TAIWANESE JOURNAL OF MATHEMATICS
Vol. 27, No. 3, pp. 577–609, June 2023
DOI: 10.11650/tjm/230102

A Depth-dependent Stability Estimate in an Iterative Method for Solving a Cauchy Problem for the Laplace Equation

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Abstract. In this paper, we consider the Cauchy problem for the Laplace operator. We construct approximate solutions by using the iterative method proposed by Bastay, Kozlov and Turesson. In the iterative method, we solve the corresponding boundary value problems repeatedly. Then, we show that we construct them more stably when we choose the smaller domain where we consider the boundary value problems. Furthermore, since the iterative method is applicable to the case where we know only approximations to the exact data with error, we also deal with this case.

1. Introduction

In this paper, we consider the following Cauchy problem for the Laplace operator

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_*, \\ u = \varphi & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} = \psi & \text{on } \Gamma_0. \end{cases} \tag{1.1}$$

Here Ω_* is a bounded domain in $\mathbb{R}^2$, and its boundary is represented as $\partial\Omega_* = \Gamma_0 \cup \Gamma_*$, where Γ_0 and Γ_* are closed and disjoint (see Figure 1.1), and ν denotes the outward unit vector normal of $\partial\Omega_*$. There exist various methods for solving the problem (1.1). One way is to use iterative methods, which were proposed by Bastay, Kozlov and Turesson [1]. We remark that this iterative method is reduced to the Landweber iteration. For the Landweber iteration, see [3, Ch. 6, Sec. 1] for example. In [1], they discussed the iterative methods for Cauchy problems for parabolic equations and mentioned that their methods can be also applied to elliptic equations. The iterative methods for the elliptic equations are also discussed in [6, 8], and the ones for the parabolic equations are also discussed in [2, 7], for instance.

In the iterative method, we construct a sequence of approximate solutions to the problem (1.1) by solving the boundary value problems repeatedly. Here, we need to solve the corresponding boundary value problems in the whole of domain Ω_* if we would like to

Received July 26, 2022; Accepted January 15, 2023.
Communicated by Jenn-Nan Wang.
2020 *Mathematics Subject Classification.* 35R25.
Key words and phrases. Cauchy problem, iterative method, stability estimate.

construct the solution to the problem (1.1) in Ω_*. However, if we would like to construct the solution to the problem (1.1) in Ω^* ($\subset \Omega_*$) near Γ_0, then we can also consider the boundary value problems only in Ω satisfying $\Omega^* \subset \Omega \subset \Omega_*$ (see Figure 1.1). We then expect that we can construct approximate solutions more stably when we choose smaller domain Ω. The aim of this paper is to show this property by giving explicitly the order of convergence of approximate solutions. We remark that the order of convergence is not given in [1]. In this paper, we give the order.

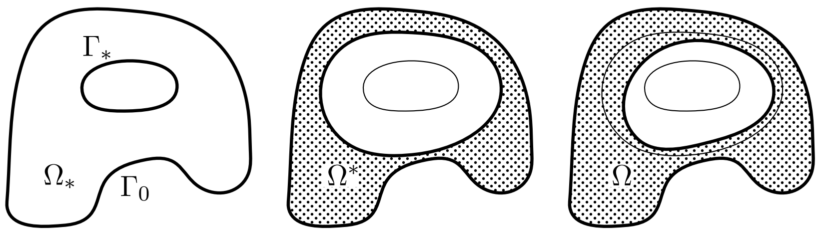

Figure 1.1: The relationship of Ω_*, Ω^* and Ω.

In order to show it, we consider the following situation. Throughout this paper, we shall use the following notation. Let r be a positive number. We shall donate by $B_r = \{x \in \mathbb{R}^2 \mid |x| < r\}$ (the circle of radius r centered at the origin) and $\partial B_r = \{x \in \mathbb{R}^2 \mid |x| = r\}$ (the circumference of radius r centered at the origin). We consider domains Ω_*, Ω^* and Ω to be as annulus $B_1 \setminus \overline{B_{\rho_*}}$, $B_1 \setminus \overline{B_{\rho^*}}$ and $B_1 \setminus \overline{B_\rho}$ respectively, where $0 < \rho_* \le \rho \le \rho^* < 1$ (see Figure 1.2). Instead of the problem (1.1), we consider the following Cauchy problem

$$(1.2) \qquad \begin{cases} \Delta u = 0 & \text{in } B_1 \setminus \overline{B_{\rho_*}}, \\ u = \varphi & \text{on } \partial B_1, \\ \frac{\partial u}{\partial \nu} = \psi & \text{on } \partial B_1. \end{cases}$$

We give a strict definition of a solution to the problem (1.2) in Definition 2.6.

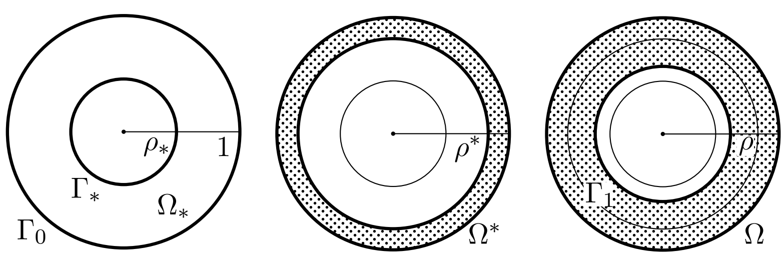

Figure 1.2: The relationship of annulus Ω_*, Ω^* and Ω.

Throughout this paper, we impose the following assumption.

Assumption 1.1. *Given positive numbers M_0 and M. We suppose that φ and ψ are real-valued functions with*

$$\|\varphi\|^2_{L^2(\partial B_1)} + \|\psi\|^2_{L^2(\partial B_1)} \le M_0^2. \tag{1.3}$$

We assume that there exists a solution u to the Cauchy problem (1.2) *in $B_1 \setminus \overline{B_{\rho_*}}$ and its restriction on ∂B_{ρ_*} be in L^2. We suppose that*

$$\|u\|_{L^2(\partial B_{\rho_*})} \le M. \tag{1.4}$$

We apply the iterative method in [1] to construct a sequence of approximate solutions to the problem (1.2). As we describe in Section 2.2, in the iterative method, we solve the boundary value problem in $B_1 \setminus \overline{B_\rho}$ at each iteration step. We now state our main theorem. We here remark that, in Sections 2.2 and 2.5, we introduce an approximate solution $u^{(\ell)}$ which appears in the following theorem.

Theorem 1.2. *Let Assumption* 1.1 *hold. Let $u^{(\ell)}$ be an approximate solution obtained by the iterative procedure with $\eta^{(0)} = 0$ and $\gamma = \rho$. Then, for $\ell \ge 2$ and $\rho \in [\rho_*, \rho^*]$, the following estimate holds:*

$$\|u^{(\ell)} - u\|^2_{L^2(B_1 \setminus \overline{B_{\rho^*}})} \le C \left(\frac{\log \ell}{\ell}\right)^{\min\left\{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}, 1\right\}} \tag{1.5}$$

where a positive constant C depends only on M_0, M, ρ_* and ρ^*.

Remark 1.3. We have

$$\min\left\{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}, 1\right\} = \begin{cases} \frac{\log(\rho^*/\rho_*)}{\log(1/\rho)} & \text{for } \rho_* \le \rho \le \frac{\rho_*}{\rho^*}, \\ 1 & \text{for } \frac{\rho_*}{\rho^*} < \rho \le \rho^*. \end{cases}$$

In addition, there exists no ρ such that $\rho_*/\rho^* < \rho \le \rho^*$ in the case where $\sqrt{\rho_*} > \rho^*$. Namely, we have

$$\|u^{(\ell)} - u\|^2_{L^2(B_1 \setminus \overline{B_{\rho^*}})} \le C \left(\frac{\log \ell}{\ell}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}}$$

for any ρ in this case.

The larger ρ we choose, the larger the power $\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}$ on the right-hand side of the estimate (1.5) is. This means that we obtain the better stability as we choose ρ larger. Moreover, with regard to the optimality of the estimate (1.5), we have the following theorem.

Theorem 1.4. *Let $\varepsilon > 0$ be given. Under the assumptions in Theorem* 1.2, *there exists no positive constant $\widehat{C}$ depending only on M_0, M, ρ_* and ρ^* such that*

$$\|u^{(\ell)} - u\|^2_{L^2(B_1 \setminus \overline{B_{\rho^*}})} \le \widehat{C} \left(\frac{\log \ell}{\ell}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)} + \varepsilon}$$

holds for any $\ell \ge 2$ and $\rho \in [\rho_*, \rho^*]$.

By Theorem 1.4, we know that the estimate (1.5) is optimal in the case where $\rho_* \le \rho \le \rho_*/\rho^*$. Furthermore, the estimate (1.5) is always optimal especially in the case where $\sqrt{\rho_*} > \rho^*$.

Since the Landweber iteration works with inexact data, the iterative method proposed in [1] also works. We next consider the case where we know only approximations φ^δ and ψ^δ to the exact data with error δ in $L^2(\partial B_1)$-norm.

Assumption 1.5. *We suppose that* φ^δ *and* ψ^δ *are real-valued functions with*

$$\|\varphi^\delta - \varphi\|_{L^2(\partial B_1)} \le \delta, \quad \|\psi^\delta - \psi\|_{L^2(\partial B_1)} \le \delta. \tag{1.6}$$

Theorem 1.6. *Let Assumptions* 1.1 *and* 1.5 *hold. Let* $u^{(\ell),\delta}$ *be an approximation obtained by the iterative procedure with* $\eta^{(0)} = 0$ *and* $\gamma = \rho$ *for Cauchy data* φ^δ *and* ψ^δ*. Then, for* $0 < \delta \le 1/e^3$ *and* $\rho \in [\rho_*, \rho^*]$*, we have*

$$\|u^{(\ell(\delta,\rho)),\delta} - u\|^2_{L^2(B_1 \setminus \overline{B_{\rho^*}})} \le \widetilde{C} \begin{cases} \left(\delta^2 \log \frac{1}{\delta}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)+\log(\rho^*/\rho_*)}} & \text{for } \rho_* \le \rho \le \frac{\rho_*}{\rho^*}, \\ \left(\delta^2 \log \frac{1}{\delta}\right)^{1/2} & \text{for } \frac{\rho_*}{\rho^*} < \rho \le \rho^*, \end{cases}$$

where $\ell(\delta,\rho)$ *is the minimum integer satisfying* $\ell(\delta,\rho) \ge \ell_0(\delta,\rho)$ *with*

$$\ell_0(\delta,\rho) := \begin{cases} \left(\frac{1}{\delta}\right)^{\frac{2\log(1/\rho)}{\log(1/\rho)+\log(\rho^*/\rho_*)}} \left(\log \frac{1}{\delta}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)+\log(\rho^*/\rho_*)}} & \text{for } \rho_* \le \rho \le \frac{\rho_*}{\rho^*}, \\ \frac{1}{\delta}\left(\log \frac{1}{\delta}\right)^{1/2} & \text{for } \frac{\rho_*}{\rho^*} < \rho \le \rho^*, \end{cases}$$

and a positive constant $\widetilde{C}$ *depends only on* M_0*,* M*,* ρ_* *and* ρ^**.*

The rest of this paper is organized as follows. In Section 2, we introduce the iterative procedure presented by Bastay, Kozlov and Turesson [1] and its properties. In Section 3, we give the explicit solution formulae to respective problems. In Section 4, we prove the estimate (1.5) in Theorem 1.2. Section 5 is devoted to the proof of Theorem 1.4. Finally, we show Theorem 1.6 in Section 6.

2. The iterative method for the Cauchy problem

In this section, we introduce the iterative method proposed by Bastay, Kozlov and Turesson [1]. In [1], they proposed the iterative method for the Cauchy problems for parabolic problems and mentioned that their methods can be also applied to elliptic equations.

In Section 2.1, we prove lemmas from which it follows that the boundary value problem we solve in the algorithm is well-posed. This section corresponds to [1, Sec. 2]. In Section 2.2, we state the iterative procedure for the Cauchy problem (1.2). It is an analogy to the iterative procedure in [1, Sec. 3.1.1]. In the iterative method, we solve the corresponding boundary value problems repeatedly. Since we now consider the problem (1.2) in the annulus, we can obtain the explicit solution formula to the boundary value problem by introducing the polar coordinates. Then, Section 2.3 is devoted to giving the explicit solution formula to the boundary value problem. We describe its properties in Section 2.4. Finally, in Section 2.5, we state iterative method for the problem (1.2) more concisely.

2.1. A well-posedness for the boundary value problem

At each step in the algorithm, we solve the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \setminus \overline{B_\rho}, \\ u = \eta & \text{on } \partial B_\rho, \\ \frac{\partial u}{\partial \nu} = \psi & \text{on } \partial B_1. \end{cases} \tag{2.1}$$

In this case, we observe that the adjoint equation to the Laplace equation is the same as the Laplace equation.

We can define a weak solution $u \in L^2(B_1 \setminus \overline{B_\rho})$ to the problem (2.1) for $\eta \in L^2(\partial B_\rho)$ and $\psi \in L^2(\partial B_1)$.

Definition 2.1 (Weak solution)**.** Assume that $\eta \in L^2(\partial B_\rho)$ and $\psi \in L^2(\partial B_1)$. Then, $u \in L^2(B_1 \setminus \overline{B_\rho})$ is a weak solution to the problem (2.1) if u satisfies

$$\int_{B_1 \setminus \overline{B_\rho}} u \Delta g \, dx + \int_{\partial B_1} \psi g \, d\sigma - \int_{\partial B_\rho} \eta \frac{\partial g}{\partial \nu} \, d\sigma = 0 \tag{2.2}$$

for every $g \in H^2(B_1 \setminus \overline{B_\rho})$ subject to

$$\begin{cases} g = 0 & \text{on } \partial B_\rho, \\ \frac{\partial g}{\partial \nu} = 0 & \text{on } \partial B_1. \end{cases} \tag{2.3}$$

Here, ν is the outward unit vector normal of $\partial(B_1 \setminus \overline{B_\rho})$. We remark that ν on ∂B_ρ points toward the center of B_ρ, whereas ν on ∂B_1 points outward.

We next show the problem (2.1) is well-posed, that is, it has a unique solution that depends continuously on the data. This proof is presented in Appendix A.

Theorem 2.2. *The problem* (2.1) *has a unique solution u that satisfies*

$$\|u\|_{L^2(B_1 \setminus \overline{B_\rho})} \le C' \big(\|\eta\|_{L^2(\partial B_\rho)} + \|\psi\|_{L^2(\partial B_1)} \big). \tag{2.4}$$

Finally, using the Green function, we shall show that the restriction $u\big|_{\partial B_1}$ exists and is in $L^2(\partial B_1)$. We now give a definition of the Green function.

Definition 2.3 (Green function)**.** A function $G(x,y)$ is called the Green function for the problem

$$\begin{cases} \Delta w = f & \text{in } B_1 \setminus \overline{B_\rho}, \\ w = \eta & \text{on } \partial B_\rho, \\ \frac{\partial w}{\partial \nu} = \psi & \text{on } \partial B_1 \end{cases}$$

if $G(x,y)$ satisfies

$$\begin{cases} \Delta_x G(x,y) = -\delta(x-y), & x \in B_1 \setminus \overline{B_\rho}, \\ G(x,y) = 0, & x \in \partial B_\rho, \\ \frac{\partial G}{\partial \nu_x}(x,y) = 0, & x \in \partial B_1. \end{cases}$$

Then, we have the following lemma.

Lemma 2.4. *The solution to the problem* (2.1) *is given by*

$$u(x) = \int_{\partial B_1} \psi(y) G(x,y)\, d\sigma(y) - \int_{\partial B_\rho} \eta(y) \frac{\partial G}{\partial \nu_y}(x,y)\, d\sigma(y). \tag{2.5}$$

It is also well-known that the Green function $G(x,y)$ has the estimates

$$0 < G(x,y) \le C'' \max\left\{1, \big|\log|x-y|\big|\right\} \tag{2.6}$$

and

$$|\nabla G(x,y)| \le \frac{C'''}{|x-y|}. \tag{2.7}$$

We remark that the Green function $G(x,y)$ is smooth for $x \neq y$. The second term of (2.5) has enough smoothness near ∂B_1 since $\frac{\partial G}{\partial \nu_y}(x,y)$ is smooth for x near ∂B_1 and $y \in \partial B_\rho$. On the other hand, if $\eta = 0$, then the solution to the problem (2.1) is given by

$$u(x) = \int_{\partial B_1} \psi(y) G(x,y)\, d\sigma(y),$$

which has enough smoothness near ∂B_ρ since the Green function $G(x,y)$ is smooth for x near ∂B_ρ and $y \in \partial B_1$. Thus, using the solution formula (2.5) and estimates (2.6) and (2.7), we obtain the following lemma.

Lemma 2.5. *Let u be the solution to the problem* (2.1)*. Then, the restriction $u\big|_{\partial B_1}$ is in $L^2(\partial B_1)$. Moreover, in the case where $\eta = 0$, the restriction $\frac{\partial u}{\partial \nu}\big|_{\partial B_\rho}$ is in $L^2(\partial B_\rho)$.*

We now give a strict definition of a solution to the Cauchy problem (1.2).

Definition 2.6. A solution u to the problem (1.2) is a weak solution to the problem (2.1) for some $\eta \in L^2(\partial B_1)$ such that $u\big|_{\partial B_1} = \varphi$.

2.2. Description of the iterative method

We now state the iterative procedure.

- Choose an arbitrary function $\eta^{(0)} \in L^2(\partial B_\rho)$.
- The first approximation $u^{(0)}$ to the weak solution u is obtained by solving the boundary value problem (2.1) with $\eta = \eta^{(0)}$ on ∂B_ρ.
- Then, we find the auxiliary function $v^{(0)}$, which is given by the weak solution to the problem (2.1) with $\eta = 0$ and $\psi = \zeta^{(0)}$, where $\zeta^{(0)} = u^{(0)} - \varphi$ on ∂B_1.
- When the solutions $u^{(\ell-1)}$ and $v^{(\ell-1)}$ have been constructed, the approximation $u^{(\ell)}$ is the weak solution to the problem (2.1) with $\eta = \eta^{(\ell)}$ on ∂B_ρ, where

$$\eta^{(\ell)} = u^{(\ell-1)} + \gamma \frac{\partial v^{(\ell-1)}}{\partial \nu}$$

  and γ is a fixed positive number.
- The auxiliary function $v^{(\ell)}$ is the weak solution to the problem (2.1) with $\eta = 0$ and $\psi = \zeta^{(\ell)}$, where $\zeta^{(\ell)} = u^{(\ell)} - \varphi$ on ∂B_1.

Remark 2.7. It follows from Lemma 2.5 that the functions $\eta^{(\ell)}$ and $\zeta^{(\ell)}$ are well-defined. Moreover, we obtain $\eta^{(\ell)} \in L^2(\partial B_\rho)$ and $\zeta^{(\ell)} \in L^2(\partial B_1)$.

2.3. The solution formula to the boundary value problem

Introducing the polar coordinates, we give the explicit solution formula to the boundary value problem (2.1). We now make the change of variables

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta.$$

Let

$$\widetilde{\eta}(\theta) := \eta(\rho\cos\theta, \rho\sin\theta) \quad \text{for } \eta \in L^2(\partial B_\rho)$$

and

$$\widetilde{\psi}(\theta) := \psi(\cos\theta, \sin\theta) \quad \text{for } \psi \in L^2(\partial B_1).$$

They are periodic functions which repeat at intervals of 2π radians. Hence, they can be expressed through Fourier series expansions as

$$\widetilde{\eta}(\theta) = \sum_{k\in\mathbb{Z}} \eta_k e^{ik\theta}, \quad \widetilde{\psi}(\theta) = \sum_{k\in\mathbb{Z}} \psi_k e^{ik\theta}$$

and we have the following lemma.

Lemma 2.8. *The solution to the problem* (2.1) *is given by*

$$\widetilde{u}(r,\theta) = \sum_{k\neq 0} \left\{ \frac{k\eta_k + \psi_k \rho^{-k}}{k(\rho^k + \rho^{-k})} r^k + \frac{k\eta_k - \psi_k \rho^k}{k(\rho^k + \rho^{-k})} r^{-k} \right\} e^{ik\theta} + \eta_0 + \psi_0 \log \frac{r}{\rho}, \tag{2.8}$$

where $\widetilde{u}(r,\theta) = u(r\cos\theta, r\sin\theta)$.

Proof. Introducing the polar coordinates, the problem (2.1) is equivalent to

$$\begin{cases} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) \widetilde{u}(r,\theta) = 0, & \rho \le r \le 1,\ 0 \le \theta < 2\pi, \\ \widetilde{u}(\rho,\theta) = \widetilde{\eta}(\theta), & 0 \le \theta < 2\pi, \\ \frac{\partial \widetilde{u}}{\partial r}(1,\theta) = \widetilde{\psi}(\theta), & 0 \le \theta < 2\pi. \end{cases}$$

Then, we have the solution formula to the problem (2.1). □

2.4. Reduction to the Landweber iteration

In accordance with [1, Sec. 3.1.2], we define operators in order to show that $u^{(\ell)}$ constructed above converges to the solution u to the Cauchy problem (1.2).

We first introduce the operator $K\colon L^2(\partial B_\rho) \to L^2(\partial B_1)$ through

$$(K\eta)(\cos\theta, \sin\theta) = \sum_{k\in\mathbb{Z}} \frac{2\eta_k}{\rho^k + \rho^{-k}} e^{ik\theta}. \tag{2.9}$$

By (2.8), we know that (2.9) corresponds to

$$K\eta = z_1\big|_{\partial B_1} \quad \text{for } \eta \in L^2(\partial B_\rho), \tag{2.10}$$

where z_1 is the solution to the boundary value problem (2.1) with $\psi = 0$. It follows from Lemma 2.5 that $z_1\big|_{\partial B_1}$ is in $L^2(\partial B_1)$. Similarly, we define the operator $K_1\colon L^2(\partial B_1) \to L^2(\partial B_1)$ by

$$(K_1\psi)(\cos\theta, \sin\theta) = \sum_{k\neq 0} \frac{\psi_k(\rho^{-k} - \rho^k)}{k(\rho^k + \rho^{-k})} e^{ik\theta} - \psi_0 \log\rho. \tag{2.11}$$

Then, using (2.8), we know that (2.11) corresponds to

$$K_1\psi = z_2\big|_{\partial B_1} \quad \text{for } \psi \in L^2(\partial B_1), \tag{2.12}$$

where z_2 is the solution to the problem (2.1) with $\eta = 0$.

We here consider the equation

$$K\eta = \varphi - K_1\psi. \tag{2.13}$$

We remark that solving the equation (2.13) is equivalent to solving the problem (1.2). Indeed, if η is a solution to the equation (2.13), the solution to the problem (2.1) satisfies $u = \varphi$ on ∂B_1 and thus solves the problem (1.2). Conversely, if u solves the problem (1.2), then $\eta = u\big|_{\partial B_\rho}$ is a solution to the equation (2.13).

Correspondingly to [1, Sec. 3.1.2], we next see properties of the operator K.

Lemma 2.9. *The operator K is injective.*

Proof. We suppose that $K\eta = 0$. Then, the solution to the problem (2.1) with $\psi = 0$ also solves the Cauchy problem

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \setminus \overline{B_\rho}, \\ u = 0 & \text{on } \partial B_1, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1. \end{cases}$$

Using the uniqueness theorem (see [5, Theorem 3.2.1], for example), we have $u = 0$. Hence, from the problem (2.1) with $\psi = 0$, we obtain $\eta = 0$. □

Lemma 2.10. *The adjoint $K^*\colon L^2(\partial B_1) \to L^2(\partial B_\rho)$ to the operator K is given by*

$$(K^*\zeta)(\rho\cos\theta, \rho\sin\theta) = \sum_{k\in\mathbb{Z}} \frac{2\zeta_k}{\rho(\rho^k + \rho^{-k})} e^{ik\theta}. \tag{2.14}$$

Proof. We first show that we have

$$K^*\zeta = -\left(\frac{\partial v}{\partial \nu}\right)\bigg|_{\partial B_\rho} \quad \text{for } \zeta \in L^2(\partial B_1), \tag{2.15}$$

where v is the solution to the problem (2.1) with $\eta = 0$ and $\psi = \zeta$. The formula (2.15) is equivalent to

$$(K^*\zeta, \eta)_{L^2(\partial B_\rho)} = \left(-\frac{\partial v}{\partial \nu}, \eta\right)_{L^2(\partial B_\rho)}$$

for all $\eta \in L^2(\partial B_\rho)$. Since we have

$$(K^*\zeta, \eta)_{L^2(\partial B_\rho)} = (\zeta, K\eta)_{L^2(\partial B_1)},$$

it is enough to show

$$\int_{\partial B_1} \zeta K\eta \, d\sigma = -\int_{\partial B_\rho} \frac{\partial v}{\partial \nu}\eta \, d\sigma \tag{2.16}$$

instead of showing (2.15). Moreover, we remark that it is sufficient to prove (2.16) for $\eta \in C^\infty(\partial B_\rho)$ and $\zeta \in C^\infty(\partial B_1)$. Now, let u be the solution to the problem (2.1) with $\psi = 0$. We here remark that we have

$$\begin{aligned} &\int_{B_1\setminus\overline{B_\rho}} (\Delta u)v \, dx - \int_{B_1\setminus\overline{B_\rho}} u\Delta v \, dx \\ &= \int_{\partial B_1} \frac{\partial u}{\partial \nu} v \, d\sigma - \int_{\partial B_1} u\frac{\partial v}{\partial \nu} \, d\sigma + \int_{\partial B_\rho} \frac{\partial u}{\partial \nu} v \, d\sigma - \int_{\partial B_\rho} u\frac{\partial v}{\partial \nu} \, d\sigma \end{aligned} \tag{2.17}$$

by using the divergence theorem. Then, using (2.17) and equations which u and v satisfy, we have

$$0 = \int_{\partial B_1} u\zeta \, d\sigma + \int_{\partial B_\rho} \eta \frac{\partial v}{\partial \nu} \, d\sigma. \tag{2.18}$$

Substituting (2.10) into the first term on the right-hand side of (2.18), we get (2.16). Moreover, it follows from (2.16) that the relationship (2.15) holds. Furthermore, using (2.8) and (2.15), we obtain the formula (2.14). □

Finally, using the Green function, we obtain the lemma about compactness.

Lemma 2.11. *The operators K, K_1 and K^* are compact.*

Proof. Using the solution formula (2.5) and the relationships (2.10) and (2.12), we have

$$\begin{aligned} K\eta(x) &= -\int_{\partial B_\rho} \eta(y) \frac{\partial G}{\partial \nu_y}(x,y) \, d\sigma(y) \Big|_{\partial B_1}, \\ K_1\psi(x) &= \int_{\partial B_1} \psi(y) G(x,y) \, d\sigma(y) \Big|_{\partial B_1} \end{aligned} \tag{2.19}$$

and

$$K^*\zeta(x) = -\int_{\partial B_1} \zeta(y) \frac{\partial G}{\partial \nu_x}(x,y) \, d\sigma(y) \Big|_{\partial B_\rho}. \tag{2.20}$$

We first see $\frac{\partial G}{\partial \nu_y}(x,y)$ in (2.19). Since $\frac{\partial G}{\partial \nu_y} \in L^2(\partial B_1 \times \partial B_\rho)$, see [4, (2.35) Claim], K is a Hilbert–Schmidt integral operator. Hence, K is compact, see [4, (0.45) Theorem]. Similar to $\frac{\partial G}{\partial \nu_x}(x,y)$ in (2.20), K^* is compact. Finally, by the estimate (2.6), we know that K_1 is compact in the same way as the proof of [4, (3.11) Proposition]. □

Similar to [1, Theorem 3.1], we have the following theorem.

Theorem 2.12. *Let $u \in L^2(B_1 \setminus \overline{B_\rho})$ be the solution to the problem* (1.2). *We suppose that γ satisfies $0 < \gamma < \frac{2}{\|K\|^2_{L^2(\partial B_\rho) \to L^2(\partial B_1)}}$. Let $u^{(\ell)}$ be the ℓ-th approximate solution in the iterative procedure. Then, we have*

$$\|u^{(\ell)} - u\|_{L^2(B_1 \setminus \overline{B_\rho})} \to 0 \quad \text{as } \ell \to \infty$$

for every initial data function $\eta^{(0)} \in L^2(\partial B_\rho)$.

Proof. From the algorithm described in Section 2.2 and (2.15), we obtain

$$\begin{aligned} \eta^{(\ell)} &= u^{(\ell-1)}\big|_{\partial B_\rho} + \gamma \left(\frac{\partial v^{(\ell-1)}}{\partial \nu} \right)\Big|_{\partial B_\rho} = \eta^{(\ell-1)} + \gamma(-K^*\zeta^{(\ell-1)}) \\ &= \eta^{(\ell-1)} + \gamma\big\{ -K^*(u^{(\ell-1)} - \varphi)\big|_{\partial B_1}\big\} = \eta^{(\ell-1)} - \gamma K^*(K\eta^{(\ell-1)} + K_1\psi - \varphi) \\ &= (1 - \gamma K^*K)\eta^{(\ell-1)} + \gamma K^*(\varphi - K_1\psi). \end{aligned} \tag{2.21}$$

Since the linear operator K is injective and compact from Lemmas 2.9 and 2.11, we know that (2.21) is the Landweber iteration for the equation (2.13). Hence, we have

$$\|\eta^{(\ell)} - \eta\|_{L^2(\partial B_\rho)} \to 0 \quad \text{as } \ell \to \infty.$$

We now remark that u satisfies (2.1). Using (2.4), we obtain

$$\|u^{(\ell)} - u\|_{L^2(B_1 \setminus \overline{B_\rho})} \le C' \|\eta^{(\ell)} - \eta\|_{L^2(\partial B_\rho)} \to 0 \quad \text{as } \ell \to \infty.$$

The proof is completed. □

We remark that our aim is to show the order of this convergence.

2.5. Reformulation of the iterative method

Based on above discussions, we would like to reformulate the iterative method. A function $\eta^{(\ell)}$ in the iterative procedure is defined on ∂B_ρ; thus, let

$$\widetilde{\eta^{(\ell)}}(\theta) := \eta^{(\ell)}(\rho\cos\theta, \rho\sin\theta).$$

We express it as

$$\widetilde{\eta^{(\ell)}}(\theta) = \sum_{k\in\mathbb{Z}} \eta_k^{(\ell)} e^{ik\theta}$$

by the Fourier expression.

We first give the explicit formula of $\widetilde{u^{(\ell)}}(r,\theta)$, where $\widetilde{u^{(\ell)}}$ is an approximate solution obtained by the iterative procedure. From the algorithm described in Section 2.2, we remark that $u^{(\ell)}$ is the solution to the boundary value problem (2.1) with $\eta = \eta^{(\ell)}$ on ∂B_ρ.

Lemma 2.13. *An approximate solution $u^{(\ell)}$ obtained by the iterative procedure is given by*

$$\widetilde{u^{(\ell)}}(r,\theta) = \sum_{k\neq 0} \left\{ \frac{k\eta_k^{(\ell)} + \psi_k \rho^{-k}}{k(\rho^k + \rho^{-k})} r^k + \frac{k\eta_k^{(\ell)} - \psi_k \rho^{k}}{k(\rho^k + \rho^{-k})} r^{-k} \right\} e^{ik\theta} + \eta_0^{(\ell)} + \psi_0 \log \frac{r}{\rho}. \tag{2.22}$$

Proof. Using (2.8), we have (2.22). □

We next state the reformulated procedure for the Cauchy problem (1.2).

- Fix a positive constant γ and choose an arbitrary function $\eta^{(0)} \in L^2(\partial B_\rho)$.
- Using (2.21), $\eta^{(1)} \in L^2(\partial B_\rho)$ is defined.
- Then, we get the first approximation $u^{(1)} \in L^2(B_1 \setminus \overline{B_\rho})$ by (2.22).
- When $\eta^{(\ell-1)}$ has been defined, we find $\eta^{(\ell)} \in L^2(\partial B_\rho)$ by using (2.21).

- Using (2.22), we obtain the approximation $u^{(\ell)} \in L^2(B_1 \setminus \overline{B_\rho})$.

Remark 2.14. Since we now consider the annulus as a domain, it follows from the formulae (2.9), (2.11) and (2.14) that the linear operators

$$K\colon H^\alpha(\partial B_\rho) \to H^\alpha(\partial B_1), \quad K_1\colon H^\alpha(\partial B_1) \to H^{\alpha+1}(\partial B_1)$$

and

$$K^*\colon H^\alpha(\partial B_1) \to H^\alpha(\partial B_\rho)$$

are bounded for each $\alpha \geq 0$. If $\varphi \in H^{1/2}(\partial B_1)$ and $\psi \in L^2(\partial B_1)$, we get the approximation $u^{(\ell)} \in H^1(B_1 \setminus \overline{B_\rho})$ by choosing $\eta^{(0)} \in H^{1/2}(\partial B_\rho)$ since each $\eta^{(\ell)}$ obtained by the iterative procedure is in $H^{1/2}(\partial B_\rho)$. Indeed, the solution u to the problem (2.1) is in $H^1(B_1 \setminus \overline{B_\rho})$ for $\varphi \in H^{1/2}(\partial B_1)$ and $\psi \in L^2(\partial B_1)$.

3. Solution formulae

Since we consider the annulus as a domain, we can obtain the explicit solution formulae to the problems by introducing the polar coordinates. In addition to $\widetilde{\eta}(\theta)$ and $\widetilde{\psi}(\theta)$ in Section 2.3, let $\widetilde{\varphi}(\theta) := \varphi(\cos\theta, \sin\theta)$. Hence, it can be expressed through Fourier series expansions as

$$\widetilde{\varphi}(\theta) = \sum_{k\in\mathbb{Z}} \varphi_k e^{ik\theta}.$$

We then have the following lemma.

Lemma 3.1. *The solution to the Cauchy problem* (1.2) *is given by*

$$\widetilde{u}(r,\theta) = \sum_{k\neq 0} \left(\frac{k\varphi_k + \psi_k}{2k} r^k + \frac{k\varphi_k - \psi_k}{2k} r^{-k} \right) e^{ik\theta} + \varphi_0 + \psi_0 \log r, \tag{3.1}$$

where $\widetilde{u}(r,\theta) = u(r\cos\theta, r\sin\theta)$.

Proof. Introducing the polar coordinates, the problem (1.2) is equivalent to

$$\begin{cases} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) \widetilde{u}(r,\theta) = 0 & \rho_* < r < 1,\ 0 \leq \theta < 2\pi, \\ \widetilde{u}(1,\theta) = \widetilde{\varphi}(\theta) & 0 \leq \theta < 2\pi, \\ \frac{\partial \widetilde{u}}{\partial r}(1,\theta) = \widetilde{\psi}(\theta) & 0 \leq \theta < 2\pi. \end{cases}$$

Then, we have the solution formula to the problem (1.2). □

We now assume $\eta^{(0)} = 0$. From (2.9), (2.11), (2.14) and (2.21), we obtain the recurrence relation

$$\eta_k^{(\ell)} = \left\{ 1 - \frac{4\gamma}{\rho(\rho^k + \rho^{-k})^2} \right\} \eta_k^{(\ell-1)} + \eta_k^{(1)}. \tag{3.2}$$

Solving (3.2), we have the following lemma.

Lemma 3.2. *The Fourier coefficients* $\eta_k^{(\ell)}$ *are given by*

$$\eta_k^{(\ell)} = -\frac{\rho(\rho^k+\rho^{-k})^2}{4\gamma}\eta_k^{(1)}\left[\left\{1-\frac{4\gamma}{\rho(\rho^k+\rho^{-k})^2}\right\}^\ell - 1\right], \tag{3.3}$$

where

$$\eta_k^{(1)} = \begin{cases} \frac{\gamma}{\rho}(\varphi_0+\psi_0\log\rho) & \text{if } k=0, \\ \frac{2\gamma}{\rho(\rho^k+\rho^{-k})}\left\{\varphi_k + \frac{\psi_k(\rho^k-\rho^{-k})}{k(\rho^k+\rho^{-k})}\right\} & \text{if } k\neq 0. \end{cases}$$

By Theorem 2.12, we have to choose $0<\gamma<\frac{2}{\|K\|^2_{L^2(\partial B_\rho)\to L^2(\partial B_1)}}$ in order for $\eta^{(\ell)}$ to converge. We now find $\|K\|_{L^2(\partial B_\rho)\to L^2(\partial B_1)}$.

Lemma 3.3. $\|K\|_{L^2(\partial B_\rho)\to L^2(\partial B_1)} = \frac{1}{\sqrt{\rho}}$.

Proof. Since we have

$$\|K\eta\|^2_{L^2(\partial B_1)} = \sum_{k\in\mathbb{Z}}\frac{8\pi|\eta_k|^2}{(\rho^k+\rho^{-k})^2} \le 2\pi\sum_{k\in\mathbb{Z}}|\eta_k|^2 = \frac{1}{\rho}\|\eta\|^2_{L^2(\partial B_\rho)}$$

from (2.9), we obtain

$$\|K\|_{L^2(\partial B_\rho)\to L^2(\partial B_1)} \le \frac{1}{\sqrt{\rho}}.$$

On the other hand, since we have

$$\|K1\|^2_{L^2(\partial B_1)} = 2\pi, \quad \|1\|^2_{L^2(\partial B_\rho)} = 2\pi\rho$$

from (2.9), we know that the equality holds. □

We here give the explicit formula of $\widetilde{u^{(\ell)}}(r,\theta)$, where $\widetilde{u^{(\ell)}}$ is an approximate solution obtained by the iterative procedure with $\eta^{(0)}=0$.

Lemma 3.4. *An approximate solution* $u^{(\ell)}$ *obtained by the iterative procedure with* $\eta^{(0)}=0$ *is given by*

$$\begin{aligned} &\widetilde{u^{(\ell)}}(r,\theta) \\ &= -\frac{1}{2}\sum_{k\neq 0}\left\{\varphi_k + \frac{\psi_k(\rho^k-\rho^{-k})}{k(\rho^k+\rho^{-k})}\right\}\left[\left\{1-\frac{4\gamma}{\rho(\rho^k+\rho^{-k})^2}\right\}^\ell - 1\right](r^k+r^{-k})e^{ik\theta} \\ &\quad + \sum_{k\neq 0}\frac{\psi_k}{k(\rho^k+\rho^{-k})}(\rho^{-k}r^k - \rho^k r^{-k})e^{ik\theta} \\ &\quad - (\varphi_0+\psi_0\log\rho)\left\{\left(1-\frac{\gamma}{\rho}\right)^\ell - 1\right\} + \psi_0\log\frac{r}{\rho}. \end{aligned} \tag{3.4}$$

Proof. Substituting (3.3) into (2.22), we obtain (3.4). □

We now give the explicit formula of $\|u^{(\ell)} - u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})}$.

Lemma 3.5. *Let u be a solution to the problem (1.2) for real-valued Cauchy data φ and ψ, and $u^{(\ell)}$ be an approximation obtained by the iterative procedure with $\eta^{(0)} = 0$. Then, we have*

$$\begin{aligned}
\|u^{(\ell)} - u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} &= \pi\left(1-\frac{\gamma}{\rho}\right)^{2\ell}|\varphi_0 + \psi_0\log\rho|^2\{1-(\rho^*)^2\} \\
&\quad + \pi\left\{1-\frac{4\gamma}{\rho(\rho+\rho^{-1})^2}\right\}^{2\ell}\left|\varphi_1 + \frac{\psi_1(\rho-\rho^{-1})}{\rho+\rho^{-1}}\right|^2 \\
&\quad \times\left[\frac{5}{4} - \log\rho^* - (\rho^*)^2\left\{1+\frac{1}{4}(\rho^*)^2\right\}\right] \\
&\quad + \pi\sum_{k=2}^{\infty}\left\{1-\frac{4\gamma}{\rho(\rho^k+\rho^{-k})^2}\right\}^{2\ell}\left|\varphi_k + \frac{\psi_k(\rho^k-\rho^{-k})}{k(\rho^k+\rho^{-k})}\right|^2 \\
&\quad \times\left\{\frac{(\rho^*)^{-2(k-1)}-1}{2(k-1)} + 1 - (\rho^*)^2 + \frac{1-(\rho^*)^{2(k+1)}}{2(k+1)}\right\}.
\end{aligned} \tag{3.5}$$

Proof. Using (3.1) and (3.4), we have

$$\widetilde{u^{(\ell)}}(r,\theta) - \widetilde{u}(r,\theta) = \sum_{k\neq 0} w_k(r^k + r^{-k})e^{ik\theta} - (\varphi_0 + \psi_0\log\rho)\left(1-\frac{\gamma}{\rho}\right)^{\ell},$$

where

$$w_k = -\frac{1}{2}\left\{\varphi_k + \frac{\psi_k(\rho^k-\rho^{-k})}{k(\rho^k+\rho^{-k})}\right\}\left\{1-\frac{4\gamma}{\rho(\rho^k+\rho^{-k})^2}\right\}^{\ell}.$$

Then, we have

$$\begin{aligned}
&\|u^{(\ell)} - u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} \\
&= \int_{B_1\setminus\overline{B_{\rho^*}}} |u^{(\ell)}(x) - u(x)|^2\,dx \\
&= \int_0^{2\pi}\int_{\rho^*}^1 \left|\widetilde{u^{(\ell)}}(r,\theta) - \widetilde{u}(r,\theta)\right|^2 r\,drd\theta \\
&= 2\pi\sum_{k\neq 0}|w_k|^2\int_{\rho^*}^1 (r^k + r^{-k})^2 r\,dr + 2\pi\left|(\varphi_0+\psi_0\log\rho)\left(1-\frac{\gamma}{\rho}\right)^{\ell}\right|^2\int_{\rho^*}^1 r\,dr \\
&= 2\pi\sum_{k\neq 0}\left|\frac{1}{2}\left\{\varphi_k + \frac{\psi_k(\rho^k-\rho^{-k})}{k(\rho^k+\rho^{-k})}\right\}\left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{\ell}\right|^2\int_{\rho^*}^1 (r^k+r^{-k})^2 r\,dr \\
&\quad + \pi\left(1-\frac{\gamma}{\rho}\right)^{2\ell}|\varphi_0+\psi_0\log\rho|^2\{1-(\rho^*)^2\}.
\end{aligned}$$

Since φ and ψ are real-valued functions, we have the formula (3.5). □

Substituting $\gamma = \rho$ into (3.5), we have the following corollary.

Corollary 3.6. *Under the assumptions in Lemma* 3.5*, if* $\gamma = \rho$*, then we have*

$$\begin{aligned}\|u^{(\ell)} - u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} &= \pi\left\{1 - \frac{4}{(\rho+\rho^{-1})^2}\right\}^{2\ell}\left|\varphi_1 + \frac{\psi_1(\rho-\rho^{-1})}{\rho+\rho^{-1}}\right|^2 \\ &\quad\times\left[\frac{5}{4} - \log\rho^* - (\rho^*)^2\left\{1 + \frac{1}{4}(\rho^*)^2\right\}\right] \\ &\quad + \pi\sum_{k=2}^{\infty}\left\{1 - \frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{2\ell}\left|\varphi_k + \frac{\psi_k(\rho^k-\rho^{-k})}{k(\rho^k+\rho^{-k})}\right|^2 \\ &\quad\times\left\{\frac{(\rho^*)^{-2(k-1)} - 1}{2(k-1)} + 1 - (\rho^*)^2 + \frac{1-(\rho^*)^{2(k+1)}}{2(k+1)}\right\}.\end{aligned} \tag{3.6}$$

Remark 3.7. We have

$$1 - \frac{4}{(\rho^k+\rho^{-k})^2} = \left(\frac{\rho^{-k}-\rho^k}{\rho^{-k}+\rho^k}\right)^2 = \left\{\tanh\left(k\log\frac{1}{\rho}\right)\right\}^2 \tag{3.7}$$

and the hyperbolic tangent function is monotone increasing. Moreover, we have

$$0 \le \left\{1 - \frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{\ell} \le 1. \tag{3.8}$$

Finally, we rewrite the conditions imposed in Assumption 1.1.

Lemma 3.8. *If real-valued functions* φ *and* ψ *satisfy* (1.3)*, then we have*

$$\sum_{k=1}^{\infty}\left(|\varphi_k|^2 + |\psi_k|^2\right) \le \widetilde{M_0}^2, \tag{3.9}$$

where $\widetilde{M_0} = \frac{M_0}{2\sqrt{\pi}}$*. Moreover, we have*

$$\sum_{k=1}^{\infty}\left|\frac{\varphi_k}{2} \pm \frac{\psi_k}{2k}\right|^2 \le \frac{1}{2}\widetilde{M_0}^2. \tag{3.10}$$

Proof. Since φ and ψ are real-valued functions, we have

$$\|\varphi\|^2_{L^2(\partial B_1)} = 2\pi\sum_{k\in\mathbb{Z}}|\varphi_k|^2 = 2\pi\left(2\sum_{k=1}^{\infty}|\varphi_k|^2 + |\varphi_0|^2\right)$$

and

$$\|\psi\|^2_{L^2(\partial B_1)} = 2\pi\sum_{k\in\mathbb{Z}}|\psi_k|^2 = 2\pi\left(2\sum_{k=1}^{\infty}|\psi_k|^2 + |\psi_0|^2\right).$$

Then, using (1.3), we have

$$4\pi\sum_{k=1}^{\infty}\left(|\varphi_k|^2 + |\psi_k|^2\right) \le M_0^2.$$

Hence, we obtain (3.9). Moreover, using (3.9), we have

$$\sum_{k=1}^{\infty}\left|\frac{\varphi_k}{2}\pm\frac{\psi_k}{2k}\right|^2\leq 2\sum_{k=1}^{\infty}\left(\left|\frac{\varphi_k}{2}\right|^2+\left|\frac{\psi_k}{2k}\right|^2\right)\leq\frac{1}{2}\sum_{k=1}^{\infty}\left(|\varphi_k|^2+|\psi_k|^2\right)\leq\frac{1}{2}\widetilde{M_0}^2.$$

Therefore, the lemma follows. □

Lemma 3.9. *If Assumption* 1.1 *holds, then we have*

$$\sum_{k=1}^{\infty}\left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2\rho_*^{-2k}\leq\widetilde{M}^2, \tag{3.11}$$

where $\widetilde{M}^2=\frac{M^2}{2\pi\rho_*}+\widetilde{M_0}^2$.

Proof. Since we have

$$\|u\|^2_{L^2(\partial B_{\rho_*})}=2\pi\rho_*\left(2\sum_{k=1}^{\infty}\left|\frac{k\varphi_k+\psi_k}{2k}\rho_*^k+\frac{k\varphi_k-\psi_k}{2k}\rho_*^{-k}\right|^2+|\varphi_0+\psi_0\log\rho_*|^2\right)$$

by (3.1), from (1.4), we remark that we get

$$\sum_{k=1}^{\infty}\left|\frac{k\varphi_k+\psi_k}{2k}\rho_*^k+\frac{k\varphi_k-\psi_k}{2k}\rho_*^{-k}\right|^2\leq\frac{M^2}{4\pi\rho_*}. \tag{3.12}$$

Hence, using (3.10) and (3.12), we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2\rho_*^{-2k}&\leq 2\sum_{k=1}^{\infty}\left|\frac{k\varphi_k+\psi_k}{2k}\rho_*^k+\frac{k\varphi_k-\psi_k}{2k}\rho_*^{-k}\right|^2+2\sum_{k=1}^{\infty}\left|\frac{k\varphi_k+\psi_k}{2k}\rho_*^k\right|^2\\&\leq\frac{M^2}{2\pi\rho_*}+\widetilde{M_0}^2=\widetilde{M}^2.\end{aligned}$$

We complete the proof. □

4. Proof of Theorem 1.2

We would like to estimate $\|u^{(\ell)}-u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})}$, which is given by (3.6). Since

$$\frac{(\rho^*)^{-2(k-1)}-1}{2(k-1)}+1-(\rho^*)^2+\frac{1-(\rho^*)^{2(k+1)}}{2(k+1)}\leq\frac{(\rho^*)^{-2(k-1)}}{2}+1+\frac{1}{6}\leq\frac{5}{3}(\rho^*)^{-2(k-1)}$$

holds for $k\geq 2$, we get

$$\|u^{(\ell)}-u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})}\leq C_1\sum_{k=1}^{\infty}\left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{2\ell}\left|\varphi_k+\frac{\psi_k(\rho^k-\rho^{-k})}{k(\rho^k+\rho^{-k})}\right|^2(\rho^*)^{-2(k-1)},$$

where

$$C_1 := \pi \max\left\{\frac{5}{4} - \log \rho^* - (\rho^*)^2 \left\{1 + \frac{1}{4}(\rho^*)^2\right\}, \frac{5}{3}\right\}.$$

We now remark that we have

$$\begin{aligned}\left|\varphi_k + \frac{\psi_k(\rho^k - \rho^{-k})}{k(\rho^k + \rho^{-k})}\right|^2 &= 4\left|\left(\frac{\varphi_k}{2} + \frac{\psi_k}{2k}\right)\rho^k + \left(\frac{\varphi_k}{2} - \frac{\psi_k}{2k}\right)\rho^{-k}\right|^2 \frac{1}{(\rho^k + \rho^{-k})^2} \\ &\le 8\left(\left|\frac{\varphi_k}{2} + \frac{\psi_k}{2k}\right|^2 \rho^{2k} + \left|\frac{\varphi_k}{2} - \frac{\psi_k}{2k}\right|^2 \rho^{-2k}\right)\frac{1}{(\rho^k + \rho^{-k})^2}.\end{aligned}$$

Moreover, since we have

$$\frac{(\rho^*)^{-2(k-1)}}{(\rho^k + \rho^{-k})^2} \le (\rho^*)^2 \left(\frac{\rho}{\rho^*}\right)^{2k} \le 1, \quad \frac{\rho^{-2k}}{(\rho^k + \rho^{-k})^2} \le 1,$$

we can estimate

$$\|u^{(\ell)} - u\|^2_{L^2(B_1 \setminus \overline{B_{\rho^*}})} \le 8C_1(I_1 + I_2), \tag{4.1}$$

where

$$I_1 := \sum_{k=1}^{\infty}\left\{1 - \frac{4}{(\rho^k + \rho^{-k})^2}\right\}^{2\ell} (\rho^*)^{-2(k-1)} \left|\frac{\varphi_k}{2} - \frac{\psi_k}{2k}\right|^2$$

and

$$I_2 := \sum_{k=1}^{\infty}\left\{1 - \frac{4}{(\rho^k + \rho^{-k})^2}\right\}^{2\ell} \left|\frac{\varphi_k}{2} + \frac{\psi_k}{2k}\right|^2 \rho^{2k}.$$

Our proof is based on the idea in [9].

We first evaluate I_1. Let $0 < \lambda < \widetilde{M}$ be given. Let N_1 be the minimum integer satisfying $(\rho_*/\rho^*)^{N_1}\widetilde{M} \le \lambda$, namely

$$N_1 - 1 < \frac{\log(\lambda/\widetilde{M})}{\log(\rho_*/\rho^*)} \le N_1. \tag{4.2}$$

We now divide I_1 into two parts:

$$\begin{aligned}I_1 &= \sum_{k=1}^{N_1-1}\left\{1 - \frac{4}{(\rho^k + \rho^{-k})^2}\right\}^{2\ell} (\rho^*)^{-2(k-1)} \left|\frac{\varphi_k}{2} - \frac{\psi_k}{2k}\right|^2 \\ &\quad + \sum_{k=N_1}^{\infty}\left\{1 - \frac{4}{(\rho^k + \rho^{-k})^2}\right\}^{2\ell} (\rho^*)^{-2(k-1)} \left|\frac{\varphi_k}{2} - \frac{\psi_k}{2k}\right|^2 \\ &=: I_{11} + I_{12}.\end{aligned} \tag{4.3}$$

We can estimate

$$\begin{aligned}I_{12} &\le \sum_{k=N_1}^{\infty}\left|\frac{\varphi_k}{2} - \frac{\psi_k}{2k}\right|^2 (\rho^*)^{-2k} = \sum_{k=N_1}^{\infty}\left|\frac{\varphi_k}{2} - \frac{\psi_k}{2k}\right|^2 \rho_*^{-2k}\left(\frac{\rho_*}{\rho^*}\right)^{2k} \\ &\le \left(\frac{\rho_*}{\rho^*}\right)^{2N_1} \sum_{k=N_1}^{\infty}\left|\frac{\varphi_k}{2} - \frac{\psi_k}{2k}\right|^2 \rho_*^{-2k} \le \left(\frac{\rho_*}{\rho^*}\right)^{2N_1} \widetilde{M}^2 \le \lambda^2\end{aligned} \tag{4.4}$$

by (3.8), (3.11) and (4.2). On the other hand, using (3.7), (3.10) and (4.2), we have

$$
\begin{aligned}
I_{11} &= \sum_{k=1}^{N_1-1} \left\{\tanh\left(k\log\frac{1}{\rho}\right)\right\}^{4\ell} (\rho^*)^{-2(k-1)} \left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2 \\
&\le \left\{\tanh\left((N_1-1)\log\frac{1}{\rho}\right)\right\}^{4\ell} (\rho^*)^{-2(N_1-1)} \sum_{k=1}^{N_1-1} \left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2 \\
&\le \frac{1}{2}\widetilde{M_0}^2 (\rho^*)^{-2(N_1-1)} \left\{\tanh\left((N_1-1)\log\frac{1}{\rho}\right)\right\}^{4\ell} \\
&< \frac{1}{2}\widetilde{M_0}^2 \left(\frac{1}{\rho^*}\right)^{2\frac{\log(\lambda/\widetilde{M})}{\log(\rho_*/\rho^*)}} \left[\tanh\left(\frac{\log(\lambda/\widetilde{M})}{\log(\rho_*/\rho^*)}\log\frac{1}{\rho}\right)\right]^{4\ell} \\
&= C_2\left(\frac{1}{\rho^*}\right)^{-\frac{2\log\lambda}{\log(\rho^*/\rho_*)}} \left\{\tanh\left(b\log\frac{\widetilde{M}}{\lambda}\right)\right\}^{4\ell} \\
&= C_2\lambda^{-\mu}\{\tanh(a-b\log\lambda)\}^{4\ell} = C_2\lambda^{-\mu}\left(\frac{e^a\lambda^{-b}-e^{-a}\lambda^b}{e^a\lambda^{-b}+e^{-a}\lambda^b}\right)^{4\ell},
\end{aligned} \tag{4.5}
$$

where

$$
C_2 := \frac{1}{2}\widetilde{M_0}^2\left(\frac{1}{\rho^*}\right)^{\frac{2\log\widetilde{M}}{\log(\rho^*/\rho_*)}}, \quad \mu := \frac{2\log(1/\rho^*)}{\log(\rho^*/\rho_*)} > 0, \quad b := \frac{\log(1/\rho)}{\log(\rho^*/\rho_*)} > 0, \quad a := b\log\widetilde{M}.
$$

Combining (4.3), (4.4) and (4.5), we obtain

$$
I_1 \le F(\lambda), \tag{4.6}
$$

where

$$
F(\lambda) := C_2\lambda^{-\mu}\left(\frac{e^a\lambda^{-b}-e^{-a}\lambda^b}{e^a\lambda^{-b}+e^{-a}\lambda^b}\right)^{4\ell} + \lambda^2. \tag{4.7}
$$

Now, let us choose λ such that $F(\lambda)$ is as small as possible. We choose λ_0 satisfying

$$
\left(\frac{e^a\lambda_0^{-b}-e^{-a}\lambda_0^b}{e^a\lambda_0^{-b}+e^{-a}\lambda_0^b}\right)^{4\ell} = \ell^{-\omega_1}, \tag{4.8}
$$

that is, we define

$$
\lambda_0 = \left\{\frac{e^{2a}\left(1-\ell^{-\frac{\omega_1}{4\ell}}\right)}{1+\ell^{-\frac{\omega_1}{4\ell}}}\right\}^{1/(2b)} = \widetilde{M}\left(\frac{1-J_1}{1+J_1}\right)^{1/(2b)}, \tag{4.9}
$$

where we put $J_1 := \ell^{-\frac{\omega_1}{4\ell}}$ for short and define $\omega_1 > 0$ later. Moreover, we remark that J_1 tends to 1 as $\ell \to \infty$ and $J_1 < 1$ for large enough ℓ. We shall see the order of $1 - J_1$. Since

$$
\frac{1-J_1}{\frac{\log\ell}{\ell}} = \frac{1-\ell^{-\frac{\omega_1}{4\ell}}}{\frac{\log\ell}{\ell}} \to \frac{\omega_1}{4} \quad \text{as } \ell\to\infty,
$$

we have

$$J_1 = 1 - \frac{\omega_1}{4}\frac{\log \ell}{\ell} + o\left(\frac{\log \ell}{\ell}\right) \quad \text{as } \ell \to \infty.$$

Then, we get

$$\frac{1-J_1}{1+J_1} = \frac{1}{2+o(1)}\left\{\frac{\omega_1}{4}\frac{\log \ell}{\ell} + o\left(\frac{\log \ell}{\ell}\right)\right\} = \frac{\omega_1}{8}\frac{\log \ell}{\ell}\{1+o(1)\}. \tag{4.10}$$

Therefore, using (4.9) and (4.10), we obtain

$$\lambda_0 = \widetilde{M}\left(\frac{\omega_1}{8}\right)^{1/(2b)}\left(\frac{\log \ell}{\ell}\right)^{1/(2b)}\{1+o(1)\}^{1/(2b)}. \tag{4.11}$$

Thus, using (4.7), (4.8) and (4.11), we get

$$\begin{aligned} F(\lambda_0) &= C_2\lambda_0^{-\mu}\ell^{-\omega_1} + \lambda_0^2 \\ &= C_2\widetilde{M}^{-\mu}\left(\frac{\omega_1}{8}\right)^{-\mu/(2b)}\left(\frac{\log \ell}{\ell}\right)^{-\mu/(2b)}\{1+o(1)\}^{-\mu/(2b)}\ell^{-\omega_1} \\ &\quad + \widetilde{M}^2\left(\frac{\omega_1}{8}\right)^{1/b}\left(\frac{\log \ell}{\ell}\right)^{1/b}\{1+o(1)\}^{1/b} \\ &= C_2\widetilde{M}^{-\mu}\left(\frac{\omega_1}{8}\right)^{-\mu/(2b)}\ell^{-\omega_1+\mu/(2b)}(\log \ell)^{-\mu/(2b)}\{1+o(1)\} \\ &\quad + \widetilde{M}^2\left(\frac{\omega_1}{8}\right)^{1/b}\left(\frac{\log \ell}{\ell}\right)^{1/b}\{1+o(1)\}. \end{aligned} \tag{4.12}$$

Since we can choose sufficiently large ω_1, by using (4.6) and (4.12), we obtain

$$I_1 \le F(\lambda_0) \le C_3\left(\frac{\log \ell}{\ell}\right)^{1/b} \tag{4.13}$$

for $\ell \ge 2$, where C_3 depends only on M, M_0, ρ^* and ρ_*. We here remark that C_3 can be chosen independently of ρ due to the observation

$$0 < \frac{\log(1/\rho^*)}{\log(\rho^*/\rho_*)} < b = \frac{\log(1/\rho)}{\log(\rho^*/\rho_*)} < \frac{\log(1/\rho_*)}{\log(\rho^*/\rho_*)}.$$

We next evaluate I_2. Let $0 < \kappa < 1$ be given. Let N_2 be the minimum integer satisfying $\rho^{N_2} \le \kappa$, namely

$$N_2 - 1 < \frac{\log \kappa}{\log \rho} \le N_2. \tag{4.14}$$

We now divide I_2 into two parts:

$$\begin{aligned} I_2 &= \sum_{k=1}^{N_2-1}\left\{1 - \frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{2\ell}\left|\frac{\varphi_k}{2} + \frac{\psi_k}{2k}\right|^2\rho^{2k} \\ &\quad + \sum_{k=N_2}^{\infty}\left\{1 - \frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{2\ell}\left|\frac{\varphi_k}{2} + \frac{\psi_k}{2k}\right|^2\rho^{2k} \\ &=: I_{21} + I_{22}. \end{aligned} \tag{4.15}$$

Then, using (3.8), (3.10) and (4.14), we have

$$I_{22} \le \rho^{2N_2} \sum_{k=N_2}^{\infty} \left| \frac{\varphi_k}{2} + \frac{\psi_k}{2k} \right|^2 \le \frac{1}{2} \widetilde{M_0}^2 \rho^{2N_2} \le \frac{1}{2} \widetilde{M_0}^2 \kappa^2. \tag{4.16}$$

On the other hand, by (3.7), (3.10) and (4.14), we obtain

$$\begin{aligned} I_{21} &= \sum_{k=1}^{N_2-1} \left\{ \tanh \left(k \log \frac{1}{\rho} \right) \right\}^{4\ell} \left| \frac{\varphi_k}{2} + \frac{\psi_k}{2k} \right|^2 \rho^{2k} \\ &\le \left\{ \tanh \left((N_2 - 1) \log \frac{1}{\rho} \right) \right\}^{4\ell} \sum_{k=1}^{N_2-1} \left| \frac{\varphi_k}{2} + \frac{\psi_k}{2k} \right|^2 \\ &\le \frac{1}{2} \widetilde{M_0}^2 \left\{ \tanh \left((N_2 - 1) \log \frac{1}{\rho} \right) \right\}^{4\ell} \\ &< \frac{1}{2} \widetilde{M_0}^2 \{\tanh(-\log \kappa)\}^{4\ell} = \frac{1}{2} \widetilde{M_0}^2 \left(\frac{\kappa^{-1} - \kappa}{\kappa^{-1} + \kappa} \right)^{4\ell}. \end{aligned} \tag{4.17}$$

Combining (4.15), (4.16) and (4.17), we obtain

$$I_2 \le G(\kappa), \tag{4.18}$$

where

$$G(\kappa) := \frac{1}{2} \widetilde{M_0}^2 \left\{ \left(\frac{\kappa^{-1} - \kappa}{\kappa^{-1} + \kappa} \right)^{4\ell} + \kappa^2 \right\}. \tag{4.19}$$

Now, let us choose κ such that $G(\kappa)$ is as small as possible. We choose κ_0 satisfying

$$\left(\frac{\kappa_0^{-1} - \kappa_0}{\kappa_0^{-1} + \kappa_0} \right)^{4\ell} = \ell^{-\omega_2}, \tag{4.20}$$

that is, we define

$$\kappa_0 = \left(\frac{1 - J_2}{1 + J_2} \right)^{1/2}, \tag{4.21}$$

where we put $J_2 := \ell^{-\frac{\omega_2}{4\ell}}$ for short and define $\omega_2 > 0$ later. Then, we remark that J_2 behaves like J_1. Therefore, from (4.21), we have

$$\kappa_0 = \left(\frac{\omega_2}{8} \right)^{1/2} \left(\frac{\log \ell}{\ell} \right)^{1/2} \{1 + o(1)\}. \tag{4.22}$$

Thus, using (4.19), (4.20) and (4.22), we get

$$\begin{aligned} G(\kappa_0) &= \frac{1}{2} \widetilde{M_0}^2 (\ell^{-\omega_2} + \kappa_0^2) \\ &= \frac{1}{2} \widetilde{M_0}^2 \left[\ell^{-\omega_2} + \frac{\omega_2}{8} \frac{\log \ell}{\ell} \{1 + o(1)\} \right]. \end{aligned} \tag{4.23}$$

Since we can choose sufficiently large ω_2, by using (4.18) and (4.23), we obtain

$$I_2 \le G(\kappa_0) \le C_4 \frac{\log \ell}{\ell} \tag{4.24}$$

for $\ell \ge 2$, where C_4 depends only on M_0.

In the case where $\rho_* \le \rho \le \rho_*/\rho^*$, we remark that we have $0 \le 1/b \le 1$. Hence, using (4.1), (4.13) and (4.24), we obtain

$$\|u^{(\ell)} - u\|^2_{L^2(B_1 \setminus \overline{B_{\rho^*}})} \le C \left(\frac{\log \ell}{\ell} \right)^{1/b}$$

for $\ell \ge 2$, where $C > 0$ depends only on M, M_0, ρ_* and ρ^*. On the other hand, in the case where $\rho_*/\rho^* < \rho \le \rho^*$, we remark that we have $1/b > 1$. Hence, using (4.1), (4.13) and (4.24), we obtain

$$\|u^{(\ell)} - u\|^2_{L^2(B_1 \setminus \overline{B_{\rho^*}})} \le C \frac{\log \ell}{\ell}$$

for $\ell \ge 2$, where $C > 0$ depends only on M, M_0, ρ_* and ρ^*.

5. Proof of Theorem 1.4

As the Cauchy data, we especially choose

$$\widetilde{\varphi}(\theta) = \varphi_s(e^{is\theta} + e^{-is\theta}), \quad \widetilde{\psi}(\theta) = 0,$$

where $s \ne 0, \pm 1$ and φ_s is a positive number. Then, we have

$$\|\varphi\|^2_{L^2(\partial B_1)} = 4\pi |\varphi_s|^2. \tag{5.1}$$

Moreover, using (3.1), we obtain the explicit solution formula to the Cauchy problem (1.2). It is given by

$$\widetilde{u}(r,\theta) = \frac{\varphi_s}{2}(r^s + r^{-s})(e^{is\theta} + e^{-is\theta}) \tag{5.2}$$

and we have

$$\|u\|^2_{L^2(\partial B_{\rho_*})} = \pi |\varphi_s|^2 \rho_* (\rho_*^s + \rho_*^{-s})^2. \tag{5.3}$$

Furthermore, an approximate solution $u^{(\ell)}$ obtained by the iteration procedure with $\eta^{(0)} = 0$ and $\gamma = \rho$ is given by

$$\widetilde{u^{(\ell)}}(r,\theta) = -\frac{\varphi_s}{2} \left[\left\{ 1 - \frac{4}{(\rho^s + \rho^{-s})^2} \right\}^{\ell} - 1 \right] (r^s + r^{-s})(e^{is\theta} + e^{-is\theta}). \tag{5.4}$$

We now choose s such that

$$s \geq \max\left\{2, \frac{\log \frac{2M}{M_0\sqrt{\rho_*}}}{\log \frac{1}{\rho_*}}\right\}. \tag{5.5}$$

Then, we remark that we have

$$\frac{M^2}{\pi\rho_*(\rho_*^s + \rho_*^{-s})^2} \leq \frac{M_0^2}{4\pi} \tag{5.6}$$

by using (5.5). We here put

$$\varphi_s := \frac{M}{\sqrt{\pi\rho_*}(\rho_*^s + \rho_*^{-s})}. \tag{5.7}$$

Then, using (5.1), (5.3), (5.6) and (5.7), we know that conditions (1.3) and (1.4) are satisfied.

On the other hand, since we have

$$\varphi_s \geq \frac{M}{2\sqrt{\pi}}\rho_*^{s-1/2},$$

using (5.2) and (5.4), we obtain

$$\begin{aligned}
\|u^{(\ell)} - u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} &= \pi\varphi_s^2\left(\frac{1-\rho^{2s}}{1+\rho^{2s}}\right)^{4\ell}\left\{\frac{(\rho^*)^{-2(s-1)}-1}{2(s-1)} + 1 - (\rho^*)^2 + \frac{1-(\rho^*)^{2(s+1)}}{2(s+1)}\right\} \\
&> \pi\varphi_s^2\left(\frac{1-\rho^{2s}}{1+\rho^{2s}}\right)^{4\ell}\frac{(\rho^*)^{-2(s-1)}-1}{2(s-1)} \\
&\geq \frac{M^2}{4}\rho_*^{2s-1}\left(\frac{1-\rho^{2s}}{1+\rho^{2s}}\right)^{4\ell}\frac{(\rho^*)^{-2(s-1)}\{1-(\rho^*)^2\}}{2s} \\
&= \frac{M^2(\rho^*)^2}{8\rho_*}\{1-(\rho^*)^2\}\frac{1}{s}\left(\frac{\rho_*}{\rho^*}\right)^{2s}\left(\frac{1-\rho^{2s}}{1+\rho^{2s}}\right)^{4\ell}.
\end{aligned}$$

Since $\frac{M^2(\rho^*)^2}{8\rho_*}\{1-(\rho^*)^2\}$ is independent of s, in order to see the optimality, it is sufficient to show the following lemma.

Lemma 5.1. *There exists no* $\widehat{C} > 0$ *such that*

$$\frac{1}{s}\left(\frac{\rho_*}{\rho^*}\right)^{2s}\left(\frac{1-\rho^{2s}}{1+\rho^{2s}}\right)^{4\ell} \leq \widehat{C}\left(\frac{\log \ell}{\ell}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}+2\varepsilon} \tag{5.8}$$

for any s *satisfying* (5.5), $\ell \geq 2$ *and* $\rho \in [\rho_*, \rho^*]$.

Proof. By taking the logarithm of both sides, (5.8) is equivalent to

$$X(\ell, s) \leq \log\widehat{C},$$

where

$$X(\ell,s) := -\log s + 2s\log\frac{\rho_*}{\rho^*} + 4\ell\big\{\log(1-\rho^{2s}) - \log(1+\rho^{2s})\big\} - \left(\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)} + 2\varepsilon\right)\log\frac{\log\ell}{\ell}.$$

We now show that

$$X(\ell,\widehat{c}\log\ell) \to \infty \quad \text{as } \ell\to\infty \tag{5.9}$$

holds by taking $\widehat{c}$ such that

$$\frac{1}{2\log(1/\rho)} < \widehat{c} < \frac{1}{2\log(1/\rho)} + \frac{\varepsilon}{\log(\rho^*/\rho_*)} \tag{5.10}$$

for any $\varepsilon > 0$. Since we have

$$\log(1-t) - \log(1+t) \geq -3t \quad \text{for } 0\leq t\leq \frac{1}{\sqrt{3}},$$

we get

$$\begin{aligned} X(\ell,\widehat{c}\log\ell) &\geq -\log(\widehat{c}\log\ell) + 2\widehat{c}\log\ell\log\frac{\rho_*}{\rho^*} - 12\ell\rho^{2\widehat{c}\log\ell} \\ &\quad - \left\{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)} + 2\varepsilon\right\}\log\frac{\log\ell}{\ell} \\ &= -\log\widehat{c} + \log\frac{\ell^{2\left\{\widehat{c}\log\frac{\rho_*}{\rho^*} + \frac{\log(\rho^*/\rho_*)}{2\log(1/\rho)} + \varepsilon\right\}}}{(\log\ell)^{1+\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}+2\varepsilon}} - 12\ell\rho^{2\widehat{c}\log\ell} \\ &= -\log\widehat{c} + \log\frac{\ell^{2\left\{\widehat{c}\log\frac{\rho_*}{\rho^*} + \frac{\log(\rho^*/\rho_*)}{2\log(1/\rho)} + \varepsilon\right\}}}{(\log\ell)^{1+\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}+2\varepsilon}} - 12\ell^{1+2\widehat{c}\log\rho} \end{aligned} \tag{5.11}$$

for $0\leq \rho^{2\widehat{c}\log\ell} \leq \frac{1}{\sqrt{3}}$, that is, $\ell \geq 3^{\frac{1}{4\widehat{c}\log(1/\rho)}}$. Since

$$1 + 2\widehat{c}\log\rho < 0$$

holds by (5.10), the third term on the right-hand side of (5.11) tends to zero as $\ell\to\infty$. On the other hand, the second term on the right-hand side of (5.11) diverges to positive infinity as $\ell\to\infty$ since

$$\widehat{c}\log\frac{\rho_*}{\rho^*} + \frac{\log(\rho^*/\rho_*)}{2\log(1/\rho)} + \varepsilon > 0$$

holds by (5.10). Hence, we obtain (5.9).

Strictly speaking, both ℓ and s are natural numbers now. Then, let us take a natural number m such that

$$m > \left(\frac{\rho^*}{\rho_*}\right)^{1/\varepsilon}.$$

Let $\ell = m^q$, where q is a natural number. We remark that (5.10) is equivalent to

$$\frac{\log m}{2\log(1/\rho)} < \widehat{c}\log m < \frac{\log m}{2\log(1/\rho)} + \frac{\varepsilon \log m}{\log(\rho^*/\rho_*)}, \tag{5.12}$$

where $\frac{\varepsilon \log m}{\log(\rho^*/\rho_*)} > 1$ holds. Hence, there exists $\widehat{c}$ satisfying (5.12) such that $\widehat{c}\log m$ is a natural number. Then, both $\ell = m^q$ and $s = \widehat{c}\log \ell = \widehat{c}q\log m$ are natural numbers, and

$$X(m^q, \widehat{c}q\log m) \to \infty \quad \text{as } q \to \infty.$$

Therefore, the lemma follows. □

6. Proof of Theorem 1.6

We first remark that the condition (1.6) is equivalent to

$$2\pi|\varphi_0^\delta - \varphi_0|^2 + 4\pi\sum_{k=1}^{\infty}|\varphi_k^\delta - \varphi_k|^2 \le \delta^2 \tag{6.1}$$

and

$$2\pi|\psi_0^\delta - \psi_0|^2 + 4\pi\sum_{k=1}^{\infty}|\psi_k^\delta - \psi_k|^2 \le \delta^2 \tag{6.2}$$

when we express

$$\varphi^\delta(\cos\theta, \sin\theta) = \sum_{k\in\mathbb{Z}} \varphi_k^\delta e^{ik\theta}, \quad \psi^\delta(\cos\theta, \sin\theta) = \sum_{k\in\mathbb{Z}} \psi_k^\delta e^{ik\theta}.$$

Moreover, using (3.4), an approximate solution $u^{(\ell),\delta}$ obtained by the iteration procedure with $\eta^{(0)} = 0$ and $\gamma = \rho$ for Cauchy data φ^δ and ψ^δ is given by

$$\begin{aligned} &\widetilde{u^{(\ell),\delta}}(r,\theta) \\ &= -\frac{1}{2}\sum_{k\neq 0}\left\{\varphi_k^\delta + \frac{\psi_k^\delta(\rho^k - \rho^{-k})}{k(\rho^k + \rho^{-k})}\right\}\left[\left\{1 - \frac{4}{(\rho^k + \rho^{-k})^2}\right\}^\ell - 1\right](r^k + r^{-k})e^{ik\theta} \\ &\quad + \sum_{k\neq 0}\frac{\psi_k^\delta}{k(\rho^k + \rho^{-k})}(\rho^{-k}r^k - \rho^k r^{-k})e^{ik\theta} + \varphi_0^\delta + \psi_0^\delta \log r. \end{aligned} \tag{6.3}$$

Furthermore, we have

$$\|u^{(\ell),\delta} - u\|_{L^2(B_1\setminus\overline{B_{\rho^*}})} \le \|u^{(\ell),\delta} - u^{(\ell)}\|_{L^2(B_1\setminus\overline{B_{\rho^*}})} + \|u^{(\ell)} - u\|_{L^2(B_1\setminus\overline{B_{\rho^*}})}. \tag{6.4}$$

We now evaluate the first term on the right-hand side of (6.4).

Lemma 6.1. *Under the assumptions in Theorem* 1.6, *we have*

$$\|u^{(\ell),\delta} - u^{(\ell)}\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} \le \widetilde{C_0}\ell\delta^2, \tag{6.5}$$

where a positive constant $\widetilde{C_0}$ *depends only on* ρ^*.

Proof. Using (3.4) and (6.3), we have

$$\widetilde{u^{(\ell),\delta}}(r,\theta) - \widetilde{u^{(\ell)}}(r,\theta) = \sum_{k\in\mathbb{Z}} \widehat{w_k}(r)e^{ik\theta}, \tag{6.6}$$

where

$$\widehat{w_k}(r) := \begin{cases} -\frac{1}{2}\left\{\varphi_k^\delta - \varphi_k + \frac{(\psi_k^\delta-\psi_k)(\rho^k-\rho^{-k})}{k(\rho^k+\rho^{-k})}\right\}\left[\left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^\ell - 1\right](r^k+r^{-k}) & \\ \quad + \frac{\psi_k^\delta-\psi_k}{k(\rho^k+\rho^{-k})}(\rho^{-k}r^k - \rho^k r^{-k}) & \text{if } k\neq 0, \\ \varphi_0^\delta - \varphi_0 + (\psi_0^\delta - \psi_0)\log r & \text{if } k = 0. \end{cases}$$

Then, using (6.6), we have

$$\|u^{(\ell),\delta} - u^{(\ell)}\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} = 2\pi\sum_{k\neq 0}\int_{\rho^*}^1 |\widehat{w_k}(r)|^2 r\,dr + 2\pi\int_{\rho^*}^1 |\widehat{w_0}(r)|^2 r\,dr. \tag{6.7}$$

We first evaluate $|\widehat{w_k}(r)|^2$ for $k\neq 0$. We have

$$\begin{aligned} &|\widehat{w_k}(r)|^2 \\ &\le \frac{1}{2}\left\{|\varphi_k^\delta - \varphi_k| + \frac{|\psi_k^\delta - \psi_k|(\rho^k-\rho^{-k})}{k(\rho^k+\rho^{-k})}\right\}^2\left[\left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^\ell - 1\right]^2(r^k+r^{-k})^2 \\ &\quad + 2\frac{|\psi_k^\delta-\psi_k|^2}{k^2(\rho^k+\rho^{-k})^2}(\rho^{-k}r^k - \rho^k r^{-k})^2. \end{aligned} \tag{6.8}$$

Here, since

$$1-(1-s)^\ell \le \ell s$$

holds for $0\le s\le 1$, by (3.8), we remark that we have

$$1 - \left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^\ell \le \frac{4}{(\rho^k+\rho^{-k})^2}\ell. \tag{6.9}$$

Moreover, using (3.8) and (6.9), we have

$$\begin{aligned} 1 - \left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^\ell &\le \sqrt{1 - \left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^\ell} \\ &\le \sqrt{\frac{4}{(\rho^k+\rho^{-k})^2}\ell} = \frac{2}{\rho^k+\rho^{-k}}\sqrt{\ell}. \end{aligned} \tag{6.10}$$

Combining (6.8) and (6.10), we obtain

$$|\widehat{w_k}(r)|^2 \le 2\left\{|\varphi_k^\delta - \varphi_k| + \frac{|\psi_k^\delta - \psi_k|(\rho^k - \rho^{-k})}{k(\rho^k + \rho^{-k})}\right\}^2 \frac{\ell(r^k + r^{-k})^2}{(\rho^k + \rho^{-k})^2} + 2\frac{|\psi_k^\delta - \psi_k|^2}{k^2(\rho^k + \rho^{-k})^2}(\rho^{-k}r^k - \rho^k r^{-k})^2 \tag{6.11}$$

for $k \neq 0$. On the other hand, we get

$$|\widehat{w_0}(r)|^2 \le 2\big\{|\varphi_0^\delta - \varphi_0|^2 + |\psi_0^\delta - \psi_0|^2(\log r)^2\big\}. \tag{6.12}$$

Then, using (6.7), (6.11) and (6.12), since φ, ψ, φ^δ and ψ^δ are real-valued functions, we obtain

$$\|u^{(\ell),\delta} - u^{(\ell)}\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} \le Z_1 + Z_2 + Z_3, \tag{6.13}$$

where

$$\begin{aligned}
Z_1 := 8\pi\ell \sum_{k=2}^{\infty} &\left\{|\varphi_k^\delta - \varphi_k| + \frac{|\psi_k^\delta - \psi_k|(\rho^k - \rho^{-k})}{k(\rho^k + \rho^{-k})}\right\}^2 \frac{1}{(\rho^k + \rho^{-k})^2} \\
&\times \left\{\frac{(\rho^*)^{-2(k-1)} - 1}{2(k-1)} + 1 - (\rho^*)^2 + \frac{1 - (\rho^*)^{2(k+1)}}{2(k+1)}\right\} \\
&+ 8\pi\ell \left\{|\varphi_1^\delta - \varphi_1| + \frac{|\psi_1^\delta - \psi_1|(\rho - \rho^{-1})}{\rho + \rho^{-1}}\right\}^2 \frac{1}{(\rho + \rho^{-1})^2} \\
&\times \left[\frac{5}{4} - \log\rho^* - (\rho^*)^2\left\{1 + \frac{1}{4}(\rho^*)^2\right\}\right], \\
Z_2 := 8\pi \sum_{k=2}^{\infty} &\frac{|\psi_k^\delta - \psi_k|^2}{k^2(\rho^k + \rho^{-k})^2}\left\{\frac{(\rho^*)^{-2(k-1)} - 1}{2(k-1)}\rho^{2k} - 1 + (\rho^*)^2 + \frac{1 - (\rho^*)^{2(k+1)}}{2(k+1)}\rho^{-2k}\right\} \\
&+ 8\pi\frac{|\psi_1^\delta - \psi_1|^2}{(\rho + \rho^{-1})^2}\left[\frac{1}{4\rho^2}\big\{1 - (\rho^*)^4\big\} - 1 + (\rho^*)^2 - \rho^2\log\rho^*\right]
\end{aligned}$$

and

$$Z_3 := 2\pi|\varphi_0^\delta - \varphi_0|^2\big\{1 - (\rho^*)^2\big\} + 2\pi|\psi_0^\delta - \psi_0|^2\left[\frac{1}{2} - (\rho^*)^2\left\{(\log\rho^*)^2 - \log\rho^* + \frac{1}{2}\right\}\right].$$

We first evaluate Z_1. Since

$$\begin{aligned}
&\frac{(\rho^*)^{-2(k-1)} - 1}{2(k-1)} + 1 - (\rho^*)^2 + \frac{1 - (\rho^*)^{2(k+1)}}{2(k+1)} \\
&\le \frac{(\rho^*)^{-2(k-1)}}{2} + 1 + \frac{1}{6} \le \frac{5}{3}(\rho^*)^{-2(k-1)}
\end{aligned}$$

holds for $k \geq 2$, we have

$$
\begin{aligned}
Z_1 &\leq \widetilde{C_{01}} \ell \sum_{k=1}^{\infty} \left\{ \varphi_k^\delta - \varphi_k + \frac{(\psi_k^\delta - \psi_k)(\rho^k - \rho^{-k})}{k(\rho^k + \rho^{-k})} \right\}^2 \frac{(\rho^*)^{-2(k-1)}}{(\rho^k + \rho^{-k})^2} \\
&\leq 2\widetilde{C_{01}} \ell \sum_{k=1}^{\infty} \left(|\varphi_k^\delta - \varphi_k|^2 + |\psi_k^\delta - \psi_k|^2 \right) \frac{(\rho^*)^2}{(\rho^*)^{2k}(\rho^k + \rho^{-k})^2} \\
&\leq 2\widetilde{C_{01}} \ell \sum_{k=1}^{\infty} \left(|\varphi_k^\delta - \varphi_k|^2 + |\psi_k^\delta - \psi_k|^2 \right) \\
&\leq 2\widetilde{C_{01}} \ell \left(\frac{\delta^2}{4\pi} + \frac{\delta^2}{4\pi} \right) = \frac{\widetilde{C_{01}}}{\pi} \ell \delta^2
\end{aligned} \tag{6.14}
$$

by (6.1) and (6.2), where

$$
\widetilde{C_{01}} := 8\pi \max \left\{ \frac{5}{4} - \log \rho^* - (\rho^*)^2 \left\{ 1 + \frac{1}{4} (\rho^*)^2 \right\}, \frac{5}{3} \right\}.
$$

We next evaluate Z_2. Since

$$
\begin{aligned}
&\frac{(\rho^*)^{-2(k-1)} - 1}{2(k-1)} \rho^{2k} - 1 + (\rho^*)^2 + \frac{1 - (\rho^*)^{2(k+1)}}{2(k+1)} \rho^{-2k} \\
&\leq \frac{1}{2(k-1)} (\rho^*)^2 \left(\frac{\rho}{\rho^*} \right)^{2k} + (\rho^*)^2 + \frac{1}{2(k+1)} \rho^{-2k} \\
&\leq \frac{1}{2} (\rho^*)^2 + (\rho^*)^2 + \frac{1}{6} \rho^{-2k} \leq \frac{5}{3} \rho^{-2k}
\end{aligned}
$$

holds for $k \geq 2$ and we have

$$
\begin{aligned}
\frac{1}{4\rho^2} \{1 - (\rho^*)^4\} - 1 + (\rho^*)^2 - \rho^2 \log \rho^* &= \frac{1}{\rho^2} \left[\frac{1 - (\rho^*)^4}{4} - \rho^2 \{1 - (\rho^*)^2\} + \rho^4 \log \frac{1}{\rho^*} \right] \\
&\leq \frac{1}{\rho^2} \left(\frac{1}{4} + \log \frac{1}{\rho^*} \right),
\end{aligned}
$$

we get

$$
\begin{aligned}
Z_2 &\leq \widetilde{C_{02}} \sum_{k=1}^{\infty} \frac{|\psi_k^\delta - \psi_k|^2}{k^2(\rho^k + \rho^{-k})^2} \rho^{-2k} \leq \widetilde{C_{02}} \sum_{k=1}^{\infty} |\psi_k^\delta - \psi_k|^2 \frac{1}{(\rho^{2k} + 1)^2} \\
&\leq \widetilde{C_{02}} \sum_{k=1}^{\infty} |\psi_k^\delta - \psi_k|^2 \leq \frac{\widetilde{C_{02}}}{4\pi} \delta^2
\end{aligned} \tag{6.15}
$$

by (6.2), where

$$
\widetilde{C_{02}} := 8\pi \max \left\{ \frac{1}{4} + \log \frac{1}{\rho^*}, \frac{5}{3} \right\}.
$$

Finally, we evaluate Z_3. Since we have

$$
(\log \rho^*)^2 - \log \rho^* + \frac{1}{2} = \left(\log \rho^* - \frac{1}{2} \right)^2 + \frac{1}{4} \geq \frac{1}{4},
$$

we get

$$Z_3 \le 2\pi|\varphi_0^\delta - \varphi_0|^2 + \pi|\psi_0^\delta - \psi_0|^2 \le \delta^2 + \frac{1}{2}\delta^2 = \frac{3}{2}\delta^2 \tag{6.16}$$

by (6.1) and (6.2). Therefore, combining (6.13), (6.14), (6.15) and (6.16), we obtain

$$\|u^{(\ell),\delta} - u^{(\ell)}\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} \le \frac{\widetilde{C_{01}}}{\pi}\ell\delta^2 + \frac{\widetilde{C_{02}}}{4\pi}\delta^2 + \frac{3}{2}\delta^2 \le \widetilde{C_0}\ell\delta^2$$

for $\ell \ge 1$, where

$$\widetilde{C_0} := \frac{\widetilde{C_{01}}}{\pi} + \frac{\widetilde{C_{02}}}{4\pi} + \frac{3}{2}.$$

The proof is completed. □

Lastly, we prove Theorem 1.6.

Proof of Theorem 1.6. We first consider the case where $\rho_* \le \rho \le \rho_*/\rho^*$. Using (1.5), (6.4) and (6.5), we have

$$\begin{aligned}\|u^{(\ell),\delta} - u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} &\le 2\big(\|u^{(\ell),\delta} - u^{(\ell)}\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} + \|u^{(\ell)} - u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})}\big)\\ &\le 2\widetilde{C_0}\ell\delta^2 + 2C\left(\frac{\log\ell}{\ell}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}} \le \widetilde{C_{11}}H_1(\ell),\end{aligned} \tag{6.17}$$

where $\widetilde{C_{11}} := 2\max\{\widetilde{C_0}, C\}$ and

$$H_1(\ell) := \ell\delta^2 + \left(\frac{\log\ell}{\ell}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}}.$$

Let $\ell = \ell(\delta,\rho)$, where the definition of $\ell(\delta,\rho)$ is given in Theorem 1.6. We now remark that we have

$$\frac{\ell}{2} < \ell - 1 < \ell_0(\delta,\rho) \le \ell \tag{6.18}$$

for $\ell \ge 3$, and

$$0 \le \log\log\frac{1}{\delta} < \log\frac{1}{\delta}. \tag{6.19}$$

We also remark that $\ell_0(\delta,\rho)$ is monotone decreasing for $0 < \delta < e^{\frac{\log(\rho^*/\rho_*)}{2\log(1/\rho)}}$. Hence, we have

$$\ell_0(\delta,\rho) \ge \ell_0(e^{-3},\rho) = (e^3)^{\frac{2\log(1/\rho)}{\log(1/\rho)+\log(\rho^*/\rho_*)}} \cdot 3^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)+\log(\rho^*/\rho_*)}} > 3$$

since we now consider $0 < \delta \le e^{-3}$. Then, we obtain $\ell \ge 4$. Therefore, using (6.18) and (6.19), we obtain

$$\ell\delta^2 = 2\frac{\ell}{2}\delta^2 < 2\ell_0(\delta,\rho)\delta^2 = 2\left(\delta^2\log\frac{1}{\delta}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)+\log(\rho^*/\rho_*)}} \tag{6.20}$$

and

$$\begin{aligned}\frac{\log \ell}{\ell} &\leq \frac{\log \ell_0(\delta,\rho)}{\ell_0(\delta,\rho)} \\ &< \frac{2\log(1/\rho)+\log(\rho^*/\rho_*)}{\log(1/\rho)+\log(\rho^*/\rho_*)}\left(\delta^2\log\frac{1}{\delta}\right)^{\frac{\log(1/\rho)}{\log(1/\rho)+\log(\rho^*/\rho_*)}} \\ &< 2\left(\delta^2\log\frac{1}{\delta}\right)^{\frac{\log(1/\rho)}{\log(1/\rho)+\log(\rho^*/\rho_*)}}.\end{aligned} \tag{6.21}$$

Using (6.17), (6.20) and (6.21), we have

$$\begin{aligned}\|u^{(\ell),\delta}-u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} &\leq \widetilde{C_{11}}\left(2+2^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}}\right)\left(\delta^2\log\frac{1}{\delta}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)+\log(\rho^*/\rho_*)}} \\ &\leq \widetilde{C_1}\left(\delta^2\log\frac{1}{\delta}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)+\log(\rho^*/\rho_*)}},\end{aligned}$$

where

$$\widetilde{C_1} := \widetilde{C_{11}}\left(2+2^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho_*)}}\right).$$

We next consider the case where $\rho_*/\rho^* < \rho \leq \rho^*$. Similarly, using (1.5), (6.4) and (6.5), we have

$$\begin{aligned}\|u^{(\ell),\delta}-u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} &\leq 2\left(\|u^{(\ell),\delta}-u^{(\ell)}\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})}+\|u^{(\ell)}-u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})}\right) \\ &\leq 2\widetilde{C_0}\ell\delta^2+2C\frac{\log\ell}{\ell} \leq \widetilde{C_{21}}H_2(\ell),\end{aligned} \tag{6.22}$$

where $\widetilde{C_{21}} := 2\max\left\{\widetilde{C_0}, C\right\}$ and

$$H_2(\ell) := \ell\delta^2+\frac{\log\ell}{\ell}.$$

Let $\ell = \ell(\delta,\rho)$. We remark that $\ell_0(\delta,\rho)$ is monotone decreasing for $0<\delta<\sqrt{e}$. Hence, we have

$$\ell_0(\delta,\rho) \geq \ell_0(e^{-3},\rho) = \sqrt{3}e^3 > 3$$

since we now consider $0<\delta\leq e^{-3}$. Then, we obtain $\ell \geq 4$. Therefore, using (6.18) and (6.19), we obtain

$$\ell\delta^2 = 2\frac{\ell}{2}\delta^2 < 2\ell_0(\delta,\rho)\delta^2 = 2\left(\delta^2\log\frac{1}{\delta}\right)^{1/2} \tag{6.23}$$

and

$$\frac{\log\ell}{\ell} \leq \frac{\log\ell_0(\delta,\rho)}{\ell_0(\delta,\rho)} < \frac{3}{2}\left(\delta^2\log\frac{1}{\delta}\right)^{1/2}. \tag{6.24}$$

Using (6.22), (6.23) and (6.24), we have

$$\|u^{(\ell),\delta} - u\|^2_{L^2(B_1\setminus\overline{B_{\rho^*}})} \le \widetilde{C_2}\left(\delta^2 \log\frac{1}{\delta}\right)^{1/2},$$

where

$$\widetilde{C_2} := \frac{7}{2}\widetilde{C_{21}}.$$

The proof is completed. □

A. A solvability of the weak solution

In this appendix, we prove the solvability of the weak solution.

Proof of Theorem 2.2. Let us first assume that $\eta \in C^\infty(\partial B_\rho)$ and $\psi \in C^\infty(\partial B_1)$. Then, the boundary value problem (2.1) has a solution $u \in C^\infty(\overline{B_1 \setminus \overline{B_\rho}})$. Moreover, u also solves the problem (2.1) in the weak sense defined in Definition 2.1. Indeed, multiplying the equation in the problem (2.1) by g and using (2.3) and the divergence theorem, we arrive at (2.2).

We first prove the inequality (2.4). Denote by g the solution to the equation

$$\Delta g = u \quad \text{in } B_1 \setminus \overline{B_\rho} \tag{A.1}$$

subject to (2.3). This problem has a unique solution $g \in H^2(B_1 \setminus \overline{B_\rho})$ which satisfies

$$\|g\|_{H^2(B_1\setminus\overline{B_\rho})} \le C_1' \|u\|_{L^2(B_1\setminus\overline{B_\rho})}. \tag{A.2}$$

We now take g as a test function in (2.2). By (2.2), we have

$$\int_{B_1\setminus\overline{B_\rho}} u\Delta g\,dx = -\int_{\partial B_1} \psi g\,d\sigma + \int_{\partial B_\rho} \eta\frac{\partial g}{\partial \nu}\,d\sigma. \tag{A.3}$$

Moreover, from (A.1), we get

$$\int_{B_1\setminus\overline{B_\rho}} u\Delta g\,dx = \int_{B_1\setminus\overline{B_\rho}} u^2\,dx = \|u\|^2_{L^2(B_1\setminus\overline{B_\rho})}. \tag{A.4}$$

Using the Cauchy–Schwarz inequality, we have

$$\int_{\partial B_1} \psi g\,d\sigma \le \|\psi\|_{L^2(\partial B_1)}\|g\|_{L^2(\partial B_1)} \tag{A.5}$$

and

$$\int_{\partial B_\rho} \eta\frac{\partial g}{\partial \nu}\,d\sigma \le \|\eta\|_{L^2(\partial B_\rho)}\left\|\frac{\partial g}{\partial \nu}\right\|_{L^2(\partial B_\rho)}. \tag{A.6}$$

Hence, combining (A.3), (A.4), (A.5) and (A.6), we obtain

$$\|u\|^2_{L^2(B_1\setminus\overline{B_\rho})} \le \|\psi\|_{L^2(\partial B_1)}\|g\|_{L^2(\partial B_1)} + \|\eta\|_{L^2(\partial B_\rho)}\left\|\frac{\partial g}{\partial\nu}\right\|_{L^2(\partial B_\rho)}. \tag{A.7}$$

From the well-known trace inequality for g, we have

$$\|g\|_{L^2(\partial B_1)} \le C_2'\|g\|_{H^2(B_1\setminus\overline{B_\rho})} \tag{A.8}$$

and

$$\left\|\frac{\partial g}{\partial\nu}\right\|_{L^2(\partial B_\rho)} \le C_3'\|g\|_{H^2(B_1\setminus\overline{B_\rho})}. \tag{A.9}$$

Therefore, combining (A.7), (A.8) and (A.9), we obtain

$$\|u\|^2_{L^2(B_1\setminus\overline{B_\rho})} \le C_4'\big(\|\psi\|_{L^2(\partial B_1)} + \|\eta\|_{L^2(\partial B_\rho)}\big)\|g\|_{H^2(B_1\setminus\overline{B_\rho})}, \tag{A.10}$$

where

$$C_4' := \max\{C_2', C_3'\}.$$

Thus, the inequality (2.4) follows from (A.2) and (A.10), where $C' := C_1'C_4'$.

To handle the general case, we approximate η and ψ in the appropriate L^2-norm by smooth functions $\eta_j \in C^\infty(\partial B_\rho)$ and $\psi_j \in C^\infty(\partial B_1)$, that is,

$$\|\eta_j - \eta\|_{L^2(\partial B_\rho)} \to 0 \quad \text{as } j \to \infty, \tag{A.11}$$

and

$$\|\psi_j - \psi\|_{L^2(\partial B_1)} \to 0 \quad \text{as } j \to \infty. \tag{A.12}$$

Let u_j be the solution to the problem (2.1) with data η_j and ψ_j. We then note that we have

$$\|u_j\|_{L^2(B_1\setminus\overline{B_\rho})} \le C'\big(\|\eta_j\|_{L^2(\partial B_\rho)} + \|\psi_j\|_{L^2(\partial B_1)}\big). \tag{A.13}$$

Using (A.11), (A.12) and (A.13), we get

$$\|u_j - u_k\|_{L^2(B_1\setminus\overline{B_\rho})} \le C'\big(\|\eta_j - \eta_k\|_{L^2(\partial B_\rho)} + \|\psi_j - \psi_k\|_{L^2(\partial B_1)}\big) \to 0 \quad \text{as } j,k \to \infty. \tag{A.14}$$

It follows from (A.14) that $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $L^2(B_1 \setminus \overline{B_\rho})$. Hence, $\{u_j\}_{j=1}^\infty$ converges to $u \in L^2(B_1 \setminus \overline{B_\rho})$. Moreover, we remark that we have

$$\int_{B_1\setminus\overline{B_\rho}} u_j \Delta g\, dx + \int_{\partial B_1} \psi_j g\, d\sigma - \int_{\partial B_\rho} \eta_j \frac{\partial g}{\partial\nu}\, d\sigma = 0 \tag{A.15}$$

for every $g \in C^\infty(\overline{B_1 \setminus \overline{B_\rho}})$ subject to (2.3). By the Cauchy–Schwarz inequality, we have

$$\left| \int_{B_1 \setminus \overline{B_\rho}} (u_j - u)\Delta g \, dx \right| \le \|u_j - u\|_{L^2(B_1 \setminus \overline{B_\rho})} \|\Delta g\|_{L^2(B_1 \setminus \overline{B_\rho})} \to 0 \quad \text{as } j \to \infty.$$

Evaluating the second and third terms on the left-hand of (A.15) in the same way, we have (2.2). Furthermore, since $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence, we have

$$\|u_j\|_{L^2(B_1 \setminus \overline{B_\rho})} \to \|u\|_{L^2(B_1 \setminus \overline{B_\rho})} \quad \text{as } j \to \infty. \tag{A.16}$$

Thus, using (A.11), (A.12), (A.13) and (A.16), we obtain (2.4).

We finally show that the solution to the problem (2.1) is unique. Let u_1 and u_2 be solutions to the problem (2.1). We now put $u = u_1 - u_2$. Then, by linearity, we have (2.1) with $\eta = \psi = 0$. Hence, it suffices to show that if

$$\int_{B_1 \setminus \overline{B_\rho}} u \Delta g \, dx = 0 \tag{A.17}$$

for all test functions g, then $u = 0$. Now, let g be the test function that we used earlier. It follows from (A.17) that

$$0 = \int_{B_1 \setminus \overline{B_\rho}} u \Delta g \, dx = \|u\|^2_{L^2(B_1 \setminus \overline{B_\rho})}.$$

Therefore, we obtain $u = 0$. □

Acknowledgments

The author expresses gratitude to Professor Sei Nagayasu (University of Hyogo) for many valuable discussions and comments. My deep appreciation goes to the anonymous referees for invaluable comments.

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