TAIWANESE JOURNAL OF MATHEMATICS
Vol. 27, No. 6, pp. 1041–1052, December 2023
DOI: 10.11650/tjm/230805

Subeulerian Oriented Graphs

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Abstract. A graph is subeulerian if it is a spanning subgraph of an eulerian graph. All subeulerian graphs were characterized by Boesch, Suffel, Tindell in 1977. Later, a simple proof of their theorem was given by Jaeger. A digraph D is eulerian if and only if D is connected and $d^+(x) = d^-(x)$ for every vertex $x \in V(D)$. An orientation $\overrightarrow{G}$ of a graph G is a digraph obtained from G by replacing each edge xy of G with an arc xy or yx. An oriented graph is an orientation of a simple graph. An oriented graph is said to be subeulerian if it is a spanning subdigraph of an eulerian oriented graph.

In this paper, we initiated the study of subeulerian oriented graphs. We give a necessary and sufficient condition for an orientated digraph to be subeulerian. We refine this condition in order to give necessary and sufficient condition for an orientation of a forest to be subeulerian. Furthermore, we prove that if G is a graph of order n with $n \geq \max\{4\Delta(G) - 1, 3\}$, then every orientation of G is subeulerian. In particular, we show that if G is a graph of odd order n with $\Delta(G) \leq n/4$, then every orientation of G is a spanning subdigraph of a regular tournament.

1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph. Its order and size are respectively denoted by $|V(G)|$ and $|E(G)|$. If $uv \in E(G)$, then we say that u is a *neighbor* of v in G and vice verse. For a vertex $v \in V(G)$, the *degree* of a vertex v is denoted by $d_G(v)$. Its *neighborhood*, denoted by $N_G(v)$, is $\{u : u \in V(G), uv \in E(G)\}$. The *closed neighborhood* of v, denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. If there is no ambiguity, the above symbols can be simplified as $d(v)$, $N(v)$ and $N[v]$. The *minimum degree* and the *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. In particular, G is *k-regular* for some nonnegative integer k if $\Delta(G) = \delta(G) = k$. We use $c(G)$ to denote the number of component of G. For a simple graph G, its complement $\overline{G}$ is the simple graph with $V(\overline{G}) = V(G)$, in which two vertices are adjacent if and only if they are nonadjacent in G.

Received December 3, 2022; Accepted August 21, 2023.
Communicated by Daphne Der-Fen Liu.
2020 *Mathematics Subject Classification.* 05C20, 05C45.
Key words and phrases. orientation, oriented graphs, subeulerian oriented graphs.
This research was supported by NSFC (No. 12061073) and Open Project of Key Laboratory in Xinjiang Uygar Autonomous Region, China (2023D04026) and the Danish research council for independent research (DFF 7014-00037B).
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We say that a graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and write $H \subseteq G$. In this case, we say that G is a *supergraph* of H. A connected acyclic graph is called a *tree*. A disjoint collection of trees is called a *forest*. Let m and n be two positive integers. We use K_n, P_n, C_n to denote the complete graph, the path, the cycle of order n, respectively. The symbol $K_{m,n}$ present the *complete bipartite graph* with its two parts having m and n vertices respectively. We say that $K_{m,n}$ is *odd* if both m and n are odd. The *double star* $S_{m,n}$ is a tree obtained by joining the centres of $K_{1,m-1}$ and $K_{1,n-1}$. For graph theoretical terms and notations not explicitly defined here, readers are referred to [9].

Throughout this paper all digraphs are finite without loops or multiple arcs. Let $D = (V(D), A(D))$ be a digraph. For a vertex $x \in V(D)$. The symbols $N_D^+(x)$ and $N_D^-(x)$ present the *out-neighborhood* and *in-neighborhood* of x in D, respectively. The *neighbourhood* of x, denoted by $N_D(x)$, is $N_D^+(x) \cup N_D^-(x)$. Similarly, $d_D^+(x)$ and $d_D^-(x)$ denote the *out-degree* and *in-degree* of x in D, respectively. The *degree* of x, $d_D(x) = d_D^+(x) + d_D^-(x)$. A vertex $x \in V(D)$ is said to be *source* (or *sink*) if $d_D^+(x) = d_D(x)$ (or $d_D^-(x) = d_D(x)$). The minimum out-degree and in-degree are denoted by $\delta^+(D)$ and $\delta^-(D)$, respectively. Similarly, the *maximum out-degree* and the *maximum in-degree* of D are denoted by $\Delta^+(D)$ and $\Delta^-(D)$, respectively. The *minimum semi-degree* of D is denoted by $\delta^0(D)$, $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$. The *maximum semi-degree* of D is $\Delta^0(D) = \max\{\Delta^+(D), \Delta^-(D)\}$. We say that D is regular if $\delta^0(D) = \Delta^0(D)$. In this case, D is also called $\delta^0(D)$-regular. When the digraph D is understood from the context, we often omit the subscript. We use $U(D)$ to denote the underlying undirected graph of D, the graph obtained from D by erasing all orientation on the arcs of D. For a set $X \subseteq V(D)$, $D[X] = D - (V(D) \setminus X)$. For more digraph theory terminologies, we refer the readers to [5].

An *Euler tour* of a graph is a closed trial containing all edges. A graph is said to *eulerian* if it has an Euler tour. It is well-known that a graph is eulerian if and only if it is connected and degrees of all vertices are even. A graph G is *subeulerian* if it is a spanning subgraph of an eulerian graph. Boesch, Suffel, Tindell [8] characterized all subeulerian graphs in the following two theorems. For a short proof of the above results, we refer to Jaeger [16].

Theorem 1.1. (see [8, Theorem 1]) *A connected graph G is subeulerian if and only if no spanning subgraph of G is isomorphic to an odd complete bipartite graph.*

Theorem 1.2. (see [8, Theorem 3]) *A disconnected graph G is subeulerian if and only if G is not $K_1 \cup K_{2m+1}$.*

For a given graph G (or a digraph D), the subeulerian problem seeks whether one can obtain an eulerian graph (or digraph) by adding edges (arcs). Dually, for a given graph

G (or a digraph D), the supereulerian problem seeks whether one can obtain an eulerian graph (or digraph) by deleting edges (arcs). Boesch, Suffel and Tindell in [8] first raised the supereulerian problem. Pulleyblank [17] proved that the problem of determining if a graph is supereulerian is NP-complete. The supereulerian problem can also be raised for digraphs. A digraph is said to be supereulerian if it has an eulerian spanning subdigraph, otherwise, *nonsupereulerain*. Some sufficient conditions were established for a digraph being supereulerian in [1–4, 6, 7, 10–15, 18].

A digraph D is *weakly eulerian* if $d^+(x) = d^-(x)$ for every $x \in V(D)$. Moreover, D is *eulerian* if it is weakly eulerian and $U(D)$ is connected. It is well-known that, a digraph D is eulerian if and only if D is connected and $d^+(x) = d^-(x)$ for every $x \in V(D)$.

An *orientation* $\overrightarrow{G}$ of a graph G is a digraph obtained from G by replacing each edge xy of G with an arc xy or yx. An *oriented* graph is an orientation of a simple graph. An orientation of a complete graph is called a *tournament*. Clearly, an oriented graph D is eulerian if and only if D is connected and $d^+(x) = d^-(x)$ for every vertex $x \in V(D)$. Similarly, an orientation of a graph is said to be *subeulerian* if it is a spanning subdigraph of an eulerian oriented graph, otherwise, *nonsubeulerian*.

Let $\overleftrightarrow{K_n}$ be a digraph such that there exist symmetric arcs between any distinct vertices. Notice that any digraph D of order n is a spanning subdigraph of $\overleftrightarrow{K_n}$, that is to say any digraph is a spanning subdigraph of an eulerian digraph. But not all oriented graphs are spanning subdigraphs of an eulerian oriented graph. It is an interesting problem to decide whether an oriented graph is a spanning subdigraph of an eulerian oriented graph or not. This is our main topic here.

Let $\overrightarrow{G}$ be a subeulerian orientation of a graph G of order n, and let D be an eulerian oriented graph spanned by $\overrightarrow{G}$. It is easy to see that $\Delta^0(D) \leq (n-1)/2$ and G is subeulerian. By Theorem 1.1, we have

Corollary 1.3. *Let $\overrightarrow{G}$ be a orientation of a graph G of order n. If $\overrightarrow{G}$ is subeulerian, then $\Delta^0(\overrightarrow{G}) \leq (n-1)/2$ and G has no spanning subgraph isomorphic to an odd complete bipartite graph.*

For digraph D, the *converse* of D, denoted by $\overleftarrow{D}$, is the digraph obtained by reversing each arc of D. The proof of the following lemma is omitted.

Lemma 1.4. *An oriented graph D is subeulerian if and only if $\overleftarrow{D}$ is subeulerian.*

In this paper, we give a necessary and sufficient condition for an orientated digraph to be subeulerian. We refine this condition in order to give necessary and sufficient condition for an orientation of a forest being subeulerian. Furthermore, we prove that if G is a graph of order n with $n \geq \max\{4\Delta(G) - 1, 3\}$, then every orientation of G is subeulerian.

In particular, we show that if G is a graph of odd order n with $\Delta(G) \leq n/4$, then every orientation of G is a spanning subdigraph of a regular tournament.

2. Preliminaries

We say that (X, Y) partitions a set S if X and Y are non-empty and $X \cup Y = S$ and $X \cap Y = \emptyset$. We say that (X, Y) is a (x, y)-partition of S if (X, Y) is a partition of S and $x \in X$ and $y \in Y$.

Let G be any oriented graph. We let $e_G(X, Y)$ denote the number of edges in the cut (X, Y). That is, the number of edges $xy \in E(G)$ with $x \in X$ and $y \in Y$. We let $\overline{e_G}(X, Y)$ denote the number of non-edges in the cut (X, Y). That is, $\overline{e_G}(X, Y) = |X| \cdot |Y| - e_G(X, Y)$. Let D be a digraph or multi-digraph. Let $a_D(X, Y)$ denote the number of arcs going from X to Y in D. That is, the number of arcs $xy \in A(D)$ with $x \in X$ and $y \in Y$. If $X \subseteq V(D)$ then we denote the complement of X by $\overline{X} = V(D) \setminus X$. If the graph or digraph is clear from the context then we omit the subscript. The following lemma is well known and follows from the fact that any eulerian tour has to leave any set equally many times as it returns to the set.

Lemma 2.1. *If D is an eulerian digraph, then $a(X, Y) = a(Y, X)$ for all partitions (X, Y) of $V(D)$.*

We say that the deficiency of a set X in D is $\text{def}(X) = |a_D(X, \overline{X}) - a_D(\overline{X}, X)|$. Note that Lemma 2.1 is equivalent to saying that the deficiency of every set in a Eulerain digraph is zero. Let $\mathcal{Q}$ denote all digraphs obtained from a regular tournament by adding an isolated vertex.

Theorem 2.2. *Let D be any oriented digraph of order at least 3. Then D is subeulerian if and only if $\text{def}(X) \leq \overline{e_{UG[D]}}(X, \overline{X})$ for all $X \subseteq V(D)$ and $D \notin \mathcal{Q}$.*

Proof. Let D be any digraph of order at least 3. First assume that $D \in \mathcal{Q}$ and x is the isolated vertex in D (which implies that $D - x$ is a regular tournament). If D is a spanning subdigraph of some eulerian oriented digraph D^*, then some arc, ux, must enter x in D^*, which implies that $d^+_{D^*}(u) > d^-_{D^*}(u)$, a contradiction to D^* being eulerian. So, if $D \in \mathcal{Q}$ then D is not subeulerian.

Now assume that D is an oriented digraph and that there is some $X \subseteq V(D)$ with $\text{def}(X) > \overline{e_{UG[D]}}(X, \overline{X})$ and that D is a spanning subdigraph of some digraph D^*. Then the following

$$\text{def}_{D^*}(X) \geq \text{def}_D(X) - \overline{e_{UG[D]}}(X, \overline{X}) > 0$$

holds, which proves that D^* is not eulerian, by Lemma 2.1.

This proves one direction of the theorem. Now assume that $\mathrm{def}(X) \le \overline{e_{UG[D]}}(X, \overline{X})$ for all $X \subseteq V(D)$ and $D \notin \mathcal{Q}$.

Let H be the multi-digraph with vertex set $V(H) = \{s,t\} \cup V(D)$ and arc set defined as follows. For all $xy \notin UG[D]$ we add the 2-cycle xyx to H. Then, for all $u \in V(G)$ we add $\max\{0, d_D^-(u) - d_D^+(u)\}$ arcs from s to u and we add $\max\{0, d_D^+(u) - d_D^-(u)\}$ arcs from u to t. This completes the construction of H. Let $X_1 = N_H^+(s)$ and let $X_2 = V(D) \setminus X_1$.

The following two claims now complete the proof (where we note that $d_H^+(s)$ denotes the number of arcs out of s in a multi-digraph).

Claim A. *There exists $d_H^+(s)$ arc-disjoint (s,t)-paths in H.*

Proof of Claim A. We will show that for any (s,t)-partition (S,T) of $V(H)$ we have $a_H(S,T) \ge d_H^+(s)$. This will complete the proof of Claim A by Mengers Theorem. Let $S_1 = S \cap X_1$ and $S_2 = S \cap X_2$ and let $S^* = S_1 \cup S_2 = S \setminus \{s\}$ and let $T^* = T \setminus \{t\}$. With these definitions, direct computation yield

$$\begin{aligned}
\mathrm{def}_D(S_2) &= |a_D(S_2, S_1 \cup T^*) - a_D(S_1 \cup T^*, S_2)| \\
&\ge a_D(S_2, S_1) + a_D(S_2, T^*) - a_D(S_1, S_2) - a_D(T^*, S_2), \\
\mathrm{def}_D(T^*) &= |a_D(T^*, S_1 \cup S_2) - a_D(S_1 \cup S_2, T^*)| \\
&\ge a_D(T^*, S_1) + a_D(T^*, S_2) - a_D(S_1, T^*) - a_D(S_2, T^*).
\end{aligned}$$

Adding the two equations above implies the following, as by the definition of S_1 we have $d_D^-(u) - d_D^+(u) > 0$ for all $u \in S_1$ and therefore $\mathrm{def}(S_1) = \sum_u d_D^-(u) - d_D^+(u) = a_D(S_2 \cup T^*, S_1) - a_D(S_1, S_2 \cup T^*)$,

$$\mathrm{def}_D(S_2) + \mathrm{def}_D(T^*) \ge a_D(S_2, S_1) + a_D(T^*, S_1) - a_D(S_1, S_2) - a_D(S_1, T^*) = \mathrm{def}_D(S_1).$$

This implies that the following holds, as $a_H(s, S_1) = \mathrm{def}_D(S_1)$ and $a_H(S_2, t) = \mathrm{def}_D(S_2)$ and $\mathrm{def}_D(X) \le \overline{e_{UG[D]}}(X, \overline{X}) = a_H(V(D) \setminus X, X) = a_H(X, V(D) \setminus X)$ for all $X \subseteq V(D)$,

$$\begin{aligned}
d_H^+(s) &= a_H(s, S_1) + a_H(s, T^*) = \mathrm{def}_D(S_1) + a_H(s, T^*) \\
&\le \mathrm{def}_D(S_2) + \mathrm{def}_D(T^*) + a_H(s, T^*) = a_H(S_2, t) + \mathrm{def}_D(T^*) + a_H(s, T^*) \\
&\le a_H(S_2, t) + a_H(S_1 \cup S_2, T^*) + a_H(s, T^*) = a_H(S, T).
\end{aligned}$$

This completes the proof of Claim A.

Claim B. *If there exist $d_H^+(s)$ arc-disjoint (s,t)-paths in H, then D is subeulerian.*

Proof of Claim B. Let $P_1, P_2, \ldots, P_k$ be $d_H^+(s)$ arc-disjoint (s,t)-paths in H ($k = d_H^+(s)$) after removing s and t (and the arcs incident with s and t) from each path. Adding the paths $P_1, P_2, \ldots, P_k$ to D results in a digraph, which we denote by F, with $d^+_F(u) = d^-_F(u)$ for all $u \in V(F)$. If there exists 2-cycles in F, then we remove both the arcs of the 2-cycle from F. For convenience, the resulting graph still denoted by F. Note that we still have $d^+_F(u) = d^-_F(u)$ for all $u \in V(H)$ and that D is a spanning subdigraph of F (as no arc of D is part of a 2-cycle in F).

We have now proved that D is a spanning subdigraph of a weakly eulerian oriented digraph. Let R be a weakly eulerian oriented digraph containing D as a spanning subdigraph such that R contains as few connected components as possible. If R is connected, then we are done, so assume that R is not connected. If there exists $a_1, a_2, b_1, b_2 \in V(R)$ such that no connected component contains a vertex a_i and a vertex b_j, then adding the arcs $a_1b_1a_2b_2a_2$ to R gives us a weakly eulerian oriented digraph containing D as a subdigraph with fewer connected components than R, a contradiction. Thus there must exist a vertex $r \in V(R)$ such that $R - r$ is a connected component of R and r is an isolated vertex in R. As $|V(R)| = |V(G)| \geq 3$ we note that $|V(R-r)| \geq 2$ and as $R-r$ is oriented and eulerian we note that $|V(R-r)| \geq 3$.

If $D[V(R) \setminus \{r\}]$ is a regular tournament then $D \in \mathcal{Q}$, a contradiction, so $D[V(R) \setminus \{r\}]$ is not a regular tournament. If $D[V(R) \setminus \{r\}]$ is a tournament, then $R-r = D[V(R) \setminus \{r\}]$ is a non-regular tournament, a contradiction to R being weakly eulerian (as r is isolated in R). So, $D[V(R) \setminus \{r\}]$ is not a tournament and there exists two vertices $q, w \in V(R-r)$ which are non-adjacent in D. If $qw \in A(R)$ then replace qw with the path qrw and if $wq \in A(R)$ then replace wq with the path wrq and if q and w are non-adjacent in R then add the 3-cycle $qwrq$. In all cases we obtain a contradiction to the minimality of the number of connected components in R. This completes the proof. □

Corollary 2.3. *Let G be any graph of order at least 3. Then every orientation of G is subeulerian if and only if $e(X,Y) \leq \overline{e}(X,Y)$ for all partitions (X,Y) of $V(G)$.*

Proof. Let G be any graph of order at least 3. If $e_G(X,\overline{X}) > \overline{e_G}(X,\overline{X})$ for some subset $X \subseteq V(G)$, then let D be any orientation of G that orients all edges in the cut $(X,\overline{X})$ from X to $\overline{X}$. In this case $\mathrm{def}(X) > \overline{e_G}(X,\overline{X})$ and so D is not subeulerian by Theorem 2.2.

If $e_G(X,\overline{X}) \leq \overline{e_G}(X,\overline{X})$ for all subsets $X \subseteq V(G)$ and D is any orientation of G, then $D \notin \mathcal{Q}$ (as $e_G(\{x\}, V(G) \setminus \{x\}) < \overline{e}(\{x\}, V(G) \setminus \{x\})$ for all $x \in V(G)$) and therefore D is subeulerian by Theorem 2.2 (as $\mathrm{def}_D(X) \leq e_G(X,\overline{X})$ for all X). This completes the proof. □

3. Orientation of a forest

For convenience, a trivial orientation of a connected bipartite graph $G[X,Y]$ is an orientation of G, in which all edges of G are oriented from X to Y, or from Y to X. Thus, there are two kinds of trivial orientations of a connected bipartite graph $G[X, Y]$, one of which is the converse of the other.

For a positive integer k, B_k denotes the tree obtained from identifying two leaves of two vertex-disjoint stars $K_{1,k}$. Clearly, the order of B_k is $2k+1$. In particular, $B_1 \cong P_3$ and $B_2 \cong P_5$. Note that there is a unique trivial orientation of P_6 up to isomorphism. For an example of a trivial orientation of P_6 and B_4, see Figure 3.1.

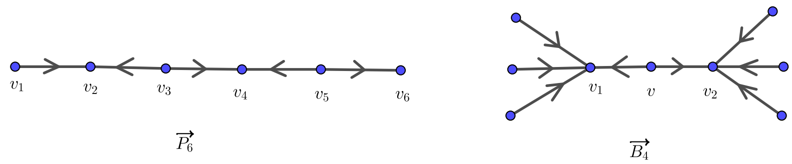

Figure 3.1: A trivial orientation of P_6 and B_4.

Lemma 3.1. *Each trivial orientation of P_6 and B_k is nonsubeulerian, where $k \geq 1$.*

Proof. Let $V(P_6) = \{v_1, \ldots, v_6\}$ and $\overrightarrow{P_6}$, a trivial orientation of P_6. In view of Lemma 1.4, we may assume that $A(\overrightarrow{P_6}) = \{v_1v_2, v_3v_2, v_3v_4, v_5v_4, v_5v_6\}$. Suppose that D is an eulerian oriented graph with $V(D) = V(\overrightarrow{P_6})$ and $A(D) \supseteq A(\overrightarrow{P_6})$. By Corollary 1.3, $\Delta^0(D) = \Delta^-(D) = 2$. Thus v_2 and v_4 are nonadjacent in D. Since $N_D^-(v_2) = \{v_1, v_3\}$ and $N_D^-(v_4) = \{v_3, v_5\}$, we have $N_D^+(v_2) = \{v_5, v_6\}$ and $N_D^+(v_4) = \{v_1, v_6\}$. It forces that $v_6 \in N_D^+(v_2) \cap N_D^+(v_4)$. This gives $d_D^-(v_6) \geq 3$, contradicting that $d_D^+(v_6) = d_D^-(v_6) \leq (5-1)/2$.

Since $B_1 \cong P_3$, clearly, any trivial orientation of P_3 is nonsubeulerian. Now let $k \geq 2$. Again by Lemma 1.4, let $\overrightarrow{B_k}$ be a trivial orientation of B_k such that $\Delta^0(\overrightarrow{B_k}) = \Delta^-(\overrightarrow{B_k})$. Let v_1 and v_2 be the two centers of $K_{1,k}$ constructing B_k. Let $N(v) = \{v_1, v_2\}$. Note that $d^-(v_i) = k$ and $|V \setminus (N(v_i) \cup \{v_1, v_2\})| = k-1$ for each $i \in \{1, 2\}$. If D is an eulerian oriented graph with $V(D) = V(B_k)$, then $\Delta^0(D) = k$. In particular, $d_D^+(v_i) = d_D^-(v_i) = k$ for each $i \in \{1, 2\}$. It forces each edge joining v_1 and a vertex in $N[v_2] \setminus \{v\}$ is oriented from v_1 to $N[v_2] \setminus \{v\}$ and each edge joining v_2 and a vertex in $N[v_1] \setminus \{v\}$ is oriented from v_2 to $N[v_1] \setminus \{v\}$. But, this is not possible for the orientation of v_1v_2. □

Let $\mathcal{F}$ be a digraph class contains exactly trivial orientation of P_6 and all trivial orientation of B_k where $k \geq 2$.

Theorem 3.2. *Let D be any orientation of some forest F. Then D is subeulerian if and only if $D \notin \mathcal{F}$ and $\Delta^0(D) \leq (|V(D)|-1)/2$.*

Proof. By Corollary 1.3 and Lemma 3.1, it remains to prove its sufficiency.

Now assume that D is any orientation of some forest, F, with $\Delta^0(D) \leq (|V(D)|-1)/2$ and such that D is not subeulerian. We will show that $D \in \mathcal{F}$, which will complete the proof. By Theorem 2.2 there must exist a subset $X \subseteq V(D)$ such that $\mathrm{def}(X) > \overline{e_{UG[D]}}(X, \overline{X})$. Without loss of generality assume that $|X| \leq |\overline{X}|$ (otherwise consider $\overline{X}$ instead of X, as $\mathrm{def}(X) = \mathrm{def}(\overline{X})$). Also, without loss of generality assume that $a_D(X, \overline{X}) \geq a_D(\overline{X}, X)$ (otherwise we can reverse all arcs). Now $\mathrm{def}(X) = a_D(X, \overline{X}) - a_D(\overline{X}, X)$ and $\overline{e_{UG[D]}}(X, \overline{X}) = |X|(|V(D)| - |X|) - (a_D(X, \overline{X}) + a_D(\overline{X}, X))$. This implies

$$0 > \overline{e_{UG[D]}}(X, \overline{X}) - \mathrm{def}(X) \geq |X|(|V(D)| - |X|) - 2a_D(X, \overline{X}). \tag{3.1}$$

We now consider the following cases.

Case 1*:* $|X| = 1$. Let $X = \{x\}$. By (3.1) we note that $2a_D(X, \overline{X}) > 1 \cdot (|V(D)| - 1)$. Therefore $d^+_D(x) > (|V(D)| - 1)/2$, a contradiction.

Case 2*:* $|X| = 2$. By (3.1) we note that $2a_D(X, \overline{X}) > 2(|V(D)| - 2)$. So $a_D(X, \overline{X}) = |V(D)| - 1$. That is, all arcs of D go from X to $\overline{X}$ and $d^+(x) = (|V(D)| - 1)/2$ for each $x \in X$, which implies that $D \in \mathcal{F}$. This completes the case when $|X| = 2$.

Case 3*:* $|X| = 3$. By (3.1) we note that $2a_D(X, \overline{X}) > 3(|V(D)| - 3)$. If $a_D(X, \overline{X}) \leq |V(D)| - 2$ or if $|V(D)| \geq 7$ then we obtain a contradiction to the above (as $|V(D)| \geq 6$, as $3 = |X| \leq |\overline{X}|$). So we must have $a_D(X, \overline{X}) = |V(D)| - 1$ and $|V(D)| = 6$, which, as $\Delta^0(D) \leq (|V(D)| - 1)/2$, implies that $D \in \mathcal{F}$. This completes the case when $|X| = 3$.

Case 4*:* $|X| \geq 4$. By (3.1) we note that $2(|V(D)| - 1) \geq 2a_D(X, \overline{X}) > 4(|V(D)| - 4)$, which implies that $14 > 2|V(D)|$. As $4 = |X| \leq |\overline{X}|$, we note that $|V(D)| \geq 8$, a contradiction to $14 > 2|V(D)|$. □

4. Orientation of C_n

For any two vertices $x, y \in V(G)$, $d_G(x, y)$ denote the distance between x and y in G.

Theorem 4.1. *An orientation, D_n, of C_n is subeulerian if and only if one of the following hold.*

(1) *D_n is a directed cycle when $3 \leq n \leq 4$;*

(2) *D_n contains no trivial orientation of P_n and $5 \leq n \leq 6$;*

(3) *$n \geq 7$.*

Proof. Part (1) is immediately seen and part (3) follows from Theorem 5.1. In order to prove part (2) we first consider the case when n = 5. If D_n contains a trivial orientation of P_n, then letting X be the sources of the trivial orientation of P_n we note that $a(X, V(D_5) \setminus X) \geq 4 > 3 = |X| \times |V(D_5) \setminus X|/2$. So D_n is not subeulerian by Lemma 2.1. Now assume that D_n contains no trivial orientation of P_n. There are only three possible orientations of C_5 (see Figure 4.1) that contain no trivial orientation of P_5 and it is not difficult to check that each of these are subdigraphs of the unique 2-regular tournament of order five (which is eulerian).

So now consider $n = 6$. If D_6 contains a trivial orientation of P_6, then letting X be the sources of the trivial orientation of P_n we note that $a(X, V(D_6) \setminus X) \geq 5 > 4.5 = |X| \times |V(D_6) \setminus X|/2$. So D_6 is not subeulerian by Lemma 2.1.

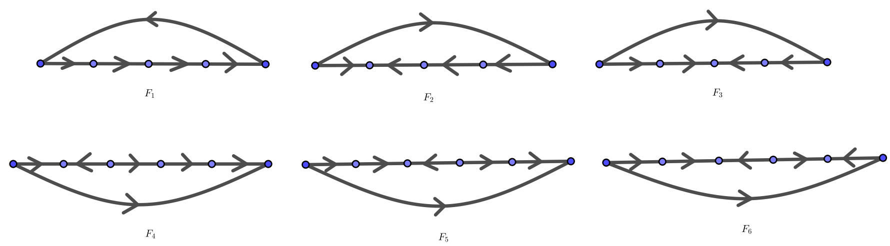

Figure 4.1: $D_5 \cong F_i$ where $i \in \{1, 2, 3\}$ and $D_6 \cong F_i$ where $i \in \{4, 5, 6\}$.

Now assume that D_6 contains no trivial orientation of P_6. Let X contain all sources in D_6. Let G^* denote the complement of the underlying graph of D_6. So G^* is 3-regular and 3-edge-connected. If $|X| = 3$ then D_6 contains a trivial orientation of P_6, a contradiction. If $|X| = 2$ then let $X = \{x_1, x_2\}$ and note that D_6 is unique if $d_{C_6}(x_1, x_2) = 2$ as show in Figure 4.1, F_4 and that there are two possible options for D_6 if $d_{C_6}(x_1, x_2) = 3$ (see Figure 4.1, F_5 and F_6). One can easily check that D_6 is subeulerian in all three cases. If $|X| = 1$ then let $X = \{x\}$ let y be the unique sink in D_n and let P_1 and P_2 be two edge-disjoint (y, x)-paths in G^* (which exist as G^* is 3-edge-connected). Orient P_1 and P_2 from y to x and add these arcs to D_n. This results in a eulerain digraph, showing that D_6 is subeulerian. If $|X| = 0$ then D_6 is an oriented cycle and therefore eulerian (and subeulerian). This completes the proof. □

5. Orientation of graphs with order $n \geq \max\{4\Delta - 1, 3\}$

In this section, we prove that if G is a graph of order n with $n \geq \max\{4\Delta - 1, 3\}$, then every orientation of G is subeulerian. In particular, we show that if G is a graph of odd order n with $\Delta(G) \leq n/4$, then every orientation of G is a spanning subdigraph of a regular tournament.

Theorem 5.1. *If G is a graph of order n with $n \geq \max\{4\Delta(G) - 1, 3\}$, then every orientation of G is subeulerian.*

Proof. Let G be a graph of order n with $n \geq \max\{4\Delta(G) - 1, 3\}$ and (X, Y) be any partition of $V(G)$. Without loss of generality assume that $|Y| \geq |X|$, which implies that $|Y| \geq \lceil n/2 \rceil \geq 2\Delta(G)$. As every vertex in X has at most $\Delta(G)$ neighbours in Y

(and therefore at least $\Delta(G)$ non-neighbours in Y), we note that $e(X,Y) \leq \overline{e}(X,Y)$. By Corollary 2.3 we note that every orientation of G is subeulerian. □

Furthermore, we prove that the order and maximum degree condition in Theorem 5.1 is tight by the following lemma.

Lemma 5.2. *For any positive integer k, a trivial orientation $\overrightarrow{G}$ of a k-regular bipartite graph G of order $n = 4k - 2$ is nonsubeulerian.*

Proof. Let $\overrightarrow{G}$ be a trivial orientation of $G = G[X,Y]$ with $X = \{x_1, \ldots, x_{2k-1}\}$ and $Y = \{y_1, \ldots, y_{2k-1}\}$. Without loss of generality, we may assume that

$$d^+_{\overrightarrow{G}}(x_i) = k \quad \text{for each } i \in \{1, \ldots, 2k-1\}.$$

If D is an eulerian oriented graph spanned by $\overrightarrow{G}$, then for each i, $d^+_D(x_i) = d^-_D(x_i)$. Let $D[X,Y] = D - A(D[X]) - A(D[Y])$. Since G is regular, $|X| = |Y| = n/2 = 2k-1$. Since for each i, $d^+_{D[X,Y]}(x_i) \geq d^+_{\overrightarrow{G}}(x_i) = k$ and $d^+_{D[X,Y]}(x_i) + d^-_{D[X,Y]}(x_i) \leq |Y| = 2k-1$, it follows that $d^+_{D[X,Y]}(x_i) > d^-_{D[X,Y]}(x_i)$. Since $d^+_{D[X,Y]}(x_i) + d^+_{D[X]}(x_i) = d^+_D(x_i) = d^-_D(x_i) = d^-_{D[X,Y]}(x_i) + d^-_{D[X]}(x_i)$, we have $d^+_{D[X]}(x_i) < d^-_{D[X]}(x_i)$ for each i. This is not possible. □

Theorem 5.3. *If G is a graph of odd order n with $\Delta(G) \leq n/4$, then every orientation of G is a spanning subdigraph of a regular tournament.*

Proof. Let G is a graph of odd order n with $\Delta(G) \leq n/4$. As $n \geq 4\Delta(G) \geq 4\Delta(G) - 1$, we note that Theorem 5.1 implies that there exists an eulerian oriented graph D^* such that D is a spanning subdigraph of D^*. Let G^* be the underlying graph of D^* and let $\overline{G^*}$ denote the complement of G^*. As D^* is eulerian (and contains no 2-cycles) we note that all degrees in G^* are even and therefore that all degrees in $\overline{G^*}$ are even (as n is odd). While there exists any edge in $\overline{G^*}$ we can therefore pick a cycle, C, in $\overline{G^*}$ and orient it (in one of the two ways) and add it to D^*. Doing this as long as $\overline{G^*}$ contains edges, will result in the desired tournament D^*. □

Acknowledgments

The authors are very grateful to the editor and reviewer for their useful suggestions and comments, which are very helpful to improving the presentation of this paper.

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