x,t) \quad \text{on } \partial_p D_{H_3}, \tag{3.4}$$

$$\overline{u}(x,t) = H_1^2 + c \geq w(x,t) \geq u_-(x,t) \quad \text{on } \partial_p D_{H_1}. \tag{3.5}$$

By simple computation, we have

$$\begin{gathered}
U'(h) = 2(h^{k-l} + b)^{\frac{1}{k-l}}, \quad U''(h) = 2(h^{k-l} + b)^{\frac{1}{k-l}-1} h^{k-l-1}, \\
h_t = -\frac{1}{2h}, \quad h_r = \frac{ar}{2h}, \quad h_{rr} = \frac{2ah^2 - a^2r^2}{4h^3}, \\
(u_-)_t = -\frac{1}{h}(h^{k-l} + b)^{\frac{1}{k-l}}, \quad (u_-)_r = (h^{k-l} + b)^{\frac{1}{k-l}} \cdot \frac{ar}{h}, \\
(u_-)_{rr} = (h^{k-l} + b)^{\frac{1}{k-l}-1} h^{k-l-1} \cdot \frac{a^2r^2}{2h^2} + (h^{k-l} + b)^{\frac{1}{k-l}} \cdot \frac{2ah^2 - a^2r^2}{2h^3}.
\end{gathered}$$

Then we have

$$\begin{aligned}
&S_{k,l}(D^2 u_-) \\
&= \frac{C_{n-1}^{k}\left(\frac{(u_-)_r}{r}\right)^{k} + (u_-)_{rr} C_{n-1}^{k-1}\left(\frac{(u_-)_r}{r}\right)^{k-1}}{C_{n-1}^{l}\left(\frac{(u_-)_r}{r}\right)^{l} + (u_-)_{rr} C_{n-1}^{l-1}\left(\frac{(u_-)_r}{r}\right)^{l-1}} \\
&= \left(\frac{(u_-)_r}{r}\right)^{k-l} \frac{C_{n-1}^{k}\left(\frac{(u_-)_r}{r}\right) + (u_-)_{rr} C_{n-1}^{k-1}}{C_{n-1}^{l}\left(\frac{(u_-)_r}{r}\right) + (u_-)_{rr} C_{n-1}^{l-1}}
\end{aligned}$$

$$
\begin{aligned}
&= (h^{k-l}+b)\left(\frac{a}{h}\right)^{k-l} \frac{C_{n-1}^{k}(h^{k-l}+b)^{\frac{1}{k-l}}\cdot\frac{a}{h}+C_{n-1}^{k-1}(u_-)_{rr}}{C_{n-1}^{l}(h^{k-l}+b)^{\frac{1}{k-l}}\cdot\frac{a}{h}+C_{n-1}^{l-1}(u_-)_{rr}}\\
&= (h^{k-l}+b)\left(\frac{a}{h}\right)^{k-l} \frac{C_{n-1}^{k}(h^{k-l}+b)+C_{n-1}^{k-1}\left[h^{k-l-1}\cdot\frac{ar^2}{2h}+(h^{k-l}+b)\cdot\frac{2h^2-ar^2}{2h^2}\right]}{C_{n-1}^{l}(h^{k-l}+b)+C_{n-1}^{l-1}\left[h^{k-l-1}\cdot\frac{ar^2}{2h}+(h^{k-l}+b)\cdot\frac{2h^2-ar^2}{2h^2}\right]}\\
&= (h^{k-l}+b)\left(\frac{a}{h}\right)^{k-l} \frac{C_{n-1}^{k}(h^{k-l}+b)+C_{n-1}^{k-1}\left[(h^{k-l}+b)-\frac{abr^2}{2h^2}\right]}{C_{n-1}^{l}(h^{k-l}+b)+C_{n-1}^{l-1}\left[(h^{k-l}+b)-\frac{abr^2}{2h^2}\right]}\\
&= (h^{k-l}+b)\left(\frac{a}{h}\right)^{k-l} \frac{C_{n}^{k}(h^{k-l}+b)-C_{n-1}^{k-1}\frac{abr^2}{2h^2}}{C_{n}^{l}(h^{k-l}+b)-C_{n-1}^{l-1}\frac{abr^2}{2h^2}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
-(u_-)_t S_{k,l}(D^2u_-) &= \frac{1}{h}(h^{k-l}+b)^{\frac{1}{k-l}}(h^{k-l}+b)\left(\frac{a}{h}\right)^{k-l} \frac{C_{n}^{k}(h^{k-l}+b)-C_{n-1}^{k-1}\frac{abr^2}{2h^2}}{C_{n}^{l}(h^{k-l}+b)-C_{n-1}^{l-1}\frac{abr^2}{2h^2}}\\
&\geq (h^{k-l}+b)\left(\frac{a}{h}\right)^{k-l} \frac{C_{n}^{k}(h^{k-l}+b)-C_{n}^{k}\frac{abr^2}{2h^2}}{C_{n}^{l}(h^{k-l}+b)}\\
&= \left(\frac{a}{h}\right)^{k-l} \frac{C_{n}^{k}\left(h^{k-l}-\frac{bt}{h^2}\right)}{C_{n}^{l}}\\
&\geq a^{k-l}\frac{C_n^k}{C_n^l} = 1.
\end{aligned}
\tag{3.6}
$$

By simple computation, we have

$$
\begin{aligned}
U(h) &= 2\int_{H_2}^{h}(s^{k-l}+b)^{\frac{1}{k-l}}\,ds + \inf_{D_{H_2}} w\\
&= 2\int_{H_2}^{h} s\left[\left(1+\frac{b}{s^{k-l}}\right)^{\frac{1}{k-l}}-1\right]ds + h^2 - H_2^2 + \inf_{D_{H_2}} w\\
&= h^2 + O(h^{l-k+2}) + 2\int_{H_2}^{+\infty} s\left[\left(1+\frac{b}{s^{k-l}}\right)^{\frac{1}{k-l}}-1\right]ds - H_2^2 + \inf_{D_{H_2}} w
\end{aligned}
$$

as h tends to infinity. Then

$$
u_-(x,t) = -t + \frac{1}{2}a|x|^2 + \mu(b) + O\left(\left(-t+\frac{1}{2}a|x|^2\right)^{\frac{l-k+2}{2}}\right)
$$

as $|x|^2 - t$ tends to infinity, where

$$
\mu(b) = 2\int_{H_2}^{+\infty} s\left[\left(1+\frac{b}{s^{k-l}}\right)^{\frac{1}{k-l}}-1\right]ds - H_2^2 + \inf_{D_{H_2}} w.
$$

We can see that $\mu(b)$ is continuous, strictly increasing in $(0,+\infty)$, and

$$
\lim_{b\to+\infty}\mu(b) = +\infty.
$$

Then there exists c^* large enough such that for any $c > c^*$, there exists $b(c)$ satisfying $\mu(b(c)) = c$. Therefore, as $|x|^2 - t \to +\infty$,

$$u_-(x,t) = \overline{u}(x,t) = -t + \frac{1}{2}a|x|^2 + c + O\Big(\Big(-t + \frac{1}{2}a|x|^2\Big)^{\frac{l-k+2}{2}}\Big). \tag{3.7}$$

By (3.5), (3.6), (3.7) and Lemma 2.5, we know that

$$u_- \le \overline{u} \quad \text{in } \mathbb{R}^{n+1}_- \setminus D_{H_1}. \tag{3.8}$$

Now we want to show that u_- is k-convex. From the computation above, we have

$$(u_-)_i = U'h_i = U'\frac{ax_i}{2h}$$

and

$$(u_-)_{ij} = U'\frac{a\delta_{ij}2h - ax_i\frac{ax_j}{h}}{4h^2} + U''\frac{a^2x_ix_j}{4h^2} = U'\frac{a}{2h}\delta_{ij} - \Big(U'\frac{a^2}{4h^3} - U''\frac{a^2}{4h^2}\Big)x_ix_j.$$

By Proposition 2.9, for any $1 \le m \le k-1$,

$$\begin{aligned}
\sigma_m(\lambda(D^2u_-)) &= C_n^m\Big(U'\frac{a}{2h}\Big)^m - \Big(U'\frac{a^2}{4h^3} - U''\frac{a^2}{4h^2}\Big)\sum_{i=1}^n x_i^2 C_{n-1}^{m-1}\Big(U'\frac{a}{2h}\Big)^{m-1} \\
&= U'^{m-1}\frac{a^m}{(2h)^m}\Big[C_n^m U' - C_{n-1}^{m-1}\Big(U'\frac{a}{2h^2} - U''\frac{a}{2h}\Big)|x|^2\Big] \\
&= U'^{m-1}\frac{a^m}{(2h)^m}\Big[C_{n-1}^m U' + C_{n-1}^{m-1}\frac{-t}{h^2} + C_{n-1}^{m-1}U''\frac{a}{2h}|x|^2\Big] \\
&> 0.
\end{aligned}$$

4. Proof of Theorem 1.1

For $c > c^*$, define

$$\underline{u}(x,t) = \begin{cases} \max\{w(x,t), u_-(x,t)\}, & (x,t) \in D_{H_3} \setminus D_{H_1}, \\ u_-(x,t), & (x,t) \in \mathbb{R}^{n+1}_- \setminus D_{H_3}. \end{cases}$$

By (3.3), $\underline{u} \in C^0(\mathbb{R}^{n+1}_- \setminus D_{H_1})$. By (3.2), (3.6) and Lemma 2.4, $\underline{u}$ satisfies

$$-(\underline{u})_t S_{k,l}(D^2\underline{u}) \ge 1 \quad \text{in } \mathbb{R}^{n+1}_- \setminus D$$

in the viscosity sense. By (3.1) and (3.3), we know that

$$\underline{u} = w = \varphi \quad \text{on } \partial_p D. \tag{4.1}$$

Then $\underline{u}$ is a subsolution of (1.1) and (1.2). By (3.7), as $|x|^2 - t \to +\infty$,

$$\underline{u}(x,t) = -t + \frac{1}{2}a|x|^2 + c + O\bigg(\Big(-t + \frac{1}{2}a|x|^2\Big)^{\frac{l-k+2}{2}}\bigg). \tag{4.2}$$

Furthermore, by (3.2), (3.4), (3.5) and Lemma 2.5,

$$w \le \overline{u} \quad \text{in } D_{H_3} \setminus D_{H_1}.$$

Then combining with (3.8),

$$\underline{u} \le \overline{u} \quad \text{in } \mathbb{R}^{n+1}_{-} \setminus D_{H_1}.$$

Let S_c denote the set of functions v which are weak viscosity subsolutions of (1.1) and (1.2) satisfying

$$v \le \overline{u} \quad \text{in } \mathbb{R}^{n+1}_{-} \setminus D.$$

By the arguments above, $\underline{u} \in S_c$. So $S_c \neq \emptyset$. Define

$$u_c(x,t) = \sup\{v(x,t) : v \in S_c\}, \quad (x,t) \in \mathbb{R}^{n+1}_{-} \setminus D.$$

By the definition of u_c, we know that $\underline{u} \le u_c \le \overline{u}$. Then by (4.2), as $|x|^2 - t \to +\infty$,

$$u_c(x,t) = -t + \frac{1}{2}a|x|^2 + c + O\bigg(\Big(-t + \frac{1}{2}a|x|^2\Big)^{\frac{l-k+2}{2}}\bigg).$$

For any $(\xi,\tau) \in \partial_p D$, on the one hand, by (4.1),

$$\liminf_{(x,t)\to(\xi,\tau)} u_c(x,t) \ge \liminf_{(x,t)\to(\xi,\tau)} \underline{u}(x,t) = \varphi(\xi,\tau).$$

On the other hand, we have

$$\liminf_{(x,t)\to(\xi,\tau)} u_c(x,t) \le \varphi(\xi,\tau).$$

Indeed, for every $v \in S_c$, by the definition of viscosity solutions, v^* satisfies

$$\begin{cases} -(v_t)^* + \Delta v^* \ge 0 & \text{in } D_{H_2} \setminus \overline{D}, \\ v^* \le \varphi & \text{on } \partial_p D, \\ v^* \le \sup_{\partial_p D_{H_2}} \overline{u} =: B & \text{on } \partial_p D_{H_2}. \end{cases}$$

Let $\overline{v} \in C^{2,1}(D_{H_2} \setminus \overline{D}) \cap C^0(\overline{D_{H_2}} \setminus D)$ be the solution of the problem [7]

$$\begin{cases} -\overline{v}_t + \Delta \overline{v} = 0 & \text{in } D_{H_2} \setminus \overline{D}, \\ \overline{v} = \varphi & \text{on } \partial_p D, \\ \overline{v} = \sup_{\partial_p D_{H_2}} \overline{u} =: B & \text{on } \partial_p D_{H_2}. \end{cases}$$

By the comparison principle for the heat conduction equation, which can be proved directly by the definition of viscosity solutions, we have $v \leq v^* \leq \overline{v}$ on $\overline{D_{H_2}} \setminus \overline{D}$. So $u_c \leq \overline{v}$ on $\overline{D_{H_2}} \setminus \overline{D}$, and

$$\limsup_{(x,t)\to(\xi,\tau)} u_c(x,t) \leq \limsup_{(x,t)\to(\xi,\tau)} \overline{v}(x,t) = \varphi(\xi,\tau).$$

Thus, $u_c = \varphi$ on $\partial_p D$.

By the definition of u_c and Lemma 2.8, we can prove that u_c is a weak viscosity solution of (1.1). Then by Lemma 2.5 and the asymptotic behavior, $u_c^* \leq u_{c*}$. By the definition of u_c^* and u_{c*}, $u_c^* \geq u_{c*}$. Thus, $u_c^* = u_{c*}$. Then u_c is continuous and a viscosity solution.

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