TAIWANESE JOURNAL OF MATHEMATICS
Vol. 27, No. 4, pp. 737–757, August 2023
DOI: 10.11650/tjm/230401

Blow-up Phenomena for a Reaction-diffusion Equation with Nonlocal Gradient Terms

Su-Cheol Yi and Zhong Bo Fang*

Abstract. In this paper, we investigate blow-up phenomena of the solution to a reaction-diffusion equation with nonlocal gradient absorption terms under Robin boundary condition on a bounded star-shaped region. Based on the method of auxiliary function and the technique of modified differential inequality, we establish some conditions on the nonlinearities for which the solution exists globally or blows up at finite time, when the sign of the constant σ is either positive or negative. Moreover, upper and lower bounds for a blow-up time are derived under appropriate measure in higher dimensional spaces.

1. Introduction

Consider the following reaction-diffusion equation with nonlocal gradient terms:

$$u_t = \Delta u + f(u, |\nabla u|), \quad (x,t) \in \Omega \times (0,t^*), \tag{1.1}$$

under the null Robin boundary and initial conditions

$$\frac{\partial u}{\partial \nu} + \sigma u = 0, \quad (x,t) \in \partial\Omega \times (0,t^*), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded star-shaped region with smooth boundary $\partial\Omega$, ν is the unit outward normal vector on $\partial\Omega$, σ is a nonzero constant, and t^* is a possible blow-up time when blow-up occurs, otherwise $t^* = +\infty$. The nonlinearity $f(u, |\nabla u|)$ is assumed to be a continuous function that satisfies appropriate conditions and contains nonlocal gradient terms such as the form $u^m \big(\int_\Omega |\nabla u^{n/2}|^2 \, dx \big)^q$. Moreover, the initial data $u_0(x)$ is assumed to be a positive C^1-function satisfying an appropriate compatibility condition. When σ is a positive constant, it follows from the parabolic maximum principle that the

Received December 4, 2022; Accepted April 6, 2023.
Communicated by François Hamel.
2020 *Mathematics Subject Classification.* 35K20, 35B40, 35B44.
Key words and phrases. reaction-diffusion equation, nonlocal gradient terms, Robin boundary condition, bounds for blow-up time.
*Corresponding author.

solution of problem (1.1)–(1.3) will be nonnegative. However, it cannot be assured that the solution is nonnegative when σ is negative.

The gradient model (1.1) is often referred to as a viscous Hamilton–Jacobi equation and appears in many natural phenomena such as explosion model, compressible reactant gas model, population dynamics theories and some biological species with a human-controlled distribution model, see [1,3,6] and references therein. The formulation (1.1) also describes the evolution of some biological population u on a certain occupied region and whose growth is governed by the law of f, see [7]. Moreover, when the coefficient σ is zero, i.e., the well-known Neumann boundary condition, the distribution of u on the boundary of the region maintains constant through the time. When σ is positive, the population u enters the region with rate σ, whereas the population u gets out of the region with rate σ when σ is negative.

During the past decades, there have been many works to deal with existence and nonexistence of global solutions, blow-up of solutions, blow-up rates, blow-up sets, life span, and asymptotic behavior of the solutions to reaction-diffusion equations and systems, (cf. [9,24]). Among those topics, it is important to investigate whether the solution of the reaction-diffusion equation blows up and when blow-up occurs in the sense of appropriate measure. In particular, Quittner and Souplet [24, Chapters 2, 4 and 5] detailed a series of research progresses on the reaction-diffusion equations with local nonlinear terms $f(u)$ and $f(u, \nabla u)$, and with nonlocal terms under Dirichlet boundary condition. In a sense, the nonlocal models are closer to practical problems than the local models, and many local theories are no longer valid for the nonlocal models. Hence, the nonlocal models are more challenging and difficult to deal with. In this paper, we investigate upper and lower bounds for a blow-up time of the solution to a nonlocal reaction-diffusion model with competition between inner source and absorption terms. As far as we know, a variety of methods have been used to study upper bounds for blow-up times (cf. [10]). However, lower bounds for blow-up times may be harder to be determined. Recently, some researchers such as Payne, Schaefer and Philippin provided pioneering works on determining lower bounds for blow-up times, and there have been many new progresses on that issue of lower bounds for blow-up times of the solutions to the models without gradient term under Robin boundary condition. One can refer to literatures [4, 5, 19, 21] for local models and [14, 26, 27] for nonlocal models. Moreover, about the study of problems with nonlocal boundary flows and time-varying coefficients, we can see [16].

However, there are only few works on lower bounds for blow-up times of the solution to the gradient diffusion model. The salient feature of the gradient model is that boundary or internal gradient blow-up may or may not occur under some conditions (cf. [8, 23, 25]). In particular, Quittner and Souplet [23, 25] studied the reaction-diffusion equation with inner source and gradient absorption terms

$$u_t = \Delta u + \lambda u^p - |\nabla u|^q, \quad (x,t) \in \Omega \times (0,t^*), \tag{1.4}$$

and the reaction-diffusion equation with nonlocal gradient term

$$u_t = \Delta u + u^m \left(\int_\Omega |\nabla u|^2 \, dx \right)^q, \quad (x,t) \in \Omega \times (0,t^*),$$

where $m \geq 1$ and $q > 0$. The authors pointed out that gradient blow-up never occurs, whereas blow-up occurs in L^∞-norm under either Dirichlet or Neumann boundary condition. Payne and Song [22] firstly derived some lower bounds for a blow-up time of the solution to the constant coefficient gradient damping model (1.4) in three-dimensional space, when blow-up occurs. For higher dimensional case ($N \geq 3$), one can refer to [11]. Liu et al. [13] studied lower bounds for the blow-up time of the solution to the reaction-diffusion equation (1.4) with gradient absorption terms on a three-dimensional bounded convex domain under nonlinear boundary flux. Recently, Marras et al. [18] extended the results of [13] to the problem with time-dependent coefficients on a three-dimensional bounded star-shaped region. Meanwhile, for the progress of parabolic equations with local gradient sources and porous medium equations with local gradient terms under different boundary conditions, we can see [12, 15, 17].

To the best of our knowledge, any research on the blow-up phenomena to problem (1.1)–(1.3) with competition between nonlocal gradient damping and local source terms under Robin boundary condition has not been started yet. At a glance, the main difficulties are to deal with the gradient terms and it cannot be assured that the solution is nonnegative, when σ is negative. Motivated by these observations, we will establish some conditions for which the solution of problem (1.1)–(1.3) exists globally or blows up by using the method of auxiliary function and the technique of modified differential inequality, and derive some upper and lower bounds for blow-up time, when the constant σ is either positive or negative.

The remainder of this paper is organized as follows: In Section 2, we investigate the existence of global solution and lower and upper bounds for a blow-up time of the solution to problem (1.1)–(1.3) when σ is positive, and the existence of global solution and lower bound for a blow-up time of the solution to the problem are examined when σ is negative in Section 3.

2. The case that $\sigma > 0$

In this section, we investigate the existence of the global solution and upper and lower bounds for a blow-up time of the solution to problem (1.1)–(1.3) with $\sigma > 0$.

2.1. The global existence

In this subsection, we present some conditions on the nonlinearity f for which a global solution exists. In order to prove our main conclusions, we introduce the following lemmas on a bounded star-shaped region with smooth boundary.

Lemma 2.1. [20] *Suppose that $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded star-shaped region with smooth boundary $\partial\Omega$. Then for any nonnegative C^1-function w and nonnegative constant θ, we have the inequality*

$$\int_{\partial\Omega} w^\theta \, ds \leq \frac{N}{\rho_0} \int_\Omega w^\theta \, dx + \frac{\theta d}{\rho_0} \int_\Omega w^{\theta-1} |\nabla w| \, dx, \tag{2.1}$$

where $\rho_0 = \min_{x\in\partial\Omega}(x\cdot\nu) > 0$ and $d = \max_{x\in\overline{\Omega}} |x|$. Note that if Ω is a bounded star-shaped region centered at origin, then d clearly exists, and that if Ω is a bounded star-shaped region centered at $x_0 \neq 0$, then one can also have the inequality (2.1) by using the technique of translation, in which

$$\rho_0 = \min_{x\in\partial\Omega}((x - x_0) \cdot \nu) \quad \textit{and} \quad d = \max_{x\in\overline{\Omega}} |x - x_0|.$$

Lemma 2.2. [12] *Suppose that $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded star-shaped region with smooth boundary $\partial\Omega$. If $\xi_1(\sigma)$ is the first positive eigenvalue of the Robin boundary problem*

$$\begin{cases} \Delta w + \xi(\sigma) w = 0, & x \in \Omega, \\ \frac{\partial w}{\partial \nu} + \sigma w = 0, & x \in \partial\Omega, \end{cases}$$

and the geometry of Ω is chosen so that

$$\frac{\xi_1(\beta\sigma)}{\beta\sigma} > \frac{N + d}{\rho_0}, \quad \beta \geq 1, \ \sigma > 0, \tag{2.2}$$

then for any nonnegative C^1-function w, we have the inequality

$$\int_\Omega |\nabla w^\beta|^2 \, dx \geq C \int_\Omega w^{2\beta} \, dx,$$

where $C = \eta(\beta\sigma)$ and $\eta(\sigma) = \frac{\rho_0 \xi_1(\sigma) - \sigma(N+d)}{\rho_0 + \sigma d} > 0$.

Remark 2.3. In fact, after the first version of our manuscript was completed, we found that the proof of Lemma 2.2 (the case of $\sigma > 0$) has been given in the latest literature [12]. However, for the sake of the application of Lemma 3.1 (the case of $\sigma < 0$) in Section 3, we present the detailed proof.

Remark 2.4. Similar to the literature [12, page 4], there exists a triple (σ, β, Ω) such that the geometric condition (2.2) in Lemma 2.2 is satisfied. For example, let us fix $\beta \geq 1$, take dimensional $N = 2$ and the 2-dimensional rectangle-like domain $\mathcal{R}^2_{\{L_1,L_2\}}(0)$ with center at the origin and sizes $2L_i$. Then we have

$$\rho_0 = \min_{i=1,2} L_i, \quad d = \sqrt{\sum_{i=1}^{2} L_i^2}.$$

Meanwhile, applying the standard method of separation of variables, we let $w(x) = w(x_1, x_2) = X_1(x_1)X_2(x_2)$, substitute it into the Robin eigenvalue problem in Lemma 2.2 and by a direct calculation, we arrive at

$$-\sum_{i=1}^{2} \frac{X_i''(x_i)}{X_i(x_i)} = \xi_1(\sigma), \quad \pm X_i'(\pm L_i) + \sigma X_i(\pm L_i) = 0, \quad \forall\, i = 1, 2. \tag{2.3}$$

Obviously, this system is composed of 2 independent second-order ordinary differential problems, hence, for $i = 1, 2$, it can be rewritten as

$$-\frac{X_i''(x_i)}{X_i(x_i)} = \Lambda_i, \quad \pm X_i'(\pm L_i) + \sigma X_i(\pm L_i) = 0, \tag{2.4}$$

where $\Lambda_i = \Lambda_i(\sigma) > 0$ is exactly the corresponding eigenvalue ($\Lambda_i \le 0$ is not compatible with $\sigma > 0$ and $X_i \not\equiv 0$). By applying the characteristic solution, we obtain the general solution of (2.3) in the form

$$X_i(x_i) = C_1 \cos\big(\sqrt{\Lambda_i} x_i\big) + C_2 \sin\big(\sqrt{\Lambda_i} x_i\big),$$

where C_1, C_2 are constants. Moreover, at the boundary point $x_i = \pm L_i$, it must satisfy

$$\big(\sigma C_1 \pm \sqrt{\Lambda_i} C_2\big) \cos\big(\sqrt{\Lambda_i} x_i\big) + \big(\sigma C_2 \mp \sqrt{\Lambda_i} C_1\big) \sin\big(\sqrt{\Lambda_i} x_i\big) = 0.$$

In order to ensure that the above system in the unknown (C_1, C_2) admits a nontrivial solution, we set its determinant equal to zero. It yields, for $z_i = L_i\sqrt{\Lambda_i}$,

$$\left(\sigma \cos(z_i) - \frac{z_i}{L_i} \sin(z_i)\right) \left(\frac{z_i}{L_i} \cos(z_i) + \sigma \sin(z_i)\right) = 0,$$

dividing by $\cos(z_i)$, we obtain

$$\tan(z_i) = -\frac{z_i}{L_i\sigma} \quad \text{or} \quad \tan(z_i) = \frac{L_i\sigma}{z_i}. \tag{2.5}$$

Since we are dealing with the smallest positive eigenvalue Λ_i to (2.4), and it is seen that the first positive eigenvalue $\widehat{z}_i^1$ of the first equation in (2.5) satisfies $\widehat{z}_i^1 \in (\pi/2, 3\pi/2)$. Also similar to the second equation in (2.5), $\widehat{z}_i^2 \in (0, \pi/2)$. Finally, the same reasons apply for each independent problem given in (2.4), so that by superposition we obtain the first eigenvalue of (2.3) is

$$\xi_1(\sigma) = \sum_{i=1}^{2} \Lambda_i = \sum_{i=1}^{2} \left(\frac{\widehat{z}_i^1}{L_i}\right)^2 \text{ or } \left(\frac{\widehat{z}_1^1}{L_1}\right)^2 + \left(\frac{\widehat{z}_2^2}{L_2}\right)^2 \text{ or } \left(\frac{\widehat{z}_1^2}{L_1}\right)^2 + \left(\frac{\widehat{z}_2^1}{L_2}\right)^2 \text{ or } \sum_{i=1}^{2} \left(\frac{\widehat{z}_i^2}{L_i}\right)^2.$$

Now, we can present some specific examples.

Case 1: $\xi_1(\sigma) = \sum_{i=1}^{2} \Lambda_i = \sum_{i=1}^{2} \big(\frac{\widehat{z}_i^1}{L_i}\big)^2$. Selecting the triple $\big(2, 3/2, \mathcal{R}^2_{\{1,8/9\}}(0)\big)$, by employing MATLAB, we obtain $\xi_1(\sigma\beta) \cong 13.3644$, and then

$$\frac{\xi_1(\sigma\beta)}{\sigma\beta} - \frac{N+d}{\rho_0} = \frac{\xi_1(\sigma\beta)}{\sigma\beta} - \left(\frac{2}{L_2} + \frac{\sqrt{L_1^2 + L_2^2}}{L_2}\right) \cong 0.6996 > 0;$$

Case 2: $\xi_1(\sigma) = \big(\frac{\widehat{z}_1^1}{L_1}\big)^2 + \big(\frac{\widehat{z}_2^2}{L_2}\big)^2$. Selecting the triple $\big(1/2, 2, \mathcal{R}^2_{\{7/8,8/7\}}(0)\big)$, by employing MATLAB, we obtain $\xi_1(\sigma\beta) \cong 5.7742$, and then

$$\frac{\xi_1(\sigma\beta)}{\sigma\beta} - \frac{N+d}{\rho_0} = \frac{\xi_1(\sigma\beta)}{\sigma\beta} - \left(\frac{2}{L_1} + \frac{\sqrt{L_1^2 + L_2^2}}{L_1}\right) \cong 1.8435 > 0;$$

Case 3: $\xi_1(\sigma) = \big(\frac{\widehat{z}_1^2}{L_1}\big)^2 + \big(\frac{\widehat{z}_2^1}{L_2}\big)^2$. Selecting the triple $\big(1, 2, \mathcal{R}^2_{\{5/6,6/7\}}(0)\big)$, by employing MATLAB, we obtain $\xi_1(\sigma\beta) \cong 8.2511$, and then

$$\frac{\xi_1(\sigma\beta)}{\sigma\beta} - \frac{N+d}{\rho_0} = \frac{\xi_1(\sigma\beta)}{\sigma\beta} - \left(\frac{2}{L_1} + \frac{\sqrt{L_1^2 + L_2^2}}{L_1}\right) \cong 0.2910 > 0;$$

Case 4: $\xi_1(\sigma\beta) = \sum_{i=1}^{2} \big(\frac{\widehat{z}_i^2}{L_i}\big)^2$. Unfortunately, we can not find a suitable example, mainly because $\widehat{z}_i^2 \in (0, \pi/2)$ is too small.

Proof. For the variational definition of ξ_1, one can have the inequality

$$\xi_1(\sigma) \int_\Omega w^2\,dx \le \int_\Omega |\nabla w|^2\,dx + \sigma \int_{\partial\Omega} w^2\,ds. \tag{2.6}$$

It then follows from (2.6) and Young's inequality with exponent 1/2 that

$$\xi_1(\sigma) \int_\Omega w^2\,dx \le \sigma \frac{N+d}{\rho_0} \int_\Omega w^2\,dx + \left(1 + \frac{\sigma d}{\rho_0}\right) \int_\Omega |\nabla w|^2\,dx,$$

and hence we have the inequality

$$\int_\Omega |\nabla w|^2\,dx \ge \eta(\sigma) \int_\Omega w^2\,dx \tag{2.7}$$

since $\sigma > 0$, where $\eta(\sigma) = \frac{\rho_0 \xi_1(\sigma) - \sigma(N+d)}{\rho_0 + \sigma d} > 0$. Now, the function $\varphi := w^\beta$ satisfies the equation $\frac{\partial \varphi}{\partial \nu} + \beta\sigma\varphi = 0$, and hence one can have

$$\int_\Omega |\nabla w^\beta|^2\,dx = \int_\Omega |\nabla \varphi|^2\,dx \ge \eta(\beta\sigma) \int_\Omega \varphi^2\,dx = C \int_\Omega w^{2\beta}\,dx$$

from (2.7), where $C = \eta(\beta\sigma) = \frac{\rho_0 \xi_1(\beta\sigma) - \beta\sigma(N+d)}{\rho_0 + \beta\sigma d} > 0$. □

A result on the existence of the global solution of problem (1.1)–(1.3) is stated below.

Theorem 2.5. *Suppose that $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded star-shaped region with smooth boundary $\partial\Omega$ and that the nonlinearity f is such that*

$$f(s, |\nabla s|) \leq a_1 s^p - a_2 s^m \left(\int_\Omega |\nabla s^{n/2}|^2\, dx\right)^q, \quad s \geq 0, \tag{2.8}$$

where $a_1 \geq 0$, $a_2 > 0$, $q > 0$, $n \geq 2$ and $m > p > 1$. Then the nonnegative classical solution $u(x,t)$ of problem (1.1)–(1.3) does not blow up; that is, $u(x,t)$ exists for all $t > 0$.

Proof. Define an auxiliary function as

$$\Phi(t) = \int_\Omega u^n\, dx.$$

With (1.1), (1.2), (2.8) and Green's formula, it can be seen that

$$\begin{aligned} \Phi'(t) &= n \int_\Omega u^{n-1} \big[\Delta u + f(u, |\nabla u|)\big]\, dx \\ &\leq n \int_\Omega u^{n-1} \left[\Delta u + a_1 u^p - a_2 u^m \left(\int_\Omega |\nabla u^{n/2}|^2\, dx\right)^q\right] dx \\ &= -n\sigma \int_{\partial\Omega} u^n\, ds - n(n-1) \int_\Omega u^{n-2} |\nabla u|^2\, dx + n a_1 \int_\Omega u^{n+p-1}\, dx \\ &\quad - n a_2 \int_\Omega u^{n+m-1}\, dx \left(\int_\Omega |\nabla u^{n/2}|^2\, dx\right)^q. \end{aligned} \tag{2.9}$$

Now, taking $\omega = u$ and $\beta = n/2$ in Lemma 2.2, we obtain

$$\int_\Omega |\nabla u^{n/2}|^2\, dx \geq C \int_\Omega u^n\, dx. \tag{2.10}$$

From (2.9) and (2.10), we obtain the inequality

$$\Phi'(t) \leq n a_1 \int_\Omega u^{n+p-1}\, dx - n a_2 C^q \Phi^q \int_\Omega u^{n+m-1}\, dx. \tag{2.11}$$

Applying Hölder's inequality we obtain

$$\int_\Omega u^{n+p-1}\, dx \leq \left(\int_\Omega u^{n+m-1}\, dx\right)^{\frac{n+p-1}{n+m-1}} |\Omega|^{\frac{m-p}{n+m-1}}, \tag{2.12}$$

$$\int_\Omega u^n\, dx \leq \left(\int_\Omega u^{n+m-1}\, dx\right)^{\frac{n}{n+m-1}} |\Omega|^{\frac{m-1}{n+m-1}}. \tag{2.13}$$

Substituting (2.12) and (2.13) into (2.11), one can have that

$$\begin{aligned} \Phi'(t) &\leq n a_1 |\Omega|^{\frac{m-p}{n+m-1}} \left(\int_\Omega u^{n+m-1}\, dx\right)^{\frac{n+p-1}{n+m-1}} - n a_2 C^q \Phi^q \int_\Omega u^{n+m-1}\, dx \\ &= \int_\Omega u^{n+m-1}\, dx \left[n a_1 |\Omega|^{\frac{m-p}{n+m-1}} \left(\int_\Omega u^{n+m-1}\, dx\right)^{\frac{p-m}{n+m-1}} - n a_2 C^q \Phi^q\right] \\ &\leq \int_\Omega u^{n+m-1}\, dx \big[P_1 \Phi^{\frac{p-m}{n}} - P_2 \Phi^q\big], \end{aligned} \tag{2.14}$$

where $P_1 = na_1|\Omega|^{\frac{(1-m-n)(-m+p)}{n(n+m-1)}} > 0$, $P_2 = na_2C^q > 0$, $\frac{p-m}{n} < 0$ and $q > 0$.

From (2.14), it can be seen that $\Phi(t)$ remains bounded for all time under the conditions stated in Theorem 2.5. In fact, if $u(x,t)$ blows up at finite time t^*, then $\Phi(t)$ is unbounded near t^*. In view of (2.14), $p < m$ and $q > 0$, which implies $\Phi(t)$ is decreasing in some interval $[t_0, t^*)$. Hence, we have $\Phi(t) \le \Phi(t_0)$ in $[t_0, t^*)$, which means that $\Phi(t)$ is bounded in $[t_0, t^*)$. This leads to a contradiction. Therefore, $u(x,t)$ exists for all $t > 0$, which completes the proof. □

2.2. Blow-up and upper bound of t^*

In this subsection, the domain Ω only needs to be a bounded region with smooth boundary instead of star-shaped one. We establish a sufficient condition for which the solution of problem (1.1)–(1.3) blows up at finite time t^* and derive an upper bound for t^*. Our result can be summarized as follows:

Theorem 2.6. *Suppose that $u(x,t)$ is a nonnegative classical solution of problem* (1.1)–(1.3) *and that the integrable function f is such that*

$$\xi f(\xi, |\nabla \xi|) \ge 2(1+\alpha)F(\xi), \quad \xi \ge 0, \tag{2.15}$$

where $F(\xi) = \int_0^\xi f(s, |\nabla s|)\, ds$ and $\alpha \ge 0$. Define a function $\Theta(t)$ as

$$\Theta(t) = -\sigma \int_{\partial\Omega} u^2\, ds - \int_\Omega |\nabla u|^2\, dx + 2\int_\Omega F(u)\, dx,$$

and assume that $\Theta(0) > 0$. Then the solution $u(x,t)$ of problem (1.1)–(1.3) *blows up in L^2-norm at some finite time $t^* \le T$, where*

$$T = \frac{\Psi(0)}{2\alpha(1+\alpha)\Theta(0)} \ \text{ for } \alpha > 0 \quad \text{and} \quad \Psi(t) = \int_\Omega u^2\, dx, \ \ \Psi(0) > 0,$$

and if $\alpha = 0$ then $T = \infty$.

Proof. With (1.1), (1.2), (2.15) and Green's formula, one can see that

$$\begin{aligned} \Psi'(t) &= 2\int_\Omega u u_t\, dx \\ &= -2\sigma \int_{\partial\Omega} u^2\, ds - 2\int_\Omega |\nabla u|^2\, dx + 2\int_\Omega u f(u, |\nabla u|)\, dx \\ &\ge 2(1+\alpha)\left[-\sigma \int_{\partial\Omega} u^2\, ds - \int_\Omega |\nabla u|^2\, dx + 2\int_\Omega F(u)\, dx\right] \\ &\ge 2(1+\alpha)\Theta(t). \end{aligned} \tag{2.16}$$

By using Green's formula, it can be seen that

$$\begin{aligned}\Theta'(t) &= -2\sigma \int_{\partial\Omega} u u_t\, ds - 2\int_\Omega \nabla u \cdot \nabla u_t\, dx + 2\int_\Omega f(u, |\nabla u|) u_t\, dx \\ &= -2\sigma \int_{\partial\Omega} u u_t\, ds + 2\int_\Omega u_t \Delta u\, dx + 2\sigma \int_{\partial\Omega} u u_t\, ds + 2\int_\Omega f(u, |\nabla u|) u_t\, dx \\ &= 2\int_\Omega u_t \big[\Delta u + f(u, |\nabla u|)\big]\, dx = 2\int_\Omega u_t^2\, dx \geq 0,\end{aligned}$$

which implies $\Theta(t) > 0$ for all $t \in (0, t^*)$ since $\Theta(0) > 0$. Moreover, it follows from Schwarz's inequality that

$$\begin{aligned}2(1+\alpha)\Psi'(t)\Theta(t) \leq (\Psi'(t))^2 &= 4\left(\int_\Omega u u_t\, dx\right)^2 \leq 4\int_\Omega u^2\, dx \int_\Omega u_t^2\, dx \\ &= 2\Psi(t)\Theta'(t).\end{aligned}$$

From the inequality above, one can have the inequality

$$\big(\Theta\Psi^{-(1+\alpha)}\big)' \geq 0. \tag{2.17}$$

Integrating (2.17) from 0 to t and noticing that $\Psi(0) > 0$, one can see that

$$\Theta(t)(\Psi(t))^{-(1+\alpha)} \geq \Theta(0)(\Psi(0))^{-(1+\alpha)} := M > 0;$$

that is,

$$\Theta(t) \geq M(\Psi(t))^{1+\alpha}. \tag{2.18}$$

Substituting (2.18) into (2.16), one can have the differential inequality

$$\Psi'(t) \geq 2M(1+\alpha)(\Psi(t))^{1+\alpha}. \tag{2.19}$$

If $\alpha > 0$, (2.19) leads to

$$\big[(\Psi(t))^{-\alpha}\big]' = -\alpha(\Psi(t))^{-(1+\alpha)}\Psi'(t) \leq -2M\alpha(1+\alpha). \tag{2.20}$$

By (2.16), $\Theta(t) > 0$ and $\Psi(0) > 0$, one can easily see that $\Psi(t) > 0$ for all $t \geq 0$. Integrating (2.20) from 0 to t again, we obtain the inequalities

$$0 < (\Psi(t))^{-\alpha} \leq (\Psi(0))^{-\alpha} - 2M\alpha(1+\alpha)t. \tag{2.21}$$

Obviously, (2.21) cannot be hold for all time. Therefore, (2.21) leads to

$$t^* \leq T = \frac{\Psi(0)}{2\alpha(1+\alpha)\Theta(0)}.$$

If $\alpha = 0$, by (2.19), one can see that

$$\Psi(t) \geq \Psi(0)e^{2\Theta(0)(\Psi(0))^{-1}t}$$

is valid for all $t > 0$, which implies $t^* = \infty$; that is, blow-up occurs in infinite time. This completes the proof. □

Remark 2.7. We observe that the function $f(\xi, |\nabla\xi|) = a_1\xi^p - a_2\xi^m\big(\int_\Omega |\nabla\xi^{n/2}|^2\,dx\big)^q$ with $n \geq 2$, $q > 0$, $p \geq 2\alpha + 1 \geq m > 1$, satisfies (2.15).

Remark 2.8. By Theorem 2.6 and $L^n(\Omega) \subset L^2(\Omega)$, one can see that the solution $u(x,t)$ of problem (1.1)–(1.3) blows up in L^n-norm at some finite time t^* for $n \geq 2$.

2.3. Lower bounds for t^*

In this subsection, we make some appropriate assumptions on the nonlinearity f to seek lower bounds for the blow-up time t^* in high dimensional spaces ($N \geq 3$).

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded star-shaped region with smooth boundary $\partial\Omega$. Suppose the function f satisfies* (2.8) *with $a_1, a_2, q > 0$, $p, m > 1$ and $p \geq m + nq$. Meanwhile, we give the same auxiliary function*

$$\Phi(t) := \int_\Omega u^n\,dx, \quad n > \max\{2(N-2)(p-1), 2\}$$

in Theorem 2.5. *If the nonnegative classical solution $u(x,t)$ of the problem* (1.1)–(1.3) *blows up in the measure of $\Phi(t)$ at finite time t^*, then the blow-up time t^* is bounded below, i.e.,*

$$\int_{\Phi(0)}^{+\infty} \frac{d\xi}{Q_1\xi^{\frac{3(N-2)}{3N-8}} + Q_4} \leq t^*$$

where $\Phi(0) = \int_\Omega u_0^n\,dx$ and Q_1, Q_4 are some positive constants given in the proof.

Proof. By using a similar argument as in Theorem 2.5, one can see that

$$\begin{aligned}
\Phi'(t) &= n\int_\Omega u^{n-1}\big[\Delta u + f(u, |\nabla u|)\big]\,dx \\
&\leq -n\sigma\int_{\partial\Omega} u^n\,ds - n(n-1)\int_\Omega u^{n-2}|\nabla u|^2\,dx + na_1\int_\Omega u^{n+p-1}\,dx \\
&\quad - na_2\int_\Omega u^{n+m-1}\,dx\left(\int_\Omega |\nabla u^{n/2}|^2\,dx\right)^q \\
&\leq -\frac{4(n-1)}{n}\int_\Omega |\nabla u^{n/2}|^2\,dx + na_1\int_\Omega u^{n+p-1}\,dx \\
&\quad - na_2\int_\Omega u^{n+m-1}\,dx\left(\int_\Omega |\nabla u^{n/2}|^2\,dx\right)^q.
\end{aligned} \tag{2.22}$$

By Lemma 2.2 and applying Hölder's inequality to the last term on the right-hand side of (2.22), we obtain the inequalities

$$\int_\Omega |\nabla u^{n/2}|^2\,dx \geq C\int_\Omega u^n\,dx, \tag{2.23}$$

$$\int_\Omega u^{n+m-1}\,dx \geq \left(\int_\Omega u^n\,dx\right)^{1+\frac{m-1}{n}} |\Omega|^{\frac{-m+1}{n}}. \tag{2.24}$$

Combining (2.23) and (2.24) with (2.22), one can have the inequality

$$\Phi'(t) \le -\frac{4(n-1)}{n}\int_\Omega |\nabla u^{n/2}|^2\,dx + na_1\int_\Omega u^{n+p-1}\,dx - na_2C^q|\Omega|^{-\frac{m-1}{n}}\Phi^{1+q+\frac{m-1}{n}}. \tag{2.25}$$

We now consider the integral in the second term on the right-hand side of (2.25) and it can be shown that

$$\begin{aligned}\int_\Omega u^{n+p-1}\,dx &= \int_\Omega u^{\frac{n(2N-3)}{2(N-2)}\cdot\frac{2(N-2)(n+p-1)}{n(2N-3)}}\,dx\\ &\le |\Omega|^{1-m_1}\left(\int_\Omega u^{\frac{n(2N-3)}{2(N-2)}}\,dx\right)^{m_1}\\ &\le (1-m_1)|\Omega| + m_1\int_\Omega u^{\frac{n(2N-3)}{2(N-2)}}\,dx,\end{aligned} \tag{2.26}$$

by using Hölder's and Young's inequalities, where

$$m_1 := \frac{2(N-2)(n+p-1)}{n(2N-3)} \in (0,1).$$

By applying Schwarz's inequality to the integral in (2.26), we have

$$\begin{aligned}\int_\Omega u^{\frac{n(2N-3)}{2(N-2)}}\,dx &\le \left(\int_\Omega u^n\,dx\right)^{1/2}\left(\int_\Omega u^{\frac{n(N-1)}{N-2}}\,dx\right)^{1/2}\\ &= \left(\int_\Omega u^n\,dx\right)^{3/4}\left(\int_\Omega (u^{n/2})^{\frac{2N}{N-2}}\,dx\right)^{1/4}.\end{aligned} \tag{2.27}$$

To bound $\int_\Omega (u^{n/2})^{\frac{2N}{N-2}}\,dx$, we use the Sobolev inequality ($N \ge 3$) given in [2] and, with the inequality, one can obtain the inequalities

$$\begin{aligned}\|u^{n/2}\|_{L^{\frac{2N}{N-2}}(\Omega)}^{\frac{N}{2(N-2)}} &\le (c_s)^{\frac{N}{2(N-2)}}\|u^{n/2}\|_{W^{1,2}(\Omega)}^{\frac{N}{2(N-2)}}\\ &\le c_b\left(\|\nabla u^{n/2}\|_{L^2(\Omega)}^{\frac{N}{2(N-2)}} + \|u^{n/2}\|_{L^2(\Omega)}^{\frac{N}{2(N-2)}}\right),\end{aligned} \tag{2.28}$$

where c_s is the Sobolev constant depending on Ω and N, and

$$c_b := \begin{cases} 2^{1/2}(c_s)^{3/2}, & N = 3,\\ (c_s)^{\frac{N}{2(N-2)}}, & N > 3.\end{cases} \tag{2.29}$$

Substituting (2.28) into (2.27) and using Young's inequality, one can see that

$$\begin{aligned}\int_\Omega u^{\frac{n(2N-3)}{2(N-2)}}\,dx &\le c_b\left(\int_\Omega u^n\,dx\right)^{3/4}\left(\int_\Omega |\nabla u^{n/2}|^2\,dx\right)^{\frac{N}{4(N-2)}} + c_b\left(\int_\Omega u^n\,dx\right)^{\frac{2N-3}{2(N-2)}}\\ &\le \frac{c_b^{\frac{4(N-2)}{3N-8}}(3N-8)}{4(N-2)}\varepsilon_1^{-\frac{N}{3N-8}}\Phi^{\frac{3(N-2)}{3N-8}}(t) + \frac{N\varepsilon_1}{4(N-2)}\int_\Omega |\nabla u^{n/2}|^2\,dx\\ &\quad + c_b\left(\int_\Omega u^n\,dx\right)^{\frac{2N-3}{2(N-2)}},\end{aligned} \tag{2.30}$$

where ε_1 is a positive constant to be determined later. It can be seen that

$$\begin{aligned}\Phi'(t) \leq\ & \frac{na_1m_1c_b^{\frac{4(N-2)}{3N-8}}(3N-8)}{4(N-2)}\varepsilon_1^{-\frac{N}{3N-8}}\Phi^{\frac{3(N-2)}{3N-8}}(t) + na_1m_1c_b\left(\int_\Omega u^n\,dx\right)^{\frac{2N-3}{2(N-2)}} \\ & + \left(\frac{na_1m_1N\varepsilon_1}{4(N-2)} - \frac{4(n-1)}{n}\right)\int_\Omega |\nabla u^{n/2}|^2\,dx - na_2C^q|\Omega|^{-\frac{m-1}{n}}\Phi^{1+q+\frac{m-1}{n}} \\ & + na_1(1-m_1)|\Omega|, \end{aligned} \tag{2.31}$$

by substituting (2.26) and (2.30) into (2.25), and it follows from Young's inequality that

$$\left(\int_\Omega u^n\,dx\right)^{\frac{2N-3}{2(N-2)}} \leq m_2\varepsilon_2^{-\frac{m_3}{m_2}}\left(\int_\Omega u^n\,dx\right)^{\frac{3(N-2)}{3N-8}} + m_3\varepsilon_2\left(\int_\Omega u^n\,dx\right)^{\frac{m+n-1}{n}+q}, \tag{2.32}$$

where

$$\begin{aligned} m_2 &:= \frac{(3N-8)\big[(2N-3)n - 2q(N-2)n - 2(N-2)(m+n-1)\big]}{2(N-2)\big[3(N-2)n - q(3N-8)n - (3N-8)(m+n-1)\big]}, \\ m_3 &:= \frac{n\big[6(N-2)^2 - (3N-8)(2N-3)\big]}{2(N-2)\big[3(N-2)n - q(3N-8)n - (3N-8)(m+n-1)\big]}, \end{aligned}$$

and ε_2 is a positive constant to be determined later. Since $p \geq m+nq$ and $n > \max\{2(p-1)(N-2), 2\}$, the constants m_2 and m_3 are in $(0,1)$.

Combining (2.31) and (2.32) yields the inequality

$$\Phi'(t) \leq Q_1\Phi^{\frac{3(N-2)}{3N-8}} + Q_2\Phi^{1+q+\frac{m-1}{n}} + Q_3\int_\Omega |\nabla u^{n/2}|^2\,dx + Q_4, \tag{2.33}$$

where

$$\begin{aligned} Q_1 &= \frac{na_1m_1c_b^{\frac{4(N-2)}{3N-8}}(3N-8)}{4(N-2)\varepsilon_1^{\frac{N}{3N-8}}} + na_1m_1c_bm_2\varepsilon_2^{-\frac{m_3}{m_2}} > 0, \\ Q_2 &= na_1m_1c_bm_3\varepsilon_2 - na_2|\Omega|^{-\frac{m-1}{n}}C^q, \\ Q_3 &= \frac{na_1m_1N\varepsilon_1}{4(N-2)} - \frac{4(n-1)}{n}, \\ Q_4 &= na_1(1-m_1)|\Omega| > 0. \end{aligned}$$

By choosing appropriate positive constants ε_1 and ε_2 so that Q_2 and Q_3 are zero, inequality (2.33) can be reduced to

$$\Phi'(t) \leq Q_1\Phi^{\frac{3(N-2)}{3N-8}} + Q_4. \tag{2.34}$$

Integrating (2.34) from 0 to t, we have the inequality

$$\int_{\Phi(0)}^{\Phi(t)} \frac{d\xi}{Q_1\xi^{\frac{3(N-2)}{3N-8}} + Q_4} \leq t,$$

and letting $t \to t^{*-}$, we obtain the inequality

$$\int_{\Phi(0)}^{+\infty} \frac{d\xi}{Q_1 \xi^{\frac{3(N-2)}{3N-8}} + Q_4} \le t^*. \qquad \square$$

Remark 2.10. If $u(x,t)$ is a nonnegative solution of problem (1.1)–(1.3) with $\sigma > 0$, and $u_N(x,t)$ and $u_D(x,t)$ are nonnegative solutions of equation (1.1) satisfying the boundary conditions

$$\frac{\partial u_N}{\partial \nu} = 0, \qquad x \in \partial\Omega, \ \ t \in (0, t_N^*),$$
$$u_D = 0, \qquad x \in \partial\Omega, \ \ t \in (0, t_D^*),$$

respectively, and the initial condition (1.3), then one can have the inequalities

$$u_N(x,t) \ge u(x,t) \ge u_D(x,t),$$

on their common existence interval and

$$t_N^* \le t^* \le t_D^*,$$

by the comparison principle, when blow-up occurs, where t_N^* and t_D^* are the corresponding possible blow-up times. This shows that the maximal existence times of the solutions to equation (1.1) satisfying the three different boundary conditions can be ordered.

3. The case that $\sigma < 0$

In this section, we investigate the existence of the global solution and a lower bound for the blow-up time of the solution to problem (1.1)–(1.3) with $\sigma < 0$.

3.1. The global existence

In this subsection, we present some conditions on the nonlinearity f for which a global solution exists. In order to prove our main conclusion, we introduce a lemma on a bounded star-shaped region with smooth boundary.

Lemma 3.1. *Suppose that $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a bounded star-shaped region with smooth boundary $\partial\Omega$. If $\zeta_1(\sigma)$ is the first positive eigenvalue of the Robin boundary problem*

$$\begin{cases} \Delta w + \zeta(\sigma) w = 0, & x \in \Omega, \\ \frac{\partial w}{\partial \nu} + \sigma w = 0, & x \in \partial\Omega, \end{cases}$$

where $\sigma < 0$ and $\beta \ge 1$, then for any nonnegative C^1-function w, we have

$$\int_\Omega |\nabla w^\beta|^2 \, dx \ge H \int_\Omega w^{2\beta} \, dx,$$

where $H = \zeta_1(\beta\sigma) > 0$.

Proof. For the variational definition of ζ_1, one can see that

$$\zeta_1(\sigma)\int_\Omega w^2\,dx \le \int_\Omega |\nabla w|^2\,dx + \sigma\int_{\partial\Omega} w^2\,ds \le \int_\Omega |\nabla w|^2\,dx,$$

and since the function $\chi = w^\beta$ satisfies the equation $\frac{\partial\chi}{\partial\nu} + \beta\sigma\chi = 0$, it can be easily seen that

$$\int_\Omega |\nabla w^\beta|^2\,dx = \int_\Omega |\nabla\chi|^2\,dx \ge \zeta_1(\beta\sigma)\int_\Omega \chi^2\,dx = H\int_\Omega w^{2\beta}\,dx,$$

where $H = \zeta_1(\beta\sigma) > 0$. □

When σ is negative, the solution of problem (1.1)–(1.3) is not negative, and so we assume that the nonlinearity f is such that

$$sf(s,|\nabla s|) \le a_3|s|^{p+1} - a_4|s|^{m+1}\left(\int_\Omega |\nabla s^{n/2}|^2\,dx\right)^q, \tag{3.1}$$

and introduce an auxiliary function $\phi(t) := \int_\Omega |u|^n\,dx$ where $n \ge 2$.

For convenience, we set $\phi(t) = \phi_+(t) + \phi_-(t)$, where

$$\begin{aligned}
\phi_+(t) &:= \int_{\Omega_+} u^n\,dx, \qquad \Omega_+ = \{x\in\Omega \mid u(x,t) > 0\}, \quad t\in(0,t^*),\\
\phi_-(t) &:= \int_{\Omega_-} |u|^n\,dx, \qquad \Omega_- = \{x\in\Omega \mid u(x,t) < 0\}, \quad t\in(0,t^*).
\end{aligned}$$

Theorem 3.2. *Let $\Omega\subset\mathbb{R}^N$ $(N\ge 2)$ be a bounded star-shaped region with smooth boundary $\partial\Omega$. If the nonlinearity f satisfies* (3.1)*, where $a_3 \ge 0$, $a_4, q > 0$ and $m > p > 1$, then the classical solution $u(x,t)$ of problem* (1.1)–(1.3) *does not blow up in the measure $\phi(t)$, that is, $u(x,t)$ exists for all $t > 0$.*

Proof. It follows from (1.1), (1.2), (3.1) and Green's formula that

$$\begin{aligned}
\phi_-'(t) &= -n\int_{\Omega_-} |u|^{n-1}u_t\,dx = -n\int_{\Omega_-} |u|^{n-1}\big[\Delta u + f(u,|\nabla u|)\big]\,dx\\
&= -n\sigma\int_{\partial\Omega_-} |u|^n\,ds - n(n-1)\int_{\Omega_-} |u|^{n-2}|\nabla u|^2\,dx\\
&\quad + n\int_{\Omega_-} |u|^{n-2}uf(u,|\nabla u|)\,dx\\
&\le -n\sigma\int_{\partial\Omega_-} |u|^n\,ds - n(n-1)\int_{\Omega_-} |u|^{n-2}|\nabla u|^2\,dx + na_3\int_{\Omega_-} |u|^{n+p-1}\,dx\\
&\quad - na_4\int_{\Omega_-} |u|^{n+m-1}\,dx\left(\int_{\Omega_-} |\nabla u^{n/2}|^2\,dx\right)^q,
\end{aligned} \tag{3.2}$$

and similarly it can be shown that

$$\begin{aligned}\phi_+'(t) &= n\int_{\Omega_+}|u|^{n-1}u_t\,dx = n\int_{\Omega_+}|u|^{n-1}\big[\Delta u + f(u,|\nabla u|)\big]\,dx\\ &\le -n\sigma\int_{\partial\Omega_+}|u|^n\,ds - n(n-1)\int_{\Omega_+}|u|^{n-2}|\nabla u|^2\,dx + na_3\int_{\Omega_+}|u|^{n+p-1}\,dx\\ &\quad - na_4\int_{\Omega_+}|u|^{n+m-1}\,dx\left(\int_{\Omega_+}|\nabla u^{n/2}|^2\,dx\right)^q.\end{aligned}\tag{3.3}$$

Adding (3.2) and (3.3), we obtain the inequality

$$\begin{aligned}\phi'(t) \le& -n\sigma\int_{\partial\Omega}|u|^n\,ds - n(n-1)\int_{\Omega}|u|^{n-2}|\nabla u|^2\,dx + na_3\int_{\Omega}|u|^{n+p-1}\,dx\\ &- na_4\int_{\Omega}|u|^{n+m-1}\,dx\left(\int_{\Omega}|\nabla u^{n/2}|^2\,dx\right)^q.\end{aligned}\tag{3.4}$$

From Lemma 2.1, one can have the inequality

$$\int_{\partial\Omega}|u|^n\,ds \le \frac{N}{\rho_0}\int_{\Omega}|u|^n\,dx + \frac{nd}{\rho_0}\int_{\Omega}|u|^{n-1}|\nabla u|\,dx,\tag{3.5}$$

and, by applying Schwarz's and Young's inequalities, we can obtain the inequalities

$$\begin{aligned}\int_{\Omega}|u|^{n-1}|\nabla u|\,dx &\le \left(\int_{\Omega}|u|^{n-2}|\nabla u|^2\,dx\right)^{1/2}\left(\int_{\Omega}|u|^n\,dx\right)^{1/2}\\ &\le \frac{\mu}{2}\int_{\Omega}|u|^{n-2}|\nabla u|^2\,dx + \frac{1}{2\mu}\int_{\Omega}|u|^n\,dx,\end{aligned}\tag{3.6}$$

where μ is a positive constant to be determined later, and it follows from Lemma 3.1 that

$$\int_{\Omega}|\nabla u^{n/2}|^2\,dx \ge H\int_{\Omega}|u|^n\,dx.\tag{3.7}$$

From (3.4)–(3.7), one can obtain the inequality

$$\begin{aligned}\phi'(t) \le& \left(\frac{-\sigma nN}{\rho_0} - \frac{\sigma n^2 d}{2\rho_0\mu}\right)\int_{\Omega}|u|^n\,dx + na_3\int_{\Omega}|u|^{n+p-1}\,dx\\ &- na_4H^q\phi^q\int_{\Omega}|u|^{n+m-1}\,dx + \left(\frac{-2\sigma d\mu}{\rho_0} - \frac{4(n-1)}{n}\right)\int_{\Omega}|\nabla u^{n/2}|^2\,dx.\end{aligned}\tag{3.8}$$

Choosing $\mu = -\frac{2(n-1)\rho_0}{\sigma dn}$ so that $\frac{-2\sigma d\mu}{\rho_0} - \frac{4(n-1)}{n} = 0$, inequality (3.8) can be reduced to

$$\begin{aligned}\phi'(t) \le& \left(\frac{-\sigma nN}{\rho_0} - \frac{\sigma n^2 d}{2\rho_0\mu}\right)\int_{\Omega}|u|^n\,dx + na_3\int_{\Omega}|u|^{n+p-1}\,dx\\ &- na_4H^q\phi^q\int_{\Omega}|u|^{n+m-1}\,dx.\end{aligned}\tag{3.9}$$

With Hölder's inequality, we obtain the inequalities

$$\int_\Omega |u|^{n+p-1}\,dx \le \left(\int_\Omega |u|^{n+m-1}\,dx\right)^{\frac{n+p-1}{n+m-1}} |\Omega|^{\frac{m-p}{n+m-1}}, \tag{3.10}$$

$$\int_\Omega |u|^{n}\,dx \le \left(\int_\Omega |u|^{n+m-1}\,dx\right)^{\frac{n}{n+m-1}} |\Omega|^{\frac{m-1}{n+m-1}}. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), one can see that

$$\begin{aligned}\phi'(t) &\le \int_\Omega |u|^{n+m-1}\,dx\Bigg[\left(\frac{-\sigma nN}{\rho_0}-\frac{\sigma n^2 d}{2\rho_0\mu}\right)|\Omega|^{\frac{m-1}{n+m-1}}\left(\int_\Omega |u|^{n+m-1}\,dx\right)^{\frac{-m+1}{n+m-1}}\\ &\qquad + na_3|\Omega|^{\frac{m-p}{n+m-1}}\left(\int_\Omega |u|^{n+m-1}\,dx\right)^{\frac{p-m}{n+m-1}} - na_4H^q\phi^q\Bigg]\\ &\le \int_\Omega |u|^{n+m-1}\,dx\big[J_1\phi^{\frac{-m+1}{n}} + J_2\phi^{\frac{-m+p}{n}} - J_3\phi^q\big],\end{aligned} \tag{3.12}$$

where

$$J_1 = \left(\frac{-\sigma nN}{\rho_0}-\frac{\sigma n^2 d}{2\rho_0\mu}\right)|\Omega|^{\frac{(1-m)(1-m-n)}{n(n+m-1)}} > 0,$$
$$J_2 = na_3|\Omega|^{\frac{(p-m)(1-m-n)}{n(n+m-1)}} > 0, \quad J_3 = na_4H^q > 0,$$

and $\frac{1-m}{n} < 0$, $\frac{p-m}{n} < 0$ and $q > 0$.

We conclude from (3.12) that $\phi(t)$ remains bounded for all time under the conditions stated in Theorem 3.2. In fact, if $u(x,t)$ blows up at finite time t^*, then $\phi(t)$ is unbounded near t^*. In view of (3.12), $m > 1$, $p < m$ and $q > 0$, which implies $\phi(t)$ is decreasing in some interval $[t_0, t^*)$, and hence we have $\phi(t) \le \phi(t_0)$ in $[t_0, t^*)$, which means that $\phi(t)$ is bounded in $[t_0, t^*)$. This leads to a contradiction. Therefore, $u(x,t)$ exists for all $t > 0$, which completes the proof. □

3.2. Lower bounds for t^*

When σ is negative, we can derive a blow-up result for the nonnegative solution of problem (1.1)–(1.3) by a similar argument as in Theorem 2.6 under the same conditions. The difference is that the blow-up time t^* will decrease, which means that the blow-up time for the solution of problem (1.1)–(1.3) with $\sigma < 0$ is ahead of that for the solution to the problem with $\sigma > 0$. Therefore, one can assume that the solution of problem (1.1)–(1.3) blows up at a finite time. However, the main difficulty is that the nonpositivity of the solution cannot be assured, when σ is negative. In this subsection, we obtain a lower bound for the blow-up time t^* in high dimensional spaces ($N \ge 3$).

Theorem 3.3. *Suppose the function f satisfies* (3.1) *with $a_3, a_4 > 0$, $q > 0$, $p, m > 1$, $p \ge m + nq$ and $n > \max\{2(N-2)(p-1), 2\}$. Meanwhile, we give the same auxiliary function*

$\phi(t)$ in Theorem 3.2. If the nonnegative classical solution $u(x,t)$ of the problem (1.1)–(1.3) blows up in the measure of $\phi(t)$ at finite time t^*, then the blow-up time t^* is bounded below, i.e.,

$$\int_{\phi(0)}^{+\infty} \frac{d\vartheta}{I_1\vartheta + I_2\vartheta^{\frac{3(N-2)}{3N-8}} + I_5} \le t^*,$$

where $\phi(0) = \int_\Omega u_0^n\,dx$ and I_1, I_2, I_5 are some computable positive constants.

Proof. By a similar argument as in Theorem 3.2, we have

$$\begin{aligned} \phi'(t) \le& -n\sigma \int_{\partial\Omega} |u|^n\,ds - n(n-1)\int_\Omega |u|^{n-2}|\nabla u|^2\,dx + na_3\int_\Omega |u|^{n+p-1}\,dx \\ & - na_4\int_\Omega |u|^{n+m-1}\,dx\left(\int_\Omega |\nabla u^{n/2}|^2\,dx\right)^q. \end{aligned} \tag{3.13}$$

By Lemma 2.1 and applying Young's inequality, we obtain the inequalities

$$\begin{aligned} \int_{\partial\Omega} |u|^n\,ds &\le \frac{N}{\rho_0}\int_\Omega |u|^n\,dx + \frac{nd}{\rho_0}\int_\Omega |u|^{n-1}|\nabla u|\,dx \\ &\le \frac{nd\delta_1}{2\rho_0}\int_\Omega |u|^{n-2}|\nabla u|^2\,dx + \left(\frac{N}{\rho_0} + \frac{nd}{2\rho_0\delta_1}\right)\int_\Omega |u|^n\,dx, \end{aligned} \tag{3.14}$$

where δ_1 is a positive constant to be determined later. It follows from Hölder's inequality that

$$\int_\Omega |u|^{n+m-1}\,dx \ge \left(\int_\Omega |u|^n\,dx\right)^{1+\frac{m-1}{n}} |\Omega|^{\frac{-m+1}{n}}. \tag{3.15}$$

Substituting (3.14), (3.15) and (3.7) into (3.13), one can have the inequality

$$\phi'(t) \le \widetilde{I}_1\int_\Omega |u|^n\,dx + \widetilde{I}_2\int_\Omega |u|^{n+p-1}\,dx - \widetilde{I}_3\left(\int_\Omega |u|^n\,dx\right)^{1+q+\frac{m-1}{n}} + \widetilde{I}_4\int_\Omega |\nabla u^{n/2}|^2\,dx, \tag{3.16}$$

where

$$\begin{aligned} &\widetilde{I}_1 = \frac{-\sigma nN}{\rho_0} - \frac{\sigma n^2 d}{2\rho_0\delta_1} > 0, && \widetilde{I}_2 = na_3 > 0, \\ &\widetilde{I}_3 = na_4H^q|\Omega|^{\frac{-m+1}{n}} > 0, && \widetilde{I}_4 = \frac{-2\sigma d\delta_1}{\rho_0} - \frac{4(n-1)}{n}. \end{aligned}$$

We now consider the second term on the right-hand side of (3.16). By using Hölder's and Young's inequalities, we have the inequalities

$$\begin{aligned} \int_\Omega |u|^{n+p-1}\,dx &\le |\Omega|^{1-m_1}\left(\int_\Omega |u|^{\frac{n(2N-3)}{2(N-2)}}\,dx\right)^{m_1} \\ &\le (1-m_1)|\Omega| + m_1\int_\Omega |u|^{\frac{n(2N-3)}{2(N-2)}}\,dx, \end{aligned} \tag{3.17}$$

where m_1 is a number given in (2.23). By applying Schwarz's inequality to the second term on the right-hand side of (3.17), one can obtain the inequality

$$\int_\Omega |u|^{\frac{n(2N-3)}{2(N-2)}}\,dx \le \left(\int_\Omega |u|^n\,dx\right)^{3/4}\left(\int_\Omega \left(|u|^{n/2}\right)^{\frac{2N}{N-2}}\,dx\right)^{1/4}. \tag{3.18}$$

To bound $\int_\Omega (|u|^{n/2})^{\frac{2N}{N-2}}\,dx$, we use the Sobolev inequality ($N \ge 3$) given in [2] and, with the inequality, we can obtain the inequality

$$\begin{aligned}\left\||u|^{n/2}\right\|^{\frac{N}{2(N-2)}}_{L^{\frac{2N}{N-2}}(\Omega)} &\le (c_s)^{\frac{N}{2(N-2)}}\left\||u|^{n/2}\right\|^{\frac{N}{2(N-2)}}_{W^{1,2}(\Omega)}\\ &\le c_b\left(\left\|\nabla|u|^{n/2}\right\|^{\frac{N}{2(N-2)}}_{L^2(\Omega)} + \left\||u|^{n/2}\right\|^{\frac{N}{2(N-2)}}_{L^2(\Omega)}\right),\end{aligned} \tag{3.19}$$

where c_s is the Sobolev constant and c_b is a number given in (2.26). By substituting (3.19) into (3.18) and using Young's inequality, one can see that

$$\begin{aligned}&\int_\Omega |u|^{\frac{n(2N-3)}{2(N-2)}}\,dx\\ &\le c_b\left(\int_\Omega |u|^n\,dx\right)^{3/4}\left(\int_\Omega |\nabla u^{n/2}|^2\,dx\right)^{\frac{N}{4(N-2)}} + c_b\left(\int_\Omega |u|^n\,dx\right)^{\frac{2N-3}{2(N-2)}}\\ &\le \frac{c_b^{\frac{4(N-2)}{3N-8}}(3N-8)}{4(N-2)}\delta_2^{-\frac{N}{3N-8}}\phi^{\frac{3(N-2)}{3N-8}}(t) + \frac{N\delta_2}{4(N-2)}\int_\Omega |\nabla u^{n/2}|^2\,dx\\ &\quad + c_b\left(\int_\Omega |u|^n\,dx\right)^{\frac{2N-3}{2(N-2)}},\end{aligned} \tag{3.20}$$

where δ_2 is a positive constant to be determined later. We then obtain the inequality

$$\begin{aligned}\phi(t) \le \widetilde{I}_1\phi &+ \frac{\widetilde{I}_2 m_1 c_b^{\frac{4(N-2)}{3N-8}}(3N-8)}{4(N-2)}\delta_2^{-\frac{N}{3N-8}}\phi^{\frac{3(N-2)}{3N-8}}(t) + \widetilde{I}_2 m_1 c_b \phi^{\frac{2N-3}{2(N-2)}}\\ &+ \left(\widetilde{I}_4 + \frac{\widetilde{I}_2 m_1 N\delta_2}{4(N-2)}\right)\int_\Omega |\nabla u^{n/2}|^2\,dx - \widetilde{I}_3\phi^{1+q+\frac{m-1}{n}} + (1-m_1)\widetilde{I}_2|\Omega|,\end{aligned} \tag{3.21}$$

by substituting (3.17) and (3.20) into (3.16). It follows from Young's inequality that

$$\phi^{\frac{2N-3}{2(N-2)}} \le m_2\delta_3^{-\frac{m_3}{m_2}}\phi^{\frac{3(N-2)}{3N-8}} + m_3\delta_3\phi^{\frac{m+n-1}{n}+q}, \tag{3.22}$$

where m_2 and m_3 are the numbers given in (2.29) and (2.30), respectively, and δ_3 is a positive constant to be determined later. From (3.21) and (3.22), it can be seen that

$$\phi'(t) \le I_1\phi + I_2\phi^{\frac{3(N-2)}{3N-8}} + I_3\phi^{1+q+\frac{m-1}{n}} + I_4\int_\Omega |\nabla u^{n/2}|^2\,dx + I_5, \tag{3.23}$$

where

$$I_1 = \widetilde{I}_1 = \frac{-\sigma n N}{\rho_0} - \frac{\sigma n^2 d}{2\rho_0 \delta_1} > 0, \quad I_2 = \frac{\widetilde{I}_2 m_1 c_b^{\frac{4(N-2)}{3N-8}}(3N-8)}{4(N-2)\delta_2^{\frac{N}{3N-8}}} + \widetilde{I}_2 m_1 c_b m_2 \delta_3^{-\frac{m_3}{m_2}} > 0,$$

$$I_3 = \widetilde{I}_2 m_1 c_b m_3 \delta_3 - \widetilde{I}_3, \;\; I_4 = \frac{n a_3 m_1 N \delta_2}{4(N-2)} - \frac{2\sigma d \delta_1}{\rho_0} - \frac{4(n-1)}{n}, \;\; I_5 = \widetilde{I}_2(1-m_1)|\Omega| > 0.$$

Choosing appropriate $\delta_1, \delta_2, \delta_3 > 0$ so that $I_3, I_4 = 0$, (3.23) can be reduced to

$$\phi'(t) \leq I_1 \phi + I_2 \phi^{\frac{3(N-2)}{3N-8}} + I_5. \tag{3.24}$$

Integrating (3.24) from 0 to t and letting $t \to t^{*-}$, we obtain the inequality

$$\int_{\phi(0)}^{+\infty} \frac{d\vartheta}{I_1 \vartheta + I_2 \vartheta^{\frac{3(N-2)}{3N-8}} + I_5} \leq t^*. \qquad \square$$

Acknowledgments

Yi was supported by the National Research Foundation of Korea (NRF) funded by the Ministry of Science and ICT (2021R1F1A1059539) and Fang was supported by the Natural Science Foundation of Shandong Province of China (No. ZR2019MA072). The authors would like to deeply thank all the reviewers for their insightful and constructive comments.

References

[1] J. Bebernes and A. Bressan, *Thermal behavior for a confined reactive gas*, J. Differential Equations **44** (1982), no. 1, 118–133.

[2] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[3] À. Calsina, C. Perelló and J. Saldaña, *Non-local reaction-diffusion equations modelling predator-prey coevolution*, Publ. Mat. **38** (1994), no. 2, 315–325.

[4] J. Ding and X. Shen, *Blow-up analysis for a class of nonlinear reaction diffusion equations with Robin boundary conditions*, Math. Methods Appl. Sci. **41** (2018), no. 4, 1683–1696.

[5] C. Enache, *Blow-up phenomena for a class of quasilinear parabolic problems under Robin boundary condition*, Appl. Math. Lett. **24** (2011), no. 3, 288–292.

[6] J. Furter and M. Grinfeld, *Local vs. nonlocal interactions in population dynamics*, J. Math. Biol. **27** (1989), no. 1, 65–80.

[7] M. E. Gurtin and R. C. MacCamy, *On the diffusion of biological populations*, Math. Biosci. **33** (1977), no. 1-2, 35–49.

[8] M. Hesaaraki and A. Moameni, *Blow-up of positive solutions for a family of nonlinear parabolic equations in general domain in* $\mathbb{R}^N$, Michigan Math. J. **52** (2004), no. 2, 375–389.

[9] B. Hu, *Blow-up Theories for Semilinear Parabolic Equations*, Lecture Notes in Mathematics **2018**, Springer, Heidelberg, 2011.

[10] H. A. Levine, *Nonexistence of global weak solutions to some properly and improperly posed problems of mathematical physics: The method of unbounded Fourier coefficients*, Math. Ann. **214** (1975), no. 3, 205–220.

[11] H. Li, W. Gao and Y. Han, *Lower bounds for the blowup time of solutions to a nonlinear parabolic problem*, Electron. J. Differential Equations **2014**, No. 20, 6 pp.

[12] T. Li, N. Pintus and G. Viglialoro, *Properties of solutions to porous medium problems with different sources and boundary conditions*, Z. Angew. Math. Phys. **70** (2019), no. 3, Paper No. 86, 18 pp.

[13] Y. Liu, S. Luo and Y. Ye, *Blow-up phenomena for a parabolic problem with a gradient nonlinearity under nonlinear boundary conditions*, Comput. Math. Appl. **65** (2013), no. 8, 1194–1199.

[14] L. Ma and Z. B. Fang, *Blow-up analysis for a nonlocal reaction-diffusion equation with Robin boundary conditions*, Taiwanese J. Math. **21** (2017), no. 1, 131–150.

[15] M. Marras, *Bounds for blow-up time in nonlinear parabolic systems under various boundary conditions*, Numer. Funct. Anal. Optim. **32** (2011), no. 4, 453–468.

[16] M. Marras and S. Vernier Piro, *Reaction-diffusion problems under non-local boundary conditions with blow-up solutions*, J. Inequal. Appl. **2014** (2014), no. 167, 11 pp.

[17] M. Marras, S. Vernier Piro and G. Viglialoro, *Estimates from below of blow-up time in a parabolic system with gradient term*, Int. J. Pure Appl. Math. **93** (2014), no. 2, 297–306.

[18] ———, *Lower bounds for blow-up time in a parabolic problem with a gradient term under various boundary conditions*, Kodai Math. J. **37** (2014), no. 3, 532–543.

[19] L. E. Payne and G. A. Philippin, *Blow-up in a class of non-linear parabolic problems with time-dependent coefficients under Robin type boundary conditions*, Appl. Anal. **91** (2012), no. 12, 2245–2256.

[20] L. E. Payne, G. A. Philippin and S. Vernier Piro, *Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition II*, Nonlinear Anal. **73** (2010), no. 4, 971–978.

[21] L. E. Payne and P. W. Schaefer, *Blow-up in parabolic problems under Robin boundary conditions*, Appl. Anal. **87** (2008), no. 6, 699–707.

[22] L. E. Payne and J. C. Song, *Lower bounds for blow-up time in a nonlinear parabolic problem*, J. Math. Anal. Appl. **354** (2009), no. 1, 394–396.

[23] P. Quittner, *On global existence and stationary solutions for two classes of semilinear parabolic problems*, Comment. Math. Univ. Carolin. **34** (1993), no. 1, 105–124.

[24] P. Quittner and P. Souplet, *Superlinear Parabolic Problems: Blow-up, global existence and steady states*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 2007.

[25] P. Souplet, *Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions*, Differential Integral Equations **15** (2002), no. 2, 237–256.

[26] H. Tian and L. Zhang, *Global and blow-up solutions for a nonlinear reaction diffusion equation with Robin boundary conditions*, Bound. Value Probl. **2020**, Paper No. 68, 19 pp.

[27] Y. Wang, Z. B. Fang and S.-C. Yi, *Lower bounds for blow-up time in nonlocal parabolic problem under Robin boundary conditions*, Appl. Anal. **98** (2019), no. 8, 1403–1414.

Su-Cheol Yi
Department of Mathematics, Changwon National University, Changwon 51140, South Korea
E-mail address: scyi@changwon.ac.kr

Zhong Bo Fang
School of Mathematical Sciences, Ocean University of China, Qingdao 266100, China
E-mail address: fangzb7777@hotmail.com