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Existence of Solutions for Asymptotically Periodic Fractional p-Laplacian Equations

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Abstract. In this paper we study a class of asymptotically periodic fractional p-Laplacian equations. Under the suitable conditions, the existence of ground state solutions are obtained via the variational method.

1. Introduction

The aim of this paper is to consider the existence of ground state solutions for the following fractional p-Laplace equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $p \in [2,\infty)$, $N > sp$, $V \in C(\mathbb{R}^N,\mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R},\mathbb{R})$ are asymptotically periodic in x, the fractional p-Laplacian $(-\Delta)_p^s$ is the nonlinear nonlocal operator defined on smooth functions by

$$(-\Delta)_p^s u(x) = 2\lim_{r\to 0}\int_{\mathbb{R}^N\setminus B_r(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,dy, \quad x \in \mathbb{R}^N.$$

Nonlocal fractional operators arise in many different contexts, such as finance, quantum mechanics, game theory and so on, see [5, 7, 17] and the references therein. In particular, when $p = 2$, (1.1) gives back to a fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

here $(-\Delta)^s$ is the so-called fractional Laplacian operator of order s, which can be characterized as $(-\Delta)^s u = \Im^{-1}(|\xi|^{2s}\Im u)$, $\Im$ denotes the usual Fourier transform in $\mathbb{R}^N$. The fractional Schrödinger equation originated from the study of particle problems in random fields driven by Lévy processes in quantum mechanics. Specifically, it comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. Solutions of (1.2) are related to the existence of standing wave solutions for the time-dependent fractional Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = (-\Delta)^s\psi + (V(x)+\hbar)\psi - f(x,\psi) \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

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that is solutions of the form $\psi(x,t) = e^{-i\hbar t}u(x)$, where $\hbar$ is a constant. In recent years, under various assumptions on the potential and the nonlinearity, (1.2) has been widely investigated by many authors, see, for instance [6, 8, 10–12, 14, 15, 21, 23, 26, 27, 29] and references therein. It is particularly noteworthy that Zhang et al. [27] showed the existence of ground state solutions for problem (1.2) with asymptotically periodic terms under the Nehari type condition

$$\text{(Ne)} \qquad t \mapsto \frac{f(x,t)}{|t|} \text{ is strictly increasing on } (-\infty, 0) \text{ and } (0, +\infty).$$

Moreover, for the case of $s = 1$, the authors [28, 30] considered the concentration and multiplication of ground states for the double phase problems including the p-Laplacian and the asymptotic potentials.

At present, the research on the solutions of (1.1) has become one of the hot issues in the field of nonlinear analysis. In [9], Cheng and Tang proved the existence of nontrivial solutions for the case V and f are allowed to be sign-changing. After that, Torres [24] obtained the radially symmetric solutions when $V > 0$ satisfies coercive condition and f is p-superlinear. Furthermore, Ambrosio et al. [2] got the existence and concentration of positive solutions for p-fractional Schrödinger equations. For more results about fractional p-Laplacian problems, we refer the readers to [3, 4, 13, 18–20] and the references therein.

Motivated by the works mentioned above, this paper investigates the existence of ground state solutions for the problem (1.1) without the so-called p-Ambrosetti–Rabinowitz growth condition (see [1]): there is $\eta > p$ such that

$$\text{(AR)} \qquad 0 < \eta F(x,t) \le f(x,t)t \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \neq 0.$$

At the same time, we require that V and f are asymptotically periodic at infinity in x, and problem (1.1) no longer requires the (Ne) condition. In conclusion, the work of this article is different from the above papers, and it also develops the relevant results of [27]. Before proving our results, we denote by $\mathcal{F}$ the class of functions $g \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ such that the set $|\{x \in \mathbb{R}^N : |g(x)| \ge \varepsilon\}| < \infty$ for any $\varepsilon > 0$, where $|\cdot|$ is the Lebesgue measure. Moreover, if $t \neq 0$, let

$$F(x,t) = \int_0^t f(x,\tau)\,d\tau \ge 0 \quad \text{and} \quad \widetilde{F}(x,t) = \frac{1}{p} f(x,t)t - F(x,t) > 0.$$

We assume that V and f satisfy the following conditions:

(V) there exist a constant $a_0 > 0$ and a function $V_0 \in C(\mathbb{R}^N, \mathbb{R})$, 1-periodic in x_i, $1 \le i \le N$, such that $V_0 - V \in \mathcal{F}$ and

$$V_0(x) \ge V(x) \ge a_0 \quad \text{for all } x \in \mathbb{R}^N;$$

(f_1) $f(x,t) = o(|t|^{p-2}t)$ as $|t| \to 0$ uniformly for $x \in \mathbb{R}^N$;

(f_2) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^p} = +\infty$ uniformly for $x \in \mathbb{R}^N$;

(f_3) there exist constants $a_1 > 0$ and $q \in (p, p_s^*)$ such that

$$|f(x,t)| \le a_1(1 + |t|^{q-1}) \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $p_s^* = \frac{Np}{N-sp}$ is the fractional critical exponent;

(f_4) there exist $a_2 > 0$, $\delta > 0$, and $w \in \mathbb{R}$ with $|w| \le \delta$, such that

$$|f(x,t+w) - f(x,t)| \le a_2|w|(1 + |t|^{q-1}) \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(f_5) there exist $q \in (p, p_s^*)$ and functions $g \in \mathcal{F}$, $f_0 \in C(\mathbb{R}^N \times \mathbb{R})$, 1-periodic in x_i, $1 \le i \le N$, such that

(i) $F(x,t) \ge F_0(x,t) = \int_0^t f_0(x,\tau)\, d\tau$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$;

(ii) $|f(x,t) - f_0(x,t)| \le g(x)|t|^{q-1}$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$;

(iii) the map $t \mapsto \frac{f_0(x,t)}{|t|^{p-1}}$ is increasing on $(-\infty, 0)$ and $(0, +\infty)$;

(f_6) there exist $a_3 > 0$ and $\beta > \frac{N(q-p)}{sp}$ such that $\widetilde{F}(x,t) \ge a_3|t|^\beta$;

(f_6') there exist $\sigma, r > 0$ and $\kappa > \frac{N}{sp}$ such that

$$|f(x,t)|^\kappa \le \sigma |t|^{(p-1)\kappa} \widetilde{F}(x,t) \quad \text{for all } |t| \ge r.$$

Our main results are as follows:

Theorem 1.1. *Suppose that* (V) *and* (f_1)–(f_6) *hold. Then problem* (1.1) *has at least one ground state solution.*

Theorem 1.2. *Suppose that* (V), (f_1)–(f_5) *and* (f_6') *hold. Then problem* (1.1) *has at least one ground state solution.*

Remark 1.3. (i) Obviously, (f_2) is weaker than (AR) condition. For any $t \ge 0$, an example of $f(x,t)$ satisfying (f_1)–(f_6) and (f_6') but not (AR) condition is

$$f(x,t) = \psi(x)t^{p-1}\ln(1+t) + \frac{1}{e^{|x|^2}} t^{p-1}[\ln(1+t) + 1 - \cos t],$$
$$f_0(x,t) = \psi(x)t^{p-1}\ln(1+t),$$

where $p \ge 2$ and $\psi(x) \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ is some positive bounded 1-periodic in the x-variables. Moreover, $\frac{f(x,t)}{t^{p-1}}$ is oscillatory, not increasing, so the Nehari manifold method in [27] cannot be applied.

(ii) (f_4) was introduced in [16], which is used to prove Lemma 3.3. One can deduce that (f_4) is equivalent to f being locally Lipschitzian in t and satisfying

$$|f_t'(x,t)| \leq \mathcal{C}(1+|t|^{q-1}) \quad \text{for some } \mathcal{C}>0.$$

The paper is organized as follows. In Section 2, we present a variational framework and give some lemmas. The proofs of Theorems 1.1 and 1.2 are given in Section 3.

Hereafter, we denote the usual norm of $L^q(\mathbb{R})$ by $\|\cdot\|_q$, $C_1, C_2, \ldots$ stand for different positive constants. For any $r>0$ and $x \in \mathbb{R}^N$, $B_r(x) := \{y \in \mathbb{R}^N : |y-x|<r\}$. Moreover, we use $o(1)$ to denote any quantity which tends to zero when $n \to \infty$.

2. Variational setting and preliminaries

In this section, we will establish the variational framework for problem (1.1) and give some useful lemmas. Let the working space

$$E = \left\{ u \in W^{s,p}(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy + \int_{\mathbb{R}^N} V(x)|u|^p\,dx < \infty \right\}$$

be equipped with the norm

$$\|u\|^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy + \int_{\mathbb{R}^N} V(x)|u|^p\,dx.$$

As a consequence, the energy functional for problem (1.1) is defined by

$$\begin{aligned}\mathcal{J}(u) &= \frac{1}{p}\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy + \int_{\mathbb{R}^N} V(x)|u|^p\,dx\right) - \int_{\mathbb{R}^N} F(x,u)\,dx \\ &= \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} F(x,u)\,dx.\end{aligned}$$

It is easy to see that $\mathcal{J} \in C^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle \mathcal{J}'(u), v\rangle = &\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dxdy \\ &+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\,dx - \int_{\mathbb{R}^N} f(x,u)v\,dx\end{aligned}$$

for all $u, v \in E$. Thus, critical points of $\mathcal{J}$ are weak solutions of problem (1.1) as a standard argument. A solution $u_0 \in E$ of problem (1.1) is called a ground state solution if

$$\mathcal{J}(u_0) = \min\{\mathcal{J}(u) : u \in E \setminus \{0\}, \mathcal{J}'(u) = 0\}.$$

Lemma 2.1. [13] *E is continuously embedded in $L^q(\mathbb{R})$ for any $q \in [p, p_s^*]$ and compactly in $L^q_{\mathrm{loc}}(\mathbb{R})$ for any $q \in [p, p_s^*)$. In particular, there exists a constant $C_q > 0$ such that $\|u\|_q \leq C_q\|u\|$ for all $q \in [p, p_s^*]$ and $u \in E$.*

We recall that $\{u_n\} \subset X$ is called a Cerami sequence (in short $(\mathrm{Ce})_c$ sequence) at the level $c \in \mathbb{R}$ if $\mathcal{I}(u_n) \to c$ and $(1 + \|u_n\|_X)\|\mathcal{I}'(u_n)\|_{X^*} \to 0$, where X is a Banach space and $\mathcal{I} \in C^1(X, \mathbb{R})$. We introduce the following lemma, the proof is standard.

Lemma 2.2. *Suppose that* (f_1)–(f_3) *are satisfied. Then* $\mathcal{J}$ *satisfies the mountain pass geometry:*

(I_1) *there exist* $\alpha, \rho > 0$ *such that* $\mathcal{J}(u) \geq \alpha$ *if* $\|u\| = \rho$*;*

(I_2) *there is* $e \in E$ *such that* $\|e\| > \rho$ *and* $\mathcal{J}(e) < 0$.

Proof. In fact, (f_1) and (f_3) imply that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon |t|^{p-1} + C_\varepsilon |t|^{q-1} \quad \text{and} \quad |F(x,t)| \leq \frac{\varepsilon}{p}|t|^p + \frac{C_\varepsilon}{q}|t|^q \tag{2.1}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Thus, together with (2.1) and Lemma 2.1, we obtain

$$\begin{aligned} \mathcal{J}(u) &\geq \frac{1}{p}\|u\|^p - \frac{\varepsilon}{p}\int_{\mathbb{R}^N} |u|^p \, dx - \frac{C_\varepsilon}{q}\int_{\mathbb{R}^N} |u|^q \, dx \\ &\geq \frac{1 - \varepsilon C_p^p}{p}\|u\|^p - \frac{C_\varepsilon C_q^q}{q}\|u\|^q. \end{aligned}$$

If ε is small enough such that $\frac{1-\varepsilon C_p^p}{p} > 0$, then there exist $\alpha, \rho > 0$ such that $\mathcal{J}(u) \geq \alpha$ for all $\|u\| = \rho$.

On the other hand, we fix $\widehat{u} \in E$, by using (f_2), we have for t is large that

$$\frac{\mathcal{J}(t\widehat{u})}{t^p} = \frac{1}{p}\|\widehat{u}\|^p - \int_{\mathbb{R}^N} \frac{F(x, t\widehat{u})}{(t\widehat{u})^p}\widehat{u}^p \, dx \to -\infty,$$

then there certainly exists $t_0 > 0$ such that $\|t_0\widehat{u}\| > \rho$ and $\mathcal{J}(t_0\widehat{u}) < 0$. We can take $e = t_0\widehat{u}$. $\square$

Define

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{J}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \|\gamma(1)\| > \rho, \mathcal{J}(\gamma(1)) \leq 0\}$. By Lemma 2.2 and the variant of the mountain pass theorem (see [22]) with Cerami condition, there exists a $(\mathrm{Ce})_c$ sequence $\{u_n\} \subset E$ satisfying

$$\mathcal{J}(u_n) \to c \geq \alpha > 0 \quad \text{and} \quad (1 + \|u_n\|)\|\mathcal{J}'(u_n)\|_{E^*} \to 0. \tag{2.2}$$

Lemma 2.3. [22] *Suppose that* $\mathcal{J}$ *satisfies* $\mathcal{J}(0) = 0$, (I_1) *and* (I_2)*. Let* $K_c = \{u \in E : \mathcal{J}'(u) = 0$ *and* $\mathcal{J}(u) = c\}$*. If there exists* $\gamma_0 \in \Gamma$ *such that*

$$c = \max_{t \in [0,1]} \mathcal{J}(\gamma_0(t)),$$

then $\mathcal{J}$ *has a nontrivial critical point* $u \in K_c \cap \gamma_0([0,1])$.

Lemma 2.4. *Suppose that* (f_1), (f_3) *and* (f_6) *are satisfied. Then any* $(\mathrm{Ce})_c$ *sequence of* $\mathcal{J}$ *at the level* $c > 0$ *is bounded in* E.

Proof. By using (2.2) and (f_6), we have

$$c + o(1) = \mathcal{J}(u_n) - \frac{1}{p}\langle \mathcal{J}'(u_n), u_n\rangle = \int_{\mathbb{R}^N} \widetilde{F}(x, u_n)\, dx \geq a_3\|u_n\|_\beta^\beta,$$

which implies that $\|u_n\|_\beta^\beta \leq C_1$. Now, recall the following interpolation inequality

$$\|u\|_q \leq \|u\|_\beta^t \|u\|_\gamma^{1-t}, \quad u \in L^\beta(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N),$$

where $0 < \beta \leq q \leq \gamma$, $\frac{1}{q} = \frac{t}{\beta} + \frac{1-t}{\gamma}$ and $t \in [0, 1]$. Without loss of generality, we assume that $\beta < q$ and $\gamma = p_s^*$, then one has

$$\|u_n\|_q^q \leq \|u_n\|_\beta^{tq} \|u_n\|_{p_s^*}^{(1-t)q}. \tag{2.3}$$

By (2.1), (2.2) and Lemma 2.1, we have

$$\begin{aligned} \frac{1}{p}\|u_n\|^p &= \int_{\mathbb{R}^N} F(x, u_n)\, dx + \mathcal{J}(u_n) \\ &\leq \frac{\varepsilon}{p} \int_{\mathbb{R}^N} |u_n|^p\, dx + \frac{C_\varepsilon}{q} \int_{\mathbb{R}^N} |u_n|^q\, dx + C_2 \\ &\leq \frac{\varepsilon C_p^p}{p}\|u_n\|^p + \frac{C_\varepsilon}{q}\|u_n\|_q^q + C_2. \end{aligned}$$

Taking $\varepsilon > 0$ small such that $\frac{1-\varepsilon C_p^p}{p} > 0$ and using (2.3) we deduce that

$$\frac{1-\varepsilon C_p^p}{p}\|u_n\|^p \leq \frac{C_\varepsilon}{q}\|u_n\|_q^q + C_2 \leq \frac{C_\varepsilon}{q}\|u_n\|_\beta^{tq}\|u_n\|_{p_s^*}^{(1-t)q} + C_2 \leq \widetilde{C}_\varepsilon \|u_n\|^{(1-t)q} + C_2. \tag{2.4}$$

Since $\beta > \frac{N(q-p)}{sp}$, we conclude that $p > (1-t)q$. Therefore, (2.4) implies that $\{u_n\}$ is bounded in E. □

Lemma 2.5. *Suppose that* (f_1)–(f_3) *and* (f_6') *are satisfied. Then any* $(\mathrm{Ce})_c$ *sequence of* $\mathcal{J}$ *at the level* $c > 0$ *is bounded in* E.

Proof. Let $\{u_n\} \subset E$ be a $(\mathrm{Ce})_c$ sequence of $\mathcal{J}$. To check the boundedness of $\{u_n\}$, let us argue by contradiction. Suppose that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Passing to a subsequence, we can assume that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ $(\forall\, q \in [p, p_s^*))$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. Then, we can deduce that

$$o(1) = \frac{\langle \mathcal{J}'(u_n), u_n\rangle}{\|u_n\|^p} = 1 - \int_{\mathbb{R}^N} \frac{f(x, u_n)v_n}{\|u_n\|^{p-1}}\, dx$$

and therefore

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{f(x,u_n)v_n}{\|u_n\|^{p-1}}\,dx = 1. \tag{2.5}$$

Using (f_2) and (f_6'), we have

$$\sigma\widetilde{F}(x,t) \geq \left|\frac{f(x,t)t}{t^p}\right|^{\kappa} \geq \left|\frac{pF(x,t)}{t^p}\right|^{\kappa} \to +\infty \quad \text{as } t\to+\infty.$$

For any $r>0$, we set $h(r) = \inf\{\widetilde{F}(x,t) : x\in\mathbb{R}^N, |t|\geq r\}$. Then $h(r)>0$ for all $r>0$ and $h(r)\to+\infty$ as $r\to+\infty$. By (f_6') and for $0\leq a<b$, we set $\Omega_n(a,b) = \{x\in\mathbb{R}^N : a\leq |u(x)|<b\}$ and $C_a^b = \inf\left\{\frac{\widetilde{F}(x,t)}{|t|^p} : x\in\mathbb{R}^N, a\leq|t|<b\right\}>0$. From (2.2) and the above definitions, we obtain

$$\begin{aligned} c+o(1) &= \mathcal{J}(u_n) - \frac{1}{p}\langle \mathcal{J}'(u_n), u_n\rangle \\ &= \int_{\Omega_n(0,a)}\widetilde{F}(x,u_n)\,dx + \int_{\Omega_n(a,b)}\widetilde{F}(x,u_n)\,dx + \int_{\Omega_n(b,+\infty)}\widetilde{F}(x,u_n)\,dx \\ &\geq C_a^b\int_{\Omega_n(a,b)}|u_n|^p\,dx + h(b)|\Omega_n(b,+\infty)|. \end{aligned}$$

Hence, there exists $C_3>0$ such that

$$\max\left\{C_a^b\int_{\Omega_n(a,b)}|u_n|^p\,dx, h(b)|\Omega_n(b,+\infty)|\right\}\leq C_3. \tag{2.6}$$

By virtue of $h(r)\to+\infty$ as $r\to+\infty$, we have $|\Omega_n(b,+\infty)|\to 0$ as $b\to+\infty$, and

$$\int_{\Omega_n(b,+\infty)}|v_n|^{\lambda}\,dx \leq \left(\int_{\Omega_n(b,+\infty)}|v_n|^{p_s^*}\,dx\right)^{\frac{\lambda}{p_s^*}}|\Omega_n(b,+\infty)|^{\frac{p_s^*-\lambda}{p_s^*}}\to 0 \tag{2.7}$$

for any $\lambda\in[p,p_s^*)$, as $b\to+\infty$ uniformly in n. Moreover, from (2.6) we infer that

$$\int_{\Omega_n(a,b)}|v_n|^p\,dx = \frac{1}{\|u_n\|^p}\int_{\Omega_n(a,b)}|u_n|^p\,dx\to 0 \quad \text{as } n\to\infty. \tag{2.8}$$

By (f_1), for any $\varepsilon>0$, there exists $a_\varepsilon>0$ such that $|f(x,t)|\leq\frac{\varepsilon}{3}|t|^{p-1}$ for all $|t|\leq a_\varepsilon$. Thus

$$\int_{\Omega_n(0,a_\varepsilon)}\frac{f(x,u_n)v_n}{\|u_n\|^{p-1}}\,dx \leq \int_{\Omega_n(0,a_\varepsilon)}\frac{\varepsilon|u_n|^{p-1}|v_n|}{3\|u_n\|^{p-1}}\,dx = \frac{\varepsilon}{3}\int_{\Omega_n(0,a_\varepsilon)}|v_n|^p\,dx\leq\frac{\varepsilon}{3} \quad \text{for all } n. \tag{2.9}$$

By using (f_1) and (f_3), there exist $C_4>0$ and $b_\varepsilon>a_\varepsilon$ such that $|f(x,t)|\leq C_4|t|^{p-1}$ for all $x\in\Omega_n(a_\varepsilon,b_\varepsilon)$. Combining with (2.8), there exists $N_0>0$ such that

$$\int_{\Omega_n(a_\varepsilon,b_\varepsilon)}\frac{f(x,u_n)v_n}{\|u_n\|^{p-1}}\,dx\leq C_4\int_{\Omega_n(a_\varepsilon,b_\varepsilon)}|v_n|^p\,dx\leq\frac{\varepsilon}{3} \quad \text{for all } n\geq N_0. \tag{2.10}$$

Set $\kappa' = \frac{\kappa}{\kappa-1} \in \left(1, \frac{N}{N-sp}\right)$. It follows from ($\mathrm{f}_6'$), (2.6), (2.7) and Hölder's inequality that

$$
\begin{aligned}
&\int_{\Omega_n(b_\varepsilon,+\infty)} \frac{f(x,u_n)v_n}{\|u_n\|^{p-1}}\,dx \\
&\le \left[\int_{\Omega_n(b_\varepsilon,+\infty)} \left(\frac{|f(x,u_n)|}{|u_n|^{p-1}}\right)^{\kappa} dx\right]^{\frac{1}{\kappa}} \left(\int_{\Omega_n(b_\varepsilon,+\infty)} |v_n|^{p\kappa'}\,dx\right)^{\frac{1}{\kappa'}} \\
&\le \left(\int_{\Omega_n(b_\varepsilon,+\infty)} \sigma\widetilde{F}(x,u_n)\,dx\right)^{\frac{1}{\kappa}} \left(\int_{\Omega_n(b_\varepsilon,+\infty)} |v_n|^{p\kappa'}\,dx\right)^{\frac{1}{\kappa'}} \\
&\le (\sigma C_4)^{\frac{1}{\kappa}} \left(\int_{\Omega_n(b_\varepsilon,+\infty)} |v_n|^{p\kappa'}\,dx\right)^{\frac{1}{\kappa'}} < \frac{\varepsilon}{3}
\end{aligned}
\tag{2.11}
$$

as $b_\varepsilon \to +\infty$ and for all $n \ge N_0$. By virtue of (2.9)–(2.11), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,u_n)v_n}{\|u_n\|^{p-1}}\,dx = 0,$$

which contradicts (2.5). Then $\{u_n\}$ is bounded in E. $\square$

3. Proofs of the main results

In this section, in order to prove the main results, we need to introduce the following two technical results, the proofs may be found in [22, 27].

Lemma 3.1. *Suppose that* (V) *and* (f_5) *are satisfied. Let $\{u_n\} \subset E$ be a bounded sequence and $\varphi_n = \varphi(\,\cdot - x_n)$, for any $\varphi \in E$ and $\{x_n\} \subset \mathbb{R}^N$. If $|x_n| \to \infty$, then we have*

$$
\begin{aligned}
\int_{\mathbb{R}^N} (V_0(x) - V(x))|u_n|^{p-2}u_n\varphi_n\,dx &= o(1), \\
\int_{\mathbb{R}^N} (f_0(x,u_n) - f(x,u_n))\varphi_n\,dx &= o(1), \\
\int_{\mathbb{R}^N} (F_0(x,u_n) - F(x,u_n))\,dx &= o(1).
\end{aligned}
$$

Lemma 3.2. *Suppose that $\phi \in \mathcal{F}$ and $\mu \in [p, p_s^*]$. If $\{u_n\} \subset E$ and $u_n \rightharpoonup u$ in E, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \phi|u_n|^{\mu}\,dx = \int_{\mathbb{R}^N} \phi|u|^{\mu}\,dx.$$

It is noteworthy to obtain the existence of the ground state solutions of problem (1.1), we study firstly the following periodic equations

$$(-\Delta)_p^s u + V_0(x)|u|^{p-2}u = f_0(x,u), \quad u \in W^{s,p}(\mathbb{R}^N), \tag{3.1}$$

where $V(x) \equiv V_0(x)$ and $f(x) \equiv f_0(x)$ satisfy all the conditions of Theorems 1.1 or 1.2. We define the energy functional of problem (3.1)

$$\begin{aligned}\mathcal{J}_0(u) &= \frac{1}{p}\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy + \int_{\mathbb{R}^N} V_0(x)|u|^p\,dx\right) - \int_{\mathbb{R}^N} F_0(x,u)\,dx \\ &= \frac{1}{p}\|u\|_0^p - \int_{\mathbb{R}^N} F_0(x,u)\,dx.\end{aligned}$$

By concentration compactness principle and some standard variational arguments, it is easy to see that problem (3.1) has a ground state solution $\widetilde{u}$ satisfying

$$c_0 := \mathcal{J}_0(\widetilde{u}) = \inf\left\{\mathcal{J}_0(u) : u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}, \mathcal{J}_0'(u) = 0\right\} > 0.$$

Next, we investigate the following compactness conditions for $\mathcal{J}$ by using Lemmas 3.1 and 3.2.

Lemma 3.3. *Let $\{u_n\} \subset E$ be a bounded* $(\mathrm{Ce})_c$ *sequence for $\mathcal{J}$ and u be its weak limit. Then we have either*

(i) *$u_n \to u$ in E, or*

(ii) *there exist $k \in \mathbb{N}$, nontrivial solutions $w_1, w_2, \ldots, w_k$ of problem* (3.1) *and k sequences of points $\{y_n^i\} \subset \mathbb{R}^N$, $1 \le i \le k$, such that*

$$\begin{gathered}|y_n^i| \to +\infty, \quad |y_n^i - y_n^j| \to +\infty, \quad i \neq j,\ 1 \le j \le k, \\ u_n - \sum_{i=1}^k w_i(\,\cdot - y_n^i) \to u \quad \text{and} \quad \mathcal{J}(u_n) \to \mathcal{J}(u) + \sum_{i=1}^k \mathcal{J}_0(w_i).\end{gathered}$$

Proof. Let $\{u_n\}$ be a bounded $(\mathrm{Ce})_c$ sequence of $\mathcal{J}$ and satisfy (2.2). Passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in E. Furthermore, one has that $\mathcal{J}'(u) = 0$. If the case (i) does not hold. Set $z_{n,1} = u_n - u$, then $z_{n,1} \rightharpoonup 0$ in E, the Brézis–Lieb Lemma leads to

$$\|z_{n,1}\|^p = \|u_n\|^p - \|u\|^p + o(1).$$

By (f_4), a standard argument shows that

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,z_{n,1})\,dx &= \int_{\mathbb{R}^N} F(x,u_n)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx + o(1), \\ \int_{\mathbb{R}^N} f(x,z_{n,1})\varphi\,dx &= \int_{\mathbb{R}^N} f(x,u_n)\varphi\,dx - \int_{\mathbb{R}^N} f(x,u)\varphi\,dx + o(1)\end{aligned}$$

for all $\varphi \in E$ with $\|\varphi\| \le 1$ (see [16,25]). Thus, it is easy to see that

$$\mathcal{J}(z_{n,1}) = \mathcal{J}(u_n) - \mathcal{J}(u) + o(1) \tag{3.2}$$

and

$$\mathcal{J}'(z_{n,1}) = \mathcal{J}'(u_n) - \mathcal{J}'(u) + o(1) = o(1). \tag{3.3}$$

If

$$\xi := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |z_{n,1}|^p \, dx = 0,$$

then the compactness-Lions-type result (see Lemma 2.2 in [3]) implies that $z_{n,1} \to 0$ in $L^q(\mathbb{R}^N)$ $(\forall\, q \in (p, p_s^*))$. Thus, by (2.2) and (3.3) one has

$$o(1) = \langle \mathcal{J}'(z_{n,1}), z_{n,1}\rangle = \|z_{n,1}\|^p - \int_{\mathbb{R}^N} f(x, z_{n,1}) z_{n,1} \, dx = \|z_{n,1}\|^p + o(1),$$

which implies that $u_n \to u$, this is a contradiction. Thus $\xi > 0$. Going if necessary to a subsequence, there exists $\{y_n^1\} \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n^1)} |z_{n,1}|^p \, dx \geq \frac{\xi}{2}. \tag{3.4}$$

Set $w_{n,1} = z_{n,1}(\,\cdot + y_n^1)$. Since $\{z_{n,1}\}$ is bounded, then $\{w_{n,1}\}$ is also bounded, and we assume that $w_{n,1} \rightharpoonup w_1$ in E, $w_{n,1} \to w_1$ in L^q_{loc} $(\forall\, q \in [p, p_s^*))$ and $w_{n,1}(x) \to w_1(x)$ a.e. on $\mathbb{R}^N$. It follows from (3.4) that

$$\int_{B_1(0)} |w_{n,1}|^p \, dx \geq \frac{\xi}{2},$$

then $w_1 \neq 0$. Moreover, since $z_{n,1} \rightharpoonup 0$ in E, up to a subsequence, we can assume that $|y_n^1| \to +\infty$. Define $\varphi_n = \varphi(\,\cdot - y_n^1)$ for all $\varphi \in E$, one can get that

$$\begin{aligned}\langle \mathcal{J}'(z_{n,1}), \varphi_n\rangle - \langle \mathcal{J}_0'(z_{n,1}), \varphi_n\rangle &= \int_{\mathbb{R}^N} (V(x) - V_0(x)) |z_{n,1}|^{p-2} z_{n,1} \varphi_n \, dx \\ &\quad + \int_{\mathbb{R}^N} (f_0(x, z_{n,1}) - f(x, z_{n,1})) \varphi_n \, dx.\end{aligned}$$

By virtue of Lemma 3.1 and (3.3), we have

$$0 = \lim_{n\to\infty} \langle \mathcal{J}'(z_{n,1}), \varphi_n\rangle = \lim_{n\to\infty} \langle \mathcal{J}_0'(z_{n,1}), \varphi_n\rangle = \lim_{n\to\infty} \langle \mathcal{J}_0'(w_{n,1}), \varphi\rangle = \langle \mathcal{J}_0'(w_1), \varphi\rangle,$$

which implies that w_1 is a nontrivial critical point of $\mathcal{J}_0$. Set $z_{n,2} = u_n - w_1(\,\cdot - y_n^1) - u$. We replace $z_{n,1}$ by $z_{n,2}$ and repeat the above argument. If

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |z_{n,2}|^p \, dx = 0,$$

then $z_{n,2} \to 0$ in E, that is, $u_n - w_1(\,\cdot - y_n^1) \to u$ in E. Furthermore, since $z_{n,2} \to 0$ in E, it follows by translation that $w_{n,1} \to w_1$. Thus,

$$\mathcal{J}_0(w_{n,1}) = \mathcal{J}_0(z_{n,1}) \to \mathcal{J}_0(w_1).$$

By using the fact that $z_{n,1} \rightharpoonup 0$ and the conditions (V), (f_5) and Lemma 3.2 we have

$$\mathcal{J}(z_{n,1}) - \mathcal{J}_0(z_{n,1}) = \int_{\mathbb{R}^N} \left[\frac{1}{p}(V(x) - V_0(x))|z_{n,1}|^p + (F_0(x, z_{n,1}) - F(x, z_{n,1})) \right] dx \to 0.$$

Therefore, by (3.2) we have

$$\mathcal{J}(u_n) = \mathcal{J}(u) + \mathcal{J}_0(w_1) + o(1).$$

This implies that the case (ii) holds for $k = 1$. Otherwise, we can find $\{y_n^2\} \subset \mathbb{R}^N$ such that (3.4) holds. Then passing to a subsequence $|y_n^2| \to +\infty$ and $|y_n^1 - y_n^2| \to +\infty$ as $n \to \infty$. Similar to the above argument, let $w_{n,2} = z_{n,2}(\,\cdot + y_n^2)$, then we can find $w_2 \neq 0$ such that up to a subsequence, $w_{n,2} \rightharpoonup w_2$ in E, $w_{n,2} \to w_2$ in L^q_{loc} ($\forall\, q \in [p, p_s^*)$) and $w_{n,2}(x) \to w_2(x)$ a.e. on $\mathbb{R}^N$. Moreover, w_2 is a nontrivial critical point of $\mathcal{J}_0$.

Set $z_{n,3} = u_n - w_1(\,\cdot - y_n^1) - w_2(\,\cdot - y_n^2) - u$, then again by applying the above arguments, the case (ii) holds for $k = 2$. Since $\mathcal{J}_0(w_i) \geq c_0 > 0$ for all i and $\mathcal{J}(u_n)$ is bounded, the iteration must stop at some finite index. □

Proofs of Theorems 1.1 *and* 1.2. By virtue of Lemma 2.2 and Lemmas 2.4 or 2.5, there exists a bounded $(\mathrm{Ce})_c$ sequence $\{u_n\} \subset E$ satisfying (2.2). Up to a subsequence, we assume that $u_n \rightharpoonup u$ in E, and $\mathcal{J}'(u) = 0$. It follows that u is a critical point of $\mathcal{J}$. Next we will split the argument into two cases.

Case 1*:* $0 < c < c_0$. If $u_n \not\rightharpoonup u$ in E, applying Lemma 3.3, there exist k nontrivial solutions $w_1, w_2, \ldots, w_k$ of problem (3.1) such that

$$c = \lim_{n\to\infty} \mathcal{J}(u_n) = \mathcal{J}(u) + \sum_{i=1}^{k} \mathcal{J}_0(w_i) \geq kc_0 \geq c_0,$$

which is a contradiction. Then $u_n \to u$ in E, this shows that $\mathcal{J}$ satisfies the $(\mathrm{Ce})_c$ condition.

Case 2*:* $c \geq c_0$. Let $\mathcal{J}_0(\widetilde{u}) = c_0$ for $\widetilde{u} \in W^{s,p}(\mathbb{R}^N)$. From ($f_5$) we infer that

$$c \leq \max_{t\geq 0} \mathcal{J}(t\widetilde{u}) \leq \max_{t\geq 0} \mathcal{J}_0(t\widetilde{u}) = \mathcal{J}_0(\widetilde{u}) = c_0 \leq c.$$

Thus, $c = \max_{t\geq 0} \mathcal{J}(t\widetilde{u}) = c_0$ and there exists $\gamma_0 \in \Gamma$ such that

$$c = \max_{t\in[0,1]} \mathcal{J}(\gamma_0(t)).$$

We can invoke Lemma 2.3 to deduce that $\mathcal{J}$ has a nontrivial critical point at level $c > 0$.

In view of the above existence result, the set $\mathcal{N} := \{u \in E \setminus \{0\}, \mathcal{J}'(u) = 0\}$ is not empty. We claim that the ground state energy $m := \inf_{u\in\mathcal{N}} \mathcal{J}(u)$ is achieved. Indeed, suppose that $\{u_n\} \subset E$ is a minimizing sequence for m, that is,

$$\mathcal{J}(u_n) \to m, \quad \mathcal{J}'(u_n) \to 0 \quad \text{and} \quad u_n \neq 0.$$

Clearly, $\{u_n\}$ is a $(\mathrm{Ce})_m$ sequence of $\mathcal{J}$, from Lemmas 2.4 or 2.5 we deduce that it is bounded. Then, we can obtain that m is achieved by using the same arguments as Cases 1 and 2. Thus, we know that m is a critical level. The proof is complete. □

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