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Generalized Fractional Integral Operators Based on Symmetric Markovian Semigroups with Application to the Heisenberg Group

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Abstract. It is known that the fractional integral operator $\mathcal{I}_\alpha$ based on a symmetric Markovian semigroup with Varopoulos dimension d is bounded from L^p to L^q, if $0 < \alpha < d$, $1 < p < q < \infty$ and $-d/p + \alpha = -d/q$, like the usual fractional integral operator defined on the d dimensional Euclidean space. We introduce generalized fractional integral operators based on symmetric Markovian semigroups and extend the L^p-L^q boundedness to Orlicz spaces. We also apply the result to the semigroup associated with the diffusion process generated by the sub-Laplacian on the Heisenberg group. Moreover, we show necessary and sufficient conditions for the boundedness of the generalized fractional integral operator on the space of homogeneous type and apply them to the Heisenberg group.

1. Introduction

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space, and let I_α be the fractional integral operator of order $\alpha \in (0, d)$, that is,

$$I_\alpha(f)(x) = \frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{d/2}\Gamma(\alpha/2)} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\, dy, \quad x \in \mathbb{R}^d. \tag{1.1}$$

Then it is known as the Hardy–Littlewood–Sobolev theorem that

$$\|I_\alpha(f)\|_{L^q} \le C\|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^d) \tag{1.2}$$

if $p, q \in (1, \infty)$ and $-d/p + \alpha = -d/q$, where the constant C is dependent on d, α and p, and independent of f.

Let $\mathcal{S}$ be a locally compact space with countable base equipped with a positive Radon measure dx on $\mathcal{S}$ and $\{T_t\}_{t\ge 0}$ a strongly continuous symmetric Markovian semigroup. Furthermore, we assume that the semigroup is Feller and has the Varopoulos dimension d

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that we shall define in the next section. The fractional integral operator of order $\alpha \in (0, d)$ based on $\{T_t\}_{t\geq 0}$ is defined by

$$\mathcal{I}_\alpha(f)(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} T_t f(x)\, dt, \quad x \in \mathcal{S}. \tag{1.3}$$

If $\{T_t\}_{t\geq 0}$ is the standard heat semigroup on $\mathbb{R}^d$, then the definition of the fractional integral operator (1.3) is equal to (1.1), i.e., $I_\alpha(f) = \mathcal{I}_\alpha(f)$. For its proof, see [39, Lemma 16.6] for example.

In 2016, using the operator $\mathcal{I}_\alpha$, Kim [19] extended (1.2) to the $L^p(\mathcal{S})$-$L^q(\mathcal{S})$ boundedness. On the other hand, (1.2) was extended to Orlicz spaces by using the generalized fractional integral operator

$$I_\rho(f)(x) = \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^d} f(y)\, dy, \quad x \in \mathbb{R}^d,$$

where ρ is a function from $(0, \infty)$ to itself satisfying

$$\int_0^1 \frac{\rho(t)}{t}\, dt < \infty, \tag{1.4}$$

see [27, 28]. If $\rho(r) = r^\alpha$, $0 < \alpha < d$, then I_ρ is equivalent to the usual fractional integral operator I_α. If $\alpha > 0$ and

$$\rho(r) = \begin{cases} 1/(\log(1/r))^{\alpha+1}, & 0 < r < 1/e, \\ 1, & 1/e \leq r \leq e, \\ (\log r)^{\alpha-1}, & r > e, \end{cases} \tag{1.5}$$

then I_ρ is bounded from $\exp L^p(\mathbb{R}^d)$ to $\exp L^q(\mathbb{R}^d)$, where $p, q \in (0, \infty)$, $-1/p + \alpha = -1/q$, and $\exp L^p(\mathbb{R}^d)$ is the Orlicz space $L^\Phi(\mathbb{R}^d)$ with Young function Φ such that

$$\Phi(t) = \begin{cases} 1/\exp(1/t^p) & \text{for small } t, \\ \exp(t^p) & \text{for large } t. \end{cases}$$

We will state the definitions of the Young function and the Orlicz space in Section 5. For the generalized fractional integral operator I_ρ, see also [1, 2, 15, 18, 26, 30, 31, 40] and the references therein.

In this paper we introduce a generalized fractional integral operator based on $\{T_t\}_{t\geq 0}$;

$$\mathcal{I}_\rho(f)(x) = \int_0^\infty \frac{\rho(t^{1/2})}{t} T_t f(x)\, dt, \quad x \in \mathcal{S}, \tag{1.6}$$

and extend the $L^p(\mathcal{S})$-$L^q(\mathcal{S})$ boundedness to Orlicz spaces as follows.

Theorem 1.1 (Theorem 6.1). *Let Φ and Ψ be Young functions and $\rho\colon (0,\infty) \to (0,\infty)$ satisfy* (1.4). *Assume that there exists a positive constant C such that, for all $r \in (0,\infty)$,*

$$\int_0^r \frac{\rho(t)}{t}\,dt\,\Phi^{-1}(1/r^d) + \int_r^\infty \frac{\rho(t)\Phi^{-1}(1/t^d)}{t}\,dt \le C\Psi^{-1}(1/r^d), \tag{1.7}$$

where d is the Varopoulos dimension of $\{T_t\}_{t\ge0}$. If Φ satisfies the ∇_2-condition (see Definition 5.5*), then $\mathcal{I}_\rho$ is bounded from $L^\Phi(\mathcal{S})$ to $L^\Psi(\mathcal{S})$.*

If $\{T_t\}_{t\ge0}$ is the standard heat semigroup on $\mathbb{R}^d$, and if ρ satisfies suitable conditions, then I_ρ and $\mathcal{I}_\rho$ are equivalent, see the next section. Therefore, our result is also an extension of the result in [28] to the symmetric Markovian semigroup. For example, if ρ is as (1.5), then $\mathcal{I}_\rho$ is bounded from $\exp L^p(\mathcal{S})$ to $\exp L^q(\mathcal{S})$.

We also apply the result to the semigroup $\{T_t\}_{t\ge0}$ associated with the diffusion process generated by the sub-Laplacian on the Heisenberg group. Namely, we have the following corollary.

Corollary 1.2 (Corollary 6.2). *Let $\mathbb{H}^n$ be the Heisenberg group, and let $\{T_t\}_{t\ge0}$ be the semigroup associated with the diffusion process generated by the sub-Laplacian on $\mathbb{H}^n$ (see Section* 3 *for the definition). Let Φ and Ψ be Young functions and $\rho\colon (0,\infty) \to (0,\infty)$ satisfy* (1.4). *Assume* (1.7) *with $d = 2n+2$ for some $C \in (0,\infty)$ and for all $r \in (0,\infty)$. If $\Phi \in \nabla_2$, then $\mathcal{I}_\rho$ is bounded from $L^\Phi(\mathbb{H}^n)$ to $L^\Psi(\mathbb{H}^n)$.*

The Heisenberg group is an example of the space of homogeneous type $X = (X, d, \mu)$ in the sense of Coifman and Weiss [6, 7]. See Section 4, for the definition of the space of homogeneous type. The $L^p(X)$-$L^q(X)$ boundedness of I_α is known by [11], and the boundedness of I_ρ on the Orlicz space $L^\Phi(X)$ was also considered in [29]. In this paper we generalize and improve these results (see Theorem 6.3), and then we apply them to $\mathcal{I}_\rho$ based on the semigroup associated with the diffusion process on the Heisenberg group (see Corollary 6.7). Moreover, we show necessary conditions for the boundedness of I_ρ and $\mathcal{I}_\rho$ on the Orlicz space.

The organization of this paper is as follows: In the next section we recall the definition of the symmetric Markovian semigroup $\{T_t\}_{t\ge0}$ on $\mathcal{S}$ with the properties of Feller and Varopoulos dimension. We also investigate the relation between I_ρ and $\mathcal{I}_\rho$ in the case that $\{T_t\}_{t\ge0}$ is the standard heat semigroup on $\mathbb{R}^d$. In Section 3 we recall the definition of the Heisenberg group and the semigroup associated with the diffusion process generated by the sub-Laplacian on the Heisenberg group. In Section 4 we recall basic properties on the space of homogeneous type. In Section 5 we recall the definitions of Young functions Φ and Orlicz spaces L^Φ. Then we give the main results in Section 6 and prove them in Section 7.

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, are dependent on the subscripts. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

2. Symmetric Markovian semigroup

The space of all continuous functions on $\mathcal{S}$ vanishing at ∞ is denoted by $C_0(\mathcal{S})$. That is, $f \in C_0(\mathcal{S})$ means that, for any $\epsilon > 0$, there exists a compact set $K \subset \mathcal{S}$ such that $|f(x)| < \epsilon$ for all $x \in \mathcal{S} \setminus K$. We denote the inner product by $\langle f, g\rangle = \int_{\mathcal{S}} f(x)g(x)\,dx$ for notational convenience.

Now we recall the definition of the symmetric Markovian semigroup $\{T_t\}_{t\ge0}$ on $\mathcal{S}$ with Feller and Varopoulos dimension. We say that a semigroup $\{T_t\}_{t\ge0}$ on $\mathcal{S}$ is a symmetric Markovian semigroup if it satisfies the followings properties.

(S1) $T_t f \ge 0$ whenever $f \ge 0$;

(S2) $T_t 1 = 1$;

(S3) (Symmetry) $\langle T_t f, g\rangle = \langle f, T_t g\rangle$ for every $f, g \in L^2(\mathcal{S})$ and every $t \ge 0$;

(S4) (L^p-contraction) $\|T_t f\|_{L^p} \le \|f\|_{L^p}$ whenever $f \in L^p(\mathcal{S})$ for every $1 \le p \le \infty$.

Suppose that there exists a symmetric Markovian semigroup $\{T_t\}_{t\ge0}$ on $\mathcal{S}$. Furthermore, we assume that the semigroup is strongly continuous on $L^2(\mathcal{S})$ and a Feller semigroup. In other words, $\{T_t\}_{t\ge0}$ satisfies that

(S5) (Strong continuity) $\lim_{t\to0} \|T_t f - f\|_2 = 0$ for all $f \in L^2(\mathcal{S})$,

(S6) (Feller) for all $f \in \mathcal{C}_0(\mathcal{S})$, $T_t f \in \mathcal{C}_0(\mathcal{S})$ for all $t \ge 0$ and $\lim_{t\to0} \|T_t f - f\|_\infty = 0$.

The semigroup $\{T_t\}_{t\ge0}$ is also assumed to have the Varopoulos dimension d ($d > 0$) introduced by Varopoulos [43] in relation to Sobolev inequality, meaning that

(S7) (Varopoulos dimension) for all $p \in [1, \infty)$, there exists a positive constant c such that, for all $f \in L^p(\mathcal{S})$ and $t > 0$,

$$\|T_t f\|_\infty \le \frac{c}{t^{d/(2p)}} \|f\|_p.$$

Note that Varopoulos [43] defined the dimension $d \ge 2$ associated with the Dirichlet form attached to the semigroup $\{T_t\}_{t\ge0}$. In this case the semigroup $\{T_t\}_{t\ge0}$ satisfies (S7) for $d \ge 2$. However, we define the dimension $d > 0$ by the condition (S7). For example, the standard heat semigroup on $\mathbb{R}^d$ satisfies (S1)–(S7) with $d = 1, 2, \ldots$. The Brownian motion on fractals are other examples of semigroups with $d > 0$, see [3, 23].

In the rest of this section, we state the relation between $\mathcal{I}_\rho$ and I_ρ in the case that $\{T_t\}_{t\geq 0}$ is the standard heat semigroup on $\mathbb{R}^d$.

Firstly, we say that a function $\theta\colon (0,\infty) \to (0,\infty)$ satisfies the doubling condition if there exists a positive constant C such that, for all $r, s \in (0,\infty)$,

$$\frac{1}{C} \leq \frac{\theta(r)}{\theta(s)} \leq C \quad \text{if } \frac{1}{2} \leq \frac{r}{s} \leq 2. \tag{2.1}$$

We say that θ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $r, s \in (0,\infty)$,

$$\theta(r) \leq C\theta(s) \quad (\text{resp. } \theta(s) \leq C\theta(r)) \quad \text{if } r < s.$$

Let $\{T_t\}_{t\geq 0}$ be the standard heat semigroup on $\mathbb{R}^d$, i.e.,

$$(T_t f)(x) = \int_{\mathbb{R}^d} W(x-y,t) f(y)\, dy, \quad x \in \mathbb{R}^d,\ t > 0,$$

where

$$W(x,t) = (4\pi t)^{-d/2} \exp(-|x|^2/(4t)).$$

Then $\{T_t\}_{t\geq 0}$ satisfies (S1)–(S7) and we have the following proposition.

Proposition 2.1. *Let $\rho, \rho_0\colon (0,\infty) \to (0,\infty)$. Assume that both ρ and ρ_0 satisfy* (1.4) *and that there exists a positive constant C such that, for all $r, s, t \in (0,\infty)$,*

$$\frac{\rho(s)}{s^{d+1}} \exp\left(-\frac{1}{4}\left(\frac{r}{s}\right)^2\right) \leq C \frac{\rho(t)}{t^{d+1}} \exp\left(-\frac{1}{4}\left(\frac{r}{t}\right)^2\right) \quad \text{if } s < t < 2r. \tag{2.2}$$

If the relation

$$\rho_0(r) \sim r^d \int_r^\infty \frac{\rho(t)}{t^{d+1}}\, dt \tag{2.3}$$

holds, then

$$I_{\rho_0} f(x) \sim \int_0^\infty \frac{\rho(t^{1/2})}{t} (T_t f)(x)\, dt \quad \text{for all } f \geq 0. \tag{2.4}$$

Moreover, if $t \mapsto t^\theta \rho(t)$ is almost increasing for some $\theta \geq 0$ and $t \mapsto \rho(t)/t^{d-\epsilon}$ is almost decreasing for some $\epsilon > 0$, then

$$I_\rho f(x) \sim \int_0^\infty \frac{\rho(t^{1/2})}{t} (T_t f)(x)\, dt \quad \text{for all } f \geq 0. \tag{2.5}$$

In (2.4) *and* (2.5) *the implicit constants are independent of f and $x \in \mathbb{R}^d$.*

For example, the function ρ defined by (1.5) satisfies (2.5).

Proof of Proposition 2.1. Let $f \ge 0$. By Fubini's theorem,

$$\begin{aligned}&\int_0^\infty \frac{\rho(t^{1/2})}{t}\left(\int_{\mathbb{R}^d} W(x-y,t)f(y)\,dy\right)dt\\&= \int_{\mathbb{R}^d} f(y)\left(\int_0^\infty \frac{\rho(t^{1/2})}{t}(4\pi t)^{-d/2}\exp\left(-\frac{|x-y|^2}{4t}\right)dt\right)dy.\end{aligned}$$

Let $u = t^{1/2}$ and $r = |x-y|$. Then $t = u^2$ and

$$\int_0^\infty \frac{\rho(t^{1/2})}{t}(4\pi t)^{-d/2}\exp\left(-\frac{|x-y|^2}{4t}\right)dt = \frac{2}{(4\pi)^{d/2}}\int_0^\infty \frac{\rho(u)}{u^{d+1}}\exp\left(-\frac{1}{4}\left(\frac{r}{u}\right)^2\right)du.$$

By the assumption (2.2) we have

$$\int_0^r \frac{\rho(u)}{u^{d+1}}\exp\left(-\frac{1}{4}\left(\frac{r}{u}\right)^2\right)du \le C\int_r^{2r} \frac{\rho(u)}{u^{d+1}}\exp\left(-\frac{1}{4}\left(\frac{r}{u}\right)^2\right)du.$$

Hence, using this inequality, the inequalities $e^{-1/4} \le \exp\left(-\frac{1}{4}\left(\frac{r}{u}\right)^2\right) \le 1$ for $r \le u$, and (2.3), we have

$$\begin{aligned}\int_0^\infty \frac{\rho(u)}{u^{d+1}}\exp\left(-\frac{1}{4}\left(\frac{r}{u}\right)^2\right)du &\sim \int_r^\infty \frac{\rho(u)}{u^{d+1}}\exp\left(-\frac{1}{4}\left(\frac{r}{u}\right)^2\right)du\\&\sim \int_r^\infty \frac{\rho(u)}{u^{d+1}}\,du \sim \frac{\rho_0(r)}{r^d},\end{aligned}$$

and then

$$\int_0^\infty \frac{\rho(t^{1/2})}{t}\left(\int_{\mathbb{R}^d} W(x-y,t)f(y)\,dy\right)dt \sim \int_{\mathbb{R}^d}\frac{\rho_0(|x-y|)}{|x-y|^d}f(y)\,dy,$$

which shows (2.4).

If $t \mapsto t^\theta \rho(t)$ is almost increasing for some $\theta \ge 0$, then

$$\frac{\rho(t)}{t^{d+1}}\exp\left(-\frac{1}{4}\left(\frac{r}{t}\right)^2\right) = \frac{t^\theta\rho(t)}{r^{d+1+\theta}}\left(\frac{r}{t}\right)^{d+1+\theta}\exp\left(-\frac{1}{4}\left(\frac{r}{t}\right)^2\right)$$

satisfies (2.2). Moreover, if $t \mapsto \rho(t)/t^{d-\epsilon}$ is almost decreasing for some $\epsilon > 0$, then ρ satisfies the doubling condition and

$$\frac{\rho(r)}{r^d} \sim \int_r^{2r}\frac{\rho(t)}{t^{d+1}}\,dt \le \int_r^\infty \frac{\rho(t)}{t^{d+1}}\,dt \lesssim \frac{\rho(r)}{r^{d-\epsilon}}\int_r^\infty \frac{1}{t^{1+\epsilon}}\,dt \sim \frac{\rho(r)}{r^d}.$$

This shows (2.3) for $\rho = \rho_0$, and we have (2.5). □

3. Heisenberg group

In this section, we introduce Heisenberg group $\mathbb{H}^n$ following [4, 5]. As a topological space, $\mathbb{H}^n$ is the same as $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$, and we write $x \in \mathbb{H}^n$ as $x = (x_1, x_2, \ldots, x_{2n+1}) = (x_1 + ix_{n+1}, \ldots, x_n + ix_{2n}, x_{2n+1})$. The multiplication in $\mathbb{H}^n$ is given as

$$xx' = \left(x_1 + x_1', \ldots, x_{2n} + x_{2n}', x_{2n+1} + x_{2n+1}' - \frac{1}{2} \sum_{j=1}^{n} \operatorname{Im}(z_j \overline{z_j'}) \right),$$

where $z_j = x_j + ix_{n+j}$ and $z_j' = x_j' + ix_{n+j}'$. It is easy to see that the unit of $\mathbb{H}^n$ is 0 and $x^{-1} = -x$.

Let X_j be the left invariant vector field on $\mathbb{H}^n$ such that $X_j = \frac{\partial}{\partial x_j}$ at 0. At a general $x \in \mathbb{H}^n$,

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} x_{n+j} \frac{\partial}{\partial x_{2n+1}}, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} + \frac{1}{2} x_j \frac{\partial}{\partial x_{2n+1}}$$

where $j = 1, 2, \ldots, n$.

Let $(B_t)_{t \ge 0} = ((B_t^1, \ldots, B_t^{2n}))_{t \ge 0}$ be the canonical realization of $2n$-dimensional Brownian motion on the Wiener space $\mathbb{W}^{2n}$ such that $B_0 = 0$. Let $\Delta_{\mathbb{H}^n}$ be the sub-Laplacian on $\mathbb{H}^n$, that is,

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n} X_j^2.$$

Since the Lebesgue measure is a bi-invariant Haar measure on $\mathbb{H}^n$, we have

$$\int_{\mathbb{H}^n} (\Delta_{\mathbb{H}^n} f)(x) g(x)\, dx = \int_{\mathbb{H}^n} f(x) (\Delta_{\mathbb{H}^n} g)(x)\, dx$$

for all smooth functions f and g which have compact supports. Moreover, $-\Delta_{\mathbb{H}^n}$ is nonnegative and has the Friedrichs extension.

The diffusion process $(Z_t)_{t \ge 0}$ generated by $\frac{1}{2}\Delta_{\mathbb{H}^n}$ is given as

$$Z_t = (B_t^1, \ldots, B_t^{2n}, A_t),$$

where

$$A_t = \frac{1}{2} \sum_{j=1}^{n} \left(\int_0^t B_s^j \, dB_s^{n+j} - \int_0^t B_s^{n+j} \, dB_s^j \right),$$

see [12, Lemma 1] and [17, p. 473]. Hence, the associated semigroup $\{T_t\}_{t \ge 0}$ is given by

$$T_t f(x) = E[f(xZ_t)], \quad x \in \mathbb{H}^n, \tag{3.1}$$

where E denotes the expectation. We have that $\{T_t\}_{t \ge 0}$ satisfies (S1)–(S6) in a standard way.

For $s > 0$, dilation δ_s on $\mathbb{H}^n$ is defined by $\delta_s(x) = (sx_1, \ldots, sx_{2n}, s^2 x_{2n+1})$. By [38, Proposition I.1.10], we have the invariance of the law of Brownian motion $(B_t)_{t\geq 0} \sim (\epsilon B_{t/\epsilon^2})_{t\geq 0}$ for every $\epsilon > 0$. From this, we deduce that the distribution of Z_t equals to the distribution of $\delta_{\sqrt{t}}(Z_1)$ for every $t > 0$. Therefore, we obtain

$$T_t f(x) = E[f(x\delta_{\sqrt{t}}(Z_1))] = E[f(\delta_{\sqrt{t}}(\delta_{\sqrt{t}}^{-1}(x)Z_1))] = T_1(f \circ \delta_{\sqrt{t}})(\delta_{\sqrt{t}}^{-1}(x)).$$

At 0, the family of vector fields $\{X_j\}_{j=1}^{2n}$ satisfies the hypoellipticity condition of Hörmander type in the sense of [17, Chapter V, Definition 10.1] because $[X_j, X_{j+n}] = \frac{\partial}{\partial x_{2n+1}}$. Since Z_t is a solution to the stochastic differential equation

$$dZ_t = \sum_{j=1}^{2n} X_j(Z_t) \circ dB_t^j$$

with $Z_0 = 0$ realized on $\mathbb{W}^{2n}$, we can apply [17, Chapter V, Theorem 10.2] to obtain that $E[(\det \sigma_t)^{-p}] < \infty$ for every $p > 1$, where $\circ$ denotes the Stratonovich stochastic integral and σ_t denotes the Malliavin covariance of Z_t. By [42, Chapter 5, Theorem 5.9], the distribution of Z_t has a bounded smooth density. Hence, we have (S7) with $d = 2n + 2$ as follows:

$$\|T_t f\|_{L^\infty} = \|T_1(f \circ \delta_{\sqrt{t}})\|_{L^\infty} \leq \|T_1\|_{L^p \to L^\infty} \|f \circ \delta_{\sqrt{t}}\|_{L^p} = \frac{\|T_1\|_{L^p \to L^\infty}}{t^{(n+1)/p}} \|f\|_{L^p}.$$

Let $x, y \in \mathbb{H}^n$. The distance $d(x, y)$ is defined by

$$\begin{aligned} d(x,y) = \inf \Big\{ & T > 0; \exists\, \gamma \colon [0,T] \to \mathbb{H}^n \text{ s.t. } \gamma(0) = x, \gamma(T) = y, \\ & \gamma'(t) = \sum_{j=1}^{2n} a_j(t) X_j(\gamma(t)), \text{ where } \sum_{j=1}^{2n} a_j(t)^2 \leq 1 \Big\}. \end{aligned} \tag{3.2}$$

Then d and δ_s satisfy

$$d(\delta_s(x), \delta_s(y)) = s d(x, y). \tag{3.3}$$

By (3.3), we have

$$V(r) = r^{2n+2} V(1),$$

where $V(r)$ denotes the volume of the ball of radius r.

Note that

$$T_t f(x) = \int_{\mathbb{H}^n} p(t, x^{-1}y) f(y)\, dy,$$

where p is the fundamental solution to

$$\frac{\partial}{\partial t} p(t, x) = \frac{1}{2} \Delta_{\mathbb{H}^n} p(t, x).$$

It is known that p satisfies the following

Theorem 3.1. (see [44, Theorems IV.4.2 and IV.4.3] for example) *There exist positive constants C_1, C_2, C_3, C_4 such that*

$$\frac{C_1}{t^{n+1}} \exp\left(-\frac{d(0,x)^2}{C_2 t}\right) \le p(t,x) \le \frac{C_3}{t^{n+1}} \exp\left(-\frac{d(0,x)^2}{C_4 t}\right)$$

for all $(t,x) \in (0,\infty) \times \mathbb{H}^n$.

Following [14,45], we now introduce the generalized fractional integral operator I_ρ on $\mathbb{H}^n$. For $x = (x_1, x_2, \dots, x_{2n+1}) \in \mathbb{H}^n$, define $|x|_{\mathbb{H}^n}$ by

$$|x|_{\mathbb{H}^n} = \left\{ \left(\sum_{j=1}^{2n} x_j^2 \right)^2 + 16 x_{2n+1}^2 \right\}^{1/4}.$$

It is known that $d_{\mathbb{H}^n}(x,y) = |x^{-1}y|_{\mathbb{H}^n}$ is a metric on $\mathbb{H}^n$ equivalent to d in (3.2), see [5, p. 19] for example. Then, for $\rho\colon (0,\infty) \to (0,\infty)$ satisfying (1.4), define

$$(I_\rho f)(x) = \int_{\mathbb{H}^n} \frac{\rho(|x^{-1}y|_{\mathbb{H}^n})}{|x^{-1}y|_{\mathbb{H}^n}^{2n+2}} f(y)\, dy.$$

Thanks to Theorem 3.1 and the equivalence between d and $d_{\mathbb{H}^n}$, we have the following proposition in the same way as in Proposition 2.1.

Proposition 3.2. *Let $\rho\colon (0,\infty) \to (0,\infty)$. If $t \mapsto t^\theta \rho(t)$ is almost increasing for some $\theta \ge 0$ and $t \mapsto \rho(t)/t^{2n+2-\epsilon}$ is almost decreasing for some $\epsilon > 0$, then*

$$I_\rho f(x) \sim \int_0^\infty \frac{\rho(t^{1/2})}{t} (T_t f)(x)\, dt \quad \text{for all } f \ge 0.$$

In the above the implicit constant is independent of f and $x \in \mathbb{H}^n$.

For the fractional integral operator on the Heisenberg group, see also Frank and Lieb [10]. They obtained the sharp constant on L^p-$L^{p'}$ boundedness of I_ρ for $1 < p < 2$, $1/p + 1/p' = 1$ and $\rho(r) = r^\alpha$ with $\alpha = (2n+2)(2-p)/p$.

4. Spaces of homogeneous type

In this section we recall basic properties on the space of homogeneous type. Let $X = (X, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss [6,7], that is, X is a topological space endowed with a quasi-distance d and a nonnegative measure μ such that the following conditions holds:

(i) There exists a constant $K_1 \ge 1$ such that, for all $x, y, z \in X$,

$$d(x,y) \ge 0 \quad \text{and} \quad d(x,y) = 0 \text{ if and only if } x = y,$$

$$d(x,y) = d(y,x),$$

$$d(x,y) \le K_1 \left(d(x,z) + d(z,y)\right). \tag{4.1}$$

(ii) For every point $x \in X$, the balls $B(x,r) = \{y \in X : d(x,y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x.

(iii) The measure μ is defined on a σ-algebra of subsets of X which contains all balls.

(iv) There exists a constant $K_2 \ge 1$ such that, for all $x \in X$ and $r > 0$,

$$0 < \mu(B(x,2r)) \le K_2\, \mu(B(x,r)) < \infty.$$

In this paper we always assume that $\mu(\{x\}) = 0$ for all $x \in X$ and that the inequality

$$|d(x,z) - d(y,z)| \le K_3\,(d(x,z) + d(y,z))^{1-\theta} d(x,y)^{\theta} \tag{4.2}$$

holds for some constants θ $(0 < \theta \le 1)$ and $K_3 \ge 1$ which are independent of $x, y, z \in X$. The constant θ is called the order of the space of homogeneous type. Note that, from (4.1), it follows that there exist constants θ $(0 < \theta \le 1)$, $K_3 \ge 1$, and a quasi-distance which is equivalent to the original d such that (4.2) holds (see Macías and Segovia [24]).

If $\mu(X) < \infty$, then there exists a positive constant R_0 such that

$$X = B(x, R_0) \quad \text{for all } x \in X$$

(see [32, Lemma 5.1]).

Following [24], we shall say that a space of homogeneous type is normal if there exists a positive constant K_4 such that, for all $x \in X$ and $0 < r < \mu(X)$,

$$\frac{r}{K_4} \le \mu(B(x,r)) \le K_4 r.$$

For a space of homogeneous type (X, d, μ), let

$$\delta(x,y) = \begin{cases} \inf\{\mu(B) : \ B \text{ is a ball containing } x \text{ and } y\} & \text{if } x \ne y, \\ 0 & \text{if } x = y. \end{cases} \tag{4.3}$$

Then (X, δ, μ) is a normal space of homogeneous type, and the topologies induced on X by d and δ coincide (see [24]).

In general, X is called Q-homogeneous $(Q > 0)$, if there exists a positive constant K_5 such that, for all $x \in X$ and $0 < r < \mu(X)$,

$$\frac{r^Q}{K_5} \le \mu(B(x,r)) \le K_5 r^Q. \tag{4.4}$$

The Euclidean space $\mathbb{R}^d$ is d-homogeneous and the Heisenberg group $\mathbb{H}^n$ is $(2n+2)$-homogeneous.

Let (X, d, μ) be Q-homogeneous. For a function $\rho\colon (0,\infty) \to (0,\infty)$ satisfying (1.4), let

$$I_\rho(f)(x) = \int_X \frac{\rho(d(x,y))}{d(x,y)^Q} f(y)\,d\mu(y), \quad x \in X.$$

If $\rho(r) = r^\alpha$, $\alpha \in (0, Q)$, then we denote I_ρ by I_α. Then it is known that I_α is bounded from $L^p(X)$ to $L^q(X)$, if $p, q \in (1,\infty)$ and $-Q/p+\alpha = -Q/q$, see [11] for the case $Q = 1$. It is also known that I_ρ is bounded on Orlicz spaces under some assumptions, see [29]. In this paper we generalize and improve these results. We also give necessary conditions for the boundedness of I_ρ.

To consider I_ρ on the space of homogeneous type, we also use the following condition: There exist positive constants C, k_1 and k_2 with $k_1 < k_2$ such that, for all $r \in (0,\infty)$,

$$\sup_{r/2 \le t \le r} \rho(t) \le C \int_{k_1 r}^{k_2 r} \frac{\rho(t)}{t}\,dt. \tag{4.5}$$

The condition (4.5) was considered in [36] in the case $\mathbb{R}^d$. If ρ satisfies the doubling condition, then ρ satisfies (4.5). Let $\alpha \in (0, Q)$ and

$$\rho(r) = \begin{cases} r^\alpha, & 0 < r \le 1, \\ e^{-r+1}, & 1 < r. \end{cases}$$

Then ρ satisfies (4.5). For the conditions on ρ, see also Remark 6.5.

5. Young functions and Orlicz spaces

For an increasing (i.e., nondecreasing) function $\Phi\colon [0,\infty] \to [0,\infty]$, let

$$a(\Phi) = \sup\{t \ge 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \ge 0 : \Phi(t) = \infty\}$$

with convention $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. Then $0 \le a(\Phi) \le b(\Phi) \le \infty$.

Let $\overline{\boldsymbol{\Phi}}$ be the set of all increasing functions $\Phi\colon [0,\infty] \to [0,\infty]$ such that

$$\begin{gathered} 0 \le a(\Phi) < \infty, \quad 0 < b(\Phi) \le \infty, \\ \lim_{t \to +0} \Phi(t) = \Phi(0) = 0, \\ \Phi \text{ is left continuous on } [0, b(\Phi)), \\ \text{if } b(\Phi) = \infty, \text{ then } \lim_{t\to\infty} \Phi(t) = \Phi(\infty) = \infty, \\ \text{if } b(\Phi) < \infty, \text{ then } \lim_{t \to b(\Phi)-0} \Phi(t) = \Phi(b(\Phi)) \ (\le \infty). \end{gathered} \tag{5.1}$$

In what follows, if an increasing and left continuous function $\Phi\colon [0,\infty) \to [0,\infty)$ satisfies (5.1) and $\lim_{t\to\infty} \Phi(t) = \infty$, then we always regard that $\Phi(\infty) = \infty$ and that $\Phi \in \overline{\boldsymbol{\Phi}}$.

For $\Phi \in \overline{\boldsymbol{\Phi}}$, we recall the generalized inverse of Φ in the sense of O'Neil [33, Definition 1.2].

Definition 5.1. For $\Phi \in \overline{\boldsymbol{\Phi}}$ and $u \in [0,\infty]$, let

$$\Phi^{-1}(u) = \begin{cases} \inf\{t \ge 0 : \Phi(t) > u\}, & u \in [0,\infty), \\ \infty, & u = \infty. \end{cases}$$

Let $\Phi \in \overline{\boldsymbol{\Phi}}$. Then Φ^{-1} is finite, increasing and right continuous on $[0,\infty)$ and positive on $(0,\infty)$. If Φ is bijective from $[0,\infty]$ to itself, then Φ^{-1} is the usual inverse function of Φ. Moreover, if $\Phi \in \overline{\boldsymbol{\Phi}}$, then

$$\Phi(\Phi^{-1}(u)) \le u \le \Phi^{-1}(\Phi(u)) \quad \text{for all } u \in [0,\infty], \tag{5.2}$$

which is a generalization of Property 1.3 in [33], see Remark 5.3. For its proof see [41, Proposition 2.2].

For $\Phi, \Psi \in \overline{\boldsymbol{\Phi}}$, we write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(C^{-1}t) \le \Psi(t) \le \Phi(Ct) \quad \text{for all } t \in [0,\infty].$$

For functions $P, Q\colon [0,\infty] \to [0,\infty]$, we write $P \sim Q$ if there exists a positive constant C such that

$$C^{-1}P(t) \le Q(t) \le CP(t) \quad \text{for all } t \in [0,\infty].$$

Then, for $\Phi, \Psi \in \overline{\boldsymbol{\Phi}}$,

$$\Phi \approx \Psi \quad \Longleftrightarrow \quad \Phi^{-1} \sim \Psi^{-1}, \tag{5.3}$$

see [41, Lemma 2.3].

Now we recall the definition of the Young function and its generalization.

Definition 5.2. A function $\Phi \in \overline{\boldsymbol{\Phi}}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is convex on $[0, b(\Phi))$. Let $\boldsymbol{\Phi}_Y$ be the set of all Young functions. Let $\overline{\boldsymbol{\Phi}}_Y$ be the set of all $\Phi \in \overline{\boldsymbol{\Phi}}$ such that $\Phi \approx \Psi$ for some $\Psi \in \boldsymbol{\Phi}_Y$.

Remark 5.3. If $\Phi \in \boldsymbol{\Phi}_Y$, then (5.2) is Property 1.3 in [33]. If $\Phi \in \boldsymbol{\Phi}_Y$ and $0 < \Phi(t) < \infty$, then $\Phi^{-1}(\Phi(t)) = t$.

Next we define the Orlicz space $L^{\Phi}(\mathcal{S})$ for $\Phi \in \overline{\boldsymbol{\Phi}}_Y$. Let $L^0(\mathcal{S})$ be the set of all complex valued measurable functions on $\mathcal{S}$.

Definition 5.4. For a function $\Phi \in \overline{\boldsymbol{\Phi}}_Y$, let

$$L^{\Phi}(\mathcal{S}) = \left\{ f \in L^0(\mathcal{S}) : \int_{\mathcal{S}} \Phi(\epsilon|f(x)|)\,dx < \infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{L^\Phi} = \inf\left\{ \lambda > 0 : \int_{\mathcal{S}} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}.$$

Then $\|\cdot\|_{L^\Phi}$ is a quasi-norm and thereby $L^\Phi(\mathcal{S})$ is a quasi-Banach space. If $\Phi \in \boldsymbol{\Phi}_Y$, then $\|\cdot\|_{L^\Phi}$ is a norm and thereby $L^\Phi(\mathcal{S})$ is a Banach space. For $\Phi, \Psi \in \overline{\boldsymbol{\Phi}}_Y$, if $\Phi \approx \Psi$, then $L^\Phi(S) = L^\Psi(S)$ with equivalent quasi-norms. Orlicz spaces are introduced by [34,35]. For the theory of Orlicz spaces, see [20–22, 25, 37] for example.

Definition 5.5. A function $\Phi \in \overline{\boldsymbol{\Phi}}$ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \overline{\nabla}_2$, if there exists a constant $k > 1$ such that

$$\Phi(t) \le \frac{1}{2k}\Phi(kt) \quad \text{for all } t > 0.$$

Let $\nabla_2 = \boldsymbol{\Phi}_Y \cap \overline{\nabla}_2$.

For example, both

$$\Phi_1(t) = \begin{cases} t^2, & 0 \le t \le 1, \\ \infty, & t > 1 \end{cases} \quad \text{and} \quad \Phi_2 = \max(0, t^2 - 1)$$

are in ∇_2. For other examples, see after Remark 6.8.

At the end of this section we extend the maximal ergodic theorem to Orlicz spaces. Assume that $\{T_t\}_{t\ge 0}$ is a strongly continuous symmetric Markovian semigroup satisfying (S1)–(S6). Let

$$T^*f(x) = \sup_{t>0} |T_t f(x)|. \tag{5.4}$$

Then T^* is a sublinear operator of strong-type (p,p) for all $p \in (1,\infty]$, see [19, Proposition 2.3]. In this case we can use a Marcinkiewicz-type interpolation theorem for Orlicz spaces and get the following maximal ergodic theorem. We give its proof for readers' convenience. For a measurable function f on $\mathcal{S}$ and $t > 0$, we denote by $m(f,t)$ the measure of the set $\{x \in \mathcal{S} : |f(x)| > t\}$.

Theorem 5.6. *If $\Phi \in \overline{\nabla}_2$, then there exists a positive constant C_Φ such that*

$$\|T^*f\|_{L^\Phi} \le C_\Phi \|f\|_{L^\Phi}. \tag{5.5}$$

Proof. For every $p \in (1,\infty]$, there exists a positive constant C_p such that

$$\|T^*f\|_{L^p} \le C_p \|f\|_{L^p}. \tag{5.6}$$

Let $f \in L^\Phi(\mathcal{S})$. We may assume that $\|f\|_{L^\Phi} = 1$. For $t > 0$ and $c > 0$, let

$$f = f^t + f_t, \quad f^t(x) = \begin{cases} f(x), & |f(x)| > ct, \\ 0, & |f(x)| \le ct. \end{cases}$$

Then

$$\|T^* f_t\|_{L^\infty} \le C_\infty \|f_t\|_{L^\infty} \le C_\infty ct = t/2,$$

provided we choose $c = (2C_\infty)^{-1}$. It follows that $m(T^* f_t, t/2) = 0$. Therefore,

$$m(T^* f, t) \le m(T^* f^t, t/2) + m(T^* f_t, t/2) = m(T^* f^t, t/2).$$

Since $\Phi \in \overline{\nabla}_2$, there exists $p \in (1,\infty)$ such that $\Phi((\cdot)^{1/p}) \in \overline{\nabla}_2$ (see [41, Lemma 4.5]), that is, $\Phi((\cdot)^{1/p}) \approx \Phi_p$ for some $\Phi_p \in \nabla_2$. Then, there exists a positive constant c_0 such that

$$\int_0^r \frac{\Phi_p'(t)}{t}\,dt \le \frac{c_0 \Phi_p(c_0 r)}{r},$$

see [21, Theorem 1.2.1]. By (5.6) we have

$$m(T^* f^t, t/2) \le \frac{1}{(t/2)^p}\|T^* f^t\|_{L^p}^p \le \frac{(C_p)^p}{(t/2)^p}\|f^t\|_{L^p}^p \le \frac{(2C_p)^p}{t^p}\int_{|f|>ct} |f(x)|^p\,dx.$$

Hence

$$\begin{aligned}
\int_S \Phi_p(|T^* f(x)|^p)\,dx &= \int_0^\infty (\Phi_p(t^p))' m(T^* f, t)\,dt \\
&\le (2C_p)^p \int_0^\infty \frac{p\,\Phi_p'(t^p) t^{p-1}}{t^p}\left(\int_{|f|>ct} |f(x)|^p\,dx\right) dt \\
&= (2C_p)^p \int_S |f(x)|^p \left(\int_0^{|f(x)|/c} \frac{p\,\Phi_p'(t^p)}{t}\,dt\right) dx \\
&= (2C_p)^p \int_S |f(x)|^p \left(\int_0^{(|f(x)|/c)^p} \frac{\Phi_p'(u)}{u}\,du\right) dx \\
&\le (2C_p c)^p c_0 \int_S \Phi_p(c_0(|f(x)|/c)^p)\,dx,
\end{aligned}$$

which shows

$$\int_S \Phi\left(\frac{|T^* f(x)|}{C_\Phi}\right) dx \le \int_S \Phi(|f(x)|)\,dx$$

for some positive constant C_Φ. This means (5.5). □

Let (X, d, μ) be a space of homogeneous type, and let M be the Hardy–Littlewood maximal operator, that is,

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)|\,d\mu(y),$$

where the supremum is taken over all balls B containing x. It is known that M is bounded from $L^1(X)$ to $\mathrm{w}L^1(X)$ and from $L^p(X)$ to itself if $p \in (1,\infty]$, see [6,7]. For Orlicz spaces, the following theorem is known.

Theorem 5.7. *Let $\Phi \in \overline{\boldsymbol{\Phi}}_Y$. Then there exists a positive constant C_Φ such that*

$$\|Mf\|_{\mathrm{w}L^\Phi} \le C_\Phi \|f\|_{L^\Phi}.$$

If $\Phi \in \overline{\nabla}_2$, then

$$\|Mf\|_{L^\Phi} \le C_\Phi \|f\|_{L^\Phi}.$$

In the above $\mathrm{w}L^\Phi(X)$ is the weak Orlicz space defined by the following quasi-norm:

$$\|f\|_{\mathrm{w}L^\Phi} = \inf\left\{\lambda > 0 : \sup_{t\in(0,\infty)} \Phi(t)\mu(\{x \in X : |f(x)|/\lambda > t\}) \le 1\right\}.$$

For the proof of Theorem 5.7, for example, see [21, Theorem 1.2.1 and Lemma 1.2.4] for the case $\mathbb{R}^d$, and [13, Theorems 6.2.1 and 6.4.1] for the space of homogeneous type.

6. Main results

Assume that $\{T_t\}_{t\ge0}$ is a strongly continuous symmetric Markovian semigroup on $\mathcal{S}$ satisfying (S1)–(S7). Recall that $\mathcal{I}_\rho$ and T^* are defined by (1.6) and (5.4), respectively.

The first result in this paper is the following

Theorem 6.1 (Theorem 1.1)**.** *Let $\Phi, \Psi \in \overline{\boldsymbol{\Phi}}_Y$ and $\rho\colon (0,\infty) \to (0,\infty)$ satisfy (1.4). Assume that there exists a positive constant C such that, for all $r \in (0,\infty)$,*

$$\int_0^r \frac{\rho(t)}{t}\,dt\ \Phi^{-1}(1/r^d) + \int_r^\infty \frac{\rho(t)\,\Phi^{-1}(1/t^d)}{t}\,dt \le C\Psi^{-1}(1/r^d), \tag{6.1}$$

where d is the Varopoulos dimension of $\{T_t\}_{t\ge0}$. Then, for any positive constant C_0, there exists a positive constant C_1 such that, for all $f \in L^\Phi(\mathcal{S})$ with $f \ne 0$,

$$\Psi\left(\frac{|\mathcal{I}_\rho(f)(x)|}{C_1\|f\|_{L^\Phi}}\right) \le \Phi\left(\frac{T^*f(x)}{C_0\|f\|_{L^\Phi}}\right), \quad x \in \mathcal{S}.$$

Consequently, if $\Phi \in \overline{\nabla}_2$, then $\mathcal{I}_\rho$ is bounded from $L^\Phi(\mathcal{S})$ to $L^\Psi(\mathcal{S})$.

From the theorem above we have the following corollary immediately.

Corollary 6.2 (Corollary 1.2)**.** *Let $\mathbb{H}^n$ be the Heisenberg group, and let $\{T_t\}_{t\ge0}$ be the semigroup in (3.1) on $\mathbb{H}^n$. Let $\Phi, \Psi \in \overline{\boldsymbol{\Phi}}_Y$ and $\rho\colon (0,\infty) \to (0,\infty)$ satisfy (1.4). Assume (6.1) with $d = 2n + 2$ for some $C \in (0,\infty)$ and for all $r \in (0,\infty)$. If $\Phi \in \overline{\nabla}_2$, then $\mathcal{I}_\rho$ is bounded from $L^\Phi(\mathbb{H}^n)$ to $L^\Psi(\mathbb{H}^n)$.*

Next we state the result on the space of homogeneous type.

Theorem 6.3. *Let (X, d, μ) be a Q-homogeneous space of homogeneous type satisfying $\mu(\{x\}) = 0$ for all $x \in X$. Let $\Phi, \Psi \in \overline{\boldsymbol{\Phi}}_Y$ and $\rho\colon (0,\infty) \to (0,\infty)$ satisfy (1.4).*

(i) *If ρ satisfies* (4.5) *and there exists a positive constant C such that, for all $r \in (0,\infty)$,*

$$\text{(6.2)} \qquad \int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}(1/r^Q) + \int_r^\infty \frac{\rho(t)\,\Phi^{-1}(1/t^Q)}{t}\,dt \le C\Psi^{-1}(1/r^Q),$$

then, for any positive constant C_0, there exists a positive constant C_1 such that, for all $f \in L^\Phi(X)$ with $f \neq 0$,

$$\Psi\left(\frac{|I_\rho(f)(x)|}{C_1\|f\|_{L^\Phi}}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right), \quad x \in X.$$

Consequently, I_ρ is bounded from $L^\Phi(X)$ to $\mathrm{w}L^\Psi(X)$. Moreover, if $\Phi \in \overline{\nabla}_2$, then I_ρ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$.

(ii) *Conversely, if $r \mapsto \rho(r)/r^k$ is almost decreasing for some $k > 0$, and if I_ρ is bounded from $L^\Phi(X)$ to $\mathrm{w}L^\Psi(X)$, then there exists a positive constant C' such that, for all $r \in (0,\infty)$,*

$$\int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}(1/r^Q) \le C'\Psi^{-1}(1/r^Q).$$

Moreover, if one of the following conditions holds, then (6.2) *holds for some $C \in (0,\infty)$ and for all $r \in (0,\infty)$:*

(a) $\Phi^{-1}(1/d(x_0,\cdot)^Q) \in L^\Phi(X)$ *for some $x_0 \in X$.*

(b) *There exists a positive constant C'' such that, for all $r \in (0,\infty)$*

$$\int_r^\infty \frac{\rho(t)\Phi^{-1}(1/t^Q)}{t}\,dt \le C''\rho(r)\Phi^{-1}(1/r^Q).$$

We will give an example of Φ which satisfies the assumption (a) in Example 6.9 at the end of this section.

Remark 6.4. In Theorem 6.3(ii), for the cases $\mathbb{R}^d$ and $\mathbb{H}^n$, we do not need the almost decreasingness of $r \mapsto \rho(r)/r^k$, see Remark 7.4. For these cases, however, we need (4.5) only in the part (b).

Remark 6.5. A simple calculation shows that the following (R1) and (R3) are equivalent, and, (R2) and (R4) are equivalent:

(R1) $\exists\, k \in (0,\infty)$ such that $r \mapsto \rho(r)/r^k$ is almost decreasing.

(R2) $\exists\, k \in (0,\infty)$ such that $r \mapsto \rho(r)r^k$ is almost increasing.

(R3) $\exists\, C \in [1,\infty)$ such that, if $r < s \le 2r$, then $C\rho(r) \ge \rho(s)$.

(R4) $\exists\, C \in [1,\infty)$ such that, if $r < s \le 2r$, then $\rho(r) \le C\rho(s)$.

Hence, ρ satisfies the doubling condition (2.1) if and only if both (R1) and (R2) hold. It also follows that, if (R1) or (R2) holds, then (4.5) holds.

Remark 6.6. (see [29]) For any space of homogeneous type (X, d, μ) satisfying $\mu(\{x\}) = 0$ for all $x \in X$, let

$$\overline{I}_\rho(f)(x) = \int_X \frac{\rho(\mu(B(x, d(x, y))))}{\mu(B(x, d(x, y)))} f(y)\, d\mu(y), \quad x \in X.$$

Taking δ defined in (4.3), we can consider its normalized space of homogeneous type (X, δ, μ) and define $I_\rho^{(\delta)}$ as

$$I_\rho^{(\delta)}(f)(x) = \int_X \frac{\rho(\delta(x, y))}{\delta(x, y)} f(y)\, d\mu(y), \quad x \in X.$$

If $r \mapsto \rho(r)/r$ is almost decreasing, then $\overline{I}_\rho|f| \lesssim I_\rho^{(\delta)}|f|$, since

$$\frac{\rho(\mu(B(x, d(x, y))))}{\mu(B(x, d(x, y)))} \lesssim \frac{\rho(\delta(x, y))}{\delta(x, y)}.$$

Therefore, the boundedness of $\overline{I}_\rho$ follows from the boundedness of $I_\rho^{(\delta)}$.

In Theorem 6.3, for the case $\mathbb{R}^d$, the part (i) is already known by [8, Theorem 3, Item 1]. Its proof is also valid for any Q-homogeneous space of homogeneous type. Then we omit the proof of the part (i). The part (ii) is an improvement of [8, Theorem 3, Item 3]. We will give its proof in Section 7.

By Proposition 3.2 we have the following corollary.

Corollary 6.7. *Let $\mathbb{H}^n$ be the Heisenberg group, and let $\{T_t\}_{t \ge 0}$ be the semigroup in* (3.1) *on $\mathbb{H}^n$. Let $\Phi, \Psi \in \overline{\boldsymbol{\Phi}}_Y$ and $\rho\colon (0, \infty) \to (0, \infty)$ satisfy* (1.4). *Assume that $t \mapsto t^\theta \rho(t)$ is almost increasing for some $\theta \ge 0$ and $t \mapsto \rho(t)/t^{2n+2-\epsilon}$ is almost decreasing for some $\epsilon > 0$.*

(i) *If there exists a positive constant C such that, for all $r \in (0, \infty)$,*

$$\int_0^r \frac{\rho(t)}{t}\, dt\, \Phi^{-1}(1/r^{2n+2}) + \int_r^\infty \frac{\rho(t)\, \Phi^{-1}(1/t^{2n+2})}{t}\, dt \le C\Psi^{-1}(1/r^{2n+2}), \tag{6.3}$$

then $\mathcal{I}_\rho$ is bounded from $L^\Phi(\mathbb{H}^n)$ to $\mathrm{w}L^\Psi(\mathbb{H}^n)$. Moreover, if $\Phi \in \overline{\nabla}_2$, then $\mathcal{I}_\rho$ is bounded from $L^\Phi(\mathbb{H}^n)$ to $L^\Psi(\mathbb{H}^n)$.

(ii) *Conversely, if $\mathcal{I}_\rho$ is bounded from $L^\Phi(\mathbb{H}^n)$ to $\mathrm{w}L^\Psi(\mathbb{H}^n)$, then there exists a positive constant C' such that, for all $r \in (0, \infty)$,*

$$\int_0^r \frac{\rho(t)}{t}\, dt\, \Phi^{-1}(1/r^{2n+2}) \le C'\Psi^{-1}(1/r^{2n+2}).$$

Moreover, if one of the following conditions holds, then (6.3) *holds for some $C \in (0, \infty)$ and for all $r \in (0, \infty)$:*

(a) $\Phi^{-1}(1/|\cdot|_{\mathbb{H}^n}^{2n+2}) \in L^\Phi(X)$.

(b) *There exists a positive constant C'' such that, for all $r \in (0,\infty)$,*

$$\int_r^\infty \frac{\rho(t)\Phi^{-1}(1/t^{2n+2})}{t}\,dt \le C''\rho(r)\Phi^{-1}(1/r^{2n+2}).$$

Remark 6.8. We note that, to prove the theorems above, we may assume that $\Phi,\Psi \in \boldsymbol{\Phi}_Y$ instead of $\Phi,\Psi \in \overline{\boldsymbol{\Phi}}_Y$. Actually, if Φ and Ψ satisfy (6.1) and $\Phi \approx \Phi_1$, $\Psi \approx \Psi_1$, then Φ_1 and Ψ_1 also satisfy (6.1) by the relation (5.3). Moreover, $L^\Phi(\mathcal{S}) = L^{\Phi_1}(\mathcal{S})$ and $L^\Psi(\mathcal{S}) = L^{\Psi_1}(\mathcal{S})$ with equivalent quasi-norms.

Let ρ be as in (1.5), and let $p,q \in (0,\infty)$, $-1/p+\alpha = -1/q$ and

$$\Phi(t) = \begin{cases} 1/\exp(1/t^p), & 0 < t \le 1, \\ \exp(t^p), & t > 1, \end{cases} \qquad \Psi(t) = \begin{cases} 1/\exp(1/t^q), & 0 < t \le 1, \\ \exp(t^q), & t > 1. \end{cases}$$

Then these functions satisfy the inequality (6.1). In this case $\Phi,\Psi \in \overline{\boldsymbol{\Phi}}_Y \setminus \boldsymbol{\Phi}_Y$ and $\Phi,\Psi \in \overline{\nabla}_2 \setminus \nabla_2$. Moreover, $\Phi^{-1}(1/|\cdot|^d) \in L^\Phi(\mathbb{R}^d)$, see Example 6.9.

Example 6.9. Let $p,q \in (0,\infty)$, and let

$$\Phi(t) = \begin{cases} 1/\exp(1/t^p), & 0 < t \le 1, \\ \exp(t^q), & t > 1. \end{cases}$$

Then

$$\Phi^{-1}(t) = \begin{cases} (\log(1/t))^{-1/p}, & 0 < t < 1/e, \\ 1, & 1/e \le t < e, \\ (\log t)^{1/q}, & t \ge e. \end{cases}$$

Let $g(x) = \Phi^{-1}(1/|x|^d)$. In the following we will prove that $g \in L^\Phi(\mathbb{R}^d)$. To do this it is enough to show that $\Phi(g(x)/2)$ is integrable both on $B(0,\epsilon)$ and on $B(0,1/\epsilon)^\complement$ for a small ϵ, since $g/2$ is bounded on $B(0,1/\epsilon) \setminus B(0,\epsilon)$.

Firstly, if $|x| \ge 1/\epsilon$, then $g(x)/2$ is small and

$$\Phi(g(x)/2) = 1/\exp\left(1/\left(\frac{(\log(|x|^d))^{-1/p}}{2}\right)^p\right) = 1/\exp\left(2^p\log(|x|^d)\right) = |x|^{-2^p d},$$

which shows that $\Phi(g(x)/2)$ is integrable on $B(0,1/\epsilon)^\complement$. Secondly, if $|x| < \epsilon$, then $g(x)/2$ is large and

$$\Phi(g(x)/2) = \exp\left(\left(\frac{(\log(|x|^{-d}))^{1/q}}{2}\right)^q\right) = \exp\left(2^{-q}\log(|x|^{-d})\right) = |x|^{-2^{-q}d},$$

which shows that $\Phi(g(x)/2)$ is integrable on $B(0,\epsilon)$. Hence, $g \in L^\Phi(\mathbb{R}^d)$. This example valid for the Q-homogeneous space of homogeneous type (X,d,μ), since $1/d(x_0,\cdot)^{Q-\epsilon}$ is integrable on $B(x_0,1)$ and $1/d(x_0,\cdot)^{Q+\epsilon}$ is integrable on $B(x_0,1)^\complement$ for all $x_0 \in X$.

7. Proofs

To prove Theorem 6.1, we give the following two lemmas.

Lemma 7.1. *Let Φ be a Young function. Suppose that $\{T_t\}_{t\ge 0}$ satisfies* (S1) *and* (S2). *Then, for all $f \ge 0$ and $x \in \mathcal{S}$,*

$$\Phi(T_t f(x)) \le T_t(\Phi(f))(x).$$

Proof. We use the standard way. Let $\mathcal{L} = \{(a,b) \in \mathbb{R}^2 : au + b \le \Phi(|u|) \text{ for all } u \in \mathbb{R}\}$. Then, we have

$$\Phi(T_t f(x)) = \sup_{(a,b)\in\mathcal{L}} (aT_t f(x) + b) = \sup_{(a,b)\in\mathcal{L}} T_t(af + b)(x) \le T_t(\Phi(f))(x). \qquad \square$$

Lemma 7.2. *Let Φ be a Young function. Suppose that $\{T_t\}_{t\ge 0}$ satisfies* (S1), (S2) *and* (S7) *with $p = 1$. Then*

$$\|T_t f\|_{L^\infty} \le \Phi^{-1}\left(\frac{c}{t^{d/2}}\right) \|f\|_{L^\Phi},$$

where c is the same constant in (S7).

Proof. Let $f \in L^\Phi(\mathcal{S})$. We may assume that $\|f\|_{L^\Phi} = 1$. Then $\|\Phi(|f|)\|_{L^1} \le 1$. By Lemma 7.1, we have

$$\Phi(|T_t f(x)|) \le \Phi(T_t(|f|)(x)) \le T_t(\Phi(|f|))(x) \le \frac{c}{t^{d/2}}$$

for any $x \in \mathcal{S}$. By (5.2) we have the conclusion. $\square$

Proof of Theorem 6.1. We may assume that $\Phi, \Psi \in \boldsymbol{\Phi}_Y$ as stated in Remark 6.8.

We use the method of Hedberg [16] and [8, Proposition 2]. Fix $x \in \mathcal{S}$. We may assume that

$$0 < \frac{T^* f(x)}{C_0 \|f\|_{L^\Phi}} < \infty, \quad 0 \le \Phi\left(\frac{T^* f(x)}{C_0 \|f\|_{L^\Phi}}\right) < \infty;$$

otherwise there is nothing to prove.

Case 1. Let

$$\Phi\left(\frac{T^* f(x)}{C_0 \|f\|_{L^\Phi}}\right) = 0.$$

Then

$$0 < \frac{T^* f(x)}{C_0 \|f\|_{L^\Phi}} \le \sup\{u \ge 0 : \Phi(u) = 0\} = \Phi^{-1}(0),$$

which implies

$$0 < \Phi^{-1}(0) \int_0^\infty \frac{\rho(t)}{t}\, dt \le C\Psi^{-1}(0),$$

where C is the constant in (6.1). Hence

$$T^*f(x)\int_0^\infty \frac{\rho(t)}{t}\,dt \le C_0\|f\|_{L^\Phi}\Phi^{-1}(0)\int_0^\infty \frac{\rho(t)}{t}\,dt \le CC_0\|f\|_{L^\Phi}\Psi^{-1}(0),$$

and then

$$|\mathcal{I}_\rho f(x)| \le \int_0^\infty \frac{\rho(t^{1/2})}{t}T^*f(x)\,dt = 2T^*f(x)\int_0^\infty \frac{\rho(t)}{t}\,dt \le 2CC_0\|f\|_{L^\Phi}\Psi^{-1}(0),$$

which shows

$$\Psi\left(\frac{\mathcal{I}_\rho f(x)}{2CC_0\|f\|_{L^\Phi}}\right) = 0.$$

So, in this case the result is valid.

Case 2. Let

$$\Phi\left(\frac{T^*f(x)}{C_0\|f\|_{L^\Phi}}\right) > 0.$$

Choose $r \in (0,\infty)$ so that

$$\frac{1}{r^d} = \Phi\left(\frac{T^*f(x)}{C_0\|f\|_{L^\Phi}}\right).$$

By Lemma 7.2 and the concavity of Φ^{-1}, we have

$$\|T_tf\|_{L^\infty} \le \Phi^{-1}\left(\frac{c}{t^{d/2}}\right)\|f\|_{L^\Phi} \le \max(1,c)\Phi^{-1}\left(\frac{1}{t^{d/2}}\right)\|f\|_{L^\Phi}.$$

Then, using this inequality and (6.1), we have

$$\begin{aligned}
|\mathcal{I}_\rho f(x)| &\le \int_0^\infty \frac{\rho(t^{1/2})}{t}|T_tf(x)|\,dt \\
&= \int_0^{r^2} \frac{\rho(t^{1/2})}{t}|T_tf(x)|\,dt + \int_{r^2}^\infty \frac{\rho(t^{1/2})}{t}|T_tf(x)|\,dt \\
&\le T^*f(x)\int_0^{r^2} \frac{\rho(t^{1/2})}{t}\,dt + \max(1,c)\|f\|_{L^\Phi}\int_{r^2}^\infty \frac{\rho(t^{1/2})}{t}\Phi^{-1}(1/t^{d/2})\,dt \\
&= 2T^*f(x)\int_0^r \frac{\rho(t)}{t}\,dt + 2\max(1,c)\|f\|_{L^\Phi}\int_r^\infty \frac{\rho(t)}{t}\Phi^{-1}(1/t^d)\,dt \\
&\le 2C\left(T^*f(x)\frac{\Psi^{-1}(1/r^d)}{\Phi^{-1}(1/r^d)} + \max(1,c)\|f\|_{L^\Phi}\Psi^{-1}(1/r^d)\right),
\end{aligned}$$

where C is the constant in (6.1). Recall that $\Phi^{-1}(\Phi(r)) = r$ if $0 < \Phi(r) < \infty$. Thus $\Phi^{-1}(1/r^d) = T^*f(x)/(C_0\|f\|_{L^\Phi})$. Let $C_1 = 2C(C_0 + \max(1,c))$. Then

$$|\mathcal{I}_\rho f(x)| \le C_1\|f\|_{L^\Phi}\Psi^{-1}(1/r^d) = C_1\|f\|_{L^\Phi}\Psi^{-1}\left(\Phi\left(\frac{T^*f(x)}{C_0\|f\|_{L^\Phi}}\right)\right).$$

Hence, by (5.2) we have

$$\Psi\left(\frac{|\mathcal{I}_\rho f(x)|}{C_1\|f\|_{L^\Phi}}\right) \le \Phi\left(\frac{T^*f(x)}{C_0\|f\|_{L^\Phi}}\right).$$

This is the conclusion. $\square$

To prove Theorem 6.3, we need the following two lemmas. We assume that (X, d, μ) is Q-homogeneous. Let K_1 and K_5 be as in (4.1) and (4.4), respectively.

Lemma 7.3. *Assume that $r \mapsto \rho(r)/r^k$ is almost decreasing for some $k > 0$.*

(i) *Then there exists a positive constant C such that, for all $x_0 \in X$ and $r \in (0, \infty)$,*

$$\int_0^r \frac{\rho(t)}{t}\, dt\, \chi_{B(x_0,r/2)}(x) \le C I_\rho(\chi_{B(x_0,K_1 r)})(x),$$

where C is dependent only on k, K_5 and Q.

(ii) *If $\phi\colon (0,\infty) \to (0,\infty)$ satisfies the doubling condition, then there exists a positive constant C such that, for all $x_0 \in X$ and $r \in (0,\infty)$,*

$$\int_r^\infty \frac{\rho(t)\phi(t)}{t}\, dt\, \chi_{B(x_0,c_0 r)}(x) \le C I_\rho(g^r)(x),$$

where $c_0 = (2K_1)^{-1}(2K_5^2)^{-1/Q}$, $g^r(x) = \phi(d(x_0,x))\chi_{B(x_0,c_0 r)^\complement}(x)$, and C is dependent only on ϕ, k, K_5 and Q.

Proof. First note that $\rho(r) \gtrsim \rho(s)$ for $r < s \le 2r$ by Remark 6.5, and that $r^Q = 2r^Q - r^Q \le \mu(B(x,(2K_5)^{1/Q} r) \setminus B(x,(K_5)^{-1/Q} r))$ by (4.4). Let $L(x,r) = B(x,r) \setminus B(x,c_1 r)$ and $c_1 = (2K_5^2)^{-1/Q}$. Then

$$\int_r^{2r} \frac{\rho(t)}{t}\, dt \lesssim \rho(r) \lesssim \int_{L(x,r)} \frac{\rho(r)}{r^Q}\, d\mu(y) \lesssim \int_{L(x,r)} \frac{\rho(d(x,y))}{d(x,y)^Q}\, d\mu(y).$$

Similarly we have

$$\int_r^{2r} \frac{\rho(t)\phi(t)}{t}\, dt \lesssim \int_{L(x,r)} \frac{\rho(d(x,y))\phi(d(x,y))}{d(x,y)^Q}\, d\mu(y).$$

(i) From the above observation, for all $x \in X$,

$$\begin{aligned}\int_0^r \frac{\rho(t)}{t}\, dt &= \sum_{j=1}^\infty \int_{2^{-j}r}^{2^{-j+1}r} \frac{\rho(t)}{t}\, dt \lesssim \sum_{j=1}^\infty \int_{L(x,2^{-j}r)} \frac{\rho(d(x,y))}{d(x,y)^Q}\, d\mu(y) \\ &\lesssim \int_{B(x,2^{-1}r)} \frac{\rho(d(x,y))}{d(x,y)^Q}\, d\mu(y).\end{aligned}$$

Let $x \in B(x_0, r/2)$. Then $B(x, r/2) \subset B(x_0, K_1 r)$ and

$$\int_0^r \frac{\rho(t)}{t}\, dt \lesssim \int_{B(x_0,K_1 r)} \frac{\rho(d(x,y))}{d(x,y)^Q}\, d\mu(y) = I_\rho(\chi_{B(x_0,K_1 r)})(x).$$

(ii) Similarly, for all $x \in X$,

$$\int_r^\infty \frac{\rho(t)\phi(t)}{t}\, dt \lesssim \int_{B(x,c_1 r)^\complement} \frac{\rho(d(x,y))\phi(d(x,y))}{d(x,y)^Q}\, d\mu(y).$$

Let $c_0 = (2K_1)^{-1}c_1 = (2K_1)^{-1}(2K_5^2)^{-1/Q}$ and $x \in B(x_0, c_0 r)$. Then $B(x_0, c_0 r) \subset B(x, c_1 r)$ and $d(x_0, y) \sim d(x, y)$ for $y \notin B(x, c_1 r)$. Hence, we have

$$\begin{aligned}\int_0^r \frac{\rho(t)\phi(t)}{t}\,dt &\lesssim \int_{B(x,c_1r)^{\complement}} \frac{\rho(d(x,y))\phi(d(x,y))}{d(x,y)^Q}\,d\mu(y)\\ &\sim \int_{B(x,c_1r)^{\complement}} \frac{\rho(d(x,y))\phi(d(x_0,y))}{d(x,y)^Q}\,d\mu(y) \le I_\rho(g^r)(x).\end{aligned}$$

The proof is complete. □

Remark 7.4. For the cases $\mathbb{R}^d$ and $\mathbb{H}^n$, we can prove Lemma 7.3 without the almost decreasingness of $r \mapsto \rho(r)/r^k$. For the case $\mathbb{R}^d$, see [9, Lemmas 2.1 and 2.2]. For the case $\mathbb{H}^n$, we use Lemma 7.5 below. More precisely, substitute $\rho\chi_{(0,r)}$ or $\rho\phi\chi_{[r,\infty)}$ for ψ. Then we have the desired conclusion. Therefore, for these cases, we do not need the almost decreasingness of $r \mapsto \rho(r)/r^k$ to prove Theorem 6.3(ii). However, we need (4.5) only in the part (b) to show (7.3) below.

Lemma 7.5. *Let* $\psi\colon [0,\infty) \to [0,\infty)$. *Assume that the function* $t \mapsto \psi(t)/t$ *is integrable on the interval* $(0,\infty)$. *Then*

$$\int_{\mathbb{H}^n} \frac{\psi(|x|_{\mathbb{H}^n})}{|x|_{\mathbb{H}^n}^{2n+2}}\,dx = |\mathbb{S}^{2n-1}| \left(\int_0^\infty \frac{du}{(1+16u^2)^{(n+1)/2}}\right) \int_0^\infty \frac{\psi(t)}{t}\,dt,$$

where $|\mathbb{S}^{2n-1}|$ *denotes the volume of the* $(2n-1)$*-dimensional unit sphere.*

Proof. For $x = (x_1, x_2, \ldots, x_{2n+1}) \in \mathbb{H}^n$, let $z = (x_1, x_2, \ldots, x_{2n})$ and let $|z|$ be the usual norm of z in $\mathbb{R}^{2n}$. Then $|x|_{\mathbb{H}^n} = (|z|^4 + 16x_{2n+1}^2)^{1/4}$. By change of variables and Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{H}^n} \frac{\psi(|x|_{\mathbb{H}^n})}{|x|_{\mathbb{H}^n}^{2n+2}}\,dx &= \int_{\mathbb{R}^{2n}\times\mathbb{R}} \frac{\psi((|z|^4+16x_{2n+1}^2)^{1/4})}{(|z|^4+16x_{2n+1}^2)^{(n+1)/2}}\,dzdx_{2n+1}\\ &= \int_{\mathbb{R}^{2n}\times\mathbb{R}} \frac{\psi(|z|(1+16u^2)^{1/4})}{|z|^{2n}(1+16u^2)^{(n+1)/2}}\,dzdu\\ &= |\mathbb{S}^{2n-1}| \int_{(0,\infty)\times\mathbb{R}} \frac{\psi(r(1+16u^2)^{1/4})}{r(1+16u^2)^{(n+1)/2}}\,drdu\\ &= |\mathbb{S}^{2n-1}| \int_{(0,\infty)\times\mathbb{R}} \frac{\psi(t)}{t(1+16u^2)^{(n+1)/2}}\,dtdu\\ &= |\mathbb{S}^{2n-1}| \left(\int_{\mathbb{R}} \frac{du}{(1+16u^2)^{(n+1)/2}}\right) \int_0^\infty \frac{\psi(t)}{t}\,dt.\end{aligned}$$ □

Proof of Theorem 6.3(ii). We may assume that $\Phi, \Psi \in \boldsymbol{\Phi}_Y$ as stated in Remark 6.8.

Assume that I_ρ is bounded from $L^\Phi(X)$ to $\mathrm{w}L^\Psi(X)$. Then, using Lemma 7.3(i), we have

$$\int_0^r \frac{\rho(t)}{t}\,dt \|\chi_{B(0,r)}\|_{\mathrm{w}L^\Psi} \lesssim \|I_\rho(\chi_{B(0,K_1r)})\|_{\mathrm{w}L^\Psi} \lesssim \|\chi_{B(0,K_1r)}\|_{L^\Phi}.$$

An elementary calculation shows that $\|\chi_B\|_{L^\Phi} = \|\chi_B\|_{\mathrm{w}L^\Phi} = 1/\Phi^{-1}(1/\mu(B))$. By the doubling condition of Φ^{-1} and Ψ^{-1} we have

$$\int_0^r \frac{\rho(t)}{t}\,dt\Phi^{-1}(1/r^Q) \lesssim \Psi^{-1}(1/r^Q). \tag{7.1}$$

Next, assume that (a): $\Phi^{-1}(1/d(x_0,\cdot)^Q) \in L^\Phi(X)$. Let c_0 and g^r be as in Lemma 7.3(ii) with $\phi(r) = \Phi^{-1}(1/r^Q)$. Then we have

$$\begin{aligned}\int_r^\infty \frac{\rho(t)\Phi^{-1}(1/t^Q)}{t}\,dt\,\|\chi_{B(x_0,c_0r)}\|_{\mathrm{w}L^\Psi} &\lesssim \|I_\rho(g^r)\|_{\mathrm{w}L^\Psi} \lesssim \|g^r\|_{L^\Phi}\\ &\le \|\Phi^{-1}(1/d(x_0,\cdot)^Q)\|_{L^\Phi} \lesssim 1,\end{aligned}$$

which shows

$$\int_r^\infty \frac{\rho(t)\Phi^{-1}(1/t^d)}{t}\,dt \lesssim \Psi^{-1}(1/r^d). \tag{7.2}$$

If we assume that (b):

$$\int_r^\infty \frac{\rho(t)\Phi^{-1}(1/t^d)}{t}\,dt \lesssim \rho(r)\Phi^{-1}(1/r^d),$$

then, using the almost decreasingness of $r \mapsto \rho(r)/r^k$ (which is equivalent (R3) in Remark 6.5), we have

$$\rho(r) \lesssim \int_{r/2}^r \frac{\rho(t)}{t}\,dt \le \int_0^r \frac{\rho(t)}{t}\,dt,$$

and

$$\int_r^\infty \frac{\rho(t)\Phi^{-1}(1/t^d)}{t}\,dt \lesssim \rho(r)\Phi^{-1}(1/r^d) \lesssim \int_0^r \frac{\rho(t)}{t}\,dt\,\Phi^{-1}(1/r^d). \tag{7.3}$$

Since we already proved (7.1), we also have (7.2). The proof is complete. □

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