TAIWANESE JOURNAL OF MATHEMATICS
Vol. 27, No. 2, pp. 291–319, April 2023
DOI: 10.11650/tjm/221001

Oscillation and Nonoscillation for Two-dimensional Nonlinear Systems of Ordinary Differential Equations

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Abstract. For the two-dimensional nonlinear system

$$u' = a(t)|v|^{1/\alpha} \operatorname{sgn} v, \quad v' = -b(t)|u|^{\alpha} \operatorname{sgn} u$$

with $\alpha > 0$, $a, b \in C[t_0, \infty)$, $a(t) \geq 0$ ($t \geq t_0$), new oscillation criteria and nonoscillation criteria are given in both cases $\int_{t_0}^{\infty} a(s)\,ds = \infty$ and $\int_{t_0}^{\infty} a(s)\,ds < \infty$. One of the main results is an analogue of the Hartman–Wintner oscillation theorem. Our results generalize Li and Yeh's results for second order half-linear scalar equations.

1. Introduction

In this paper we consider the two-dimensional nonlinear system of ordinary differential equations

$$u' = a(t)|v|^{1/\alpha} \operatorname{sgn} v, \quad v' = -b(t)|u|^{\alpha} \operatorname{sgn} u, \tag{1.1}$$

where α is a positive constant, and $a(t)$ and $b(t)$ are real-valued continuous functions on $[t_0, \infty)$ and

$$a(t) \begin{cases} \geq 0 & \text{for } t \geq t_0, \\ \not\equiv 0 & \text{on } [t_0^+, \infty) \text{ for any } t_0^+ \geq t_0. \end{cases} \tag{1.2}$$

By a solution $(u(t), v(t))$ of the system (1.1) on an interval $I \subseteq [t_0, \infty)$ we mean that $u(t)$ and $v(t)$ are continuously differentiable on I and satisfy (1.1) at every point $t \in I$.

It is known (Mirzov [13, Lemma 2.1]) that all local solutions of (1.1) can be continued to t_0 and ∞, and so all solutions of (1.1) exist on the entire interval $[t_0, \infty)$. Clearly, if $(u(t), v(t))$ is a solution of (1.1), then so is $(-u(t), -v(t))$. It is also known (Mirzov [13, Lemma 1.1]) that if a solution $(u(t), v(t))$ of (1.1) satisfies

$$(u(t_1), v(t_1)) = (0, 0) \quad \text{for some } t_1 \geq t_0,$$

then $(u(t), v(t)) \equiv (0, 0)$ for $t \geq t_0$.

Received June 22, 2022; Accepted October 11, 2022.
Communicated by Cheng-Hsiung Hsu.
2020 *Mathematics Subject Classification.* 34C10.
Key words and phrases. half-linear system, oscillation, nonoscillation, Hartman–Wintner theorem.

The following remark is useful. Let $(u(t), v(t))$ be a solution of (1.1) such that

$$u(t_1) = u(t_2) = 0 \quad \text{and} \quad u(t) \neq 0 \text{ on } (t_1, t_2),$$

where $t_0 \leq t_1 < t_2 < \infty$. Then, $v(t)$ has at least one zero on (t_1, t_2). To prove this, assume the contrary that $v(t) \neq 0$ on (t_1, t_2). We may suppose that $v(t) > 0$ on (t_1, t_2). Then, since $a(t) \geq 0$ for $t \geq t_0$, it follows from the first equation in (1.1) that $u(t)$ is nondecreasing on (t_1, t_2). Since $u(t_1) = u(t_2) = 0$, this implies that $u(t) \equiv 0$ on $[t_1, t_2]$, which is a contradiction to the condition $u(t) \neq 0$ on (t_1, t_2).

Following the paper of Dosoudilová, Lomtatidze and Šremr [2], we say that a solution $(u(t), v(t))$ of the system (1.1) is *nontrivial* if $u(t) \not\equiv 0$ on any neighborhood of infinity, and that a nontrivial solution $(u(t), v(t))$ of (1.1) is *oscillatory* if $u(t)$ has a sequence of zeros tending to infinity, and *nonoscillatory* otherwise. By the preceding remark, it is easily seen that if $(u(t), v(t))$ is an oscillatory solution of (1.1), then the function $v(t)$ also has a sequence of zeros tending to infinity.

It is worth noting here that, for any nontrivial solution $(u(t), v(t))$ of (1.1), the sequence of zeros of $u(t)$ cannot have a finite cluster point. To see this, assume the contrary that $u(t)$ has a sequence of zeros $\{t_i\}_{i=1}^{\infty}$ such that $\lim t_i = t_\infty \in \mathbb{R}$ as $i \to \infty$. We may suppose that $t_0 \leq t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots$, $u(t_i) = u(t_{i+1}) = 0$ and $u(t) \neq 0$ on (t_i, t_{i+1}) $(i = 1, 2, 3, \ldots)$. Then, by the preceding remark, there are τ_i such that $t_i < \tau_i < t_{i+1}$ and $v(\tau_i) = 0$ $(i = 1, 2, 3, \ldots)$. It is clear that $u(t_\infty) = v(t_\infty) = 0$. Hence, from the result of Mirzov [13, Lemma 1.1] it follows that $(u(t), v(t)) \equiv (0, 0)$ for $t \geq t_0$. This is a contradiction.

An analogue of Sturm's comparison theorem was established by Mirzov [13, Theorem 1.1]. A simple version of the result is the following.

Theorem 1.1. (Mirzov [13]) *Consider the system* (1.1) *and another system of the same type*

$$u_1' = a_1(t)|v_1|^{1/\alpha} \operatorname{sgn} v_1, \quad v_1' = -b_1(t)|u_1|^{\alpha} \operatorname{sgn} u_1. \tag{1.3}$$

Suppose that

$$0 \leq a(t) \leq a_1(t) \quad \textit{and} \quad b(t) \leq b_1(t) \;\; \textit{for } t \geq t_0.$$

If (1.1) *has a solution* $(u(t), v(t))$ *such that*

$$u(t_1) = u(t_2) = 0 \quad \textit{and} \quad u(t) \neq 0 \;\; \textit{for } t \in (t_1, t_2),$$

then, for any solution $(u_1(t), v_1(t))$ *of* (1.3)*, the first component* $u_1(t)$ *has at least one zero on the interval* $[t_1, t_2]$.

In particular, if the system (1.1) has an oscillatory solution, then any other nontrivial solution is also oscillatory. Therefore, if the system (1.1) has a nonoscillatory solution, then any other nontrivial solution is also nonoscillatory. The system (1.1) is said to be oscillatory (resp. nonoscillatory) if all of its nontrivial solutions are oscillatory (resp. nonoscillatory).

If $a(t) > 0$ for $t \geq t_0$, then the first component $u(t)$ of a solution $(u(t), v(t))$ of the system (1.1) is a solution of the scalar differential equation

$$(a(t)^{-\alpha}|u'|^{\alpha} \operatorname{sgn} u')' + b(t)|u|^{\alpha} \operatorname{sgn} u = 0.$$

Conversely, for a solution $u(t)$ of the above scalar differential equation,

$$(u(t), v(t)) = (u(t), a(t)^{-\alpha}|u'(t)|^{\alpha} \operatorname{sgn} u'(t))$$

is a solution of the system (1.1). Putting $p(t) = a(t)^{-\alpha}$ and $q(t) = b(t)$, we rewrite the above scalar equation in the form

$$(p(t)|u'|^{\alpha} \operatorname{sgn} u')' + q(t)|u|^{\alpha} \operatorname{sgn} u = 0, \tag{1.4}$$

where $p(t)$ and $q(t)$ are continuous functions on $[t_0, \infty)$ and $p(t) > 0$ for $t \geq t_0$. The equation (1.4) is referred as "half-linear" equation. If $\alpha = 1$, then (1.4) becomes the linear equation

$$(p(t)u')' + q(t)u = 0. \tag{1.5}$$

In the last three decades, many results have been obtained in the theory of oscillatory and asymptotic behavior of solutions of the half-linear equation (1.4). It is known that basic results for the linear equation (1.5) can be generalized to the half-linear equation (1.4). The important works for (1.4) are summarized in the book of Došlý and Řehák [1]. For the recent results to the half-linear equation (1.4) we refer the papers [5, 7, 15–18, 22–24]. For the results to the nonlinear system (1.1) (including the linear system) we refer the papers [2, 6, 11–14, 19–21].

For simplicity consider the linear equation

$$u'' + q(t)u = 0, \tag{1.6}$$

which is the case of $p(t) \equiv 1$ in (1.5). The well-known oscillation criterion of Hartman–Wintner is as follows. If

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} (t-s)q(s)\,ds = \infty,$$

or if

$$-\infty < \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} (t-s)q(s)\,ds < \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} (t-s)q(s)\,ds,$$

then (1.6) is oscillatory (Hartman [3, Theorem 7.3, Chapter XI]). This result can be generalized to the half-linear equation (1.4) (see [1, Theorem 2.2.10]) and to the nonlinear system (1.1) (see Dosoudilová et al. [2]).

In this paper we present new oscillation criteria and nonoscillation criteria for the nonlinear system (1.1). The new oscillation criteria are slightly different from the results of Dosoudilová et al. [2]. For the half-linear scalar equation (1.4) it is usual to distinguish the cases

$$\int_{t_0}^{\infty} p(s)^{-1/\alpha}\, ds = \infty \tag{1.7}$$

and

$$\int_{t_0}^{\infty} p(s)^{-1/\alpha}\, ds < \infty. \tag{1.8}$$

In the system (1.1) these cases correspond to

$$\int_{t_0}^{\infty} a(s)\, ds = \infty \tag{1.9}$$

and

$$\int_{t_0}^{\infty} a(s)\, ds < \infty, \tag{1.10}$$

respectively.

For the nonlinear system (1.1) we define a number σ and a function $f(t)$ as follows. For the case where (1.9) holds

$$\sigma = +1, \quad f(t) = \int_{t_0}^{t} a(s)\, ds, \quad t \geq t_0, \tag{1.11}$$

and for the case where (1.10) holds

$$\sigma = -1, \quad f(t) = \int_{t}^{\infty} a(s)\, ds, \quad t \geq t_0. \tag{1.12}$$

It is clear that in the former case $f(t) > 0$ for all large t, and that in the latter case the condition (1.2) implies $f(t) > 0$ for all $t \geq t_0$. In either case there is a number $t_1 \geq t_0$ such that

$$f(t) > 0, \quad t \geq t_1. \tag{1.13}$$

Hereafter we suppose that $t_1 \geq t_0$ is a number satisfying (1.13). It is obvious that $f(t)^\sigma > 0$ for $t \geq t_1$ and

$$f'(t) = \sigma a(t), \;\; t \geq t_1 \quad \text{and} \quad \lim_{t\to\infty} f(t)^\sigma = \infty.$$

By making good use of the above σ and $f(t)$, we can treat the cases (1.9) and (1.10) in a unified manner.

To describe our results, we introduce the function classes $\mathcal{W}$ and $\mathcal{W}_0$ with respect to the system (1.1). Members of $\mathcal{W}$ and $\mathcal{W}_0$ will be used as weight functions. Define σ and $f(t)$ by (1.11) or (1.12) according as (1.9) or (1.10) holds, and let $t_1 \geq t_0$ be a number satisfying (1.13). Further, let λ be a number such that $\sigma(\lambda-\alpha) < 0$, and let ξ be a number such that $0 \leq \xi < 1/\alpha$. Then we denote by $\mathcal{W}$ the set of all locally integrable functions $w(t)$ on $[t_1, \infty)$ such that $w(t) \geq 0$ for $t \geq t_1$ and $a(t)w(t) \not\equiv 0$ on $[t_1^+, \infty)$ for any $t_1^+ \geq t_1$, and either

$$\int^{\infty} a(s)w(s)\Psi(s)\,ds = \infty \tag{1.14}$$

or

$$\limsup_{t\to\infty} \left(\int^t a(s)w(s)\,ds\right)^{-\xi+(1/\alpha)} \int_t^{\infty} a(s)w(s)\Psi(s)\,ds > 0, \tag{1.15}$$

where

$$\Psi(t) = \left(\int^t a(s)w(s)\,ds\right)^{\xi} \left(\int^t a(s)w(s)^{\alpha+1} f(s)^{\lambda}\,ds\right)^{-1/\alpha}.$$

Further, we denote by $\mathcal{W}_0$ the set of all locally integrable functions $w(t)$ on $[t_1, \infty)$ such that $w(t) \geq 0$ for $t \geq t_1$ and $a(t)w(t) \not\equiv 0$ on $[t_1^+, \infty)$ for any $t_1^+ \geq t_1$, and

$$\lim_{t\to\infty} \left(\int^t a(s)w(s)\,ds\right)^{-\alpha-1} \int^t a(s)w(s)^{\alpha+1} f(s)^{\lambda}\,ds = 0. \tag{1.16}$$

If $a(t) \equiv 1$ and $\lambda = 0$, then the sets $\mathcal{W}$ and $\mathcal{W}_0$ coincide with the sets $\mathfrak{I}$ and $\mathfrak{I}_0$ in Li and Yeh [9], and the sets $\mathcal{J}$ and $\mathcal{J}_0$ in Došlý and Řehák [1, pp. 91–92], respectively.

It is seen that if $w \in \mathcal{W}$, then

$$\int^{\infty} a(t)w(t)\,dt = \infty. \tag{1.17}$$

Moreover it can be proved without difficulty that

$$w \in \mathcal{W}_0 \quad\Longrightarrow\quad w \in \mathcal{W}, \quad \text{i.e., } \mathcal{W}_0 \subseteq \mathcal{W}.$$

Therefore, if $w \in \mathcal{W}_0$, then (1.17) holds. This fact can also be checked from the definition of $\mathcal{W}_0$. In general, $\mathcal{W}_0 \subsetneq \mathcal{W}$. Indeed, for the case $a(t) \equiv 1$, $\alpha = 1$ and $\lambda = 0$ we have $e^t \notin \mathcal{W}_0$ and $e^t \in \mathcal{W}$.

Let λ be a number satisfying $\sigma(\lambda - \alpha) < 0$. Let $w(t)$ be a locally integrable function on $[t_1, \infty)$ which satisfies $w(t) \geq 0$ for $t \geq t_1$ and $a(t)w(t) \not\equiv 0$ on $[t_1^+, \infty)$ for any $t_1^+ \geq t_1$, and (1.17) holds. If $w(t)$ satisfies the additional condition

$$w(t)f(t)^{\lambda/\alpha} \text{ is bounded on } [t_1, \infty),$$

or, more generally,

$$\lim_{t\to\infty} w(t)f(t)^{\lambda/\alpha} \left(\int^t a(s)w(s)\,ds\right)^{-1} = 0, \tag{1.18}$$

then $w \in \mathcal{W}_0$. Hence it is found that if $w(t)$ satisfies $w(t) \sim kf(t)^\rho$ with $k > 0$, $\sigma(\rho+1) > 0$ and $\sigma(\lambda - \alpha) < 0$, then $w \in \mathcal{W}_0$, and if $w(t)$ satisfies $w(t) \sim kf(t)^{-1}$ with $k > 0$ and $\sigma(\lambda - \alpha) < 0$, then $w \in \mathcal{W}_0$. The proofs of these facts are left to the reader.

Let σ, $f(t)$ and t_1 be as above, and suppose that λ and ξ satisfy $\sigma(\lambda - \alpha) < 0$ and $0 \le \xi < 1/\alpha$, respectively. Let $w \in \mathcal{W}$ or $w \in \mathcal{W}_0$. Then we set

$$C(t; w, \lambda) = \left(\int_{t_1}^t a(s)w(s)\,ds\right)^{-1} \int_{t_1}^t a(s)w(s) \left(\int_{t_1}^s f(r)^\lambda b(r)\,dr\right) ds$$

on a neighborhood of infinity. We will prove the following results.

Theorem 1.2. *Let σ, $f(t)$ and t_1 be as above, and suppose that λ and ξ satisfy $\sigma(\lambda-\alpha) < 0$ and $0 \le \xi < 1/\alpha$, respectively. Suppose moreover that there is a function $w \in \mathcal{W}$ such that*

$$\liminf_{t\to\infty} C(t; w, \lambda) > -\infty. \tag{1.19}$$

If there is a function $w_0 \in \mathcal{W}_0$ such that $C(t; w_0, \lambda)$ does not possess a finite limit as $t \to \infty$, then the system (1.1) *is oscillatory.*

In Theorem 1.2, taking $w = w_0 \in \mathcal{W}_0 \subseteq \mathcal{W}$, we find that if there is a function $w_0 \in \mathcal{W}_0$ such that $\liminf C(t; w_0, \lambda) > -\infty$ as $t \to \infty$, and $C(t; w_0, \lambda)$ does not possess a finite limit as $t \to \infty$, then (1.1) is oscillatory. Therefore we have the following corollary, which gives an analogue of the Hartman–Wintner oscillation theorem.

Corollary 1.3. *Let σ, $f(t)$ and t_1 be as above. Suppose that λ satisfies $\sigma(\lambda - \alpha) < 0$. If there is a function $w_0 \in \mathcal{W}_0$ such that*

$$\lim_{t\to\infty} C(t; w_0, \lambda) = \infty$$

or

$$-\infty < \liminf_{t\to\infty} C(t; w_0, \lambda) < \limsup_{t\to\infty} C(t; w_0, \lambda), \tag{1.20}$$

then the system (1.1) *is oscillatory.*

In Corollary 1.3, letting $w_0(t) = f(t)^\rho \in \mathcal{W}_0$ with $\sigma(\rho + 1) > 0$, we have the following result.

Corollary 1.4. *Let σ, $f(t)$ and t_1 be as above, and suppose that λ satisfies $\sigma(\lambda-\alpha)<0$. Suppose moreover that there is ρ such that $\sigma(\rho+1)>0$ and*

$$\lim_{t\to\infty}\frac{1}{f(t)^{\rho+1}}\int_{t_1}^{t}a(s)f(s)^{\rho}\left(\int_{t_1}^{s}f(r)^{\lambda}b(r)\,dr\right)ds=\infty$$

or

$$\begin{aligned}-\infty<\liminf_{t\to\infty}\frac{1}{f(t)^{\rho+1}}\int_{t_1}^{t}a(s)f(s)^{\rho}\left(\int_{t_1}^{s}f(r)^{\lambda}b(r)\,dr\right)ds\\<\limsup_{t\to\infty}\frac{1}{f(t)^{\rho+1}}\int_{t_1}^{t}a(s)f(s)^{\rho}\left(\int_{t_1}^{s}f(r)^{\lambda}b(r)\,dr\right)ds.\end{aligned}$$

Then the system (1.1) *is oscillatory.*

Similarly, letting $w_0(t)=1/f(t)\in\mathcal{W}_0$ in Corollary 1.3, we have the following result.

Corollary 1.5. *Let σ, $f(t)$ and t_1 be as above, and suppose that λ satisfies $\sigma(\lambda-\alpha)<0$. Suppose moreover that*

$$\lim_{t\to\infty}\frac{1}{\log f(t)^{\sigma}}\int_{t_1}^{t}\frac{a(s)}{f(s)}\left(\int_{t_1}^{s}f(r)^{\lambda}b(r)\,dr\right)ds=\infty$$

or

$$\begin{aligned}-\infty<\liminf_{t\to\infty}\frac{1}{\log f(t)^{\sigma}}\int_{t_1}^{t}\frac{a(s)}{f(s)}\left(\int_{t_1}^{s}f(r)^{\lambda}b(r)\,dr\right)ds\\<\limsup_{t\to\infty}\frac{1}{\log f(t)^{\sigma}}\int_{t_1}^{t}\frac{a(s)}{f(s)}\left(\int_{t_1}^{s}f(r)^{\lambda}b(r)\,dr\right)ds.\end{aligned}$$

Then the system (1.1) *is oscillatory.*

Corollary 1.4 has been proved by Dosoudilová et al. [2, Corollaries 2.5 and 2.11]. In the present paper a different proof from [2] is given. Corollary 1.5 seems to be new.

Theorem 1.6. *Let σ, $f(t)$ and t_1 be as above, and suppose that λ satisfies $\sigma(\lambda-\alpha)<0$.*

(I) *If there are $w_0,w_1\in\mathcal{W}_0$ such that*

$$\liminf_{t\to\infty}C(t;w_0,\lambda)<\liminf_{t\to\infty}C(t;w_1,\lambda),\tag{1.21}$$

then the system (1.1) *is oscillatory.*

(II) *If there are $w_0,w_1\in\mathcal{W}_0$ such that*

$$-\infty<\liminf_{t\to\infty}C(t;w_0,\lambda)<\limsup_{t\to\infty}C(t;w_1,\lambda),\tag{1.22}$$

then the system (1.1) *is oscillatory.*

If $a(t) \equiv 1$ and w_0, w_1 are nonnegative bounded functions satisfying

$$\int^{\infty} w_0(t)\,dt = \int^{\infty} w_1(t)\,dt = \infty,$$

then $w_0, w_1 \in \mathcal{W}_0$ with $\lambda = 0$. Therefore Theorem 1.6(I) gives an extension of the result by Li and Yeh [9, Corollary 3.2]. (Note that the scalar half-linear equation (1.4) is regarded as a special case of the system (1.1) with $a(t) = p(t)^{-1/\alpha}$ and $b(t) = q(t)$.)

In the statement (II) of Theorem 1.6, the condition (1.22) of the case $w_1 = w_0$ becomes the condition (1.20) in Corollary 1.3.

By Corollary 1.4 (or Corollary 1.5) we find that if there is a constant λ such that $\sigma(\lambda - \alpha) < 0$ and

$$\lim_{t\to\infty} \int_{t_1}^{t} f(s)^{\lambda} b(s)\,ds = \int_{t_1}^{\infty} f(s)^{\lambda} b(s)\,ds = \infty, \tag{1.23}$$

then (1.1) is oscillatory. A typical counter condition to (1.23) is

$$\lim_{t\to\infty} \int_{t_0}^{t} f(s)^{\alpha} b(s)\,ds = \int_{t_0}^{\infty} f(s)^{\alpha} b(s)\,ds \quad \text{exists and is finite,} \tag{1.24}$$

which plays an important role for the nonoscillation of (1.1). In fact, it can be proved that (1.24) is sufficient for (1.1) to be nonoscillatory.

Theorem 1.7. *Let $f(t)$ be as above. If* (1.24) *holds, then the system* (1.1) *is nonoscillatory.*

Theorems 1.2, 1.6 and 1.7 are proved in the next section. Since the scalar half-linear equation (1.4) is regarded as a special case of the system (1.1) with $a(t) = p(t)^{-1/\alpha}$ and $b(t) = q(t)$, the results for (1.1) automatically produce the corresponding ones for (1.4). In Section 3 we state the oscillatory and nonoscillatory results for (1.4). Several examples illustrating our results are presented in Section 4.

2. Proofs of theorems

Lemma 2.1. *Let $\varphi(t)$ be a continuous function on $[T_0, \infty)$ such that $\varphi(t) \geq 0$ for $t \geq T_0$ and*

$$\int_{T_0}^{\infty} \varphi(s)\,ds = \infty,$$

and let $\psi(t)$ be a continuous function on $[T_0, \infty)$. For $T \geq T_0$, define the function $F(t,T)$ on a neighborhood of infinity by

$$F(t,T) = \left(\int_T^t \varphi(s)\,ds \right)^{-1} \int_T^t \varphi(s) \left(\int_T^s \psi(r)\,dr \right) ds.$$

Let $T_1 \geq T_0$ and $T_2 \geq T_0$. Then

(i) $\lim_{t\to\infty} F(t,T_2) = L_2 \in \mathbb{R} \implies \lim_{t\to\infty} F(t,T_1) = \int_{T_1}^{T_2} \psi(s)\,ds + L_2,$

(ii) $\lim_{t\to\infty} F(t,T_2) = \infty\ [-\infty] \implies \lim_{t\to\infty} F(t,T_1) = \infty\ [-\infty],$

(iii) $\limsup_{t\to\infty} F(t,T_2) < \infty \implies \limsup_{t\to\infty} F(t,T_1) < \infty,$

(iv) $\liminf_{t\to\infty} F(t,T_2) > -\infty \implies \liminf_{t\to\infty} F(t,T_1) > -\infty.$

Proof. Since

$$\begin{aligned}F(t,T_1) = \int_{T_1}^{T_2} \psi(s)\,ds &+ \left(\int_{T_1}^{t} \varphi(s)\,ds\right)^{-1} \int_{T_1}^{T_2} \varphi(s) \left(\int_{T_2}^{s} \psi(r)\,dr\right) ds \\ &+ \left(\int_{T_1}^{t} \varphi(s)\,ds\right)^{-1} \left(\int_{T_2}^{t} \varphi(s)\,ds\right) F(t,T_2)\end{aligned}$$

for all large t, the assertions (i)–(iv) are clear. The proof is complete. □

Now, as in the preceding section, we define σ and $f(t)$ by (1.11) or (1.12) according as (1.9) or (1.10) holds, and let $t_1 \ge t_0$ be a number satisfying (1.13).

Suppose that the system (1.1) has a nonoscillatory solution $(u(t), v(t))$ such that $u(t) > 0$ for $t \ge T$ ($\ge t_1$). Define the function $R(t)$ by

$$R(t) = \frac{v(t)}{u(t)^{\alpha}}, \quad t \ge T. \tag{2.1}$$

It is easily seen that $R(t)$ satisfies the generalized Riccati differential equation

$$R'(t) = -b(t) - \alpha a(t)|R(t)|^{(\alpha+1)/\alpha}, \quad t \ge T.$$

This gives

$$\int_{\tau}^{t} f(s)^{\lambda} R'(s)\,ds = -\int_{\tau}^{t} f(s)^{\lambda} b(s)\,ds - \alpha \int_{\tau}^{t} a(s) f(s)^{\lambda} |R(s)|^{(\alpha+1)/\alpha}\,ds$$

for $t \ge \tau \ge T$. Using integration by parts on the left-hand side of the above, we find that

$$\begin{aligned}f(t)^{\lambda} R(t) = f(\tau)^{\lambda} R(\tau) &- \int_{\tau}^{t} f(s)^{\lambda} b(s)\,ds + \sigma\lambda \int_{\tau}^{t} a(s) f(s)^{\lambda-1} R(s)\,ds \\ &- \alpha \int_{\tau}^{t} a(s) f(s)^{\lambda} |R(s)|^{(\alpha+1)/\alpha}\,ds, \quad t \ge \tau \ge T.\end{aligned} \tag{2.2}$$

For brevity, we put

$$I(t,\tau) = \int_{\tau}^{t} a(s) f(s)^{\lambda} |R(s)|^{(\alpha+1)/\alpha}\,ds, \quad t \ge \tau \ge T. \tag{2.3}$$

Let λ and ξ satisfy $\sigma(\lambda-\alpha)<0$ and $0\le\xi<1/\alpha$, respectively, and let $w\in\mathcal{W}$ or $w\in\mathcal{W}_0$. Then it follows from (2.2) and (2.3) that

$$
\begin{aligned}
&\int_\tau^t a(s)w(s)f(s)^\lambda R(s)\,ds \\
&= f(\tau)^\lambda R(\tau)\int_\tau^t a(s)w(s)\,ds-\int_\tau^t a(s)w(s)\left(\int_\tau^s f(r)^\lambda b(r)\,dr\right)ds \\
&+\sigma\lambda\int_\tau^t a(s)w(s)\left(\int_\tau^s a(r)f(r)^{\lambda-1}R(r)\,dr\right)ds \\
&-\alpha\int_\tau^t a(s)w(s)I(s,\tau)\,ds, \quad t\ge\tau\ge T.
\end{aligned}
\tag{2.4}
$$

We will divide the proofs of Theorems 1.2 and 1.6 into two steps. The first step is the following lemma.

Lemma 2.2. *Let σ, $f(t)$, t_1, λ and ξ be as above. Suppose that there is a function $w\in\mathcal{W}$ such that* (1.19) *holds. If the system* (1.1) *has a nonoscillatory solution* $(u(t),v(t))$ *such that $u(t)>0$ for $t\ge T$ $(\ge t_1)$, then*

$$\int_T^\infty a(s)f(s)^\lambda|R(s)|^{(\alpha+1)/\alpha}\,ds<\infty, \tag{2.5}$$

where $R(t)$ is defined by (2.1).

Proof. Let $w(t)$ be a function which belongs to $\mathcal{W}$ and satisfies (1.19). For the meanwhile, we suppose that $\tau\ge t_1$ is an arbitrary number such that

$$a(\tau)w(\tau)>0. \tag{2.6}$$

This gives

$$\int_\tau^t a(s)w(s)\,ds>0 \quad \text{for all } t>\tau.$$

Then we define the functions $A(t,\tau)$ and $B(t,\tau)$ on $[\tau,\infty)$ by

$$A(t,\tau)=\int_\tau^t a(s)w(s)\,ds, \quad t\ge\tau,$$

and

$$B(t,\tau)=\frac{1}{A(t,\tau)}\int_\tau^t a(s)w(s)\left(\int_\tau^s f(r)^\lambda b(r)\,dr\right)ds, \quad t\ge\tau.$$

Here, the value of $B(\tau,\tau)$ is interpreted as 0.

To prove (2.5), assume on the contrary that

$$\int_T^\infty a(s)f(s)^\lambda|R(s)|^{(\alpha+1)/\alpha}\,ds=\infty. \tag{2.7}$$

Without loss of generality we can suppose that $a(T)w(T) > 0$. By using the above functions $A(t,\tau)$ and $B(t,\tau)$, the equality (2.4) is written as

$$\begin{aligned}\int_\tau^t a(s)w(s)f(s)^\lambda R(s)\,ds &= [f(\tau)^\lambda R(\tau) - B(t,\tau)]A(t,\tau)\\ &\quad + \sigma\lambda \int_\tau^t a(s)w(s)\left(\int_\tau^s a(r)f(r)^{\lambda-1}R(r)\,dr\right) ds\\ &\quad - \alpha \int_\tau^t a(s)w(s)I(s,\tau)\,ds, \quad t \ge \tau \ge T.\end{aligned} \tag{2.8}$$

Applying Lemma 2.1(iv) to the case $T_0 = T_2 = t_1$, $T_1 = T$, $\varphi(t) = a(t)w(t)$ and $\psi(t) = f(t)^\lambda b(t)$, we see that the assumption (1.19) implies $\liminf B(t,T) > -\infty$ $(t \to \infty)$. Therefore there exists a positive constant M such that

$$B(t,T) \ge -M \quad \text{for all } t \ge T. \tag{2.9}$$

As in the equality (2.2), we get

$$\begin{aligned}f(\tau)^\lambda R(\tau) &= f(T)^\lambda R(T) - \int_T^\tau f(s)^\lambda b(s)\,ds\\ &\quad + \sigma\lambda \int_T^\tau a(s)f(s)^{\lambda-1}R(s)\,ds - \alpha I(\tau,T), \quad \tau \ge T,\end{aligned} \tag{2.10}$$

where

$$I(\tau,T) = \int_T^\tau a(s)f(s)^\lambda |R(s)|^{(\alpha+1)/\alpha}\,ds.$$

By the definitions of $A(t,\tau)$ and $B(t,\tau)$ we have

$$\begin{aligned}B(t,\tau) &= \frac{A(t,T)}{A(t,\tau)}B(t,T) - \int_T^\tau f(s)^\lambda b(s)\,ds\\ &\quad - \frac{1}{A(t,\tau)}\int_T^\tau a(s)w(s)\left(\int_T^s f(r)^\lambda b(r)\,dr\right) ds, \quad t > \tau \ge T.\end{aligned} \tag{2.11}$$

Therefore it follows from (2.10) and (2.11) that

$$\begin{aligned}&f(\tau)^\lambda R(\tau) - B(t,\tau)\\ &= f(T)^\lambda R(T) + \sigma\lambda \int_T^\tau a(s)f(s)^{\lambda-1}R(s)\,ds - \alpha I(\tau,T) - \frac{A(t,T)}{A(t,\tau)}B(t,T)\\ &\quad + \frac{1}{A(t,\tau)}\int_T^\tau a(s)w(s)\left(\int_T^s f(r)^\lambda b(r)\,dr\right) ds, \quad t > \tau \ge T.\end{aligned} \tag{2.12}$$

Using (2.9) on the right-hand side of (2.12), and taking the upper limit as $t \to \infty$, we obtain

$$\begin{aligned}&\limsup_{t\to\infty}[f(\tau)^\lambda R(\tau) - B(t,\tau)]\\ &\le f(T)^\lambda |R(T)| + |\lambda| \int_T^\tau a(s)f(s)^{\lambda-1}|R(s)|\,ds - \alpha I(\tau,T) + M\end{aligned} \tag{2.13}$$

for $\tau \geq T$. By Hölder's inequality the integral of the second term on the right-hand side of (2.13) is estimated as follows:

$$\begin{aligned}
&\int_T^\tau a(s)f(s)^{\lambda-1}|R(s)|\,ds \\
&\leq \left(\int_T^\tau a(s)f(s)^{\lambda-\alpha-1}\,ds\right)^{1/(\alpha+1)} \left(\int_T^\tau a(s)f(s)^{\lambda}|R(s)|^{(\alpha+1)/\alpha}\,ds\right)^{\alpha/(\alpha+1)} \\
&= \left(\int_T^\tau a(s)f(s)^{\lambda-\alpha-1}\,ds\right)^{1/(\alpha+1)} I(\tau,T)^{\alpha/(\alpha+1)}, \quad \tau \geq T.
\end{aligned}$$

Note here that

$$0 \leq \int_T^\tau a(s)f(s)^{\lambda-\alpha-1}\,ds \leq \int_{t_1}^\infty a(s)f(s)^{\lambda-\alpha-1}\,ds = -\frac{1}{\sigma}\frac{f(t_1)^{\lambda-\alpha}}{\lambda-\alpha} < \infty.$$

Therefore we have

$$\int_T^\tau a(s)f(s)^{\lambda-1}|R(s)|\,ds \leq \left(\int_{t_1}^\infty a(s)f(s)^{\lambda-\alpha-1}\,ds\right)^{1/(\alpha+1)} I(\tau,T)^{\alpha/(\alpha+1)} \tag{2.14}$$

for $\tau \geq T$. Then, (2.13) gives

$$\begin{aligned}
&\limsup_{t\to\infty}[f(\tau)^\lambda R(\tau) - B(t,\tau)] \\
&\leq f(T)^\lambda|R(T)| + |\lambda| \left(\int_{t_1}^\infty a(s)f(s)^{\lambda-\alpha-1}\,ds\right)^{1/(\alpha+1)} I(\tau,T)^{\alpha/(\alpha+1)} \\
&\quad - \alpha I(\tau,T) + M, \quad \tau \geq T.
\end{aligned} \tag{2.15}$$

We denote the right-hand side of (2.15) by $L(\tau)$. Remark that τ in (2.15) is a number satisfying (2.6). There exists a sequence $\{\tau_i\}_{i=1}^\infty$ such that

$$a(\tau_i)w(\tau_i) > 0, \ \ i = 1, 2, 3, \dots \quad \text{and} \quad \lim_{i\to\infty} \tau_i = \infty.$$

Since (2.7) implies $I(\tau_i, T) \to \infty$ $(i \to \infty)$, we have $L(\tau_i)/I(\tau_i, T) \to -\alpha$ $(i \to \infty)$, and so $L(\tau_i) \to -\infty$ as $i \to \infty$. Therefore, for any positive number ζ, there is $\tau = \tau(\zeta) > T$ such that $a(\tau)w(\tau) > 0$ and $L(\tau) < -2\zeta$ hold. In what follows, $\tau = \tau(\zeta)$ is a number having these properties. In the last step we will let $\zeta \to \infty$. By the inequality (2.15), there is a sufficiently large number $T_1 > \tau$ such that

$$f(\tau)^\lambda R(\tau) - B(t,\tau) \leq -\zeta \quad \text{for all } t \geq T_1.$$

Then it follows from (2.8) that

$$\begin{aligned}
&\int_\tau^t a(s)w(s)f(s)^\lambda R(s)\,ds \\
&\leq -\zeta A(t,\tau) + |\lambda| \int_\tau^t a(s)w(s) \left(\int_\tau^s a(r)f(r)^{\lambda-1}|R(r)|\,dr\right) ds \\
&\quad - \alpha \int_\tau^t a(s)w(s)I(s,\tau)\,ds
\end{aligned} \tag{2.16}$$

for $t \ge T_1$. Similar to (2.14) we have

$$\int_\tau^s a(r)f(r)^{\lambda-1}|R(r)|\,dr \le \left(\int_{t_1}^\infty a(r)f(r)^{\lambda-\alpha-1}\,dr\right)^{1/(\alpha+1)} I(s,\tau)^{\alpha/(\alpha+1)}$$

for $s \ge \tau$. Hence, by (2.16),

(2.17)
$$\begin{aligned}&\int_\tau^t a(s)w(s)f(s)^\lambda R(s)\,ds\\ &\quad\le -\zeta A(t,\tau)\\ &\quad\quad+\int_\tau^t a(s)w(s)\left[|\lambda|\left(\int_{t_1}^\infty a(r)f(r)^{\lambda-\alpha-1}\,dr\right)^{1/(\alpha+1)} I(s,\tau)^{\alpha/(\alpha+1)}-\alpha I(s,\tau)\right]ds\end{aligned}$$

for $t \ge T_1$. Denote by $L(s,\tau)$ the term in the square brackets of the right-hand side of (2.17). Since $I(s,\tau) \to \infty$ $(s \to \infty)$, we have $L(s,\tau)/I(s,\tau) \to -\alpha$ as $s \to \infty$. Therefore there is $\eta > T_1$ such that

$$L(s,\tau) \le -\frac{\alpha}{2}I(s,\tau) \quad \text{for } s \ge \eta.$$

Then, by (2.17), we see that

$$\begin{aligned}\int_\tau^t a(s)w(s)f(s)^\lambda R(s)\,ds \le -\zeta A(t,\tau) &+ \int_\tau^\eta a(s)w(s)L(s,\tau)\,ds\\ &-\frac{\alpha}{2}\int_\eta^t a(s)w(s)I(s,\tau)\,ds\end{aligned} \tag{2.18}$$

for $t \ge \eta$. Remember that $w \in \mathcal{W}$ satisfies (1.17). This means that $A(t,\tau) \to \infty$ as $t \to \infty$. Therefore, there is $\theta > \eta$ such that

$$\int_\tau^\eta a(s)w(s)L(s,\tau)\,ds \le \frac{\zeta}{2}A(t,\tau) \quad \text{for } t \ge \theta.$$

Then it follows from (2.18) that

$$\int_\tau^t a(s)w(s)f(s)^\lambda R(s)\,ds \le -\frac{\zeta}{2}A(t,\tau) - \frac{\alpha}{2}\int_\eta^t a(s)w(s)I(s,\tau)\,ds \tag{2.19}$$

for $t \ge \theta$. Denote by $-G(t)$ the right-hand side of (2.19). We have

$$G(t) \ge \frac{\zeta}{2}A(t,\tau) > 0, \tag{2.20}$$

$$0 < G(t) \le -\int_\tau^t a(s)w(s)f(s)^\lambda R(s)\,ds = \left|\int_\tau^t a(s)w(s)f(s)^\lambda R(s)\,ds\right|, \tag{2.21}$$

and

$$G'(t) \geq \frac{\alpha}{2} a(t)w(t)I(t,\tau) \tag{2.22}$$

for $t \geq \theta$.

Now, using Hölder's inequality, we obtain

$$\begin{aligned}
&\left| \int_\tau^t a(s)w(s)f(s)^\lambda R(s)\, ds \right| \\
&\leq \left(\int_\tau^t a(s)w(s)^{\alpha+1} f(s)^\lambda \, ds \right)^{1/(\alpha+1)} \left(\int_\tau^t a(s)f(s)^\lambda |R(s)|^{(\alpha+1)/\alpha} \, ds \right)^{\alpha/(\alpha+1)} \\
&= \left(\int_\tau^t a(s)w(s)^{\alpha+1} f(s)^\lambda \, ds \right)^{1/(\alpha+1)} I(t,\tau)^{\alpha/(\alpha+1)}, \quad t \geq \tau,
\end{aligned} \tag{2.23}$$

and, hence, it follows from (2.21) that

$$I(t,\tau) \geq \left(\int_\tau^t a(s)w(s)^{\alpha+1} f(s)^\lambda \, ds \right)^{-1/\alpha} G(t)^{(\alpha+1)/\alpha}$$

for $t \geq \theta$. To simplify notation, we put

$$J(t,u) = \int_u^t a(s)w(s)^{\alpha+1} f(s)^\lambda \, ds, \quad t \geq u \geq T.$$

Then, (2.22) gives

$$G'(t) \geq \frac{\alpha}{2} a(t)w(t)J(t,\tau)^{-1/\alpha} G(t)^{(\alpha+1)/\alpha} \tag{2.24}$$

for $t \geq \theta$. Multiplying (2.24) by $G(t)^{\xi-[(\alpha+1)/\alpha]}$, where ξ is a number such that $0 \leq \xi < 1/\alpha$, and using (2.20), we get

$$G'(t)G(t)^{\xi-[(\alpha+1)/\alpha]} \geq \frac{\alpha}{2} \left(\frac{\zeta}{2} \right)^\xi a(t)w(t)J(t,\tau)^{-1/\alpha} A(t,\tau)^\xi$$

for $t \geq \theta$. Integrate the above inequality from t $(\geq \theta)$ to t', and let $t' \to \infty$. Since $\xi < 1/\alpha$ and $G(t) \to \infty$ as $t \to \infty$ (see (2.20)) we find that

$$\int_\theta^\infty a(s)w(s)J(s,\tau)^{-1/\alpha} A(s,\tau)^\xi \, ds < \infty \tag{2.25}$$

and

$$\left(\frac{1}{\alpha} - \xi \right)^{-1} G(t)^{\xi-(1/\alpha)} \geq \frac{\alpha}{2} \left(\frac{\zeta}{2} \right)^\xi \int_t^\infty a(s)w(s)J(s,\tau)^{-1/\alpha} A(s,\tau)^\xi \, ds \tag{2.26}$$

for $t \geq \theta$.

For the case where $w \in \mathcal{W}$ satisfies (1.14), the result (2.25) yields a contradiction. Therefore, let us consider the case where $w \in \mathcal{W}$ satisfies (1.15). Using (2.20) on the left-hand side of (2.26), we deduce that

$$
\tag{2.27}
\begin{aligned}
&\frac{2}{\alpha}\left(\frac{\zeta}{2}\right)^{-1/\alpha}\left(\frac{1}{\alpha}-\xi\right)^{-1} \\
&\geq A(t,\tau)^{-\xi+(1/\alpha)}\int_t^\infty a(s)w(s)J(s,\tau)^{-1/\alpha}A(s,\tau)^{\xi}\,ds, \quad t \geq \theta.
\end{aligned}
$$

Since

$$
\begin{aligned}
A(t,T) &= A(\tau,T) + A(t,\tau), \qquad J(s,\tau) \leq J(s,T), \\
A(s,T) &= A(\tau,T) + A(s,\tau), \qquad A(s,\tau) \geq A(t,\tau), \quad s \geq t,
\end{aligned}
$$

we easily see that

$$
\begin{aligned}
&A(t,T)^{-\xi+(1/\alpha)}\int_t^\infty a(s)w(s)J(s,T)^{-1/\alpha}A(s,T)^{\xi}\,ds \\
&\leq \left[\frac{A(\tau,T)+A(t,\tau)}{A(t,\tau)}\right]^{-\xi+(1/\alpha)}\left[\frac{A(\tau,T)}{A(t,\tau)}+1\right]^{\xi} A(t,\tau)^{-\xi+(1/\alpha)} \\
&\times \int_t^\infty a(s)w(s)J(s,\tau)^{-1/\alpha}A(s,\tau)^{\xi}\,ds, \quad t \geq \theta.
\end{aligned}
$$

Hence, taking the upper limit as $t \to \infty$ in (2.27), we get

$$
\tag{2.28}
\begin{aligned}
&\frac{2}{\alpha}\left(\frac{\zeta}{2}\right)^{-1/\alpha}\left(\frac{1}{\alpha}-\xi\right)^{-1} \\
&\geq \limsup_{t\to\infty} A(t,T)^{-\xi+(1/\alpha)}\int_t^\infty a(s)w(s)J(s,T)^{-1/\alpha}A(s,T)^{\xi}\,ds.
\end{aligned}
$$

Note that the right-hand side of (2.28) is independent of $\zeta > 0$. Since $\zeta > 0$ is arbitrary, letting $\zeta \to \infty$ in (2.28), we find that

$$
\limsup_{t\to\infty} A(t,T)^{-\xi+(1/\alpha)}\int_t^\infty a(s)w(s)J(s,T)^{-1/\alpha}A(s,T)^{\xi}\,ds = 0.
$$

This is a contradiction to (1.15). Thus we conclude that (2.5) holds. This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let σ, $f(t)$, t_1 and λ be as above. Suppose that the system* (1.1) *has a nonoscillatory solution $(u(t),v(t))$ such that $u(t) > 0$ for $t \geq T$ $(\geq t_1)$. Suppose further that the function $R(t)$ defined by* (2.1) *satisfies* (2.5)*. Then, for any $w \in \mathcal{W}_0$, the function $C(t;w,\lambda)$ has a finite limit as $t \to \infty$. The value of the limit of $C(t;w,\lambda)$ as $t \to \infty$ does not depend on $w \in \mathcal{W}_0$.*

Proof. For any $w \in \mathcal{W}_0$, we have (2.4). Note that $w(t)$ satisfies (1.17). Putting $\tau = T$ in (2.4), we get

$$\begin{aligned}
&\frac{1}{A(t,T)} \int_T^t a(s)w(s) \left(\int_T^s f(r)^\lambda b(r)\,dr \right) ds \\
&= f(T)^\lambda R(T) - \frac{1}{A(t,T)} \int_T^t a(s)w(s)f(s)^\lambda R(s)\,ds \\
&\quad + \frac{\sigma\lambda}{A(t,T)} \int_T^t a(s)w(s) \left(\int_T^s a(r)f(r)^{\lambda-1}R(r)\,dr \right) ds \\
&\quad - \frac{\alpha}{A(t,T)} \int_T^t a(s)w(s)I(s,T)\,ds
\end{aligned} \tag{2.29}$$

for all large t. Here,

$$A(t,T) = \int_T^t a(s)w(s)\,ds \quad \text{and} \quad I(t,T) = \int_T^t a(s)f(s)^\lambda |R(s)|^{(\alpha+1)/\alpha}\,ds, \quad t \geq T.$$

From the condition (2.5) it follows that

$$\lim_{t\to\infty} I(t,T) = \int_T^\infty a(s)f(s)^\lambda |R(s)|^{(\alpha+1)/\alpha}\,ds < \infty. \tag{2.30}$$

Then it is clear that

$$\lim_{t\to\infty} \frac{1}{A(t,T)} \int_T^t a(s)w(s)I(s,T)\,ds = \int_T^\infty a(s)f(s)^\lambda |R(s)|^{(\alpha+1)/\alpha}\,ds \in \mathbb{R}.$$

Analogously to (2.23) we have

$$\left| \int_T^t a(s)w(s)f(s)^\lambda R(s)\,ds \right| \leq \left(\int_T^t a(s)w(s)^{\alpha+1}f(s)^\lambda\,ds \right)^{1/(\alpha+1)} I(t,T)^{\alpha/(\alpha+1)}$$

and so

$$\begin{aligned}
&\frac{1}{A(t,T)} \left| \int_T^t a(s)w(s)f(s)^\lambda R(s)\,ds \right| \\
&\leq \frac{1}{A(t,T)} \left(\int_T^t a(s)w(s)^{\alpha+1}f(s)^\lambda\,ds \right)^{1/(\alpha+1)} I(t,T)^{\alpha/(\alpha+1)}
\end{aligned}$$

for all large t. Therefore, by (1.16) and (2.30), we get

$$\lim_{t\to\infty} \frac{1}{A(t,T)} \int_T^t a(s)w(s)f(s)^\lambda R(s)\,ds = 0.$$

As in the proof of (2.14), we have

$$\int_T^t a(s)f(s)^{\lambda-1}|R(s)|\,ds \leq \left(\int_{t_1}^\infty a(s)f(s)^{\lambda-\alpha-1}\,ds \right)^{1/(\alpha+1)} I(t,T)^{\alpha/(\alpha+1)} \tag{2.31}$$

for $t \geq T$. Since $I(t,T)$ has a finite limit as $t \to \infty$, it is bounded on $[T,\infty)$, and so (2.31) yields

$$\int_T^\infty a(s)f(s)^{\lambda-1}|R(s)|\,ds < \infty.$$

This implies that

$$\lim_{t\to\infty}\int_T^t a(s)f(s)^{\lambda-1}R(s)\,ds = \int_T^\infty a(s)f(s)^{\lambda-1}R(s)\,ds$$

exists and is finite. Hence,

$$\lim_{t\to\infty}\frac{1}{A(t,T)}\int_T^t a(s)w(s)\left(\int_T^s a(r)f(r)^{\lambda-1}R(r)\,dr\right)ds = \int_T^\infty a(s)f(s)^{\lambda-1}R(s)\,ds \in \mathbb{R}.$$

Then, by (2.29), we conclude that

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{A(t,T)}\int_T^t a(s)w(s)\left(\int_T^s f(r)^\lambda b(r)\,dr\right)ds\\ &= f(T)^\lambda R(T) + \sigma\lambda\int_T^\infty a(s)f(s)^{\lambda-1}R(s)\,ds - \alpha\int_T^\infty a(s)f(s)^\lambda|R(s)|^{(\alpha+1)/\alpha}\,ds.\end{aligned} \tag{2.32}$$

Observe that the right-hand side of (2.32) is a finite value, and it does not depend on $w \in \mathcal{W}_0$. Then, applying Lemma 2.1(i) to the case $T_0 = T_1 = t_1$, $T_2 = T$, $\varphi(t) = a(t)w(t)$ and $\psi(t) = f(t)^\lambda b(t)$, we see that

$$\begin{aligned}\lim_{t\to\infty} C(t;w,\lambda) = &\int_{t_1}^T f(s)^\lambda b(s)\,ds + f(T)^\lambda R(T)\\ &+ \sigma\lambda\int_T^\infty a(s)f(s)^{\lambda-1}R(s)\,ds - \alpha\int_T^\infty a(s)f(s)^\lambda|R(s)|^{(\alpha+1)/\alpha}\,ds.\end{aligned}$$

Thus, we deduce that $C(t;w,\lambda)$ has a finite limit as $t \to \infty$ and that the value of the limit of $C(t;w,\lambda)$ as $t \to \infty$ does not depend on $w \in \mathcal{W}_0$. The proof of Lemma 2.3 is complete. □

We are now ready to prove Theorems 1.2 and 1.6.

Proof of Theorem 1.2. Assume that the system (1.1) has a nonoscillatory solution $(u(t), v(t))$. Let $u(t) > 0$ for $t \geq T$ $(\geq t_1)$, and define the function $R(t)$ by (2.1). By Lemma 2.2 we have (2.5). Therefore, by Lemma 2.3, the function $C(t;w_0,\lambda)$ has a finite limit as $t \to \infty$ for any $w_0 \in \mathcal{W}_0$. Consequently, if there is a function $w_0 \in \mathcal{W}_0$ such that $C(t;w_0,\lambda)$ does not possess a finite limit as $t \to \infty$, then the system (1.1) is oscillatory. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.6. Assume that the system (1.1) has a nonoscillatory solution $(u(t), v(t))$. Let $u(t) > 0$ for $t \ge T$ $(\ge t_1)$, and define the function $R(t)$ by (2.1). Let ξ be a number satisfying $0 \le \xi < 1/\alpha$. For the proof of (I) note that $w_1 \in \mathcal{W}_0 \subseteq \mathcal{W}$ and

$$\liminf_{t\to\infty} C(t; w_1, \lambda) > -\infty,$$

and for the proof of (II) note that $w_0 \in \mathcal{W}_0 \subseteq \mathcal{W}$ and

$$\liminf_{t\to\infty} C(t; w_0, \lambda) > -\infty.$$

Then, by Lemma 2.2 applied to the case $w = w_1 \in \mathcal{W}$ (resp. $w = w_0 \in \mathcal{W}$) for the proof of (I) (resp. (II)), we have (2.5). Therefore it follows from Lemma 2.3 that, for any $w \in \mathcal{W}_0$, the function $C(t; w, \lambda)$ has a finite limit as $t \to \infty$ and the limit (in particular, the lower limit and the upper limit) of $C(t; w, \lambda)$ as $t \to \infty$ does not depend on $w \in \mathcal{W}_0$. This is a contradiction to the condition (1.21) (resp. (1.22)) for the proof of (I) (resp. (II)). The proof of Theorem 1.6 is complete. □

For the proof of Theorem 1.7 we use Sturm's comparison theorem (see Theorem 1.1).

Proof of Theorem 1.7. Suppose that (1.24) holds. We take a number $T \ge t_0$ such that

$$\left| \int_t^\infty f(s)^\alpha b(s)\, ds \right| \le \frac{1}{3} \quad \text{for } t \ge T.$$

Clearly we have

$$0 < \frac{1}{6} \le \sigma \int_t^\infty f(s)^\alpha b(s)\, ds + \frac{1}{2} \le \frac{5}{6} < 1, \quad t \ge T,$$

where $\sigma = +1$ for the case where (1.9) holds, and $\sigma = -1$ for the case where (1.10) holds. Define the function $R(t)$ by

$$R(t) = \sigma f(t)^{-\alpha} \left(\sigma \int_t^\infty f(s)^\alpha b(s)\, ds + \frac{1}{2} \right), \quad t \ge T.$$

It is easy to see that

$$R'(t) = -b(t) - \alpha a(t) |R(t)|^{(\alpha+1)/\alpha} \left(\sigma \int_t^\infty f(s)^\alpha b(s)\, ds + \frac{1}{2} \right)^{-1/\alpha}$$

for $t \ge T$, and so

$$R'(t) \le -b(t) - \alpha a(t) |R(t)|^{(\alpha+1)/\alpha}, \quad t \ge T. \tag{2.33}$$

Next, define the positive function $u_1(t)$ by

$$u_1(t) = \exp\left(\int_T^t a(s) |R(s)|^{1/\alpha} \operatorname{sgn} R(s)\, ds \right), \quad t \ge T.$$

Further, using the above $R(t)$ and $u_1(t)$, we put

$$v_1(t) = u_1(t)^\alpha R(t), \quad t \ge T.$$

Then,

$$u_1'(t) = a(t)|v_1(t)|^{1/\alpha} \operatorname{sgn} v_1(t), \quad t \ge T,$$

and, it follows from (2.33) that

$$v_1'(t) \le -b(t)u_1(t)^\alpha, \quad t \ge T.$$

Put

$$b_1(t) = -\frac{v_1'(t)}{u_1(t)^\alpha}, \quad t \ge T.$$

It is clear that $b_1(t) \ge b(t)$ $(t \ge T)$, and that $(u_1(t), v_1(t))$ is a solution on $[T, \infty)$ of the system

$$u_1' = a(t)|v_1|^{1/\alpha} \operatorname{sgn} v_1, \quad v_1' = -b_1(t)|u_1|^\alpha \operatorname{sgn} u_1.$$

Remember that $u_1(t) > 0$ for $t \ge T$. Then, by Theorem 1.1, we conclude that for any nontrivial solution $(u(t), v(t))$ of (1.1) the first component $u(t)$ has at most one zero in $[T, \infty)$. This shows that (1.1) is nonoscillatory. The proof of Theorem 1.7 is complete. □

3. Scalar half-linear equations

Now, let us state the oscillatory and nonoscillatory results for the scalar half-linear equation (1.4). Since (1.4) can be regarded as a special case of (1.1) with $a(t) = p(t)^{-1/\alpha}$ and $b(t) = q(t)$, the results for (1.1) yield the corresponding results for (1.4). The precise statements of the general results for (1.4) which can be derived from Theorem 1.2, Corollary 1.3 and Theorem 1.6 are omitted because they are complicated and long. We only give a remark that the result of Li and Yeh [9, Theorem 3.1] for (1.4) with $p(t) \equiv 1$ is easily derived from Theorem 1.2 of the case $\lambda = 0$. See also Theorem 1.3 in Willett [25] for the linear equation (1.6). We further note that the corresponding result to Theorem 1.6 of the case $\lambda = 0$ gives an extension of the result of Li and Yeh [9, Corollary 3.2]. See Corollary 1.2 in Willett [25] for the linear equation (1.6).

In this section we state the results for (1.4) which can be derived from Corollaries 1.4 and 1.5 and Theorem 1.7. We first consider the case where (1.7) holds. Then we set

$$P(t) = \int_{t_0}^{t} p(s)^{-1/\alpha}\, ds, \quad t \ge t_0. \tag{3.1}$$

Corollary 1.4 produces the following result.

Corollary 3.1. *Consider the equation* (1.4) *under the condition* (1.7)*. Define* $P(t)$ *by* (3.1)*, and take* $t_1 > t_0$ *so that* $P(t) > 0$ *for* $t \geq t_1$*. Suppose moreover that there are* λ *and* ρ *such that* $\lambda < \alpha$*,* $\rho > -1$ *and*

$$\lim_{t\to\infty} \frac{1}{P(t)^{\rho+1}} \int_{t_1}^{t} p(s)^{-1/\alpha} P(s)^{\rho} \left(\int_{t_1}^{s} P(r)^{\lambda} q(r)\, dr \right) ds = \infty$$

or

$$\begin{aligned} -\infty < \liminf_{t\to\infty} \frac{1}{P(t)^{\rho+1}} \int_{t_1}^{t} p(s)^{-1/\alpha} P(s)^{\rho} \left(\int_{t_1}^{s} P(r)^{\lambda} q(r)\, dr \right) ds \\ < \limsup_{t\to\infty} \frac{1}{P(t)^{\rho+1}} \int_{t_1}^{t} p(s)^{-1/\alpha} P(s)^{\rho} \left(\int_{t_1}^{s} P(r)^{\lambda} q(r)\, dr \right) ds. \end{aligned}$$

Then, (1.4) *is oscillatory.*

The classical Hartman–Wintner oscillation criterion for (1.6) is the case of $\alpha = 1$, $p(t) \equiv 1$, $\rho = 0$ and $\lambda = 0$ in Corollary 3.1.

Corollary 1.5 yields the following result.

Corollary 3.2. *Consider the equation* (1.4) *under the condition* (1.7)*. Define* $P(t)$ *by* (3.1)*, and take* $t_1 > t_0$ *so that* $P(t) > 0$ *for* $t \geq t_1$*. Suppose moreover that there exists* λ *such that* $\lambda < \alpha$ *and*

$$\lim_{t\to\infty} \frac{1}{\log P(t)} \int_{t_1}^{t} \frac{p(s)^{-1/\alpha}}{P(s)} \left(\int_{t_1}^{s} P(r)^{\lambda} q(r)\, dr \right) ds = \infty$$

or

$$\begin{aligned} -\infty < \liminf_{t\to\infty} \frac{1}{\log P(t)} \int_{t_1}^{t} \frac{p(s)^{-1/\alpha}}{P(s)} \left(\int_{t_1}^{s} P(r)^{\lambda} q(r)\, dr \right) ds \\ < \limsup_{t\to\infty} \frac{1}{\log P(t)} \int_{t_1}^{t} \frac{p(s)^{-1/\alpha}}{P(s)} \left(\int_{t_1}^{s} P(r)^{\lambda} q(r)\, dr \right) ds. \end{aligned}$$

Then, (1.4) *is oscillatory.*

Theorem 1.7 produces the following result.

Corollary 3.3. *Consider the equation* (1.4) *under the condition* (1.7)*. Define* $P(t)$ *by* (3.1)*. If*

$$\lim_{t\to\infty} \int_{t_0}^{t} P(s)^{\alpha} q(s)\, ds = \int_{t_0}^{\infty} P(s)^{\alpha} q(s)\, ds \quad \text{exists and is finite,} \tag{3.2}$$

then (1.4) *is nonoscillatory.*

The above result for the case $p(t) \equiv 1$ was shown by Li and Yeh [10, Corollary 3.3]. For case where $q(t) \geq 0$ for all large t, it is well known (see, e.g., [4,6]) that if (3.2) holds, then (1.4) has a nonoscillatory solution $u(t)$ such that

$$\lim_{t\to\infty} \frac{u(t)}{P(t)} \text{ exists and is a nonzero finite value,} \tag{3.3}$$

and, conversely, if (1.4) has a nonoscillatory solution $u(t)$ satisfying (3.3), then (3.2) holds.

Next, consider the case where (1.8) holds. We set

$$\pi(t) = \int_t^\infty p(s)^{-1/\alpha}\,ds, \quad t \geq t_0. \tag{3.4}$$

Then we can take $t_1 = t_0$. The following results can be obtained from Corollaries 1.4 and 1.5 and Theorem 1.7.

Corollary 3.4. *Consider the equation* (1.4) *under the condition* (1.8). *Define* $\pi(t)$ *by* (3.4). *Suppose that there are* λ *and* ρ *such that* $\lambda > \alpha$, $\rho < -1$ *and*

$$\lim_{t\to\infty} \frac{1}{\pi(t)^{\rho+1}} \int_{t_0}^t p(s)^{-1/\alpha}\pi(s)^\rho \left(\int_{t_0}^s \pi(r)^\lambda q(r)\,dr\right) ds = \infty$$

or

$$\begin{aligned}-\infty &< \liminf_{t\to\infty} \frac{1}{\pi(t)^{\rho+1}} \int_{t_0}^t p(s)^{-1/\alpha}\pi(s)^\rho \left(\int_{t_0}^s \pi(r)^\lambda q(r)\,dr\right) ds \\ &< \limsup_{t\to\infty} \frac{1}{\pi(t)^{\rho+1}} \int_{t_0}^t p(s)^{-1/\alpha}\pi(s)^\rho \left(\int_{t_0}^s \pi(r)^\lambda q(r)\,dr\right) ds.\end{aligned}$$

Then, (1.4) *is oscillatory.*

Corollary 3.5. *Consider the equation* (1.4) *under the condition* (1.8). *Define* $\pi(t)$ *by* (3.4). *Suppose that there exists* λ *such that* $\lambda > \alpha$ *and*

$$\lim_{t\to\infty} \frac{1}{\log(1/\pi(t))} \int_{t_0}^t \frac{p(s)^{-1/\alpha}}{\pi(s)} \left(\int_{t_0}^s \pi(r)^\lambda q(r)\,dr\right) ds = \infty$$

or

$$\begin{aligned}-\infty &< \liminf_{t\to\infty} \frac{1}{\log(1/\pi(t))} \int_{t_0}^t \frac{p(s)^{-1/\alpha}}{\pi(s)} \left(\int_{t_0}^s \pi(r)^\lambda q(r)\,dr\right) ds \\ &< \limsup_{t\to\infty} \frac{1}{\log(1/\pi(t))} \int_{t_0}^t \frac{p(s)^{-1/\alpha}}{\pi(s)} \left(\int_{t_0}^s \pi(r)^\lambda q(r)\,dr\right) ds.\end{aligned}$$

Then, (1.4) *is oscillatory.*

Corollary 3.6. *Consider the equation* (1.4) *under the condition* (1.8). *Define* $\pi(t)$ *by* (3.4). *If*

$$\lim_{t\to\infty}\int_{t_0}^{t}\pi(s)^{\alpha}q(s)\,ds=\int_{t_0}^{\infty}\pi(s)^{\alpha}q(s)\,ds \text{ exists and is finite,} \tag{3.5}$$

then (1.4) *is nonoscillatory.*

For the case where $q(t)\geq 0$ for all large t, it is known (see, e.g., [6, 8]) that if (3.5) holds, then (1.4) has a nonoscillatory solution $u(t)$ such that

$$\lim_{t\to\infty}\frac{u(t)}{\pi(t)} \text{ exists and is a nonzero finite value,} \tag{3.6}$$

and, conversely, if (1.4) has a nonoscillatory solution $u(t)$ satisfying (3.6), then (3.5) holds.

4. Examples

In this section we illustrate our results by several examples.

Example 4.1. Let $g\in C^1[t_0,\infty)$, $t_0>0$, be a function such that

$$g(t)>0 \quad\text{and}\quad g'(t)\geq 0 \text{ for } t\geq t_0, \quad\text{and}\quad \lim_{t\to\infty}g(t)=\infty,$$

and put

$$a(t)=g'(t)\quad\text{and}\quad b(t)=\frac{d}{dt}\big\{\sin\log g(t)+\log g(t)\cos\log g(t)\big\}$$

for $t\geq t_0$. For this pair of $a(t)$ and $b(t)$, the system (1.1) is oscillatory. To see this, take a function $G\in C^1[0,\infty)$ such that $G(t)=g(t)$ for $t\geq t_0$ and

$$G(0)=0,\quad G'(t)\geq 0 \text{ for } t\in[0,t_0].$$

We put $A(t)=G'(t)$ for $t\geq 0$, and so $A(t)=a(t)$ for $t\geq t_0$. Further, take a function $B\in C[0,\infty)$ such that $B(t)=b(t)$ for $t\geq t_0$. Then we consider the auxiliary system

$$u'=A(t)|v|^{1/\alpha}\operatorname{sgn}v,\quad v'=-B(t)|u|^{\alpha}\operatorname{sgn}u, \tag{4.1}$$

on the interval $[0,\infty)$. Clearly, the original system is oscillatory if and only if the auxiliary system (4.1) is oscillatory. For the system (4.1), we have $\sigma=+1$ and $f(t)=\int_0^t A(s)\,ds=G(t)$ for $t\geq 0$. Let $t_1=t_0$, and so $f(t)=g(t)>0$ for $t\geq t_1$. It is clear that

$$\int_{t_1}^{t}B(s)\,ds=\sin\log g(t)+\log g(t)\cos\log g(t)+c_1,\quad t\geq t_1,$$

where c_1 is a constant. Then we can easily check that

$$\int_{t_1}^{t}\frac{A(s)}{f(s)}\left(\int_{t_1}^{s}B(r)\,dr\right)ds=\log g(t)\sin\log g(t)+c_1\log g(t)+c_2,\quad t\geq t_1,$$

where c_2 is also a constant. Hence we have

$$-1+c_1 = \liminf_{t\to\infty} \frac{1}{\log f(t)} \int_{t_1}^{t} \frac{A(s)}{f(s)} \left(\int_{t_1}^{s} B(r)\,dr \right) ds$$
$$< \limsup_{t\to\infty} \frac{1}{\log f(t)} \int_{t_1}^{t} \frac{A(s)}{f(s)} \left(\int_{t_1}^{s} B(r)\,dr \right) ds = 1 + c_1.$$

Therefore, by Corollary 1.5 with $\lambda = 0$, the system (4.1) is oscillatory, and, in consequence, the system (1.1) under consideration is oscillatory.

Example 4.2. Consider the half-linear scalar equation (1.4) of the case

$$p(t) = t^{2\alpha} \quad \text{and} \quad q(t) = t^{2\alpha} \frac{d}{dt} \big\{ \sin \log t + \log t \cos \log t \big\}$$

for $t \ge t_0$ (> 0). In this case, $\sigma = -1$ and $\pi(t) = 1/t$ $(t \ge t_0)$. We will apply Corollary 3.5 with $\lambda = 2\alpha$. Since

$$\int_{t_0}^{t} \pi(r)^{2\alpha} q(r)\,dr = \sin \log t + \log t \cos \log t + c_1, \quad t \ge t_0,$$

we have

$$\int_{t_0}^{t} \frac{p(s)^{-1/\alpha}}{\pi(s)} \left(\int_{t_0}^{s} \pi(r)^{2\alpha} q(r)\,dr \right) ds = \log t \sin \log t + c_1 \log t + c_2, \quad t \ge t_0.$$

Here c_1 and c_2 are constants. It is easy to see that

$$-1+c_1 = \liminf_{t\to\infty} \frac{1}{\log(1/\pi(t))} \int_{t_0}^{t} \frac{p(s)^{-1/\alpha}}{\pi(s)} \left(\int_{t_0}^{s} \pi(r)^{2\alpha} q(r)\,dr \right) ds$$
$$< \limsup_{t\to\infty} \frac{1}{\log(1/\pi(t))} \int_{t_0}^{t} \frac{p(s)^{-1/\alpha}}{\pi(s)} \left(\int_{t_0}^{s} \pi(r)^{2\alpha} q(r)\,dr \right) ds = 1 + c_1.$$

Therefore, by Corollary 3.5 with $\lambda = 2\alpha$, this equation (1.4) is oscillatory.

Remark 4.3. From Example 4.1 of the case $g(t) = t$, we find that the equation (1.4) of the case

$$p(t) \equiv 1 \quad \text{and} \quad q(t) = \frac{d}{dt} \big\{ \sin \log t + \log t \cos \log t \big\}, \quad t \ge t_0 > 0,$$

is oscillatory. If this equation is written as

$$(|u'|^{\alpha} \operatorname{sgn} u')' + q(t)|u|^{\alpha} \operatorname{sgn} u = 0, \tag{4.2}$$

then the equation in Example 4.2 is

$$(t^{2\alpha}|u'|^{\alpha} \operatorname{sgn} u')' + t^{2\alpha} q(t)|u|^{\alpha} \operatorname{sgn} u = 0. \tag{4.3}$$

In general, let $q(t)$ be a continuous function on $[t_0, \infty)$, $t_0 > 0$. Then it may be guessed that there is some relation between (4.2) and (4.3) with respect to oscillatory (and nonoscillatory) properties of solutions. This is surely true for the case $\alpha = 1$ in the sense that (4.2) with $\alpha = 1$ is oscillatory (resp. nonoscillatory) if and only if (4.3) with $\alpha = 1$ is oscillatory (resp. nonoscillatory). Indeed, for a solution $u(t)$ of (4.2) with $\alpha = 1$, the function $\widetilde{u}(t) = u(t)/t$ is a solution of (4.3) with $\alpha = 1$, and, by this transformation, the oscillatory (and nonoscillatory) property does not change.

Next we give two examples illustrating Theorem 1.6.

Example 4.4. Consider the half-linear equation (1.4) of the case

$$p(t) = 1 \quad \text{and} \quad q(t) = t^2 \cos t$$

for $t \geq 0$. This equation is regarded as a special case of the system (1.1) with $a(t) = 1$ and $b(t) = t^2 \cos t$. In this case, we have $\sigma = +1$ and $f(t) = t$ $(t \geq 0)$. We will apply Theorem 1.6(I) with $\lambda = 0$, and so $\sigma(\lambda - \alpha) < 0$. Let $t_1 = 2\pi$. We have

$$\int_{t_1}^{t} b(r)\, dr = \int_{t_1}^{t} r^2 \cos r\, dr = t^2 \sin t + 2t \cos t - 2 \sin t + c_1,$$

where c_1 is a constant.

The positive constant function $w_0(t) \equiv 1$ satisfies $a(t)w_0(t) = 1 \not\equiv 0$ on $[t_2, \infty)$ for any $t_2 \geq t_1$, and (1.17) and $(1.18)_{\lambda=0}$ with $w(t)$ replaced by $w_0(t)$ hold. Therefore, $w_0 \in \mathcal{W}_0$. The function $C(t; w_0, 0)$ is given by

$$C(t; w_0, 0) = \frac{1}{t - t_1}\big(- t^2 \cos t + 4t \sin t + 6 \cos t + c_2 + c_1(t - t_1)\big),$$

where c_2 is a constant. Hence we have

$$\liminf_{t\to\infty} C(t; w_0, 0) = -\infty.$$

Next, take $w_1(t) = (\sin t)_+(\cos t)_+$, where in general $\psi(t)_+ = \max\{\psi(t), 0\}$. The function $w_1(t)$ satisfies $w_1(t) \geq 0$ for $t \geq t_1$ and $a(t)w_1(t) = (\sin t)_+(\cos t)_+ \not\equiv 0$ on $[t_2, \infty)$ for any $t_2 \geq t_1$. For $t \geq t_1 = 2\pi$, there is an $n \in \{1, 2, 3, \ldots\}$ such that $2n\pi \leq t < 2(n+1)\pi$. Then,

$$\begin{aligned} \int_{t_1}^{t} a(s)w_1(s)\, ds &= \int_{2\pi}^{t} (\sin s)_+(\cos s)_+\, ds \geq \sum_{i=1}^{n-1} \int_{2i\pi}^{2i\pi+(\pi/2)} \sin s\, \cos s\, ds \\ &= \frac{n-1}{2} > \frac{t - 4\pi}{4\pi}. \end{aligned} \tag{4.4}$$

Therefore, $w_1(t)$ satisfies (1.17) and $(1.18)_{\lambda=0}$ with $w(t)$ replaced by $w_1(t)$. Hence, $w_1 \in \mathcal{W}_0$. Since $(\sin t)_+ \geq 0$, $(\cos t)_+ \geq 0$, $\sin t\,(\sin t)_+ \geq 0$ and $\cos t\,(\cos t)_+ \geq 0$, we have

$$\begin{aligned}
&\int_{t_1}^{t} a(s)w_1(s)\left(\int_{t_1}^{s} b(r)\,dr\right) ds \\
&\geq \int_{2\pi}^{t} \big\{ -2\sin s\,(\sin s)_+(\cos s)_+ + c_1(\sin s)_+(\cos s)_+ \big\}\,ds \\
&\geq \int_{2\pi}^{t} (-2-|c_1|)\,ds = (-2-|c_1|)(t-2\pi), \quad t \geq 2\pi.
\end{aligned}$$

Therefore the inequality (4.4) gives

$$C(t; w_1, 0) \geq \frac{4\pi}{t-4\pi}(-2-|c_1|)(t-2\pi), \quad t > 4\pi.$$

Consequently, we get

$$\liminf_{t\to\infty} C(t; w_1, 0) > -\infty.$$

Thus the condition (1.21) holds. By Theorem 1.6(I), the equation in this example is oscillatory.

Example 4.5. Consider the half-linear equation (1.4) of the case

$$p(t) = t^{\alpha} \quad \text{and} \quad q(t) = (\log t)^{\beta}\cos t \quad (\beta\text{: constant})$$

for $t \geq \pi$. If $\beta > -\alpha$, then this equation is oscillatory. To see this, we take a function $Q \in C[1,\infty)$ such that $Q(t) = (\log t)^{\beta}\cos t$ for $t \geq \pi$, and consider the auxiliary equation

$$(t^{\alpha}|u'|^{\alpha}\operatorname{sgn} u')' + Q(t)|u|^{\alpha}\operatorname{sgn} u = 0 \tag{4.5}$$

on the interval $[1,\infty)$. Clearly, the original equation is oscillatory if and only if (4.5) is oscillatory. The equation (4.5) is regarded as a special case of the system (1.1) with $a(t) = 1/t$ and $b(t) = Q(t)$. We will apply Theorem 1.6(II) to (4.5), for which $\sigma = +1$ and $f(t) = \log t$ $(t \geq 1)$. Let $\lambda = -\beta$, and so $\sigma(\lambda - \alpha) < 0$. Put $t_1 = 2\pi$ (> 1). The positive constant function $w_0(t) \equiv 1$ satisfies $a(t)w_0(t) = 1/t \not\equiv 0$ on $[t_2,\infty)$ for any $t_2 \geq t_1$, and (1.17) and $(1.18)_{\lambda=-\beta}$ with $w(t)$ replaced by $w_0(t)$ hold. Therefore, $w_0 \in \mathcal{W}_0$. The function $C(t; w_0, -\beta)$ satisfies

$$C(t; w_0, -\beta) = \frac{1}{\log t - \log(2\pi)} \int_{2\pi}^{t} \frac{1}{s}\left(\int_{2\pi}^{s} \cos r\,dr\right) ds \to 0 \quad \text{as } t \to \infty.$$

Next, take $w_1(t) = (\sin t)_+$, where in general $\psi(t)_+ = \max\{\psi(t), 0\}$. This function satisfies $w_1(t) \geq 0$ on $[t_1,\infty)$ and $a(t)w_1(t) = (\sin t)_+/t \not\equiv 0$ on $[t_2,\infty)$ for any $t_2 \geq t_1$. Let

$t \geq t_1 = 2\pi$. There is an $n \in \{1, 2, 3, \ldots\}$ such that $2n\pi \leq t < 2(n+1)\pi$. Then,

$$\begin{aligned}
&\int_{t_1}^{t} a(s)w_1(s)\,ds \\
&= \int_{2\pi}^{t} \frac{(\sin s)_+}{s}\,ds \geq \sum_{i=1}^{n-1} \int_{2i\pi+(\pi/4)}^{(2i+1)\pi-(\pi/4)} \frac{\sin s}{s}\,ds \\
&\geq \sum_{i=1}^{n-1} \int_{2i\pi+(\pi/4)}^{(2i+1)\pi-(\pi/4)} \frac{\sqrt{2}/2}{(2i+1)\pi-(\pi/4)}\,ds = \frac{\sqrt{2}}{4}\sum_{i=1}^{n-1}\frac{1}{(2i+1)-(1/4)} \\
&\geq \frac{\sqrt{2}}{12}\sum_{i=1}^{n-1}\frac{1}{i} \geq \frac{\sqrt{2}}{12}\log n \geq \frac{\sqrt{2}}{12}\log\left(\frac{t-2\pi}{2\pi}\right), \quad t > 4\pi.
\end{aligned} \tag{4.6}$$

Therefore, $w_1(t)$ satisfies (1.17) and $(1.18)_{\lambda=-\beta}$ with $w(t)$ replaced by $w_1(t)$. Hence, $w_1 \in \mathcal{W}_0$. By a similar calculation to (4.6) we get

$$\begin{aligned}
&\int_{t_1}^{t} a(s)w_1(s)\left(\int_{t_1}^{s} f(r)^{\lambda} b(r)\,dr\right)ds \\
&= \int_{2\pi}^{t} \frac{\left[(\sin s)_+\right]^2}{s}\,ds \geq \sum_{i=1}^{n-1}\int_{2i\pi+(\pi/4)}^{(2i+1)\pi-(\pi/4)} \frac{\sin^2 s}{s}\,ds \geq \frac{1}{12}\log\left(\frac{t-2\pi}{2\pi}\right), \quad t > 4\pi.
\end{aligned}$$

We have

$$\int_{t_1}^{t} a(s)w_1(s)\,ds = \int_{2\pi}^{t}\frac{(\sin s)_+}{s}\,ds \leq \int_{2\pi}^{t}\frac{1}{s}\,ds = \log t - \log(2\pi), \quad t \geq 2\pi.$$

Therefore,

$$C(t; w_1, -\beta) \geq \frac{1}{\log t - \log(2\pi)}\frac{1}{12}\log\left(\frac{t-2\pi}{2\pi}\right), \quad t > 4\pi,$$

which gives

$$\limsup_{t\to\infty} C(t; w_1, -\beta) \geq \frac{1}{12}.$$

Thus,

$$-\infty < \liminf_{t\to\infty} C(t; w_0, -\beta) = 0 < \frac{1}{12} \leq \limsup_{t\to\infty} C(t; w_1, -\beta).$$

By Theorem 1.6(II), we conclude that if $\beta > -\alpha$, then the equation in this example is oscillatory. Since

$$\liminf_{t\to\infty} C(t; w_0, -\beta) = 0 < \frac{1}{12} \leq \liminf_{t\to\infty} C(t; w_1, -\beta),$$

we may apply Theorem 1.6(I).

In the equation of Example 4.5, the case $\beta < -\alpha$ is nonoscillatory.

Example 4.6. Consider the half-linear equation (1.4) with

$$p(t) = t^\alpha \quad \text{and} \quad q(t) = (\log t)^\beta \cos t \quad (\beta\text{: constant})$$

for $t \geq \pi$. If $\beta < -\alpha$, then this equation is nonoscillatory. In fact, for this equation, we have $\sigma = +1$ and $P(t) = \log t - \log \pi$ $(t \geq \pi)$, and it can be verified that if $\beta < -\alpha$, then

$$\lim_{t\to\infty} \int_\pi^t P(s)^\alpha q(s)\, ds = \int_\pi^\infty (\log s - \log \pi)^\alpha (\log s)^\beta \cos s\, ds$$

exists and is finite. Therefore, by Corollary 3.3, this equation with $\beta < -\alpha$ is nonoscillatory.

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