TAIWANESE JOURNAL OF MATHEMATICS
Vol. 28, No. 2, pp. 297–312, April 2024
DOI: 10.11650/tjm/231103

Structure of Rings Whose Potent Elements are Central

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Abstract. We study the structure of potent elements in matrix rings with same diagonals and polynomial rings, motivated by Jacobson's theorem of commutativity. A ring shall be said to be *PC* if every potent element is central. We investigate the structure of PC rings in relation to the commutativity of rings. It is proved that if R is a PC ring of prime characteristic then the polynomial ring over R is also a PC ring. Every periodic PC ring is shown to be commutative.

1. Introduction

Jacobson [8, Theorem 10.1.1] investigated the commutativity of rings, proving that if R is a ring in which for every $a \in R$ there exists an integer $n(a) > 1$ such that $a^{n(a)} = a$, then R is commutative. From this theorem, we can consider two ring theoretical conditions that for an element a of a ring R,

$$\text{there exists an integer } n(a) > 1 \text{ such that } a^{n(a)} = a \tag{1.1}$$

and

$$\text{if } a \text{ satisfies the condition (1.1) then } a \in Z(R). \tag{1.2}$$

In the literature, an element a of a ring is called *potent* if it satisfies the condition (1.1); and a ring is called *potent* if every element is potent. Hence potent rings are commutative, which is a restatement of Jacobson's theorem above. Therefore it may be interesting to study the structure of noncommutative rings in which every element satisfies the condition (1.2), i.e., every potent element is central. Thus we will call a ring *PC* if it satisfies the condition (1.2).

Throughout this article every ring is an associative ring with identity unless otherwise stated. Let R be a ring. $N(R)$, $I(R)$ and $Z(R)$ denote the set of all nilpotent elements, the set of all idempotents and the set of all central elements in R, respectively. Denote

Received July 3, 2023; Accepted November 12, 2023.
Communicated by Kunio Yamagata.
2020 *Mathematics Subject Classification.* 16U70, 16U40, 16S36.
Key words and phrases. Jacobson's theorem, (central) potent element, PC ring, matrix ring, polynomial ring.
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the n by n full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $T_n(R)$), and write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use I_n and E_{ij} for the identity matrix and the matrix with (i,j)-entry 1 and zeros elsewhere, respectively. $R[x]$ (resp., $R[[x]]$) denotes the polynomial (resp., power series) ring with an indeterminate x over R.

In Section 2, we investigate the shapes of potent elements in $D_n(R)$ (Theorem 2.1) and $R[x]$ ($R[[x]]$) (Theorem 2.2), when R is a PC ring. In Section 3, we prove that (i) if R is a PC domain with $\mathrm{ch}(R) = 0$ then $D_n(R)$ is PC (Theorem 3.4); (ii) for a ring R with prime characteristic, R is commutative if and only if $D_2(R)$ is PC, but $D_n(R)$ is not PC for every $n \geq 3$ (Proposition 3.6); (iii) if R is a periodic PC ring, then the Wedderburn radical of R coincides with $N(R)$ and R is commutative. In this paper, some useful examples of PC rings are also provided, including the Jordan extension and the Dorroh extension.

2. Jacobson's point of view in noncommutative rings

In this section, we will extend the Jacobson's point of view to noncommutative rings, considering the shapes of potent elements in $D_n(R)$ for $n \geq 2$ when R is a PC ring. Write ${}_mC_k = \frac{m!}{(m-k)!k!}$ for $m \geq k \geq 0$, and $n! = n(n-1)\cdots 2 \cdot 1$ for an integer $n \geq 2$. For a real number r, we denote by $[r]$ the unique integer satisfying $r - 1 < [r] \leq r$. Let R be a ring and let $a \in R$ be such that $a^k = a$ for some $k \geq 2$. Then $a^{k-1} \in I(R)$ by the simple calculation that $(a^{k-1})^2 = a^k a^{k-2} = aa^{k-2} = a^{k-1}$. So we will use this fact freely.

Theorem 2.1. *Let R be a PC ring, and $0 \neq A = (a_{ij}) \in D_n(R)$ for $n \geq 2$. Suppose that $A^k = A$ for some $k \geq 2$. Then $a_{ii}^k = a_{ii} \in Z(R)$, and $u!(k-1)a_{s,s+u} = 0$ for every $1 \leq s \leq n-1$ and $1 \leq u \leq n-1$, where $2 \leq s+u \leq n$.*

Proof. Let $A^k = A$ for $A = (a_{ij}) \in D_n(R)$ and $k \geq 2$. Then $A^{k-1} \in I(D_n(R))$ and $a^{k-1} \in I(R)$, where $a = a_{ii}$. Since R is PC, $a \in Z(R)$, and furthermore, we have $A^{k-1} = a^{k-1}I_n$ by [5, Lemma 1]. It then follows that

$$(a_{ij}) = A = A^k = A^{k-1}A = (a^{k-1}I_n)(a_{ij})$$

from which we see that $a_{ij} = a^{k-1}a_{ij}$ for all i, j. Moreover, since $A^{h(k-1)} = A^{k-1} = a^{k-1}I_n$ and $a^{h(k-1)} = a^{k-1}$ for every $h \geq 1$, we have $a_{ij} = a^{h(k-1)}a_{ij}$ for all i, j, and every non-diagonal entry of $A^{h(k-1)}$ is zero. We will use this result and $a \in Z(R)$ freely.

Also, from $A = A^k$, we obtain $a_{12} = {}_kC_1 a^{k-1}a_{12} = ka^{k-1}a_{12} = ka_{12}$ and $(k-1)a_{12} = 0$ follows.

Note that $\left(\begin{smallmatrix} a & a_{12} \\ 0 & a \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & a_{s,s+1} \\ 0 & a \end{smallmatrix}\right)$ have the same structure for every $s \geq 2$. This shows $(k-1)a_{s,s+1} = 0$.

From $A^{2(k-1)} = A^{k-1} = a^{k-1}I_n$, we obtain

$$\begin{aligned}
0 &= {}_{2(k-1)}C_1 a^{2(k-1)-1}a_{13} + {}_{2(k-1)}C_2 a^{2(k-1)-2}a_{12}a_{23} \\
&= 2(k-1)a^{2(k-1)-1}a_{13} + \frac{2(k-1)(2(k-1)-1)}{2}a^{2(k-1)-2}a_{12}a_{23} \\
&= 2(k-1)a^{2(k-1)-1}a_{13} + (k-1)(2(k-1)-1)a^{2k-4}a_{12}a_{23} \\
&= 2(k-1)a^{2(k-1)-1}a_{13} + (2(k-1)-1)a^{2k-4}(k-1)a_{12}a_{23} = 2(k-1)a^{2(k-1)-1}a_{13}
\end{aligned}$$

since $(k-1)a_{12} = 0$. From this, we obtain $0 = a(2(k-1)a^{2(k-1)-1}a_{13}) = 2(k-1)a^{2(k-1)}a_{13} = 2(k-1)a_{13}$ since $a^{2(k-1)}a_{13} = a_{13}$ as above. Note that $\begin{pmatrix} a & a_{12} & a_{13} \\ 0 & a & a_{23} \\ 0 & 0 & a \end{pmatrix}$ and $\begin{pmatrix} a & a_{s,s+1} & a_{s,s+2} \\ 0 & a & a_{s+1,s+2} \\ 0 & 0 & a \end{pmatrix}$ have the same structure for every $s \geq 2$. This shows $2(k-1)a_{s,s+2} = 0$.

From $A^{3\cdot 2(k-1)} = A^{k-1} = a^{k-1}I_n$, we obtain

$$\begin{aligned}
0 &= {}_mC_1 a^{m-1}a_{14} + {}_mC_2 a^{m-2}a_{12}a_{24} + {}_mC_2 a^{m-2}a_{13}a_{34} + {}_mC_3 a^{m-3}a_{12}a_{23}a_{34} \\
&= ma^{m-1}a_{14} + a^{m-2}3(m-1)(k-1)a_{12}a_{24} \\
&\quad + a^{m-2}3(m-1)a_{13}(k-1)a_{34} + a^{m-3}(m-1)(m-2)(k-1)a_{12}a_{23}a_{34} \\
&= 3\cdot 2(k-1)a^{3\cdot 2(k-1)-1}a_{14}
\end{aligned}$$

since $(k-1)a_{h,h+1} = 0$ with $h \geq 1$, ${}_mC_2 = 3(m-1)(k-1)$, ${}_mC_3 = (m-1)(m-2)(k-1)$, where $m = 3\cdot 2(k-1)$. From this, we obtain $0 = a(3\cdot 2(k-1)a^{3\cdot 2(k-1)-1}a_{14}) = 3\cdot 2(k-1)a^{3\cdot 2(k-1)}a_{14} = 3\cdot 2(k-1)a_{14}$ since $a^{3\cdot 2(k-1)}a_{14} = a_{14}$ as above. Similarly we get $3\cdot 2(k-1)a_{s,s+3} = 0$ for every $s \geq 2$.

Now, suppose by induction that

$$(u-1)!(k-1)a_{1,1+(u-1)} = 0 \quad \text{for } 3 \leq u < n.$$

Then, from the results obtained, we claim that the following formula holds for $2 \leq v \leq u$ and $2 \leq s_i < s_j < 1+u$ $(i < j)$:

$${}_{u!(k-1)}C_v a^{m-v}a_{1,s_1}a_{s_1,s_2}\cdots a_{s_q,1+u} = 0,$$

where $m = u!(k-1)$ and $q = v-1$.

For the proof of this claim, we use the well-known fact of number theory that the product of t consecutive positive integers is divisible by $t!$ (where $t \geq 2$). When $v = u-1$ we have

$${}_{u!(k-1)}C_{u-1}a^{m-(u-1)}a_{1,s_1}a_{s_1,s_2}\cdots a_{s_q,1+u} = 0$$

because the set $\{a_{1,s_1}, a_{s_1,s_2}, \ldots, a_{s_q,1+u}\}$ contains $a_{p,p+1}$ for some $p \geq 1$. So assume $u - v \geq 2$. Then $(v-1)u \geq (v-1)(v+2) = v^2 + (v-2) \geq v^2$ since $v \geq 2$, from which we see that $0 \leq \frac{vu-u-v^2}{v} = (u-v) - \frac{u}{v}$; that is, $u - v \geq \frac{u}{v}$.

Now we have

$$\begin{aligned}
&{}_mC_v a^{m-v} a_{1,s_1} a_{s_1,s_2} \cdots a_{s_q,1+u}\\
&= \frac{u!(k-1)(m-1)\cdots(m-(v-1))}{v(v-1)\cdots 2} a^{m-v} a_{1,s_1} a_{s_1,s_2} \cdots a_{s_q,1+u}\\
&= u(u-1)\cdots(v+1)(k-1)(m-1)\cdots(m-(v-1)) a^{m-v} a_{1,s_1} a_{s_1,s_2} \cdots a_{s_q,1+u}\\
&= (u-v)!b(k-1)(m-1)\cdots(m-(v-1)) a^{m-v} a_{1,s_1} a_{s_1,s_2} \cdots a_{s_q,1+u}\\
&= (m-1)\cdots(m-(v-1)) a^{m-v} (b(u-v)!(k-1)) a_{1,s_1} a_{s_1,s_2} \cdots a_{s_q,1+u} = 0
\end{aligned}$$

by the induction hypothesis because $u - v \geq \frac{u}{v}$ and the set $\{a_{1,s_1}, a_{s_1,s_2}, \ldots, a_{s_q,1+u}\}$ contains $a_{p,p+p'}$ for some $p, p' \geq 1$ with $p' \leq \left[\frac{u}{v}\right]$, where $b \geq 1$.

By these preparations we have

$$\begin{aligned}
0 &= {}_mC_1 a^{m-1} a_{1,1+u} + \sum_{v=2}^{u} \sum_{v} {}_mC_v a^{m-v} \text{ (a sum of products of } a_{ij}\text{'s)}\\
&= m a^{m-1} a_{1,1+u} = u!(k-1) a^{u!(k-1)-1} a_{1,1+u},
\end{aligned}$$

noting $A^{u!(k-1)} = A^{k-1} = a^{k-1} I_n$. This yields $0 = a(u!(k-1)a^{u!(k-1)-1}a_{1,1+u}) = u!(k-1)a^{u!(k-1)}a_{1,1+u} = u!(k-1)a_{1,1+u}$ since $a^{u!(k-1)}a_{1,1+u} = a_{1,1+u}$ as above. Similarly we get $u!(k-1)a_{s,s+u} = 0$ for every $s \geq 2$. $\square$

Next we observe the shapes of potent elements in polynomials (power series) over PC rings. Note that for an Abelian ring R (i.e., $I(R) \subseteq Z(R)$), we have the following: (i) $I(R) = I(R[x]) = I(R[[x]])$; (ii) both $R[x]$ and $R[[x]]$ are Abelian (see [11, Lemma 8]).

Theorem 2.2. *Let R be a PC ring. Then the following assertions hold.*

(1) *Suppose that $f(x)^k = f(x)$ for some $k \geq 2$ where $0 \neq f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Then $a_0^k = a_0 \in Z(R)$, $a_n \in N(R)$, and $u!(k-1)a_u = 0$ for every $1 \leq u \leq n$.*

(2) *Suppose that $f(x)^k = f(x)$ for some $k \geq 2$ where $0 \neq f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$. Then $a_0^k = a_0 \in Z(R)$ and $u!(k-1)a_u = 0$ for every $1 \leq u$.*

Proof. (1) The proof is similar to one of Theorem 2.1, but different in several main parts. So we write it in detail. Since $f(x)^k = f(x)$, $a_0^k = a_0$ (hence $a_0^{k-1} \in I(R)$) and $f(x)^{k-1} \in I(R[x])$. Thus $f(x)^{k-1} = a_0^{k-1}$ by [11, Lemma 8] and, because R is PC, we get $a_0 \in Z(R)$, from which we see that

$$f(x) = f(x)^k = f(x)^{k-1} f(x) = a_0^{k-1} \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} a_0^{k-1} a_i x^i.$$

This yields $a_i = a_0^{k-1} a_i$ for all i. From $f(x)^{k-1} = a_0^{k-1}$, we obtain $a_n^{k-1} = 0$, whence $a_n \in N(R)$.

Since $f(x)^{k-1} = f(x)^{h(k-1)}$ and $a_0^{k-1} = a_0^{h(k-1)}$ for every $h \geq 1$, we have $a_i = a_0^{h(k-1)} a_i$ for all i, and every non-constant term of $f(x)^{h(k-1)}$ is zero. We will use this result and $a_0 \in Z(R)$ freely.

From $f(x)^{k-1} = a_0^{k-1}$, we have $0 = {}_{k-1}C_1 a_0^{(k-1)-1} a_1 = (k-1) a_0^{k-2} a_1$, the coefficient of $f(x)^{k-1}$ of degree 1. This implies that $0 = a_0((k-1)a_0^{k-2}a_1) = (k-1)a_1$ since $a_0^{k-1}a_1 = a_1$ as above. Next we calculate the coefficients of $f(x)^{h(k-1)}$ of degree ≥ 2 which are also zero.

From $f(x)^{2(k-1)} = f(x)^{k-1} = a_0^{k-1}$, we obtain

$$\begin{aligned}
0 &= {}_{2(k-1)}C_1 a_0^{2(k-1)-1} a_2 + {}_{2(k-1)}C_2 a_0^{2(k-1)-2} a_1^2 \\
&= 2(k-1) a_0^{2(k-1)-1} a_2 + \frac{2(k-1)(2(k-1)-1)}{2} a_0^{2(k-1)-2} a_1^2 \\
&= 2(k-1) a_0^{2(k-1)-1} a_2 + (k-1)(2(k-1)-1) a_0^{2k-4} a_1^2 \\
&= 2(k-1) a_0^{2(k-1)-1} a_2 + (2(k-1)-1) a_0^{2k-4} (k-1) a_1^2 = 2(k-1) a_0^{2(k-1)-1} a_2
\end{aligned}$$

since $(k-1)a_1 = 0$. From this, we obtain $0 = a_0(2(k-1)a_0^{2(k-1)-1}a_2) = 2(k-1)a_0^{2(k-1)}a_2 = 2(k-1)a_2$ since $a_0^{2(k-1)}a_2 = a_2$ as above.

From $f(x)^{3!(k-1)} = f(x)^{k-1} = a_0^{k-1}$, we obtain

$$\begin{aligned}
0 &= {}_mC_1 a_0^{m-1} a_3 + {}_mC_2 a_0^{m-2} a_1 a_2 + {}_mC_2 a_0^{m-2} a_2 a_1 + {}_mC_3 a_0^{m-3} a_1^3 \\
&= m a_0^{m-1} a_3 + a_0^{m-2} 3(m-1)(k-1) a_1 a_2 + a_0^{m-2} 3(m-1) a_2 (k-1) a_1 \\
&\quad + a_0^{m-3}(m-1)(m-2)(k-1) a_1^3 \\
&= 3!(k-1) a_0^{3 \cdot 2(k-1)-1} a_3
\end{aligned}$$

since $(k-1)a_1 = 0$, ${}_mC_2 = 3(m-1)(k-1)$, ${}_mC_3 = (m-1)(m-2)(k-1)$, where $m = 3!(k-1)$. From this, we obtain $0 = a_0(3!(k-1)a_0^{3!(k-1)-1}a_3) = 3!(k-1)a_0^{3!(k-1)}a_3 = 3!(k-1)a_3$ since $a_0^{3!(k-1)}a_3 = a_3$ as above.

Now, suppose by induction that

$$(u-1)!(k-1)a_{u-1} = 0 \quad \text{for } 3 \leq u \leq n.$$

Then, from the results obtained, we claim that the following formula holds for $2 \leq v \leq u$ and $2 \leq s_i < s_j < 1 + u$ $(i < j)$:

$${}_{u!(k-1)}C_v a_0^{m-v} a_{s_1} a_{s_2} \cdots a_{s_v} = 0,$$

where $m = u!(k-1)$ and $s_1 + s_2 + \cdots + s_v = u$.

When $v = u - 1$ we have

$${}_{u!(k-1)}C_{u-1} a_0^{m-(u-1)} a_{s_1} a_{s_2} \cdots a_{s_{u-1}} = 0$$

because the set $\{a_{s_1}, a_{s_2}, \ldots, a_{s_{u-1}}\}$ contains a_1. So assume $u-v \geq 2$. Then $(v-1)u \geq (v-1)(v+2) = v^2+(v-2) \geq v^2$ since $v \geq 2$, from which we see that $0 \leq \frac{vu-u-v^2}{v} = (u-v)-\frac{u}{v}$; that is, $u - v \geq \frac{u}{v}$.

Now we have

$$\begin{aligned}
&{}_mC_v a_0^{m-v} a_{s_1} a_{s_2} \cdots a_{s_v} \\
&= \frac{u!(k-1)(m-1)\cdots(m-(v-1))}{v(v-1)\cdots 2} a_0^{m-v} a_{s_1} a_{s_2} \cdots a_{s_v} \\
&= u(u-1)\cdots(v+1)(k-1)(m-1)\cdots(m-(v-1)) a_0^{m-v} a_{s_1} a_{s_2} \cdots a_{s_v} \\
&= (u-v)!b(k-1)(m-1)\cdots(m-(v-1)) a_0^{m-v} a_{s_1} a_{s_2} \cdots a_{s_v} \\
&= (m-1)\cdots(m-(v-1)) a_0^{m-v}(b(u-v)!(k-1)) a_{s_1} a_{s_2} \cdots a_{s_v} = 0
\end{aligned}$$

by the induction hypothesis because $u - v \geq \frac{u}{v}$ and the set $\{a_{s_1}, a_{s_2}, \ldots, a_{s_v}\}$ contains a_p for some $p \geq 1$ with $p \leq \left[\frac{u}{v}\right]$, where $b \geq 1$.

By these preparations we have

$$\begin{aligned}
0 &= {}_mC_1 a_0^{m-1} a_u + \sum_{v=2}^{u} \sum_{v} {}_mC_v a_0^{m-v} \ \ \text{(a sum of products of } a_i\text{'s)} \\
&= m a_0^{m-1} a_u = u!(k-1) a_0^{u!(k-1)-1} a_u,
\end{aligned}$$

noting $f(x)^{u!(k-1)} = f(x)^{k-1} = a_0^{k-1}$. This yields $0 = a_0(u!(k-1)a_0^{u!(k-1)-1}a_u) = u!(k-1)a_0^{u!(k-1)}a_u = u!(k-1)a_u$ since $a_0^{u!(k-1)}a_u = a_u$ as above.

(2) This is shown by the proof of (1). □

From now on, let $\mathbb{Z}$ ($\mathbb{Z}_n$) denote the ring of integers (modulo n), and $\mathbb{R}$ (resp., $\mathbb{Q}$) denote the field of real (resp., rational) numbers. The characteristic of R is written by $\mathrm{ch}(R)$. Use $\mathrm{pI}(R)$ to denote the set of all potent elements; and $\overline{R} = R/I$ and $\overline{r} = r + I$ for $r \in R$, where I is an ideal of a ring R.

Notice that $I(R) \subseteq \mathrm{pI}(R)$ for any ring R. Hence if $\mathrm{pI}(R) \subseteq Z(R)$ then $I(R) \subseteq Z(R)$. This shows that PC rings are Abelian. In the following example, parts (1), (2) and (3) provide examples of noncommutative PC rings; and part (4) shows that there exists an Abelian ring but not PC.

Example 2.3. (1) Let K be any field and let $R = W_1(K)$ be the first Weyl algebra over K, that is, R is the ring of polynomials in s, t, subject to the relation that $st = ts + 1$. Then R is a right Noetherian domain by [14, Theorem 1.2.9(ii)], and so R has a right quotient division ring by [14, Theorem 2.1.15], say Q. Evidently, $k \in Z(R)$ for every $k \in K$, and so $K \subseteq Z(R)$. Let $0 \neq f \in R$ such that $f^k = f$ for $k \geq 2$. Then f^{k-1} is a nonzero idempotent by Lemma 3.1(3) to follow. But R is a domain, forcing $f^{k-1} = 1$.

From this, we obtain $f \in K$ by the construction of R, and hence $f \in Z(R)$. Thus R is a PC ring that is noncommutative. It can be similarly shown that Q is also PC.

(2) Let p be a prime number and $n \geq 1$. Let K be the splitting field of $x^{p^n} - x$ over $\mathbb{Z}_p$, and let $R = W_1(K)$. Then, by (1), R is a noncommutative right Noetherian domain that is PC. Note that $\mathrm{ch}(R) = p$ and $k^{p^n} = k$ for every $k \in K$. Note that $s^p \in Z(R)$ and $K \subsetneq Z(R)$.

(3) We consider such a ring of characteristic zero. Let K' be the splitting field of $x^{2^2} - x$ over $\mathbb{Q}$, that is, $K' = \mathbb{Q}(\omega)$ where $\omega^2 + \omega + 1 = 0$. Then $R = W_1(K')$ is also a noncommutative right Noetherian domain that is PC, by (1). Note that $\omega^4 = \omega$ and $\omega^k \neq \omega$ for every $1 \leq k \leq 3$, from which we also obtain that $(-(\omega+1))^4 = -(\omega+1)$, $(-(\omega+1))^k \neq -(\omega+1)$, $(\omega+1)^7 = \omega+1$, $(\omega+1)^h \neq \omega+1$ for any $1 \leq k \leq 3$ and $1 \leq h \leq 6$. Note that R is simple by [14, Theorem 1.3.5] and $K = Z(R)$ follows. We can note that $-(\omega+1)$ is another zero of $x^{2^2} - x$.

(4) There exist division rings which are not PC. Let $R = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_i \in F\}$ where F is a field between $\mathbb{Q}$ and $\mathbb{R}$. Then R is a division ring that is not PC. For, $i^5 = i$ and $i \notin Z(R)$. Similarly, every noncommutative domain $S = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_i \in G\}$ is not PC either, where G is a domain between $\mathbb{Z}$ and $\mathbb{R}$.

A ring R (possibly without identity) is usually called *reduced* if $N(R) = 0$. It is easy to check that reduced rings are Abelian. But reduced rings need not be PC by Example 2.3(4), and PC rings need not be reduced rings either by the existence of non-reduced commutative rings (e.g., $\mathbb{Z}_{m^n}$ with $m, n \geq 2$). So the concepts of PC and reduced are independent of each other.

3. PC ring property of $D_n(R)$ and $R[x]$

In this section we study the structure of potent elements in several ring extensions of PC rings, and the PC ring property of such extensions.

Lemma 3.1. (1) *Let R be a ring of prime characteristic. If R is PC then $N(R) \subseteq Z(R)$.*

(2) *The class of PC rings is closed under subrings and direct products.*

(3) *Let R be a ring and J be a nil ideal of R. Suppose that for $a \in R$, there exist integers $m > n \geq 1$ such that $\overline{a}^m = \overline{a}^n$ for $\overline{a} \in \overline{R} = R/J$. Then there exist $e \in I(R)$ such that $\overline{a}^{n(m-n)} = \overline{e}$.*

(4) [2, Proposition 1.7] *Let R be a ring and $n \geq 2$. $Z(D_n(R)) = \{rI_n + sE_{1n} \mid r, s \in Z(R)\}$.*

(5) [5, Lemma 1] *Let R be an Abelian ring and $n \geq 2$. If $(a_{ij}) \in I(D_n(R))$ then $(a_{ij}) = eI_n$ for some $e \in I(R)$.*

(6) *Every free algebra over a commutative domain is PC.*

Proof. (1) Let $\mathrm{ch}(R) = p$, a prime number, and $a \in N(R)$. Then $a^{p^m} = 0$ for some $m \geq 1$, and so $(1+a)^{p^m} = 1+a^{p^m} = 1$ by [7, Exercises 3.1.10(e)]. This yields $(1+a)^{p^m+1} = 1+a$, i.e., $1+a \in \mathrm{pI}(R)$. If R is PC then $1+a \in Z(R)$, entailing $a \in Z(R)$.

(2) It is obvious.

(3) From $\overline{a}^m = \overline{a}^n$, we obtain $\overline{a}^{n(m-n)} \in I(\overline{R})$ by the proof of [6, Proposition 16]. Then, by [13, Proposition 3.6.1], there exists $e \in I(R)$ such that $\overline{a}^{n(m-n)} = \overline{e}$.

(6) Let $R = S\langle X\rangle$ be a free algebra generated by a set X over a commutative domain S. If $|X| = 1$ then $R \cong S[x]$ and thus R is commutative. Assume $|X| \geq 2$. Then R is a noncommutative domain. Now let $a^k = a$ for $0 \neq a \in R$ and $k \geq 2$. Then $a^{k-1} \in I(R)$ by (3). But R is a domain and $a^{k-1} \neq 0$, we have $a^{k-1} = 1$ and this forces $a \in S$. Thus R is PC since S is clearly contained in $Z(R)$. □

Lemma 3.1(1) is not valid when the characteristic of a given PC ring is not prime. In the following we provide two kinds of examples related to this argument.

Example 3.2. (1) We refer to the construction of Abelian rings given by Antoine [1, Example 4.8]. Let K be a field with $\mathrm{ch}(K) = 0$, $n \geq 2$, and $A = K\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^2 and set $R = A/I$. Each element of A is identified with its image in R for simplicity. We will show that R is PC. Every $r \in R$ is expressed by

$$r = k + k_1b + bfb + g \quad \text{with } k, k_1 \in K,\ f \in R, \text{ and } g \in aR + Ra.$$

Note that $(k_1b+bfb)^2 = 0$, $g \neq g^m \neq 0$ for any $m \geq 2$ when $g \neq 0$, and $k+k_1b+bfb \in U(R)$, the group of units in R, when $k \neq 0$. Now let $0 \neq r = k + k_1b + bfb + g \in R$ be such that $r^n = r$ for some $n \geq 2$.

Assume $k \neq 0$. Then $k + k_1b + bfb \in U(R)$ and

$$(k + k_1b + bfb)^n + c = r^n = r = (k + k_1b + bfb) + g$$

where $c \in RgR$ and a occurs in every term of c when $c \neq 0$. Thus $(k+k_1b+bfb)^n = k+k_1b+bfb$ and, by Lemma 3.1(3), $(k+k_1b+bfb)^{n-1} \in I(R)$. Note that clearly $N(R) = Kb+bRb$, and $I(R) = \{0, 1\}$ by [12, Example 2.10], and we have $(k + k_1b + bfb)^{n-1} = 1$. But since $\mathrm{ch}(K) = 0$,

$$1 = (k + k_1b + bfb)^{n-1} = k^{n-1} + (n-1)k^{n-2}(k_1b + bfb)$$

implies that $k_1b + bfb = 0$, and so $r = k + g$. Consequently,

$$k^n + h_1 g + \cdots + h_{n-1} g^{n-1} + g^n = r^n = r = k + g \quad \text{with } h_i \in K$$

and this forces $g = 0$. Thus we get $r = k \in Z(R)$.

Assume $k = 0$, i.e., $r = k_1b + bfb + g$. Then, since $r = r^n$ ($n \geq 2$) and $(k_1b + bfb)^2 = 0$, we get $k_1b + bfb + g \in RgR$ and this forces $k_1b + bfb = 0$, obtaining $r = g$. This yields $g^n = r^n = r = g$ and this forces $g = 0$, contrary to $r \neq 0$.

From the results obtained, we now have $r = k \in K \subseteq Z(R)$ and therefore R is PC. But $b^2 = 0$, $ab \neq ba$, and $ba, ab \notin N(R)$. Thus $N(R)$ is not an ideal of R.

(2) Consider the first Weyl algebra $R = W_1(K)$ over a field K with $\mathrm{ch}(K) = 0$. Then R is a noncommutative PC domain by Example 2.3(1). Then, by Theorem 3.4(2) to follow, $D_n(R)$ ($n \geq 2$) is a noncommutative PC ring. Note that $\mathrm{ch}(D_n(R)) = 0$ and $N(D_n(R))$ is not contained in $Z(D_n(R))$. Here $N(D_n(R))$ forms an ideal of $D_n(R)$.

For a ring R, we use $N_*(R)$, $N^*(R)$ and $W(R)$ to denote the lower nilradical (i.e., prime radical), the upper nilradical (i.e., the sum of all nil ideals) and the Wedderburn radical (i.e., the sum of all nilpotent ideals) of R, respectively. It is well-known that $W(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$.

As a result of Lemma 3.1(1), we have the following: If R is a PC ring of prime characteristic, then $W(R) = N^*(R) = N_*(R) = N(R)$, and hence $N(R)[x] = W(R)[x] = W(R[x]) = N(R[x])$.

Corollary 3.3. *Let R be a ring and $n \geq 2$. Suppose that $D_n(R)$ is PC. Then R is a PC ring that satisfies the following.*

(1) *Let $n = 2$ and $A = (a_{ij}) \in D_2(R)$. If $A^k = A$ for some $k \geq 2$, then $a_{ii}^k = a_{ii} \in Z(R)$, $a_{12} \in Z(R)$ and $(k-1)a_{12} = 0$.*

(2) *Let $n \geq 3$ and $A = (a_{ij}) \in D_n(R)$. If $A^k = A$ for some $k \geq 2$, then $a_{ii}^k = a_{ii} \in Z(R)$, $a_{1n} \in Z(R)$, $(k-1)a_{1n} = 0$, and $a_{ij} = 0$ for all $i < j$, except a_{1n}.*

Proof. Since $D_n(R)$ is PC, R is PC by Lemma 3.1(2).

(1) Let $A^k = A$ for some $k \geq 2$. Then $a_{ii}^k = a_{ii} \in Z(R)$ and $(k-1)a_{12} = 0$ by Theorem 2.1. Additionally, since $D_2(R)$ is PC, $A \in Z(D_2(R))$ and so, by Lemma 3.1(4), $a_{12} \in Z(R)$.

(2) Let $A^k = A$ for some $k \geq 2$. Then $a_{ii}^k = a_{ii}$. Since $D_n(R)$ is PC, $A \in Z(D_n(R))$ and hence $a_{ii}, a_{1n} \in Z(R)$ and $a_{ij} = 0$ for all $i < j$, except a_{1n}, by Lemma 3.1(4). Hence the structure of A is same as one of $\left(\begin{smallmatrix} a_{11} & a_{1n} \\ 0 & a_{nn} \end{smallmatrix}\right)$, obtaining the desired result by (1). $\square$

In the following, we see a condition for $D_n(R)$ to be PC over a PC ring R.

Theorem 3.4. *Let R be a PC ring that is a domain with $\mathrm{ch}(R) = 0$ and $n \geq 2$. Then the following assertions hold.*

(1) *Let $0 \neq A = (a_{ij}) \in D_n(R)$ be such that $A^k = A$ for some $k \geq 2$. Then $a_{ii}^k = a_{ii} \in Z(R)$, and $a_{ij} = 0$ for all $i < j$; that is, $A = a_{ii}I_n$ with $a_{ii} \in Z(R)$.*

(2) *$D_n(R)$ is PC.*

Proof. (1) Since $A^k = A$, a_{ii} must be nonzero (otherwise, A is nilpotent). Next we have, by Theorem 2.1, that $a_{ii}^k = a_{ii} \in Z(R)$, and $(n-1)!(k-1)a_{ij} = 0$ for all $i < j$ where $1 \leq i \leq n-1$ and $2 \leq j \leq n$. But $(n-1)!(k-1) \neq 0$ because $\mathrm{ch}(R) = 0$. So we obtain $a_{ij} = 0$ from $(n-1)!(k-1)a_{ij} = 0$ since R is a domain.

(2) This is immediate from (1) and Lemma 3.1(4). □

The condition "R is a domain" in Theorem 3.4 is not superfluous by the following example.

Example 3.5. Let $R = \prod_{i=1}^{\infty} \mathbb{Z}_{3^i}$. Then R is a PC ring by Lemma 3.1(2), and $\mathrm{ch}(R) = 0$. But R is not a domain clearly. Consider $D_n(R)$ for $n \geq 3$ and let $A = aI_n + bE_{12} \in D_n(R)$ with $a = (1, 1, 0, 0, \ldots), b = (0, 3, 0, 0, \ldots) \in R$. Then $A^4 = A$ since $a^2 = a$ and $4ab = 4b = b$. However, $A \notin Z(D_n(R))$ by Lemma 3.1(4) and so $D_n(R)$ is not PC. Moreover A is not of the form aI_n.

In addition, Theorem 3.4 is not valid when the characteristic of the base ring R is nonzero, as follows.

Proposition 3.6. *Let R be a ring of prime characteristic. Then we have the following.*

(1) *R is commutative if and only if $D_2(R)$ is PC.*

(2) *$D_n(R)$ is not PC for every $n \geq 3$.*

Proof. (1) It suffices to show the sufficiency. Suppose that $D_2(R)$ is PC. Then $\left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right)$ is contained in $Z(D_2(R))$ by Lemma 3.1(1) since $\left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right) \subseteq N(D_2(R))$ and $\mathrm{ch}(D_2(R))$ is prime, from which we see that $\left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & Z(R) \\ 0 & 0 \end{smallmatrix}\right)$ by Lemma 3.1(4). Hence $R = Z(R)$.

(2) Since $n \geq 3$, $E_{12} \in N(D_n(R))$. But $E_{12} \notin Z(D_n(R))$, and so $D_n(R)$ is not PC by Lemma 3.1(1) since $\mathrm{ch}(D_n(R))$ is prime. □

Example 3.7. (1) If R is a noncommutative ring over which $D_n(R)$ $(n \geq 3)$ is PC, then $\mathrm{ch}(R)$ must not be prime by Proposition 3.6(2).

(2) Let R be the first Weyl algebra over any field K with $\mathrm{ch}(R) = 0$. Then $D_n(R)$ $(n \geq 3)$ is a noncommutative PC ring by Theorem 3.4(2) and Example 2.3(1).

The following shows a condition under which $R[x]$ can be PC over a PC ring R.

Corollary 3.8. (1) *If R is a PC ring that is a domain with* $\mathrm{ch}(R) = 0$*, then* $\mathrm{pI}(R[x]) \subseteq Z(R)$ *and $R[x]$ is PC.*

(2) *If R is a PC ring of prime characteristic, then* (i) *$R[x]$ is PC; and* (ii) *every potent polynomial is contained in* $\mathrm{pI}(R) + xW(R)[x]$.

Proof. (1) Suppose that R is a PC ring that is a domain with $\mathrm{ch}(R) = 0$. Let $f(x)^k = f(x)$ for some $k \geq 2$, where $0 \neq f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Then a_0 must be nonzero (otherwise, $f(x)$ is nilpotent) because $f(x)^{k-1} = a_0^{k-1}$ by [11, Lemma 8]. Next we have, by Theorem 2.2, that $a_0^k = a_0 \in Z(R)$, and $(n-1)!(k-1)a_i = 0$ for all $1 \leq i \leq n$. But since $\mathrm{ch}(R) = 0$, $(n-1)!(k-1) \neq 0$ and, since R is a domain, $(n-1)!(k-1)a_i = 0$ implies $a_i = 0$. Thus $f(x) = a_0 \in Z(R) \subseteq Z(R[x])$, as desired.

(2-i) Suppose that R is a PC ring of prime characteristic. Let $0 \neq f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $f(x)^k = f(x)$ for some $k \geq 2$. Then $a_0 \in Z(R)$ and $a_n \in N(R)$ by Theorem 2.2(1), from which we see $a_n \in Z(R)$ by Lemma 3.1(1).

Consider the nilpotent ideal $I_1 = (Ra_nR)[x]$ of $R[x]$ and the factor ring $R_1 = R[x]/I_1$. Then

$$\overline{f}_1(x)^k = \overline{f}(x)^k = \overline{f}(x) = \overline{f}_1(x)$$

in R_1, where $f_1(x) = \sum_{i=0}^{n-1} a_i x^i$. Here, $\overline{f}_1(x)^k = \overline{f}_1(x)$ implies $(a_0^k + \cdots + a_{n-1}^k x^{(n-1)k}) + I_1 = (a_0 + \cdots + a_{n-1}x^{n-1}) + I_1$ and, since $k \geq 2$, we get $\overline{a}_{n-1}^k = 0$ in R_1, entailing $a_{n-1} \in N(R)$. Then $a_{n-1} \in Z(R)$ by Lemma 3.1(1) again.

Next consider the nilpotent ideal $I_2 = (Ra_{n-1}R + Ra_nR)[x]$ of $R[x]$ and the factor ring $R_2 = R[x]/I_2$. Then

$$\overline{f}_2(x)^k = \overline{f}_1(x)^k = \overline{f}(x)^k = \overline{f}(x) = \overline{f}_1(x) = \overline{f}_2(x)$$

in R_2, where $f_2(x) = \sum_{i=0}^{n-2} a_i x^i$. Here, $\overline{f}_2(x)^k = \overline{f}_2(x)$ implies $(a_0^k + \cdots + a_{n-2}^k x^{(n-2)k}) + I_2 = (a_0 + \cdots + a_{n-2}x^{n-2}) + I_2$ and, since $k \geq 2$, we get $\overline{a}_{n-2}^k = 0$ in R_2, entailing $a_{n-2} \in N(R)$. Then $a_{n-2} \in Z(R)$ by Lemma 3.1(1) again.

Proceeding inductively, we finally obtain that $a_1, a_2, \ldots, a_n \in N(R) \subseteq Z(R)$. Consequently we have $f(x) \in Z(R[x])$ and therefore $R[x]$ is PC.

(2-ii) It comes from the proof of (2-i). □

Let R be a ring and J be an ideal of R. Then J is said to be *idempotent-lifting* (resp., *potent-lifting*) if idempotents (resp., potent elements) in R/J can be lifted to R. Nil ideals are idempotent-lifting by [13, Proposition 3.6.1].

Remark 3.9. (1) We provide another proof of [13, Proposition 3.6.1] when a given ring has prime characteristic. Let R be a ring of $\mathrm{ch}(R) = p$, a prime number, and J be a nil ideal of R. Suppose $\overline{a}^2 = \overline{a}$ in R/J. Then $a^2 - a \in N(R)$ and we can take $m \geq 1$ such that $(a^2 - a)^{p^m} = 0$. The prime number p divides ${}_{p^m}C_k$ for $1 \leq k \leq p^m - 1$ by [7, Exercises 3.1.10(e)], hence ${}_{p^m}C_k r = 0$ for all $r \in R$ because $\mathrm{ch}(R) = p$. So from $(a^2 - a)^{p^m} = 0$, we obtain $a^{2p^m} = a^{p^m}$, noting that $a^{p^m} = -a^{p^m}$ when $p = 2$. It then follows that $a^{p^{2m}} \in I(R)$ by the proof of [6, Proposition 16]. Moreover $\overline{a}^{p^{2m}} = \overline{a}$ since $\overline{a}^2 = \overline{a}$.

(2) Let R be a ring and J be a nil ideal of R. Suppose $\overline{a}^2 = \overline{a}$ in R/J. Then $(a^2 - a)^k = 0$ for some $k \geq 1$. This equality yields $a^k = a^k f(a)$ with $f(x) \in \mathbb{Z}[x]$.

Khurana [10, Example 12] showed that nil ideals need not be potent-lifting. We will give another such example in the following.

Example 3.10. Let $R = \mathbb{Z}[x]/I^2$ and let $N = I/I^2$, where $I = \mathbb{Z}[x](1-x^2)$. Then N is a nilpotent ideal of R. We identify every element of $\mathbb{Z}[x]$ with its image of R for simplicity.

Note that $x - x^3 = x(1-x^2) \in N$, and so $x^3 \equiv x \pmod{N}$. Assume that there exists $y \in R$ such that $y \equiv x \pmod{N}$ and $y^m = y$ for some $m \geq 2$. From the facts that $y \equiv x \pmod{N}$ and $x^4 = 2x^2 - 1$, we have $y = x + r(x)(1-x^2) = x + (a+bx)(1-x^2)$ for some $a, b \in \mathbb{Z}$, where $r(x) = a + bx + \cdots + cx^k$. We calculate $y^m = (x + (a+bx)(1-x^2))^m$. Then $y^m = \sum_{i=0}^{m} {}_mC_i x^{m-i}(a+bx)^i(1-x^2)^i$, and so we finally get

$$y^m = x^m + mx^{m-1}(a+bx)(1-x^2).$$

We first note that $x^{2n} = nx^2 - (n-1)$ for any $n \geq 1$. To prove this equality, we use the induction on n. If $n = 1$, then it holds. Assume that $x^{2k} = kx^2 - (k-1)$. Then $x^{2(k+1)} = x^{2k}x^2 = (kx^2 - (k-1))x^2 = kx^4 - (k-1)x^2 = k(2x^2-1) - (k-1)x^2 = (k+1)x^2 - k$, completing the claim. We use it freely.

Suppose that $m = 2n$. Then

$$\begin{aligned} y^{2n} &= x^{2n} + 2nx^{2n-1}(a+bx)(1-x^2) \\ &= 2nax^{2n-1} + (1+2nb)x^{2n} - 2nax^{2n+1} - 2nbx^{2(n+1)} \\ &= (1 + n(2b-1)) + 2nax + n(1-2b)x^2 - 2nax^3. \end{aligned}$$

Since $y^{2n} = y = a + (1+b)x - ax^2 - bx^3$, we have $1 + n(2b-1) = a$, $2na = 1 + b$, $n(1-2b) = -a$, $-2na = -b$. Then we have $1 + a = a$ and $1 + b = b$, which is impossible.

Suppose that $m = 2n+1$. Then

$$\begin{aligned} y^{2n+1} &= x^{2n+1} + (2n+1)x^{2n}(a+bx)(1-x^2) \\ &= (2n+1)ax^{2n} + (1 + (2n+1)b)x^{2n+1} - (2n+1)ax^{2n+2} - (2n+1)bx^{2n+3} \\ &= (2n+1)a + ((1-n) + (2n+1)b)x - (2n+1)ax^2 + (n - (2n+1)b)x^3. \end{aligned}$$

By the same argument as above, we have $(2n+1)a = a$ and $-n + (2n+1)b = b$, leading to that $2na = 0$ and $2nb = n$, which is also impossible.

Following the literature, a ring R is called *periodic* if for every $a \in R$, there exist integers $m(a) > n(a) \geq 1$ such that $a^{m(a)} = a^{n(a)}$.

Proposition 3.11. *Let R be a periodic PC ring. Then the following assertions hold.*

(1) *$W(R) = N(R) \subseteq Z(R)$.*

(2) *R is commutative.*

(3) *Let $f(x)^k = f(x)$ for $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $k \geq 2$. Then $a_i \in N(R)$ for every $1 \leq i \leq n$.*

Proof. (1) Let $a \in N(R)$. Since R is periodic, $(1+a)^m = (1+a)^n$ for some $m > n \geq 1$ and so $(1+a)^{m-n+1} = 1+a$, because $1+a$ is a unit. Since R is PC, $1+a \in Z(R)$ and so $a \in Z(R)$. Thus $N(R) \subseteq Z(R)$ and $W(R) = N(R)$ follows.

(2) R is commutative by (1) and [4, Theorem].

(3) Note first that $W(R) = N_*(R) = N(R)$ and $R/N_*(R)$ is a reduced (hence Abelian) ring by (1). By Theorem 2.2, we have $a_n \in N(R)$ since $f(x)^k = f(x)$, from which we see that $\overline{f}(x) = \overline{f}_1(x)$ in $\frac{R[x]}{N_*(R)[x]}$ $(\cong \frac{R}{N_*(R)}[x])$, where $f_1(x) = \sum_{i=0}^{n-1} a_i x^i$. It then follows that $\overline{f}_1(x)^{2(k-1)} = \overline{f}(x)^{2(k-1)} = \overline{f}(x)^{k-1} = \overline{f}_1^{k-1}$, that is, $\overline{f}_1^{k-1} \in I(R[x]/N_*(R)[x])$. Thus $\overline{a}_{n-1} \in N(R[x]/N_*(R)[x])$ by [11, Lemma 8]. Since $R[x]/N_*(R)[x]$ is reduced, we get $a_{n-1} \in N_*(R)$ and, consequently, we have $\overline{f}(x) = \overline{f}_1(x) = \overline{f}_2(x)$ in $R[x]/N_*(R)[x]$, where $f_2(x) = \sum_{i=0}^{n-2} a_i x^i$. Proceeding inductively, we obtain $a_i \in N_*(R)$ for all $1 \leq i \leq n$. $\square$

Remark 3.12. (1) $\mathrm{Mat}_n(R)$ and $T_n(R)$ over any ring R for $n \geq 2$ cannot be PC, since they are not Abelian, and so we see that the PC ring property is not Morita invariant.

(2) For a given ring R, recall that R is called *local* if $R/J(R)$ is a division ring, where $J(R)$ is the Jacobson radical of R; R is called *semilocal* if $R/J(R)$ is semisimple Artinian; and R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are clearly Abelian and semilocal. As a corollary of Lemma 3.1(2), we have the following results for a ring R.

(i) R is semiperfect and PC if and only if R is a finite direct product of local PC rings; and

(ii) For $e^2 = e \in Z(R)$, R is PC if and only if both eR and $(1-e)R$ are PC.

Proof. (i) Suppose that R is semiperfect and PC. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [13, Proposition 3.7.2], i.e., each $e_i R e_i$ is a local ring. Since R is PC, it is Abelian as noted earlier; whence $e_i R = e_i R e_i$ for each i and this implies $R = \sum_{i=1}^{n} e_i R$. But each $e_i R$ is also a PC ring by Lemma 3.1(2).

Conversely assume that R is a finite direct product of local PC rings. Then R is semiperfect since local rings are semiperfect by [13, Corollary 3.7.1], and moreover R is PC by Lemma 3.1(2).

(ii) This is immediate from Lemma 3.1(2), since $R = eR \oplus (1-e)R$ for every $e \in I(R) \cap Z(R)$. $\square$

We finally consider the PC ring property of both Jordan extension and Dorroh extension. For a ring R with a monomorphism σ, set $A(R, \sigma)$ be the subset $\{x^{-i}rx^i \mid r \in R \text{ and } i \geq 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$. Note that for $j \geq 0$, $x^j r = \sigma^j(r)x^j$ implies $rx^{-j} = x^{-j}\sigma^j(r)$ for $r \in R$, from which we see that $A(R, \sigma)$ forms a subring of $R[x, x^{-1}; \sigma]$ with the following natural operations: $x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\sigma^j(r) + \sigma^i(s))x^{i+j}$ and $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\sigma^j(r)\sigma^i(s)x^{i+j}$ for $r, s \in R$ and $i, j \geq 0$. Note that $A(R, \sigma)$ is an over-ring of R, and the map $\overline{\sigma}\colon A(R, \sigma) \to A(R, \sigma)$ defined by $\overline{\sigma}(x^{-i}rx^i) = x^{-i}\sigma(r)x^i$ is an automorphism of $A(R, \sigma)$. Jordan showed, with the use of left localization of the skew polynomial $R[x; \sigma]$ with respect to the set of powers of x, that for any pair (R, σ), such an extension $A(R, \sigma)$ always exists in [9]. This ring $A(R, \sigma)$ is said to be the *Jordan extension* of R by σ.

Theorem 3.13. *Let R be a ring with an automorphism σ. Then R is PC if and only if the Jordan extension $A = A(R, \sigma)$ of R by σ is PC.*

Proof. It is enough to show the necessity by Lemma 3.1(2). Note that $\mathrm{pI}(A) = \{x^{-i}rx^i \mid r \in \mathrm{pI}(R) \text{ and } i \geq 0\}$ clearly. Furthermore, $\sigma(r) \in Z(R)$ when $r \in Z(R)$, and so $Z(A) = \{x^{-i}rx^i \mid r \in Z(R) \text{ and } i \geq 0\}$. For,

$$(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\sigma^j(r)\sigma^i(s)x^{i+j} = x^{-(i+j)}\sigma^i(s)\sigma^j(r)x^{i+j} = (x^{-j}sx^j)(x^{-i}rx^i)$$

for $r \in Z(R)$ and $s \in R$ and $i, j \geq 0$.

Now let R be PC and suppose that $a^k = a$ for some $k \geq 2$, where $a = x^{-i}rx^i \in A$. This yields $x^{-i}r^kx^i = x^{-i}rx^i$ and $r^k = r$ follows. Since R is PC, we have $r \in Z(R)$. Thus $a \in Z(A)$ by the above argument, showing that A is PC. $\square$

Let A be an algebra over a commutative ring S. Due to Dorroh [3], the *Dorroh extension* of A by S is the Abelian group $A \times S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in S$. We use $A \times_{\mathrm{dor}} S$ to denote the Dorroh extension of A by S. Note that if A is unitary then $S \subseteq A$.

Proposition 3.14. *Let A be a unitary algebra over a commutative ring S.*

(1) *Suppose that $A \times_{\mathrm{dor}} S$ is PC. If (a, s) is potent in $A \times_{\mathrm{dor}} S$ then $a \in Z(A)$.*

(2) *Suppose that A satisfies the condition that if $a \in A$ is a root of some nonzero polynomial in $S[x]$ then $a \in Z(A)$. Then $A \times_{\mathrm{dor}} S$ is PC.*

Proof. Let $E = A \times_{\mathrm{dor}} S$.

(1) Let $(a, s)^k = (a, s)$ for some $k \geq 2$. Then $(a, s) \in Z(E)$ since E is PC, and it implies that $(a, 0) = (a, s) - (0, s) \in Z(E)$ since $(0, s) \in Z(E)$. Thus $a \in Z(A)$.

(2) Let there exists $k \geq 2$ such that $(a, s)^k = (a, s)$ for $(a, s) \in E$. Then $a \in A$ is a root of the nonzero polynomial $x^k + {}_kC_1 s x^{k-1} + \cdots + {}_kC_{k-2} s^{k-2} x^2 + {}_kC_{k-1} s^{k-1} x$, and so $a \in Z(A)$ by hypothesis. Hence $(a, s) \in Z(E)$, showing that E is PC. $\square$

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