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Labeling Trees of Small Diameters with Consecutive Integers

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Abstract. Given a simple graph G with m edges, we are looking for a bijection f from $E(G)$ to the integer set $\{k+1, k+2, \dots, k+m\}$ such that the vertex sum of each vertex v, $\phi(v)$, defined as the sum of $f(e)$ over all edges e incident to v is unique. If such a bijection f exists, we say G is k-shifted antimagic. This is a generalization of the antimagic graphs proposed by Hartsfield and Ringel [7]. In this paper, we proved that every tree of diameter four or five, except for two previous known examples, is k-shifted antimagic for every integer k.

1. Introduction

The concept of the antimagic labeling for graphs was introduced by Hartsfield and Ringel [7] in 1990. Let $G = (V, E)$ be a simple graph with m edges. We say a graph G is *antimagic* if there exists a bijection from $E(G)$ to the label set $L = \{1, 2, \dots, m\}$ such that the *vertex sum* of each vertex $v \in V(G)$, defined as the sum of the labels of the edges incident to v and denoted as $\phi(v)$, is unique. An injection f achieves the above condition is called an *antimagic labeling* for G. Some basic types of graphs, including the paths P_n, the cycles C_n, and the complete graphs K_n on $n \geq 3$ vertices, are shown to be antimagic in [7]. The following two conjectures are well-known.

Conjecture 1.1. [7] *All connected graphs except K_2 are antimagic.*

Conjecture 1.2. [7] *All trees except K_2 are antimagic.*

There are abundant research papers on tackling the two conjectures. It is remarkable that Conjecture 1.1 is true for dense graphs [1] and regular graphs [4]. We recommend the reference [6] to the readers as a comprehensive survey. There are various generalizations of the antimagic problems on graphs. For example, Matamala and Zamora [12] considered the problem of using any set of m positive numbers as the label set. In [2], Chang, Chen, Li, and Pan used the sets of consecutive integers, $[a, b] := \{a, a+1, \dots, b\}$, as the label sets, and defined the following term.

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Definition 1.3. [2] Let G be a graph with m edges. Given an integer k, if there is a bijective function f from $E(G)$ to $L = [k+1, k+m]$ such that the vertex sum $\phi(v)$ for each vertex $v \in V(G)$ is unique, then we say G is k-shifted antimagic and f is a k-shifted antimagic labeling for G.

A graph G is 0-shifted antimagic if it is antimagic. The idea of translating the label set $[1, m]$ to $[k+1, k+m]$ for positive k was proposed by Wang and Hsiao [14] in order to construct the antimagic labelings for the Cartesian product and the lexicographic product of sparse graphs. Moreover, in an early paper [8], Hefetz already mentioned a problem of labeling the graphs with consecutive integers, including both positive and negative integers. By replacing f with $-f$, we see that a graph G is k-shifted antimagic if and only if it is $-(k+m+1)$-shifted antimagic.

Although Conjecture 1.2 is not settled yet, Chang et al. [2] proved that every tree on at least three vertices is k-shifted antimagic when $|k|$ is sufficiently large. In addition, they also demonstrated the trees that are k-shifted antimagic for every integer k as well as the trees that are not k-shifted antimagic for some specific values of k, including some early discoveries of Hefetz [8]. The trees which are not k-shifted antimagic for some integers k found by Hefetz are the paths P_n and the star S_n. A path P_n on n vertices is not $-(n-2)$-shifted antimagic for $3 \leq n \leq 5$, while a star S_n on $n+1$ vertices is not $(-\lceil n/2 \rceil)$-shifted antimagic. A *double star* $S_{p,q}$ is a tree on $p+q+2$ vertices consisting of a pair of adjacent vertices which are respectively adjacent to p and q leaves. Chang et al. [2] determined the values of p, q, and k for which the double star $S_{p,q}$ is not k-shifted antimagic.

Theorem 1.4. [2] *Let p, q, and k be integers with $p \geq q \geq 1$. A double star $S_{p,q}$ is k-shifted antimagic if and only if p, q, and k do not satisfy any one of the two conditions:*

(1) *$p = 2$, $q = 1$, and $k = -2$ or -3.*

(2) *p is odd, $q = 1$, and $k = -(p+3)/2$.*

Moreover, Chang et al. [2] proved that P_5', a tree obtained by connecting a new vertex to the central vertex of a P_5, is not k-shifted antimagic when $k = -3$.

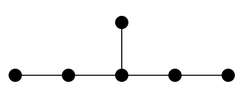


Figure 1.1: The tree P_5'.

Despite of the above instances, the path P_n is proved to be k-shifted antimagic for every integer k when $n \geq 6$. In [5], Dhananjaya and Li presented more classes of trees that are k-shifted antimagic for every integer k. An *odd tree* is a tree such that the degree of each vertex is odd. An *odd tree forest* is a forest consisting of odd trees as its components.

Theorem 1.5. [5] *An odd tree forest F not containing K_2 as a component is k-shifted antimagic for every integer k if and only if $F \notin (\{S_{2n+1} \mid n \geq 1\} \cup \{2S_3, 3S_3\})$.*

The *diameter* of a tree T, $\mathrm{diam}(T)$, is the maximum number of edges of a path in T. A tree T has $\mathrm{diam}(T) = 2$ if and only if it is a star S_n for some $n \geq 2$; while a tree T has $\mathrm{diam}(T) = 3$ if and only if T is a double star $S_{p,q}$ for some $p, q \geq 1$. It is remarkable that every known tree that is not k-shifted antimagic for some integer k has the diameter at most four. In [2], Chang et al. proposed the following question.

Question 1.6. [2] Find a tree T of diameter at least five which is not k-shifted-antimagic for some integer k.

In this paper, we investigate trees of diameter four and five. It turns out that every tree of diameter four, except the previously known trees P_5 and P_5', is k-shifted antimagic for every integer k. For trees of diameter five, we did not find any tree T and integer k such that T is not k-shifted antimagic. Therefore, we establish the following two theorems.

Theorem 1.7. *A tree T with $\mathrm{diam}(T) = 4$ is k-shifted antimagic if and only if T and k are not of the following two cases:* (1) *$T = P_5$ and $k \in \{-2, -3\}$ or* (2) *$T = P_5'$ and $k = -3$.*

Theorem 1.8. *A tree T with $\mathrm{diam}(T) = 5$ is k-shifted antimagic for every integer k.*

The paper is organized as follows. We will prove Theorem 1.7 in Section 2 and Theorem 1.8 in Section 3 by giving the methods to construct the labelings. As we mentioned earlier, a graph with m edges is k-shifted antimagic if and only if it is $-(k+m+1)$-shifted antimagic. Hence we only need to consider the label sets $L = [k+1, k+m]$ with $k \geq -(m+1)/2$, or equivalently the number of positive labels is greater than or equal to the number of negative labels in L. There is no simple labeling method which can be applied to all trees. We give the general methods that could initially induce a pair of vertices having the same vertex sum. Then we do some adjustments if the coincidence happens. Some problems on other trees related to our results will be discussed in Section 4.

2. Tree of diameter four

Let T be a tree with m edges and $\mathrm{diam}(T) = 4$. We pick a path of length four in T and view T as a rooted tree by designating the central vertex of the path as the root. The root is always the same vertex regardless of the choice of the path. For the vertices in $V(T)$, we denote the root with r, a non-leaf child of r with x, a leaf child of r with y, and a child of some x with z. Moreover, for the purpose of proof, we index x's as $x_1, x_2, \ldots, x_s$ satisfying $\deg(x_1) \leq \deg(x_2) \leq \cdots \leq \deg(x_s)$, y's as $y_1, y_2, \ldots, y_t$, and z's as $z_{i,j}$ when it is the j-th child of x_i. Note that $s \geq 2$ since $\operatorname{diam}(T) = 4$. An edge e will be indexed as e_v, where v is one of the vertices x's, y's and z's, if it is incident to v and the parent of v.

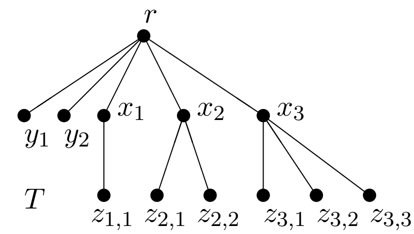


Figure 2.1: A tree of diameter 4.

2.1. Labeling with nonnegative labels

We first deal with the case that all labels in $L = [k+1, k+m]$ are nonnegative, i.e., $k \geq -1$. Let us label the edges of T by the following steps and call this labeling f:

Step 1. Assign the labels $k+1, k+2, \ldots, k+m-s-t$ to the edges $e_{z_{1,1}}, e_{z_{1,2}}, \ldots, e_{z_{1,\deg(x_1)-1}}$, $e_{z_{2,1}}, e_{z_{2,2}}, \ldots, e_{z_{2,\deg(x_2)-1}}, \ldots, e_{z_{s,\deg(x_s)-1}}$ accordingly.

Step 2. Next assign $k+m-s-t+i$ to e_{y_i} for $i \in [1, t]$.

Step 3. Final assign $k+m-s+i$ to e_{x_i} for $i \in [1, s]$.

By the degree condition and the orderings of the vertices, it is straightforward to see the vertex sums of the vertices x_i's, y_i's, $z_{i,j}$'s are all distinct. If $\phi(r) \neq \phi(x_i)$ for $i \in [1, s]$, then we already have a k-shifted antimagic labeling for T.

We claim that if $\phi(r) = \phi(x_i)$ for some i, then $\deg(x_i) \geq 3$. Recall that $s \geq 2$, so there exists some $x_{i'} \neq x_i$. If $\deg(x_i) = 2$, then $f(e_{z_{i,1}}) < f(e_{x_{i'}})$. Thus, $\phi(x_i) = f(e_{z_{i,1}}) + f(e_{x_i}) < f(e_{x_{i'}}) + f(e_{x_i}) \leq \phi(r)$, which contradicts the assumption.

Now suppose $\phi(r) = \phi(x_i)$ for some $i \geq 2$. Observe that

$$\phi(x_i) - \phi(x_{i-1}) = f(e_{x_i}) + \sum_{j=1}^{\deg(x_i)-1} f(e_{z_{i,j}}) - \left(f(e_{x_{i-1}}) + \sum_{j=1}^{\deg(x_{i-1})-1} f(e_{z_{i-1,j}}) \right) \geq 4$$

since $f(e_{x_i}) = f(e_{x_{i-1}})+1$, $f(e_{z_{i,j}}) > f(e_{z_{i-1,j'}})$ for any j, j' and $\deg(x_i) \geq 3$. We exchange the labels of e_{x_i} and $e_{x_{i-1}}$. Then the new vertex sum of x_{i-1} is increasing by one, while the new vertex sum of x_i is decreasing by one. By the above inequality, they are not equal. Note that $\phi(r)$ is not changed after the swap. Thus, the new labeling is a k-shifted antimagic labeling for T. If $\phi(r) = \phi(x_1)$, then we do the above exchange for e_{x_1} and e_{x_2}. As a consequence, every T with $\operatorname{diam}(T) = 4$ is k-shifted antimagic for all $k \geq -1$.

2.2. Some reductions for trees of diameter four

Suppose k is a negative integer and $L = [k+1, k+m]$ contains both positive and negative labels. We first provide a method to reduce the number of negative labels. The notations x_i's, y_i's, and $z_{i,j}$'s are defined the same as before in the beginning of the section. For $i = 1, 2, \ldots$, assign pairwise the labels $\pm i$ to the edges e_z, e_y, and e_x in order according to priority:

(1) First assign each pair of opposite labels to any pair of unlabeled edges $e_{z_{i,j}}$ and $e_{z_{i,j'}}$ incident to the same x_i.

(2) Then assign each pair of opposite labels to any pair of unlabeled edges e_{y_i} and e_{y_j}, or e_{y_i} and e_{x_j}, or e_{x_i} and e_{x_j}. However we can assign a label to e_{x_i} only when all $e_{z_{i,j}}$'s incident to x_i have already been labeled.

Let L_0 be the set of labels used in the above labeling process. For a vertex v, if all edges incident to v are labeled, then $\phi(v) = f(e_v) \in L_0$, otherwise the labels in L_0 contribute nothing to $\phi(v)$ and it will be determined by how we assign the remaining labels. Let us call the tree formed by the unlabeled edges the *reduced tree* and denote it with T', and denote the set of unused labels with L', i.e., $L' = L \setminus L_0$. Note that L' is nonempty since 0 is not used in the labeling process. If we can label the edges of T' with the labels in L' so that the vertex sums of all vertices in $V(T')$ are distinct and not in L_0, then it is a k-shifted antimagic labeling for T. The methods to labeling T' will vary depending on the structure of T' and the labels in L'. In the sequel, we will give a comprehensive investigation and present all labeling methods.

First suppose that L' contains no negative labels. Suppose the reduced tree $T' \notin \{P_2, P_3\}$. Pick any leaf $u \in V(T')$ and assign 0 to e_u, so $\phi(u) = 0$. Observe that $T' - u \neq P_2$ and $\operatorname{diam}(T' - u) \leq 4$. Also, the labels will be used to label $T' - u$ are in $L' \setminus \{0\} = [-k, k+m]$. By the known results of stars, double stars (see Theorem 1.4), and the result in Section 2.1, $T' - u$ is $-(k+1)$-shifted antimagic and the vertex sum of each vertex of $T' - u$ is at least $-k$, not in $L_0 \cup \{0\}$. Hence T is k-shifted antimagic.

Next, suppose $T' = P_2$. Then it is an edge incident to r. This implies that every vertex of T, except for r, has an even number of children, so T is an odd tree. Since T is not a star, by Theorem 1.5, T is k-shifted antimagic for every integer k.

When $T' = P_3$, we have $L_0 = \{\pm 1, \pm 2, \ldots, \pm(k+1)\}$ and $L' = \{0, -k\}$. The middle vertex of T' is either r or some x_i. For the former case, we may assume the two edges of T' are e_{x_i} and $e_{x_{i'}}$. It is clear that both x_i and $x_{i'}$ have even numbers, at least two, of children, and all edges $e_{z_{i,j}}$'s and $e_{z_{i',j}}$'s have been labeled. We may further assume $f(e_{z_{i,1}}) = 1$, $f(e_{z_{i,2}}) = -1$, $f(e_{z_{i',1}}) = k+1$ and $f(e_{z_{i',2}}) = -(k+1)$. The assumption of the last two edges can be made is because if they were labeled with other $\pm\ell$, we can switch them with $\pm(k+1)$ without destroying the distinctness of the vertex sums for the vertices in $V(T) \setminus V(T')$. Now we exchange the labels of $e_{z_{i,2}}$ and $e_{z_{i',2}}$, then assign $-k$ to e_{x_i} and 0 to $e_{x_{i'}}$. Thus, $\phi(r) = -k$, $\phi(x_i) = -2k$, and $\phi(x_{i'}) = k$. None of them is in L_0. See (a) and (b) in Figure 2.2. So, T is k-shifted antimagic.

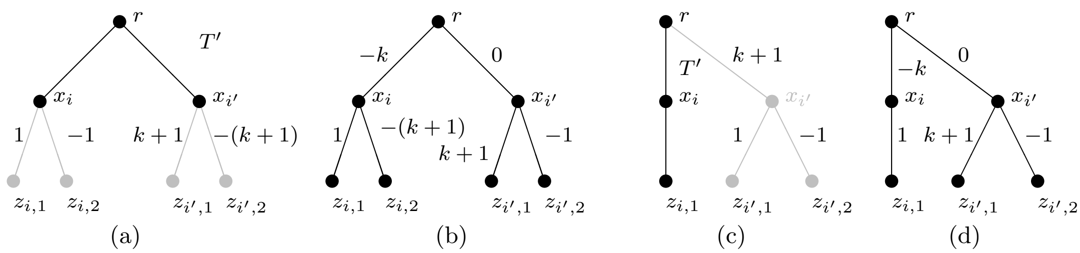


Figure 2.2: $T' = P_3$ and the middle vertex is r or some x_i.

Now consider the second case that the middle vertex of T' is some x_i. One of the edges of T' is e_{x_i}. Let the other one be $e_{z_{i,1}}$. Pick $x_{i'}$ such that $e_{x_{i'}}$ is labeled with $k+1$. We may assume $e_{z_{i',1}}$ and $e_{z_{i',2}}$ are labeled with ± 1. Now relabel $e_{x_{i'}}$ and $e_{z_{i',1}}$ with 0 and $k+1$, assign $-k$ and 1 to e_{x_i} and $e_{z_{i,1}}$, respectively. See (c) and (d) in Figure 2.2. We have $\phi(r) = -(2k+1)$, $\phi(x_i) = -(k-1)$, and $\phi(x_{i'}) = k$. For any other vertex v, $\phi(v) = f(e_v) \in L_0$. So, T is k-shifted antimagic.

2.3. Labeling the reduced tree with positive and negative labels

For a vertex of even degree, there exists at least a pair of edges incident to the vertex that cannot be labeled by the reducing method in Section 2.2. Let d be the number of x's of even degree. If the number of the negative labels is more than $(m-2d)/2$, we cannot exhaust all negative labels. Once this happens, each x has at most one child in T', and at most one child of r, either x or y, is a leaf in T', otherwise we can continue our labeling to reduce the negative labels. Thus, except r, a vertex in T' has degree one or two. A tree containing exactly one vertex of degree greater than two is called a *spider*. Our reduced tree T' above is a path P_4 or P_5, or can be viewed as a special type of spiders such that each *leg*, a maximal path with r as an endpoint in the spider, has length one or two.

As before, we need to label T' with the labels in L' and assure that the vertex sums of all vertices in T' are not only distinct but also not in L_0. Sometimes we have to exchange a pair of labels $\pm\ell \in L_0$ with some pair of labels $\pm\ell' \in L'$ to achieve our goal. The fundamental idea of our labeling is to assign two labels of the same sign to a leg of length two as possible. The negative labels will be assigned consecutively so that one endpoint of the leg has an even vertex sum while the middle vertex of the leg has an odd vertex sum. For the positive labels, we assign half of them, the larger labels, to the edges of the legs incident to r, and then assign the smaller labels accordingly to the other edges of the legs. Thus, the distinctness of the vertex sums for non-root vertices can be easily checked by direct comparing the magnitudes and parities.

Suppose that the previous reducing method labels p pairs of edges of T and initially $L_0 = \{\pm 1, \pm 2, \ldots, \pm p\}$. Let L_- and L_+ be the sets of negative and positive labels in L', respectively. We give labeling methods according to the parities of $|L_-|$ and $|L_+|$. Define the function σ on the set of integers such that $\sigma(n) = 1$ if n is odd, otherwise $\sigma(n) = 0$. In the sequel, when we say a pair of e_x and e_z, we refer to a leg consisting of e_x and e_z. Namely, z is the only child of x in T'.

Case 2.1: $|L_-| = 2a$ *and* $|L_+| = 2b+1$ *with* $1 \leq a \leq b$. Pick a pair of e_x and e_z, then assign $p+b+1$ to e_x and 0 to e_z. For any other pair of e_x and e_z, we arbitrary assign a pair of negative labels $-(p+2i-\sigma(p+1))$ to e_x and $-(p+2i-\sigma(p))$ to e_z for $i \in [1,a]$ or a pair of positive labels $p+b+1+i$ to e_x and $p+i$ to e_z for $i \in [1,b]$. It is straightforward to see that all vertex sums of the non-root vertices in T' are pairwise distinct. We only focus on the value of $\phi(r)$ in the following cases. For $\phi(r)$, we have

$$\phi(r) \geq \sum_{i=1}^{a} -(p+2i) + \sum_{i=1}^{b+1} (p+b+i) \geq p+2b+1,$$

which is greater than the vertex sum of any leaf z. Once $\phi(r) = \phi(x)$ for some x, we exchange the labels $-(p+1)$ and $-(p+2)$ so that $\phi(r)$ is changed by one, then the problem will be solved since the difference of the vertex sums of any two children of r is at least two.

Case 2.2: $|L_-| = 2a$ *and* $|L_+| = 2b$ *with* $1 \leq a \leq b$. Call w the unique leaf child of r in T'. Pick a pair of e_x and e_z, then assign $p+b+1$ to e_x and 0 to e_z. For any other pair of e_x and e_z, we arbitrary assign a pair of negative labels $-(p+2i-\sigma(p+1))$ to e_x and $-(p+2i-\sigma(p))$ to e_z for $i \in [1,a]$ or a pair of positive labels $p+b+1+i$ to e_x and $p+i$ to e_z for $i \in [1, b-1]$. In addition, assign $p+b$ to e_w. The vertex sum of r satisfies

$$\phi(r) \geq \sum_{i=1}^{a} -(p+2i) + \sum_{i=0}^{b} (p+b+i) \geq p+b+3,$$

showing that $\phi(r)$ is larger than the vertex sum of any leaf when $b \geq 3$, or when $(a,b) = (2,2)$ and p is even, or when $(a,b) = (1,2)$. Once $\phi(r) = \phi(x)$ for some x, we exchange the labels $-(p+1)$ and $-(p+2)$ so $\phi(r)$ is changed by one and the problem is solved as before. When $(a,b) = (2,2)$ and p is odd, our labeling method gives $\phi(r) = p+b+1$. Although we could exchange the labels $-(p+1)$ and $-(p+2)$ to change $\phi(r)$, we need to check the new vertex sum is not $p+b$. Note that when p is odd, $-(p+2)$ was originally assigned to an edge e_x incident to r. So exchanging $-(p+1)$ and $-(p+2)$ will increase $\phi(r)$ by one. Hence, we obtain a k-shifted antimagic labeling. When $(a,b) = (1,1)$, no matter how we assign the labels $\pm(p+1)$, $\pm(p+2)$, and 0 to the edges of T', there always exist two vertices whose vertex sums are equal. Indeed, if $p=0$, then $T=T'$ is P_5' and $L=[-2,2]$. This is the known tree T and k for which T is not k-shifted antimagic. For $p\geq 1$, we exchange the labels $\pm(p+1)\in L'$ with $\pm 2\in L_0$ whenever ± 2 have been assigned in the previous reducing process, and remove the labels ± 1 from where they were assigned for. The two edges must be both incident to r, or w, or some x. Let T_1', T_2', and T_3' be the trees formed by the unlabeled edges corresponding to the above three cases. We give the labeling methods to label T_i''s with labels $0,\pm1,\pm2$, and $\pm(p+2)$ as shown in Figure 2.3. By simple calculations we can see all the vertex sums of the vertices in each T_i' are not in $\{\pm3,\pm4,\dots,\pm(p+1)\}$.

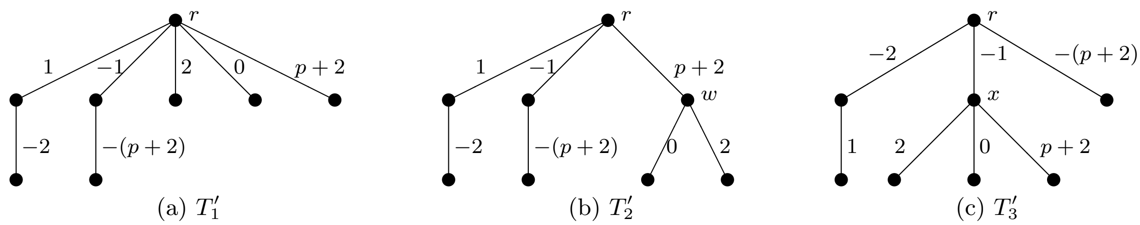


Figure 2.3: Labelings of T_1', T_2', and T_3'.

For Cases 2.3 and 2.4, we first exchange $\pm(p+1)\in L'$ with $\pm1\in L_0$ if L_0 is not empty.

Case 2.3*:* $|L_-|=2a+1$ *and* $|L_+|=2b$ *with* $0\leq a\leq b-1$. Pick a pair of e_x and e_z, then assign $p+b+1$ to e_x and 1 to e_z, and also assign -1 to e_x and 0 to e_z for another pair of e_x and e_z. For any other pair of e_x and e_z, we arbitrary assign a pair of negative labels $-(p+2i+\sigma(p+1))$ to e_x and $-(p+2i+\sigma(p))$ to e_z for $i\in[1,a]$ or a pair of positive labels $p+b+i$ to e_x and $p+i$ to e_z for $i\in[1,b]$. When $a\leq b-2$, the vertex sum of r satisfies

$$\phi(r)\geq(-1)+\sum_{i=1}^{a}-(p+2i+1)+\sum_{i=1}^{b}(p+b+i)\geq 2p+3b,$$

the largest $\phi(x)$, and equality holds if and only if $(a,b)=(0,2)$. So if $a\leq b-2$ and $(a,b)\neq(0,2)$, our method gives a k-shifted antimagic labeling for T. For $(a,b)=(0,2)$, let us call this type of tree T_4' and give a different labeling shown in Figure 2.4.

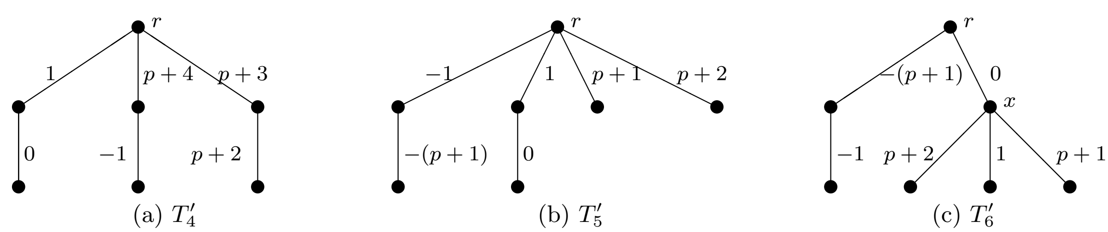


Figure 2.4: Labelings of T_4', T_5' and T_6'.

When $a = b - 1$, $\phi(r) \geq p + (b^2 + b)/2 \geq p + b$, the largest $\phi(z)$. The equality holds only when $b = 1$ or when $b = 2$ and p is even. If $\phi(r) > p + b$ but $\phi(r) = \phi(x)$ for some x, we switch the labels $-(p+2)$ and $-(p+3)$ to change the value of $\phi(r)$ by one as usual to obtain the k-shifted antimagic labeling. Now consider $\phi(r) = p + b$, the largest $\phi(z)$. For $b = 2$ and even p, the new $\phi(r)$ will increase by one when switching $-(p + 2)$ and $-(p+3)$. So we can solve the problem by the swap. For $b = 1$, the reduced tree T' is P_5. If $p = 0$, namely L_0 is empty, then $T' = T$, and the labels used to label T are in $[-1, 2]$. This is also a known case of T and k for which T is not k-shifted antimagic. For $p \geq 1$, the original labeling gives $\phi(r) = p + 1 \in L_0$. In this case, we remove the labels $\pm(p + 1)$ from where they were assigned for. The two edges are both indent to r or some x in T'. Let T'_5 and T'_6 be the trees formed by unlabeled edges corresponding to the two cases. We give the labeling methods to label T'_5 and T'_6 with labels $0, \pm1, \pm(p+1)$ and $p+2$ as shown in Figure 2.4.

Case 2.4*:* $|L_-| = 2a + 1$ *and* $|L_+| = 2b + 1$ *with* $0 \leq a \leq b$. Pick a pair of e_x and e_z, then assign $p + b + 2$ to e_x and 1 to e_z, and also assign -1 to e_x and 0 to e_z for another pair of e_x and e_z. For any other pair of e_x and e_z, we arbitrary assign a pair of negative labels $-(p+2i+\sigma(p+1))$ to e_x and $-(p+2i+\sigma(p))$ to e_z for $i \in [1, a]$ or a pair of positive labels $p + b + 1 + i$ to e_x and $p + i$ to e_z for $i \in [2, b]$. In addition, for w, the unique leaf child of r in T', we assign $p + b + 1$ to e_w. For $b \geq 2$, the vertex sum of r satisfies

$$\phi(r) \geq (-1) + \sum_{i=1}^{a} -(p + 2i + 1) + \sum_{i=1}^{b+1} (p + b + i) \geq p + b + 1,$$

and equality holds if and only if $(a, b) = (2, 2)$ and p is even. When the above inequality is strict and $\phi(r) = \phi(x)$ for some x, we can swap the labels $-(p + 2)$ and $-(p + 3)$ to change $\phi(r)$. On the other hand, if the equality holds, then $\phi(r) = \phi(w)$, and the swap of $-(p + 2)$ and $-(p + 3)$ increases the vertex sum of r by one. So the new $\phi(r)$ is not equal to the vertex sum of any other vertex. For $(a, b) = (1, 1)$, the above method does not work. We give a new method: First exchange the labels $\pm(p + 2) \in L'$ with $\pm2 \in L_0$. Then label T' with $0, \pm1, \pm2$, and $\pm(p + 3)$ as shown in Figure 2.5(a).

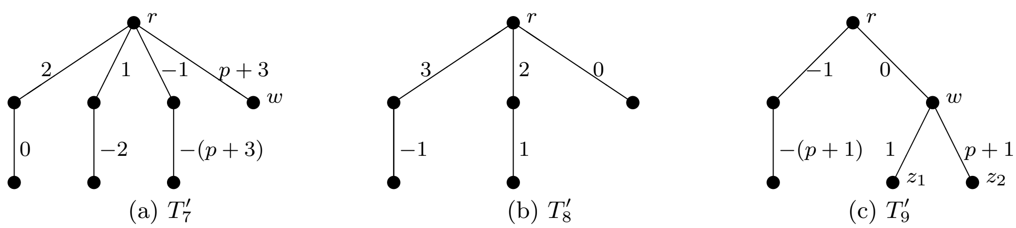


Figure 2.5: Labelings of T'_7, T'_8, and T'_9.

For $(a, b) = (0, 1)$, if $p \geq 1$, our labeling method gives a k-shifted antimagic labeling.

If $(a,b)=(0,1)$ and $p=0$, then $T'=T=P_5'$ and $k=-2$. It has been proved that P_5' is (-2)-shifted antimagic in [2]. For the sake of completeness, we demonstrate the labeling in Figure 2.5(b). Finally, for $(a,b)=(0,0)$, we have $T'=P_4$. Since $\mathrm{diam}(T)\geq 4$, we may assume w is not a leaf in T and has two children z_1 and z_2 in T such that e_{z_1} and e_{z_2} have been labeled with $\pm(p+1)$. Then relabel e_{z_1}, e_{z_2}, and the edges in $E(T')$ with 0, ± 1, and $\pm(p+1)$ as shown in Figure 2.5(c).

3. Trees of diameter five

Let T be a tree with m edges and $\mathrm{diam}(T)=5$. When deleting the middle edge of any path of length five in T, the tree always splits into the same pair of components. Each of which is a tree of diameter between two and four. We view the two components as the rooted trees T_{r_1} and T_{r_2}, where the root of each T_{r_i} is an endpoint of the middle edge. For the vertices in $V(T)$, we denote the root of T_{r_i} with r_i for $i=1,2$, a non-leaf child of r_1 or r_2 with x, a leaf child of r_1 or r_2 with y, and a child of some x with z. Note that each r_i has at least a non-leaf child since $\mathrm{diam}(T)=5$. Again, we index x's as $x_1,x_2,\ldots,x_s$ satisfying $\deg(x_1)\leq\deg(x_2)\leq\cdots\leq\deg(x_s)$, y's as $y_1,y_2,\ldots,y_t$, and z's as $z_{i,j}$ when it is the j-th child of x_i for the purpose of proof. For $v\notin\{r_1,r_2\}$, e_v is defined the same as in Section 2, and let e_r be the edge incident to r_1 and r_2.

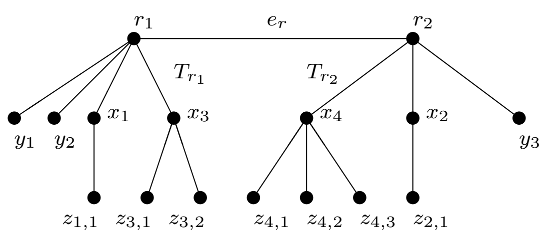


Figure 3.1: A tree of diameter 5.

3.1. Labeling with nonnegative labels

Let $k\geq -1$ so that $L=[k+1,k+m]$ contains no negative integers. We label the edges of T by the following steps and call this labeling f:

Step 1. Assign the labels $k+1,k+2,\ldots,k+m-s-t-1$ to the edges $e_{z_{1,1}},e_{z_{1,2}},\ldots,$ $e_{z_{1,\deg(x_1)-1}},e_{z_{2,1}},e_{z_{2,2}},\ldots,e_{z_{2,\deg(x_2)-1}},\ldots,e_{z_{s,\deg(x_s)-1}}$ accordingly.

Step 2. Next assign $k+m-s-t-1+i$ to e_{y_i} for $i\in[1,t]$.

Step 3. Assign $k+m-s-1+i$ to e_{x_i} for $i\in[1,s-1]$.

Step 4. First assign $k+m-1$ to e_{x_s} and $k+m$ to e_r. If $\phi(r_1) \neq \phi(r_2)$ then we assign the labels in this way. Otherwise, we label the two edges by the other way around and obtain $|\phi(r_1) - \phi(r_2)| = 1$.

It is obvious that all the vertex sums $\phi(x_i)$'s, $\phi(y_i)$'s, and $\phi(z_{i,j})$'s are distinct by the degree condition and the orderings of the vertices. Therefore, if f is not a k-shifted antimagic labeling, then we have $\phi(r_1)$ or $\phi(r_2)$ equal to $\phi(x_i)$ for some i. Moreover, if both $\phi(r_1) = \phi(x_i)$ and $\phi(r_2) = \phi(x_{i'})$, then $i \neq i'$.

Again, we show that if $\phi(r_1) = \phi(x_i)$ for some i then $\deg(x_i) \geq 3$. When x_i is a child of r_1, the argument is the same as the proof in Section 2.1, so we omit it. Suppose that x_i is a child of r_2 and $\deg(x_i) = 2$. Let $x_{i'}$ be a child of r_1. Since $\phi(r_1) \geq f(e_r) + f(e_{x_{i'}})$ and $f(e_{x_{i'}}) > f(e_{z_{i,1}})$, we have

$$f(e_r) + f(e_{x_{i'}}) \leq \phi(r_1) = \phi(x_i) = f(e_{x_i}) + f(e_{z_{i,1}}) < f(e_{x_i}) + f(e_{x_{i'}}).$$

Hence $f(e_r) < f(e_{x_i})$. This implies $f(e_r) = k + m - 1$, $f(e_{x_i}) = k + m$, and $i = s$. Furthermore,

$$0 < f(e_{x_{i'}}) - f(e_{z_{s,1}}) \leq f(e_{x_s}) - f(e_r) = 1$$

implies $f(e_{x_{i'}}) = f(e_{z_{s,1}}) + 1$, and $x_{i'}$ is the only child of r_1. Therefore,

$$\phi(r_1) = f(e_r) + f(e_{x_{i'}}) \leq f(e_r) + f(e_{x_s}) - 2 \leq \phi(r_2) - 2.$$

Recall that if we labeled e_r with $k + m - 1$ at Step 4, then $|\phi(r_1) - \phi(r_2)| = 1$, which contradicts the above inequality. As a conclusion, we have $\deg(x_i) \geq 3$ if $\phi(r_1) = \phi(x_i)$ for some i. The proof for the case of r_2 is similar.

Now assume that f is not a k-shifted antimagic labeling and we have both $\phi(r_1) = \phi(x_i)$ for some i and $\phi(r_2) = \phi(x_{i'})$ for some i'. Without loss of generality, assume $i < i'$. Since $3 \leq \deg(x_i) \leq \deg(x_{i+1}) \leq \deg(x_{i'})$, we have

$$\begin{aligned}
\phi(x_{i+1}) - \phi(x_i) &= f(e_{x_{i+1}}) + \sum_{j=1}^{\deg(x_{i+1})-1} f(e_{z_{i+1,j}}) - \left(f(e_{x_i}) + \sum_{j=1}^{\deg(x_i)-1} f(e_{z_{i,j}}) \right) \\
&= 1 + (\deg(x_i) - 1)^2 + \sum_{j=\deg(x_i)}^{\deg(x_{i+1})-1} f(e_{z_{i+1,j}}) \\
&\geq 5,
\end{aligned}$$

and similarly, $\phi(x_{i'}) - \phi(x_{i'-1}) \geq 5$. We switch the labels of $e_{z_{i,\deg(x_i)-1}}$ and $e_{z_{i+1,1}}$ so that $\phi(x_i)$ is increasing by one while $\phi(x_{i+1})$ is decreasing by one. Then do the same swap for $e_{z_{i'-1,\deg(x_{i'-1})-1}}$ and $e_{z_{i',1}}$. If $i' = i + 1$, then we only need to do the first swap which changes both the vertex sums of x_i and $x_{i'}$. For the new $\phi(x_i)$ and $\phi(x_{i'})$, we have $\phi(r_1) < \phi(x_i) < \cdots < \phi(x_{i'}) < \phi(r_2)$. Hence T is k-shifted antimagic.

Remark 3.1. The reason to exchange the labels of $e_{z_{i,\deg(x_i)-1}}$ and $e_{z_{i+1,1}}$ but not the labels of e_{x_i} and $e_{x_{i+1}}$ as in Section 2 is because exchanging the labels of e_{x_i} and $e_{x_{i+1}}$ does not change the value of $\phi(r)$ if $\operatorname{diam}(T) = 4$. For $\operatorname{diam}(T) = 5$, when $\phi(r_1) = \phi(x_i)$, if x_i is adjacent to r_1 and x_{i+1} is adjacent to r_2, then swapping the labels of e_{x_i} and $e_{x_{i+1}}$ increases both $\phi(r_1)$ and $\phi(x_i)$ by one.

Next consider $\phi(r_1) = \phi(x_i)$ for some i and $\phi(r_2) \neq \phi(x_{i'})$ for all i'. If $i < s$, we have $\phi(x_{i+1}) - \phi(x_i) \geq 5$. We can exchange $f(e_{z_{i,\deg(x_i)-1}})$ and $f(e_{z_{i+1,1}})$ to obtain $\phi(r_1) < \phi(x_i) < \phi(x_{i+1})$. If the swap causes the new $\phi(x_i)$ or $\phi(x_{i+1})$ equal to $\phi(r_2)$, then we could instead exchange $f(e_{z_{i,\deg(x_i)-1}})$ and $f(e_{z_{i+1,2}})$, so the change of each vertex sum is two, to make all vertex sums distinct. For $i = s$, if $\phi(x_s) - \phi(x_{s-1}) \geq 5$, then we do the swap for the labels $f(e_{z_{s-1,\deg(x_{s-1})-1}})$ and $f(e_{z_{s,1}})$ (or $f(e_{z_{s,2}})$). It might happen that we have $\phi(x_s) - \phi(x_{s-1}) < 5$, and the above swaps could cause $\phi(x_s) = \phi(x_{s-1})$ or one of them is equal to $\phi(r_2)$. Thus we cannot solve the coincidence by swapping the labels. However, if $\phi(x_s) - \phi(x_{s-1}) < 5$, then

$$\phi(x_s) - \phi(x_{s-1}) = 1 + (\deg(x_{s-1}) - 1)^2 + \sum_{j=\deg(x_{s-1})}^{\deg(x_s)-1} f(e_{z_{s,j}}) < 5$$

implies that $\deg(x_{s-1}) = 2$ and $\sum_{j=2}^{\deg(x_s)-1} f(e_{z_{s,j}}) < 3$. The last inequality and the fact $\deg(x_s) \geq 3$ imply that $\deg(x_s) = 3$, and moreover $f(e_{z_{s,1}}) = 1$ and $f(e_{z_{s,2}}) = 2$. Consequently, $s = 2$ and $f(e_{z_{1,1}}) = 0$. Recall that t is the number of leaves adjacent to r_1 or r_2. So

$$\phi(x_1) = f(e_{z_{1,1}}) + f(e_{x_1}) = 0 + (t+3) = t+3$$

and

$$\phi(x_2) = f(e_{z_{2,1}}) + f(e_{z_{2,2}}) + f(e_{x_2}) = 1 + 2 + f(e_{x_2})$$

is equal to $t+7$ or $t+8$. By our labeling method, when $\phi(x_2) = t+7$, we have $f(e_r) = t+5$ and

$$\phi(r_1) \geq f(e_{x_1}) + f(e_r) \geq 2t+8 > t+7 = \phi(x_2)$$

for all $t \geq 0$. This contradicts the assumption $\phi(r_1) = \phi(x_2)$. If $\phi(x_2) = t+8$, then we have $f(e_{x_2}) = t+5$, $f(e_r) = t+4$, and $\phi(r_1) \geq 2t+7$. Thus, $\phi(r_1) = \phi(x_2)$ implies $t = 1$. In other words, there is a leaf y_1 adjacent to r_1 or r_2, and $f(e_{y_1}) = t+2 = 3$. However, no matter y_1 is adjacent to r_1 or r_2, we always have $f(e_{x_2}) = t+4 = 5$ by our labeling rule. This makes a contradiction. Conclusively, we can always construct a k-shifted antimagic labeling for T when $\operatorname{diam}(T) = 5$ and $k \geq -1$.

3.2. Some reductions for trees of diameter five

Suppose k is a negative integer such that $L = [k+1, k+m]$ contains both positive and negative labels. We will use the method in Section 2.2 to reduce the number of negative labels. For $i = 1, 2, \ldots$, assign pairwise the labels $\pm i$ to the edges e_z, e_y, and e_x in order according to priority:

(1) First assign each pair of opposite labels to any pair of unlabeled edges $e_{z_{i,j}}$ and $e_{z_{i,j'}}$ incident to the same x_i.

(2) Then assign each pair of opposite labels to any pair of unlabeled edges e_{y_i} and e_{y_j}, or e_{y_i} and e_{x_j}, or e_{x_i} and e_{x_j}. However we can assign a label to e_{x_i} only when all $e_{z_{i,j}}$'s incident to x_i have already been labeled.

Notice that the edges of each T_i's could be completely labeled in the above labeling process. The reduced tree T', L_0, L', L_+, and L_- are all defined the same as before. Our goal is still to assign the labels in L' to the edges of T' so that the vertex sums of all vertices in T' are distinct and not in L_0.

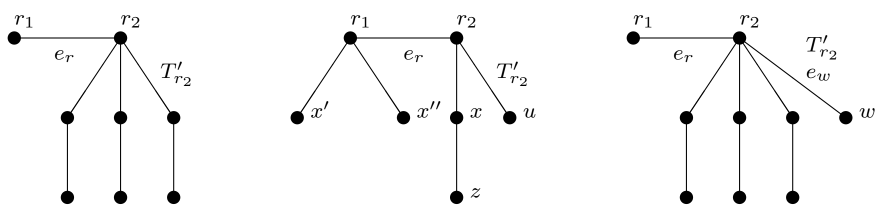
First suppose L' contains no negative labels. The following arguments are similar to the case of $\operatorname{diam}(T) = 4$. When $T' \notin \{P_2, P_3\}$, pick any leaf $u \in V(T')$ and label e_u with 0. Then $T' - u \neq P_2$ and $\operatorname{diam}(T' - u) \leq 5$. It follows from the previous results that $T' - u$ is $-(k+1)$-shifted antimagic and the vertex sum of each vertex in $T' - u$ is at least $-k$. Thus, T is k-shifted antimagic. When $T' = P_2$, the unlabeled edge is e_r. So every vertex in T_1 or T_2 has an even number of children, and each vertex in T has an odd degree. Hence T is k-shifted antimagic for every integer k by Theorem 1.5. When $T' = P_3$, one of the two edges is e_r. Assume the other edge of T' is e_{x_i} and x_i is adjacent to r_1. Let $x_{i'}$ be a child of r_2 and $e_{x_{i'}}$ is labeled with $k+1$. As before, $x_{i'}$ has at least two children $z_{i',1}$ and $z_{i',2}$ in T. Assume the edges $e_{z_{i',1}}$ and $e_{z_{i',2}}$ were labeled with 1 and -1, respectively. Now relabel $e_{z_{i',1}}$ with 0, and assign 1 to e_{x_i} and $-k$ to e_r. As a consequence, T is k-shifted antimagic.

Now consider the case that the reducing method did not use up all negative labels. Suppose that $L_0 = \{\pm 1, \pm 2, \ldots, \pm p\}$. Let T'_{r_i} be the tree formed by the unlabeled edges in T_{r_i} for $i = 1, 2$. In other words, T'_{r_1} and T'_{r_2} are the two components of $T' - e_r$. Because $|L_-| > 0$ and $|L_+| \geq |L_-|$, there are at least three unused labels and hence at least three edges in $E(T')$. Let $e(G)$ denote the number of edges of G. Since $e_r \notin E(T'_{r_i})$ for $i = 1, 2$, we have $e(T'_{r_1}) + e(T'_{r_2}) \geq 2$. Without loss of generality, we assume $e(T'_{r_1}) \leq e(T'_{r_2})$ in the sequel.

Suppose $e(T'_{r_1}) = e(T'_{r_2}) = 1$. Since $\operatorname{diam}(T) = 5$, we may assume the child of r_1 in T'_{r_1} is not a leaf in T. For simplicity, we call this vertex x and let z_1 and z_2 be two of its children in T. Assume ± 1 were assigned to e_{z_1} and e_{z_2}. The tree formed by the edges in

$\{e_{z_1}, e_{z_2}\} \cup E(T')$ is isomorphic to the reduced tree T'_9. So we use the same method to label the five edges.

Next suppose that $e(T'_{r_1}) = 0$ and $e(T'_{r_2}) \geq 2$. If r_2 has no leaf child in T'_{r_2} (see Figure 3.2(a)), then T' is indeed a reduced tree which can be labeled using the methods of Cases 2.2 and 2.4 in Section 2.3. Else, r_2 has a leaf child in T'_{r_2}, say w. Then $e(T'_{r_2})$ is odd. When $e(T'_{r_2}) = 3$, any labeling for T' with labels 0, $\pm(p+1)$, and $p+2$ will result two same vertex sums. Let x be the child of r_2 and z be the child of x in T'_{r_2}. Pick x' and x'' which are the children of r_1 in T_{r_1} and may assume $e_{x'}$ and $e_{x''}$ were labeled with $\pm p$. See Figure 3.2(b). By labeling the edges $e_{x'}$, $e_{x''}$, e_r, e_x, e_z, e_w with $-p$, $-(p+1)$, 0, $p+2$, $p+1$, and p, respectively, we obtain a k-shifted antimagic labeling for T. When $e(T'_{r_2}) \geq 5$, we can assign $\pm(p+1)$ to e_w and e_r and apply the methods of Cases 2.1 and 2.3 in Section 2.3 to label $T'_{r_2} - w$ with the labels in $L' \setminus \{\pm(p+1)\}$. See Figure 3.2(c).



(a) r_2 has no leaf child. (b) $e(T'_{r_2}) = 3$. (c) $e(T'_{r_2}) \geq 5$ and r_2 has a leaf child.

Figure 3.2: $e(T'_{r_1}) = 0$.

Suppose $e(T'_{r_1}) = 1$ and $e(T'_{r_2}) \geq 2$. Then we can just view T' as the reduced tree studied in Section 2.3, and use the previous methods to find the desired labeling. See (a) and (b) in Figure 3.3. However, there are two situations for which we cannot use the previous methods to obtain the labeling. One is $T'_{r_2} = T_{r_2} = P_3$, r_1 has only one child x in T_{r_1}, and all the labeled edges are incident to x. See Figure 3.3(c). Thus, T' is the reduced tree in Case 2.3 with $(a, b) = (0, 1)$ but we cannot remove the labels from the labeled edges to obtain trees isomorphic to T'_5 or T'_6. The solution is the following: Let z_1 and z_2 be two children of x in T_{r_1} such that e_{z_1} and e_{z_2} were labeled with $\pm p$. In addition, let x' be the child of r_2 and z' be the child of x' in T_{r_2}. Now label the edges e_{z_1}, e_{z_2}, e_x, e_r, $e_{x'}$, and $e_{z'}$ with $-(p+1)$, $-p$, 0, p, $p+2$, and $p+1$, respectively. This gives a k-shifted antimagic labeling for T. The other situation is $T'_{r_2} = T_{r_2} = P_4$, r_1 has only one child x in T_{r_1}, and all the labeled edges are incident to x. So, T' is the reduced tree in Case 2.2 with $(a, b) = (1, 1)$ but we cannot remove the labels from the labeled edges to obtain trees isomorphic to T'_1 or T'_2 or T'_3. See Figure 3.3(d). Now let w be the leaf child of r_2, and define z_1, z_2, x', and z' the same as the former case. Then label the edges e_{z_1}, e_{z_2}, e_x, e_r, e_w $e_{x'}$, and $e_{z'}$ with $-(p+2)$, $-(p+1)$, $-p$, 0, p, $p+2$, and $p+1$, respectively, and we can obtain a k-shifted antimagic labeling for T.

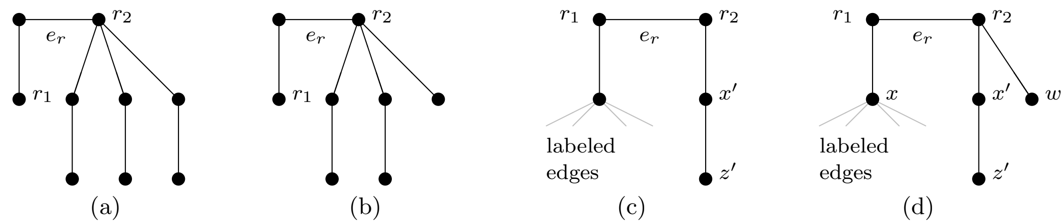


Figure 3.3: $e(T'_{r_1}) = 1$. (a) and (b): View T' as the trees in Section 2.3. (c) and (d): The trees cannot be labeled using the previous methods.

It remains to investigate the reduced trees with $e(T'_{r_1}) \geq 2$ and $e(T'_{r_2}) \geq 2$. As the trees of diameter four, we will assign a pair of adjacent e_x and e_z the labels of the same sign in general. Moreover, we will specify the labels of the edges incident to r_1 and r_2 so that the vertex sums of r_1 and r_2 can be different from others. Since the rest of the proof is lengthy, we divide it into three subsections according to the parities of $e(T'_{r_1})$ and $e(T'_{r_2})$.

3.3. $e(T'_{r_1})$ and $e(T'_{r_2})$ are even

Let $e(T'_{r_1}) = 2s_1$ and $e(T'_{r_2}) = 2s_2$ with $1 \leq s_1 \leq s_2$. Note that $|L_+|$ and $|L_-|$ have the same parity since $|L_+| + |L_-| + 1 = |L'| = e(T') = e(T'_{r_1}) + e(T'_{r_2}) + 1$.

Case 3.1*: $|L_-| = 2a$ and $|L_+| = 2b$ with $1 \leq a \leq b$.* First assign 0 to e_r. For each pair of e_x and e_z, we assign $-(p+a+j)$ to e_x and $-(p+j)$ to e_z for some $j \in [1, a]$, or alternatively $p+b+j$ to e_x and $p+j$ to e_z for some $j \in [1, b]$. The vertex sums of the non-root vertices in T' are $-(2p+a+2j)$ and $-(p+j)$ for $j \in [1, a]$, and $p+j$ and $2p+b+2j$ for $j \in [1, b]$, which are pairwise distinct. Next, let us specify the labels of the edges incident to r_1 and r_2 and evaluate $\phi(r_1)$ and $\phi(r_2)$, We write assigning a label ℓ to r_i to express assigning ℓ to an edge e_x incident to r_i for short.

(1) $s_1 = 1$: Assign $-(p+a+1)$ to r_1, and $-(p+a+j)$ for $j \in [2, a]$ and $p+b+i$ for $j \in [1, b]$ to r_2. Then $-(2p+a+2) < \phi(r_1) < -(p+a)$. For r_2, we have $\phi(r_2) = -\phi(r_1)$ and $p+b < \phi(r_2) < 2p+b+2$ if $a = b$, or $\phi(r_2) \geq (p+2b)+(p+2b-1) > 2p+3b$ if $a < b$.

In the following, assume $s_1 \geq 2$.

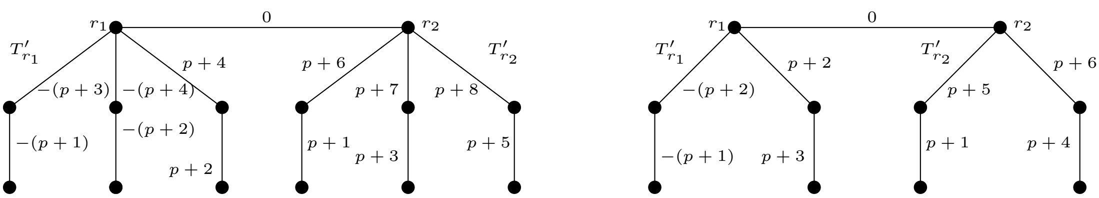
(2) $b-a \geq 4$: First assign $p+2b-3$ and $p+2b-1$ to r_1, and $p+2b-2$ and $p+2b$ to r_2. Next, assign $p+b+j$ and $-(p+a+j)$ together to r_1 or r_2 for $j \in [1, a]$. When $b-a > 4$, assign $p+b+j$ arbitrary to r_1 or r_2 for $j \in [a+1, b-4]$. So, $\phi(r_1) \geq (p+2b-3)+(p+2b-1)$ and $\phi(r_2) \geq (p+2b-2)+(p+2b)$, both greater than $2p+3b$. If $\phi(r_1) = \phi(r_2)$, we reassign $p+2b-3$ to r_2 and $p+2b-2$ to r_1 so that the new vertex sums satisfy $\phi(r_1) > \phi(r_2) > 2p+3b$.

(3) $b - a = 3$: Since $a + b = s_1 + s_2$ is odd, s_1 and s_2 have distinct parities. Let r_o (resp. r_e) be the vertex r_i if s_i is odd (resp. even). First assign $p + 2b - 2$ to r_o, and $p + 2b - 1$ and $p + 2b$ to r_e. Then assign $p + b + j$ and $-(p + a + j)$ together to r_o or r_e for $j \in [1, a]$. We have $\phi(r_e) \geq (p + 2b - 1) + (p + 2b) > 2p + 3b$, but might encounter one of the two problems, $\phi(r_o) = \phi(x)$ for some $x \in V(T')$ or $\phi(r_o) = \phi(r_e)$. Once this happens, we reassign the labels $p + 2b - 2$ to r_e and $p + 2b - 1$ to r_o. The new vertex sum of r_e satisfies $\phi(r_e) \geq (p + 2b) + (p + 2b - 2) > 2p + 3b$, and $\phi(r_o)$ is increasing by one so the first problem can be solved. For the second problem, the new vertex sums of r_o and r_e after the swap satisfy $\phi(r_o) > \phi(r_e) > 2p + 3b$.

When $b - a \leq 2$, the relations of a, b, s_1 and s_2 leads to $s_1 \leq a \leq b \leq s_2$ or $s_1 = s_2 = a + 1$.

(4) $s_1 \leq a \leq b \leq s_2$: Assign $-(p + a + j)$ for $j \in [a - s_1 + 1, a]$ to r_1. Then assign $-(p + a + j)$ for $j \in [1, a - s_1]$ and $p + b + j$ for $j \in [1, b]$ to r_2. Thus, $\phi(r_1) \leq -(p + 2a) - (p + 2a - 1) < -(2p + 3a)$ and $\phi(r_2) \geq (p + 2b) + (p + 2b - 1) > 2p + 3b$.

(5) $s_1 = s_2 = a+1$: Assign $-(p+a+j)$ for $j \in [1, a]$ to r_1 and $p+b+j$ for $j \in [2, b]$ to r_2. For the last pair of e_x and e_z in T'_{r_1}, we particularly assign $p+1$ to e_x and $p+b+1$ to e_z. For $s_1 \geq 4$, we have $\phi(r_1) \leq -(p+2a)-(p+2a-1)-(p+2a-2)+(p+1) < -(2p+3a)$ and $\phi(r_2) > (p + 2b) + (p + 2b - 1) > 2p + 3b$. For $2 \leq s_1 \leq 3$, the above labeling method fails for some values of p. We give a labeling for each T' individually shown in Figure 3.4.



(a) $a = 2$ and $b = 4$.

(b) $a = 1$ and $b = 3$.

Figure 3.4: $s_1 = 3$ and $s_1 = 2$.

Remark 3.2. For the above cases, if $\phi(r_1)$ and $\phi(r_2)$ are both positive, then the larger one of them is at least two greater than $\phi(v)$ of $v \notin \{r_1, r_2\}$.

Case 3.2*:* $|L_-| = 2a + 1$ *and* $|L_+| = 2b + 1$ *with* $0 \leq a \leq b$. When L_- only contains $-(p + 1)$, pick a pair of e_{x_0} and e_{z_0} from T'_{r_1} and assign respectively $p + 1$, $-(p + 1)$, and 0 to e_{x_0}, e_{z_0}, and e_r. So, $\phi(x_0) = 0$ and $\phi(z_0) = -(p + 1)$. Then assign $p + b + j$ and $p + j$, respectively, to each pair of e_x and e_z in T'_{r_1} for $j \in [2, s_1]$, and to each pair of e_x and e_z in T'_{r_2} for $j \in [s_1 + 1, b + 1]$. Clearly, the vertex sums of all non-root vertices are all distinct and $\phi(r_2) > \phi(v)$ for any other $v \in V(T')$. If $\phi(r_1) = \phi(x)$ for some x, we reassign $p + b + s_1$ to r_2, and $p + b + s_1 + 1$ to r_1. So the new $\phi(r_1)$ is increasing by one while the new $\phi(r_2)$ is decreasing by one, but the original difference of the two vertex sums is more than four, so they are distinct.

When $|L_-| \geq 3$, let $L'_- = L_- \setminus \{-(p+1)\}$ and $L'_+ = L_+ \setminus \{p+1\}$. Pick a pair of e_{x_0} and e_{z_0} from T'_{r_1} if $s_1 \geq 2$, or from T'_{r_2} if $s_1 = 1$. Let $T''_{r_i} = T'_{r_i} - \{x_0, z_0\}$ and $e(T''_{r_i}) = 2s'_i$ for $i = 1, 2$. Then $s'_1 < s'_2$ except when $s'_1 = s'_2 = 1$. Since $e(T''_{r_1})$, $e(T''_{r_2})$, $|L'_-|$ and $|L'_+|$ are all even, we use the methods in Case 3.1 to label T''_{r_1} and T''_{r_2}. Define $\phi'(r_i)$ as the sum of the labels in $L'_- \cup L'_+$ assigned for the edges incident to r_i for $i = 1, 2$. Note that we will not have $\phi'(r_1) = 0$, because $\phi(r_1) = 0$ happened in Case 3.1 only when $s_1 = s_2 = 2$. Next assign respectively $p+1$, $-(p+1)$, and 0 to e_r, e_{x_0}, and e_{z_0} if $\phi'(r_1) < \phi'(r_2)$, or to e_{x_0}, e_{z_0}, and e_r if $\phi'(r_2) < \phi'(r_1)$.

For $\phi'(r_1) < \phi'(r_2)$, if $e_{x_0}, e_{z_0} \in T'_{r_1}$, then $\phi(x_0) = -(p+1)$, $\phi(z_0) = 0$, $\phi(r_1) = \phi'(r_1) \neq \phi(v)$ and $\phi(r_2) = \phi'(r_2) + p + 1 > \phi(v)$ for $v \notin \{r_1, r_2\}$. Else, $e_{x_0}, e_{z_0} \in T'_{r_2}$. Then $s'_1 = s_1 = 1$. We have $\phi(x_0) = 0$, $\phi(z_0) = -(p+1)$, and $\phi(r_1) = \phi'(r_1) \neq \phi(v)$ for $v \notin \{r_1, r_2\}$. For $\phi(r_2)$, either $\phi(r_2) = \phi'(r_2) + p + 1 > \phi(v)$ for $v \notin \{r_1, r_2\}$ when $a < b$, or $\phi(r_2) = (p + b + 2) + (p + 1)$ so $p + b + 1 < \phi(r_2) < 2p + b + 4$ when $a = b$. When $\phi'(r_1) > \phi'(r_2)$, we must have $e_{x_0}, e_{z_0} \in T'_{r_1}$. Thereby, $\phi(x_0) = 0$, $\phi(z_0) = -(p+1)$, $\phi(r_1) = \phi'(r_1) + p + 1 > \phi(v)$ and $\phi(r_2) = \phi'(r_2) \neq \phi(v)$ for $v \notin \{r_1, r_2\}$.

3.4. $e(T'_{r_1})$ and $e(T'_{r_2})$ are odd

Let $e(T'_{r_1}) = 2s_1 + 1$ and $e(T'_{r_2}) = 2s_2 + 1$ with $1 \leq s_1 \leq s_2$. In this case, each r_i has a leaf child in T'_{r_i}, so we denote the leaf child of r_i in T'_{r_i} with w_i for $i = 1, 2$.

Case 3.3: $|L_-| = 2a + 1$ *and* $|L_+| = 2b + 1$ *with* $0 \leq a \leq b$. When L_- only contains $-(p+1)$, pick a pair of e_{x_0} and e_{z_0} in T'_{r_1} and assign $p+1$, $-(p+1)$, and 0, respectively to e_{w_1}, e_{x_0}, and e_{z_0}. Since $T' - \{w_1, x_0, z_0\}$ is a tree of diameter at most five, we use the methods in Section 2.1 or Section 3.1 with the labels in L'' to obtain a $(p+1)$-shifted antimagic labeling for $T' - \{w_1, x_0, z_0\}$. Thus, $\phi(w_1) = p+1$, $\phi(x_0) = -(p+1)$, $\phi(z_0) = 0$, and $\phi(v) \geq p + 2$ for any other $v \in V(T')$.

When $|L_-| \geq 3$, let $L'_- = L_- \setminus \{-(p+1)\}$ and $L'_+ = L_+ \setminus \{p+1\}$. Let $T''_{r_i} = T'_{r_i} - w_i$ for $i = 1, 2$. Use the methods in Case 3.1 to label T''_{r_1} and T''_{r_2} with the labels in $L'_- \cup L'_+$, and define $\phi'(r_1)$ and $\phi'(r_2)$ as in Case 3.2. We will assign $\pm(p+1)$ and 0 according to $\phi'(r_1)$ and $\phi'(r_2)$ as before.

If $\phi'(r_2) > \phi'(r_1)$ and $\phi'(r_1) \neq 0$, then we assign $p+1$, $-(p+1)$, and 0 to e_r, e_{w_1}, and e_{w_2}, respectively. Thus, $\phi(w_1) = -(p+1)$, $\phi(w_2) = 0$, $\phi(r_1) = \phi'(r_1) \neq \phi(v)$ and $\phi(r_2) = \phi'(r_2) + p + 1 > \phi(v)$ for any $v \notin \{r_1, r_2\}$. If $a = 1$, $b = 3$, and $s_1 = s_2 = 2$, then we have $\phi'(r_1) = 0$. Now assign $p+1$, $-(p+1)$, and 0 to e_{w_2}, e_r, and e_{w_1}, respectively. Thus, $\phi(w_1) = 0$, $\phi(w_2) = p+1$, $\phi(r_1) = -(p+1)$, and $\phi(r_2) = \phi'(r_2) > \phi(v)$ for any other v. See Figure 3.5 for more details of the labeling. If $\phi'(r_1) > \phi'(r_2)$, then we assign

$p+1$, $-(p+1)$, 0 to e_r, e_{w_2} and e_{w_1}, respectively. Thus, $\phi(w_1) = 0$, $\phi(w_2) = -(p+1)$, $\phi(r_1) = \phi'(r_1) + p + 1 > \phi(v)$ and $\phi(r_2) = \phi'(r_2) \neq \phi(v)$ for any $v \notin \{r_1, r_2\}$.

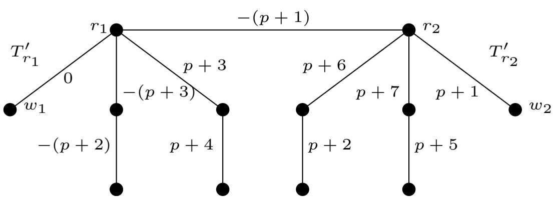


Figure 3.5: When $s'_1 = s'_2 = 2$, $\phi'(r_1) = 0$.

Case 3.4: $|L_-| = 2a$ and $|L_+| = 2b$ with $1 \leq a \leq b$. If L_- only contains $-(p+1)$ and $-(p+2)$, then pick a pair of e_{x_0} and e_{z_0} in T'_{r_1} and assign $p+1$, $p+2$, $-(p+1)$, $-(p+2)$, and 0, respectively to e_{w_1}, e_{w_2}, e_{x_0}, e_{z_0}, and e_r. So, $\phi(w_1) = p+1$, $\phi(w_2) = p+2$, $\phi(x_0) = -(2p+3)$ and $\phi(z_0) = -(p+2)$. Then assign $p+b+j$ and $p+1+j$, respectively, to each pair of e_x and e_z in T'_{r_1} for $j \in [2, s_1]$, and to each pair of e_x and e_z in T'_{r_2} for $j \in [s_1+1, b]$. It is straightforward that the vertex sums of all non-root vertices are all distinct and $\phi(r_2) > \phi(v)$ for any other $v \in V(T')$. Once $\phi(r_1) = \phi(x)$ for some x, we reassign $p+b+s_1-1$ to r_2 and $p+b+s_1$ to r_1. The new $\phi(r_1)$ is increasing by one while the new $\phi(r_2)$ is decreasing by one, but the original difference of the two vertex sums is more than four, so they are distinct.

When $|L_-| \geq 4$, let $L'_- = L_- \setminus \{-(p+1), -(p+2)\}$ and $L'_+ = L_+ \setminus \{p+1, p+2\}$. Pick a pair of e_{x_0} and e_{z_0} from T'_{r_1} if $s_1 \geq 2$ or from T'_{r_2} if $s_1 = 1$. Let $T''_{r_i} = T'_{r_i} - \{w_i, x_0, z_0\}$ and $e(T'_{r_i}) = 2s'_i$ for $i = 1, 2$. Then $s'_1 < s'_2$ except when $s'_1 = s'_2 = 1$. As before, we use the methods in Case 3.1 to label T''_{r_1} and T''_{r_2} with the labels in $L'_- \cup L'_+$, and define $\phi'(r_i)$ as in Case 3.2. Again, we have $\phi'(r_1) \neq 0$. In the following, we fix the label of e_{z_0} to be 0, so $\phi(z_0) = 0$.

Suppose $\phi'(r_1) > 0$. Then $s'_1 \geq 2$. Since if $s'_1 = 1$, then the labels assigned for T''_{r_1} are $-(p+a+2)$ and $-(p+3)$. Thus, $\phi'(r_1) = -(p+a+2) < 0$, which makes a contradiction. Consequently, $e_{x_0}, e_{z_0} \in T'_{r_1}$. Let us first assign $p+2$, $-(p+2)$, $p+1$, and $-(p+1)$ to e_{w_1}, e_{x_0}, e_r, and e_{w_2}. Then we have $\phi(w_1) = p+2$, $\phi(w_2) = -(p+1)$, $\phi(x_0) = -(p+2)$, $\phi(r_1) = \phi'(r_1) + p + 1$, and $\phi(r_2) = \phi'(r_2)$. Once this labeling causes $\phi(r_1) = \phi(r_2)$ or $\phi(r_1) = \phi(x)$ for some $x \in V(T')$, then we reassign $p+2$, $-(p+2)$, $p+1$, and $-(p+1)$ to e_r, e_{w_2}, e_{w_1}, and e_{x_0}, respectively. This changes $\phi(r_1)$ to be $\phi'(r_1) + p + 2$, $\phi(w_1) = p+1$, $\phi(w_2) = -(p+2)$, and $\phi(x_0) = -(p+1)$, which can solve both problems. The only issue is that if it is possible that $\phi(r_1) = \phi(x)$ for some $x \in V(T')$ but after the reassignment it becomes $\phi(r_2)$? In other words, we have $\phi(r_2) = \phi(x) + 1 = \phi(r_1) + 1$ for some $x \in V(T')$ at the beginning. This is impossible because by the remark after Case 3.1, the difference of the largest vertex sum and the largest $\phi(x)$ is at least two.

Next consider $\phi'(r_1) < 0$. Then either $\phi'(r_1) = -(p+a+2)$ when $s'_1 = 1$ or $\phi'(r_1) \leq -(p+2a) - (p+2a-1) < -(2p+3a+1)$ when $s'_1 \geq 2$. If $e_{x_0}, e_{z_0} \in T'_{r_1}$, then we assign $p+2$, $-(p+2)$, $p+1$, and $-(p+1)$ to e_{w_2}, e_r, e_{w_1}, and e_{x_0}, respectively. So, $\phi(w_1) = p+1$, $\phi(w_2) = p+2$, $\phi(x_0) = -(p+1)$, $\phi(r_1) = \phi'(r_1) - (p+2)$, and $\phi(r_2) = \phi'(r_2) \neq \phi(v)$ for any other $v \in V(T')$. Moreover, either $\phi(r_1) = -(p+a+2) - (p+2)$ satisfies $-(2p+a+5) < \phi(r_1) < -(p+a+1)$, or $\phi(r_1) = \phi'(r_1) - (p+2) < \phi(v)$ for any other $v \in V(T')$. If $e_{x_0}, e_{z_0} \in T'_{r_2}$, then we assign $p+2$, $-(p+2)$, $p+1$, and $-(p+1)$ to e_{w_2}, e_{x_0}, e_r, and e_{w_1}, respectively. Thus, $\phi(w_1) = -(p+1)$, $\phi(w_2) = p+2$, $\phi(x_0) = -(p+2)$, $\phi(r_1) = \phi'(r_1) = -(p+a+2)$, and $\phi(r_2) = \phi'(r_2) + p+1$. Then either $\phi(r_2) = (p+b+2)+(p+1)$ satisfies $p+b+1 < \phi(r_2) < 2p+b+5$ when $s_1 = 1$ and $a = b$, or $\phi(r_2) = \phi'(r_2) + p + 1 > \phi(v)$ for all any other $v \in V(T')$.

3.5. $e(T'_{r_1})$ and $e(T'_{r_2})$ have distinct parities

Now exact one of r_1 and r_2 has the leaf child. Denote it by w_0. Pick a pair of e_{x_0} and e_{z_0} from the T'_{r_i} containing w_0. First consider $|L_-| \leq 2$. If $L_- = \{-(p+1)\}$, assign $p+1$, $-(p+1)$, and 0 to e_{w_0}, e_{x_0}, and e_{z_0}. Then label the tree $T' - \{w_0, x_0, z_0\}$ with the labels in $L_+ \setminus \{p+1\}$ using the methods in Section 2.1 or Section 3.1. If $L_- = \{-(p+1), -(p+2)\}$, assign $p+1$, $-(p+1)$, and $-(p+2)$ to e_{w_0}, e_{x_0}, and e_{z_0}. Again, we can label $T' - \{w_0, x_0, z_0\}$ with the labels in $(L_+ \setminus \{p+1\}) \cup \{0\}$ using the methods in Section 2.1 or Section 3.1 by first assigning 0 to an arbitrary pendent edge of $T' - \{w_0, x_0, z_0\}$.

In the sequel, assume $|L_-| \geq 3$. Let $T''_{r_i} = T'_{r_i} - \{w_0, x_0, z_0\}$ and $e(T''_{r_i}) = 2s'_i$ for $i = 1, 2$. Note that it is possible for $s'_1 = 0$. The above operation leads to $e(T''_{r_1}) > e(T''_{r_2})$ if $e(T'_{r_1})$ is even and $e(T'_{r_1}) + 1 = e(T'_{r_2})$. We exchange the indices of the trees to keep the inequality $e(T''_{r_1}) \leq e(T''_{r_2})$. For example, if $e(T'_{r_1}) = 6$ and $e(T'_{r_2}) = 7$, then T'_{r_2} has the leaf child and we shall define $T''_{r_1} = T'_{r_2} - \{w_0, x_0, z_0\}$ and $T''_{r_2} = T'_{r_1}$, but if $e(T'_{r_1}) = 7$ and $e(T'_{r_2}) = 8$, then $T''_{r_1} = T'_{r_1} - \{w_0, x_0, z_0\}$ and $T''_{r_2} = T'_{r_2}$. Throughout Cases 3.5 and 3.6, we assign 0 to e_r.

Case 3.5: $|L_-| = 2a+1$ *and* $|L_+| = 2b$ *with* $1 \leq a \leq b-1$. If $s'_1 = 0$, assign $-(p+a+1)$ to e_{w_0}, $-(p+2a+1)$ to e_{x_0}, and $-(p+a)$ to e_{z_0}. For each pair of e_x and e_z in T''_{r_2}, assign $-(p+a+1+j)$ to e_x and $-(p+j)$ to e_z for some $j \in [1, a-1]$, or alternatively $p+b+j$ to e_x and $p+j$ to e_z for some $j \in [1, b]$. The vertex sums of all non-root vertices in T' are $-(p+a+1)$, $-(2p+a+1+2j)$ and $-(p+j)$ for $j \in [1, a]$, and $p+j$ and $2p+b+2j$ for $j \in [1, b]$. In addition, $\phi(r_1) = -(p+a+1) - (p+2a+1) < -(2p+3a+1)$ and $\phi(r_2) \geq (p+2b-1) + (p+2b) > 2p+3b$.

Now consider $s'_1 \geq 1$. In the following, we fix the labels of e_{w_0}, e_{x_0}, and e_{z_0} to be $-(p+1)$, $p+1$, and $p+b+1$. Then label T''_{r_1} and T''_{r_2} with labels in $L'_- = L_- \setminus \{-(p+1)\}$ and $L'_+ = L_+ \setminus \{p+1, p+b+1\}$. Our methods will be essentially the same as those in

Case 3.1. However, the positive labels are not consecutive as in Case 3.1, so we shall give more details. For each pair of e_x and e_z in $T' - \{w_0, x_0, z_0\}$, assign $-(p+a+j)$ to e_x and $-(p+j)$ to e_z for some $j \in [2, a+1]$, or alternatively assign $p+b+j$ to e_x and $p+j$ to e_z for some $j \in [2, b]$. Then we specify the labels of the edges incident to r_1 or r_2 to evaluate $\phi(r_1)$ and $\phi(r_2)$. Recall that we have $s'_1 + s'_2 = a + b - 1$, $s'_1 \le s'_2$, and $a \le b - 1$, so we give labelings according to the following conditions analogous to Case 3.1: (1) $s'_1 = 1$, (2) $b - a \ge 5$, (3) $b - a = 4$, (4) $s'_1 \le a \le b - 1 \le s'_2$, and (5) $s'_1 = s'_2 = a + 1$. For (2), (3), and (4), the labeling methods follow from those in Case 3.1.

For $s'_1 = 1$, assign $-(p+a+2)$ to r_1 so that $-(p+a+2)-(p+2) < \phi(r_1) < -(p+a+1)$ and $\phi(r_2) \ge (p+2b)+(p+2b-1) > 2p+3b$ if $a < b-1$. If $a = b-1$, we assign $-(p+a+2)$ (resp. $-(p+a+3)$) to r_1 when p and b have different parities (resp. the same parity) so that $\phi(r_2) = p + 2b$ (resp. $\phi(r_2) = p + 2b + 1$) is not equal to $2p + b + 2j$ for $j \in [2, b]$.

When $s'_1 = s'_2 = a + 1$, we have $e(T'_{r_1}) = 2s'_1$, $e(T'_{r_2}) = 2s'_1 + 3$, and $w_0 \in T'_{r_2}$. We use another method to label T'_{r_1} and T'_{r_2} with the labels in $L' \setminus \{0\}$. For T'_{r_1}, assign $-(p+1)$ to e_x and $p+1$ to e_z for some pair of e_x and e_z, and $-(p+a+j)$ and $-(p+j)$ for some $j \in [2, a+1]$ to other pairs of e_x and e_z. On the other hand, assign each pair of e_x and e_z in T'_{r_2} the labels $p+b+j$ and $p+j$ for some $j \in [2, b]$, and assign $p+b+1$ to e_{w_0}. The vertex sums of all non-root vertices in T' are 0, $p+1$, $p+b+1$, $-(2p+a+2j)$ and $-(p+j)$ for $j \in [2, a]$, and $p+j$ and $2p+b+j$ for $j \in [2, b]$. In addition, $\phi(r_1) = -(2p+4)$ so $-(p+3)-(p+2) < \phi(r_1) < -(p+2)$ when $s'_1 = 2$ and $\phi(r_1) \le -(p+2a)-(p+2a-1) < -(2p+3a)$ when $s'_1 \ge 3$. For r_2, we always have $\phi(r_2) \ge (p+2b-1)+(p+2b) > 2p+3b$.

Case 3.6*: $|L_-| = 2a$ and $|L_+| = 2b+1$ with $2 \le a \le b$.* If $s'_1 = 0$, assign $-(p+a)$ to e_{z_0}, $-(p+a+1)$ to e_{w_0}, and $-(p+2a)$ to e_{x_0}. For each pair of e_x and e_z in T''_{r_2}, assign $-(p+a+j)$ to e_x and $-(p+j)$ to e_z for some $j \in [2, a-1]$, or alternatively assign $p+b+j$ to e_x and $p+j$ to e_z for some $j \in [2, b+1]$. Also, assign $p+1$ to e_x and $-(p+1)$ to e_z for the last pair of e_x and e_z. The vertex sums of all non-root vertices in T' are 0, $-(p+1)$, $-(p+a+1)$, $-(2p+a+2j)$ and $-(p+j)$ for $j \in [2, a]$, and $2p+b+2j$ and $p+j$ for $j \in [2, b+1]$. In addition, $\phi(r_1) = -(p+2a)-(p+a+1) < -(2p+3a)$ and $\phi(r_2) \ge (p+2b+1)+(p+2b)+(p+1) > 2p+3b+2$.

Next suppose $s'_1 \ge 1$. Fix the labels of e_{w_0}, e_{x_0}, and e_{z_0} to be $p+1$, $-(p+1)$, and $-(p+a+1)$. For each pair of e_x and e_z in $T' - \{w_0, x_0, z_0\}$, assign $-(p+a+j)$ to e_x and $-(p+j)$ to e_z for some $j \in [2, a]$, or alternatively assign $p+b+j$ to e_x and $p+j$ to e_z for some $j \in [2, b+1]$. Then specify the labels of the edges incident to r_1 or r_2 to evaluate $\phi(r_1)$ and $\phi(r_2)$. We have $s'_1 + s'_2 = a + b - 1$, $s'_1 \le s'_2$, and $a \le b$, therefore we give labeling methods according to the following conditions: (1) $s'_1 = 1$, (2) $b - a \ge 3$, (3) $b - a = 2$, (4) $s'_1 \le a - 1 \le b \le s'_2$, and (5) $s'_1 = s'_2 = a$. For (1), (2), (3), and (4), the labeling methods follow from those in Case 3.1.

As Case 3.5, $s'_1 = s'_2 = a$ implies that originally $e(T'_{r_1}) = 2s'_1$, $e(T'_{r_2}) = 2s'_1 + 3$, and $w_0 \in T'_{r_2}$. For each pair of of e_x and e_z in T'_{r_1}, assign the negative labels $-(p + a + j)$ to e_x and $-(p + j)$ to e_z for some $j \in [1, a]$. On the other hand, assign each pair of e_x and e_z in T'_{r_2} the positive labels $p + b + 1 + j$ and $p + j$, respectively, for some $j \in [1, b]$ and assign e_{w_0} the label $p + b + 1$. It is straightforward to see the vertex sums are distinct for all vertices in T'.

4. Concluding remarks

Kaplan, Lev, and Roddity [10] (with a minor error in the proof) and Liang, Wong, and Zhu [11] proved that every tree on at least three vertices containing at most one vertex of degree two is antimagic. The ideas of their proofs are technically partitioning the label set $L = [1, m]$ into blocks of sizes two of tree so that the sum of the labels in each block is m or $2m$. Then view the tree as a rooted tree and assign the labels of the blocks to the edges connecting a vertex and its children. Thus, the edge connecting the vertex to its parent determines the uniqueness of the vertex sum. The way we reduce the negative labels is inspired by their method.

In Section 2.3, we obtained the reduced trees which are the special type of spiders. It has been independently proved by Huang [9] and Shang [13] that every spider is antimagic. Nonetheless, it is unknown in general that given a spider and an integer k, whether or not the spider is k-shifted antimagic. Recently, Chang, Li, Liu, and Pan [3] investigated the forests consisting of some special types of spiders. Particularly, they showed that if the length of each leg is either one or an even number for each spider in the forest, then it is k-shifted antimagic for all $k \geq 0$. Also, if the length of each leg is at least two for each spider in the forest, then it is k-shifted antimagic for all $k \geq 0$. For a spider G, if the length of each leg of G is at most three, and at most one leg of G is of length three, then $\mathrm{diam}(G)$ is equal to 4 or 5. Hence, a minor consequence from our results is that a spider G satisfying the above conditions on the lengths of the legs is k-shifted antimagic for every integer k, unless G contains exactly two legs of length two and one leg of length one. Namely, G is P'_5.

Our original motivation is to find an answer for Question 1.6, but our results eventually deny such examples of trees with diameter five. How about trees of higher diameters? For trees of diameter four, two internal vertices other than the root do not share a common edge. Our strategy assigns first the small labels to the pendent edges, then the large labels to the internal edges according to the partial vertex sums of the non-root endpoints of the edges. This gives the monotonicity and distinctness for the vertex sums of the internal vertices, and we will have at most a pair of trouble vertices, the root and some internal vertex. The labeling strategy for trees of diameter five is similar, and it causes at most two pairs of trouble vertices. For a tree of diameter greater than five, we could designate some vertex as the root and view it as the rooted tree as before. Let us call a vertex adjacent to the root but not adjacent to any leaf a *type-1* vertex, and a vertex adjacent to the leaves but not adjacent to the root a *type-2* vertex. Vertices of different types may share common edges, so their vertex sums will interfere with each other. Suppose that many type-1 vertices have degree two and many type-2 vertices have degree three. If we label the pendent edges with the small labels and the internal edges with the large labels, then we are likely to obtain a vertex of type-1 and a vertex of type-2 that have the same vertex sum. On the contrary, if we label the pendent edges with the large labels and the internal edges with the small labels, then we are likely to obtain a leaf and a vertex of type-1 that have the same vertex sum. Hence, our method cannot be generalized to trees of higher diameters without some new ideas. However, we believe that not only Conjecture 1.2 is true but also every tree of diameter at least five is k-shifted antimagic for every integer k.

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