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New Combinatorial Interpretations for the Partitions into Odd Parts Greater than One

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Abstract. In this paper, we consider $Q_1(n)$ to be the number of partitions of n into odd parts greater than one and provide new combinatorial interpretations for $Q_1(n)$. New linear relations involving Euler's partition function $p(n)$ and the overpartition function $\overline{p}(n)$ are obtained in this context.

1. Introduction

A partition of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers whose sum is n. The order of the summands is unimportant when writing the partitions of n, but for consistency, a partition of n will be written with the summands in nonincreasing order [1], i.e.,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n \quad \text{and} \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k.$$

As usual, we denote by $p(n)$ the number of integer partitions of n. Also we denote by $Q(n)$ the function which enumerates the partitions of n into odd parts. We have the generating functions

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} \quad \text{and} \quad \sum_{n=0}^{\infty} Q(n)q^n = \frac{1}{(q;q^2)_\infty}.$$

Here and throughout the paper, we use the following customary q-series notation:

$$(a;q)_n = \begin{cases} 1 & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{for } n > 0 \end{cases} \quad \text{and} \quad (a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

Moreover, we use the short notation

$$(a_1, a_2, \ldots, a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty.$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geq 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume $|q| < 1$.

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Definition 1.1. Let n be a nonnegative integer. We define $Q_1(n)$ to be the number of the partitions of n into odd parts greater than 1.

As we can see in [12, A087897], the sequence $Q_1(n)$ can be interpreted in many other ways. In this paper, we shall obtain new combinatorial interpretations for this sequence.

Definition 1.2. For a nonnegative integer n, we define

(i) $a(n)$ to be the number of the partitions λ of n such that $\lceil(\lambda_1 - 1)/2\rceil$ is a part. If $\lambda_1 = 1$, then we consider 0 to be a part of λ.

(ii) $a_e(n)$ to be the number of the partitions λ of n such that $\lambda_1/2$ is a part.

(iii) $a_o(n)$ to be the number of the partitions λ of n such that $(\lambda_1 - 1)/2$ is a part. If $\lambda_1 = 1$, then we consider 0 to be a part of λ.

It is clear that $a(n) = a_e(n) + a_o(n)$. For example, the partitions λ of 7 such that $\lambda_1/2$ is a part are

$$(4,2,1),(2,2,2,1),(2,2,1,1,1),(2,1,1,1,1,1),$$

while the partitions λ of 7 such that $(\lambda_1 - 1)/2$ is a part are

$$(5,2),(3,3,1),(3,2,1,1),(3,1,1,1,1),(1,1,1,1,1,1,1).$$

We see that $a_e(7) = 4$, $a_o(7) = 5$ and $a(7) = 9$.

The following result provides new combinatorial interpretations for the number of the partitions of n into odd parts greater than 1.

Theorem 1.3. *Let n be a nonnegative integer. Then*

(i) $a_e(n) = Q_1(2n)$;

(ii) $a_o(n) = Q_1(2n+1)$.

A famous theorem of Euler asserts that there are as many partitions of n into odd parts as there are partitions into distinct parts, i.e., $1/(q;q^2)_\infty = (-q;q)_\infty$. For a positive integer $\ell > 1$, a partition is called ℓ-regular if none of its parts is divisible by ℓ. Thus, if $b_\ell(n)$ denotes the number of ℓ-regular partitions of n, then the generating function for $b_\ell(n)$ is given by $(q^\ell;q^\ell)_\infty/(q;q)_\infty$.

Definition 1.4. Let $k \in \{1,3\}$. For a nonnegative integer n, we define $c_k(n)$ to be the number of 8-regular partitions of n with two colors, red and blue, such that the parts colored red are distinct and $\equiv \pm k \pmod 8$.

Definition 1.5. For a nonnegative integer n, we define

(i) $c_{1,1}(n)$ to be the number of 8-regular partitions of n with two colors, red and blue, such that the parts colored red are distinct and $\equiv \pm 1 \pmod 8$ and there is at least one blue 1.

(ii) $c_{1,2}(n)$ to be the number of 8-regular partitions of n with two colors, red and blue, such that the parts colored red are distinct and $\equiv \pm 1 \pmod 8$ and there is no blue 1.

We then have the following identities involving $a_e(n)$, $a_o(n)$ and $a(n)$.

Theorem 1.6. *For any nonnegative integer n, we have*

(i) $a_e(n) = c_3(n) - c_{1,1}(n)$;

(ii) $a_o(n) = c_1(n) - c_3(n)$;

(iii) $a(n) = c_{1,2}(n)$.

As a consequence of Theorem 1.6, we obtain the following generating functions for $a_e(n)$, $a_o(n)$ and $a(n)$.

Corollary 1.7. *For $|q| < 1$, we have*

(i) $$\sum_{n=0}^{\infty} a_e(n)q^n = \frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q;q)_\infty} - q\,\frac{(-q,-q^7,q^8;q^8)_\infty}{(q;q)_\infty};$$

(ii) $$\sum_{n=0}^{\infty} a_o(n)q^n = \frac{(-q,-q^7,q^8;q^8)_\infty}{(q;q)_\infty} - \frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q;q)_\infty};$$

(iii) $$\sum_{n=0}^{\infty} a(n)q^n = \frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q)_\infty}.$$

Considering that

$$Q_1(n) = Q(n) - Q(n-1),$$

by Theorem 1.3 we deduce the following combinatorial interpretation

$$a(n) = Q(2n+1) - Q(2n-1).$$

Recall that the numbers $k(3k+1)/2$ for $k \in \mathbb{Z}$ are called generalized pentagonal numbers and that m is a generalized pentagonal number if and only if $24m+1$ is a square. It is well known that $Q(n) \equiv 1 \pmod 2$ if and only if n is a generalized pentagonal number. Then, from Corollary 1.7 we easily deduce the following q-series congruences.

Corollary 1.8. (i) $$\frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q)_\infty} \equiv \sum_{48n+25 \text{ square}} (q^n + q^{n+1}) \pmod 2;$$

(ii) $\dfrac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q)_\infty} \equiv \sum_{48n+1 \text{ square}} (q^n+q^{n+1}) \pmod 2.$

Furthermore, by Theorem 1.6 we deduce the following identities involving the functions $a_e(n)$, $a_o(n)$, and $a(n)$ and the partition function $p(n)$.

Theorem 1.9. *Let n be a nonnegative integer. Then*

(i) $a_e(n) = \sum_{k=-\infty}^{\infty} (-1)^k p(n-k^2+\lfloor k/2 \rfloor)$;

(ii) $a_o(n) = \sum_{k=-\infty}^{\infty} (-1)^{k-1} p(n-k^2+\lceil k/2 \rceil)$;

(iii) $a(n) = \sum_{k=-\infty}^{\infty} p\big(n-k(4k+3)\big) - p\big(n-1-k(4k+3)\big)$.

The numbers $k(4k+3)$ for $k \in \mathbb{Z}$ are called generalized decagonal numbers. From Theorem 1.9(iii) we easily deduce the following decomposition for the number of partitions of $2n+1$ into distinct parts:

$$Q(2n+1) = \sum_{k=-\infty}^{\infty} p\big(n-k(4k+3)\big). \tag{1.1}$$

This is not a new result. Another proof of this identity can be seen in [11]. The combinatorial proof of Theorem 1.9(iii) provides a combinatorial proof of (1.1).

The properties of the partition function $p(n)$, such as its asymptotic behavior and its parity, have been an object of study for a long time. Recently, Ballantine and Merca [4] made a conjecture on when

$$\sum_{ak+1 \text{ square}} p(n-k)$$

is odd, which was proved by Hong and Zhang [9]. Taking into account that m is a generalized decagonal number if and only if $16m+9$ is a square, we easily deduce the following result.

Corollary 1.10. *Let n be a nonnegative integer. Then*

$$\sum_{16k+9 \text{ square}} p(n-k) \equiv 1 \pmod 2 \quad \Longleftrightarrow \quad 48n+25 \textit{ is a square.}$$

Linear inequalities involving Euler's partition function $p(n)$ have been the subject of recent studies [2, 3, 10]. Related to Theorem 1.9, we remark that there is a substantial amount of numerical evidence to conjecture the following inequalities.

Conjecture 1.11. *Let n be a nonnegative integer.*

(i) *For $k>1$,*

$$(-1)^k\left(a_e(n)-\sum_{j=1-k}^{k}(-1)^j p(n-j^2+\lfloor j/2\rfloor)\right)\geq 0$$

with strict inequality if $n\geq k^2+\lceil k/2\rceil$.

(ii) *For $k>0$,*

$$(-1)^{k-1}\left(a_o(n)+\sum_{j=1-k}^{k}(-1)^j p(n-j^2+\lceil j/2\rceil)\right)\geq 0$$

with strict inequality if $n\geq k^2+\lfloor k/2\rfloor$.

From an asymptotic point of view, the case k even of Conjecture 1.11(i) and the case k odd of Conjecture 1.11(ii) seem to be trivial. On the other hand, the other cases lead to interesting inequalities. For example, some special cases of Conjecture 1.11(i) for k odd are

$$\begin{gathered}
p(n)-p(n-1)\geq a_e(n),\\
p(n)-p(n-1)-p(n-2)+p(n-3)+p(n-5)-p(n-8)\geq a_e(n),\\
p(n)-p(n-1)-p(n-2)+p(n-3)+p(n-5)-p(n-8)\\
\quad -p(n-11)+p(n-14)+p(n-18)-p(n-23)\geq a_e(n).
\end{gathered}$$

Similarly, some special cases of Conjecture 1.11(ii) for k even are

$$\begin{gathered}
p(n-1)-p(n-3)\geq a_o(n),\\
p(n-1)-p(n-3)-p(n-5)+p(n-7)+p(n-10)-p(n-14)\geq a_o(n),\\
p(n-1)-p(n-3)-p(n-5)+p(n-7)+p(n-10)-p(n-14)\\
\quad -p(n-18)+p(n-22)+p(n-27)-p(n-33)\geq a_o(n).
\end{gathered}$$

In [2], while investigating the truncated Euler's pentagonal number theorem, Andrews and Merca introduced the partition function $M_k(n)$, which counts the number of partitions of n where k is the least positive integer that is not a part and there are more parts $>k$ than there are parts $<k$. For instance, we have $M_3(18)=3$ because the three partitions in question are

$$(5,5,5,2,1),(6,5,4,2,1),(7,4,4,2,1).$$

Using the notations

$$\delta_e(n)=\begin{cases}(-1)^k & \text{if } n=k^2-\lfloor k/2\rfloor,\ k\in\mathbb{Z},\\ 0 & \text{otherwise}\end{cases}$$

and

$$\delta_o(n) = \begin{cases} (-1)^k & \text{if } n = k^2 - \lceil k/2 \rceil,\ k \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain the following result.

Theorem 1.12. *Let n be a nonnegative integer. For $k > 0$, we have*

(i) $$(-1)^{k-1}\left(\sum_{j=1-k}^{k} (-1)^j a_e\big(n - j(3j-1)/2\big) - \delta_e(n)\right) = \sum_{j=-\infty}^{\infty} (-1)^j M_k(n - j^2 + \lfloor j/2 \rfloor);$$

(ii) $$(-1)^{k-1}\left(\sum_{j=1-k}^{k} (-1)^j a_o\big(n - j(3j-1)/2\big) + \delta_o(n)\right) = \sum_{j=-\infty}^{\infty} (-1)^{j+1} M_k(n - j^2 + \lceil j/2 \rceil).$$

The limiting case $k \to \infty$ of this theorem reads as follows.

Corollary 1.13. *For $n \geq 0$,*

(i) $$\sum_{j=-\infty}^{\infty} (-1)^j a_e\big(n - j(3j-1)/2\big) = \delta_e(n);$$

(ii) $$\sum_{j=-\infty}^{\infty} (-1)^j a_o\big(n - j(3j-1)/2\big) = \delta_o(n).$$

This corollary states that the partition function $p(n)$ and the sequences $a_e(n)$ and $a_o(n)$ share a common linear homogeneous recurrence relation for most values of n.

Related to Theorem 1.12, we remark that there is a substantial amount of numerical evidence to state the following conjecture.

Conjecture 1.14. *Let n be a nonnegative integer. For $k > 0$, we have*

(i) $$(-1)^{k-1}\left(\sum_{j=1-k}^{k} (-1)^j a_e\big(n - j(3j-1)/2\big) - \delta_e(n)\right) \geq 0;$$

(ii) $$(-1)^{k-1}\left(\sum_{j=1-k}^{k} (-1)^j a_o\big(n - j(3j-1)/2\big) + \delta_o(n)\right) \geq 0.$$

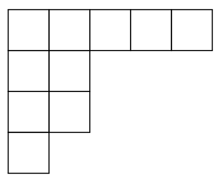
The remainder of the paper is organized as follows. In Section 2 we introduce some notation and preliminaries on partitions. In Sections 3–5, we prove Theorems 1.3, 1.6 and 1.9, respectively. For each theorem we give both analytic and combinatorial proofs. In Section 6, we prove Theorem 1.12 using generating functions. A combinatorial proof of this theorem would be welcome. In Section 7, we consider a classical theta identity of Gauss and two series that involve the half generalized pentagonal numbers in order to obtain new relations involving the partition functions $a(n)$, $a_e(n)$ and $a_o(n)$.

2. Preliminaries

In this section, we give some necessary background and notation concerning integer partitions, which were introduced in Section 1.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j > 0$ is a partition and $\sum_{i=1}^{j} \lambda_i = n$ we refer to n as the *size* of λ and to the integers λ_i as the *parts* of λ. The *length* of λ, denoted by $\ell(\lambda)$, is the number of parts of λ. We write $|\lambda|$ for the size of λ and $\lambda \vdash n$ to mean that λ is a partition of size n. We use the convention that $\lambda_k = 0$ for all $k > \ell(\lambda)$. When convenient we will also use the exponential notation for parts in a partition: the exponent of a part is the multiplicity of the part in the partition. This notation will be used mostly for rectangular partitions. We write (a^b) for the partition consisting of b parts equal to a.

The *Ferrers diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ is an array of left justified boxes such that the ith row from the top contains λ_i boxes. We abuse notation and use λ to mean a partition or its Ferrers diagram. For example, the Ferrers diagram of $\lambda = (5, 2, 2, 1)$ is



Given a partition λ, its *conjugate* λ' is the partition for which the rows in its Ferrers diagram are precisely the columns in the Ferrers diagram of λ. For example, the conjugate of $\lambda = (5, 2, 2, 1)$ is $\lambda' = (4, 3, 1, 1, 1)$.

Given two partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, by their sum we mean the partition $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_\ell + \mu_\ell)$, where $\ell = \max\{j, k\}$. The union of λ and μ is the partition $\lambda \cup \mu$ consisting of all the parts in λ and μ each with multiplicity equal to the sum of its multiplicities in λ and μ. For example, if $\lambda = (5, 2, 2, 1)$ and $\mu = (6, 2)$, then $\lambda + \mu = (11, 4, 2, 1)$ and $\lambda \cup \mu = (6, 5, 2, 2, 2, 1)$.

Given a positive integer b and an integer $0 \leq a < b$, we denote by $\ell_{a,b}(\lambda)$ the number of parts $\equiv a \pmod{b}$ in λ.

For more details on partitions, we refer the reader to [1].

3. Proof of Theorem 1.3

3.1. Analytic proof

In order to prove theorem, we consider Euler's identity (cf. [8, Eq. (18)])

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1$$

and the generating function for $Q_1(n)$ which is given by

$$\sum_{n=0}^{\infty} Q_1(n)q^n = \frac{1}{(q^3;q^2)_\infty}.$$

We can write

$$\begin{aligned}\sum_{n=0}^{\infty} Q_1(2n)q^n &= \frac{1}{2}\left(\frac{1}{(q^{3/2};q)_\infty} + \frac{1}{(-q^{3/2};q)_\infty}\right)\\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{q^{3n/2}}{(q;q)_n} + \sum_{n=0}^{\infty}\frac{(-1)^n q^{3n/2}}{(q;q)_n}\right) = \sum_{n=0}^{\infty}\frac{q^{2n+n}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} a_e(n)q^n\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} Q_1(2n+1)q^n &= \frac{1}{2q^{1/2}}\left(\frac{1}{(q^{3/2};q)_\infty} - \frac{1}{(-q^{3/2};q)_\infty}\right)\\ &= \frac{1}{2q^{1/2}}\left(\sum_{n=0}^{\infty}\frac{q^{3n/2}}{(q;q)_n} - \sum_{n=0}^{\infty}\frac{(-1)^n q^{3n/2}}{(q;q)_n}\right)\\ &= \sum_{n=0}^{\infty}\frac{q^{(2n+1)+n}}{(q;q)_{2n+1}} = \sum_{n=0}^{\infty} a_o(n)q^n.\end{aligned}$$

This concludes the analytic proof.

3.2. Combinatorial proof

Let $\mathcal{A}_e(n)$ (respectively $\mathcal{A}_o(n)$) be the set of partitions λ of n such that $\lambda_1/2$ (respectively $(\lambda_1-1)/2$) is a part of λ and let $\mathcal{Q}_1(n)$ be the set of partitions of n into odd parts greater than 1. Note that, for every partition $\lambda \in \mathcal{Q}_1(n)$ we have $\ell(\lambda) \equiv n \pmod 2$.

To prove the first identity combinatorially, we create a bijection $\varphi\colon \mathcal{A}_e(n) \to \mathcal{Q}_1(n)$ as follows. Start with $\lambda \in \mathcal{A}_e(n)$ with $\lambda_1 = 2m$ and such that m is a part of λ. Let μ be the partition obtained from λ by removing one part equal to $2m$ and one part equal to m. If μ' is the conjugate of μ, then μ' has at most $2m$ parts. Let η be the partition obtained by doubling each part of μ'. Finally, we define $\varphi(\lambda) = \eta + (3^{2m}) \in \mathcal{Q}_1(2n)$, i.e., the partition obtained by adding 3 to each part of η, where η is padded with parts equal to 0 to form a partition with $2m$ parts.

To see that φ is invertible, start with a partition $\zeta \in \mathcal{Q}_1(2n)$ with $2m$ parts. Since all parts are at least 3, we subtract 3 from each part to obtain a partition ξ with at most $2m$ parts, all even. Divide each part of ξ by 2 and take its conjugate to obtain a partition ρ with parts at most $2m$. Finally, insert a part equal to $2m$ and a part equal to m into ρ to obtain a partition $\lambda \in \mathcal{A}_e(n)$. Then, $\varphi(\lambda) = \zeta$.

The combinatorial proof of the second identity is nearly identical. To define the transformation from $\mathcal{A}_o(n)$ to $\mathcal{Q}_1(2n+1)$ start with $\lambda \in \mathcal{A}_e(n)$ with $\lambda_1 = 2m+1$ and such that m is a part of λ, remove parts $2m+1$ and m, conjugate the obtained partition and double each of its parts. Then add the partition (3^{2m+1}) to the obtained partition.

4. Proof of Theorem 1.6

4.1. Analytic proof

Elementary techniques in the theory of partitions show that the generating functions for the sequences $c_1(n)$, $c_3(n)$, $c_{1,1}(n)$ and $c_{1,2}(n)$ are

$$\begin{aligned}\sum_{n=0}^{\infty} c_1(n)q^n &= \frac{(-q,-q^7,q^8;q^8)_\infty}{(q;q)_\infty},\\ \sum_{n=0}^{\infty} c_3(n)q^n &= \frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q;q)_\infty},\\ \sum_{n=0}^{\infty} c_{1,1}(n)q^n &= q\,\frac{(-q,-q^7,q^8;q^8)_\infty}{(q;q)_\infty},\\ \sum_{n=0}^{\infty} c_{1,2}(n)q^n &= \frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q)_\infty}.\end{aligned}$$

The Jacobi triple product identity (cf. [8, Eq. (1.6.1)]) states that

$$(z,q/z,q;q)_\infty = \sum_{n=-\infty}^{\infty} (-z)^n q^{n(n-1)/2}. \tag{4.1}$$

Letting $q\to q^2$ and setting $z=\pm\sqrt{q}$ in (4.1), we obtain

$$(\pm\sqrt{q},\pm q\sqrt{q},q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} (\mp\sqrt{q})^n q^{n(n-1)}. \tag{4.2}$$

Letting $q\to q^8$ and successively setting $z=-q^3$, $z=-q$ in (4.1), we obtain

$$(-q^3,-q^5,q^8;q^8)_\infty = \sum_{n=-\infty}^{\infty} q^{4n^2-n} \tag{4.3}$$

and

$$(-q,-q^7,q^8;q^8)_\infty = \sum_{n=-\infty}^{\infty} q^{4n^2-3n}. \tag{4.4}$$

Thus, we have

$$\sum_{n=0}^{\infty} a_e(n)q^n = \sum_{n=0}^{\infty}(Q(2n)-Q(2n-1))q^n \quad \text{(by Theorem 1.3(i))}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{1}{(\sqrt{q};q)_\infty}+\frac{1}{(-\sqrt{q};q)_\infty}\right)-\frac{\sqrt{q}}{2}\left(\frac{1}{(\sqrt{q};q)_\infty}-\frac{1}{(-\sqrt{q};q)_\infty}\right)\\
&= \frac{(\sqrt{q};q)_\infty+(-\sqrt{q};q)_\infty}{2(q;q^2)_\infty}+\sqrt{q}\,\frac{(\sqrt{q};q)_\infty-(-\sqrt{q};q)_\infty}{2(q;q^2)_\infty}\\
&= \frac{(\sqrt{q},q\sqrt{q},q^2;q^2)_\infty+(-\sqrt{q},-q\sqrt{q},q^2;q^2)_\infty}{2(q;q)_\infty}\\
&\quad+\sqrt{q}\,\frac{(\sqrt{q},q\sqrt{q},q^2;q^2)_\infty-(-\sqrt{q},-q\sqrt{q},q^2;q^2)_\infty}{2(q;q)_\infty}\\
&= \frac{1}{2(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-\sqrt{q})^n q^{n(n-1)}+\sum_{n=-\infty}^{\infty}(\sqrt{q})^n q^{n(n-1)}\right)\\
&\quad+\frac{\sqrt{q}}{2(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-\sqrt{q})^n q^{n(n-1)}-\sum_{n=-\infty}^{\infty}(\sqrt{q})^n q^{n(n-1)}\right)\quad\text{(by (4.2))}\\
&= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{4n^2-n}-\frac{q}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{4n^2+3n}\\
&= \frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q;q)_\infty}-q\,\frac{(-q,-q^7,q^8;q^8)_\infty}{(q;q)_\infty}\quad\text{(by (4.3) and (4.4))}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n=0}^{\infty}a_o(n)q^n &= \sum_{n=0}^{\infty}(Q(2n+1)-Q(2n))q^n\quad\text{(by Theorem 1.3(ii))}\\
&= \frac{1}{2\sqrt{q}}\left(\frac{1}{(\sqrt{q};q)_\infty}-\frac{1}{(-\sqrt{q};q)_\infty}\right)-\frac{1}{2}\left(\frac{1}{(\sqrt{q};q)_\infty}+\frac{1}{(-\sqrt{q};q)_\infty}\right)\\
&= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{4n^2+3n}-\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{4n^2-n}\\
&= \frac{(-q,-q^7,q^8;q^8)_\infty}{(q;q)_\infty}-\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q;q)_\infty}.
\end{aligned}
$$

This concludes the proof.

4.2. Combinatorial proof

First we introduce some useful notation. We denote by $\mathcal{Q}(n)$ the set of partitions on n into odd parts.

For integers a, b with $0 \le a < b$, we denote by $\mathcal{U}_{a,b}(n)$ the set of partitions of n in which all parts are distinct and $\equiv \pm a \pmod b$, and by $\mathcal{B}_b(n)$ the set of b-regular partitions of n. By the same notation with omitted n we mean the set of partitions of any size satisfying the given properties. We denote by $\mathcal{C}_1(n)$, respectively $\mathcal{C}_3(n)$, the set of 8-regular partitions of n with two colors, red and blue, such that the parts colored red form a partition in $\mathcal{U}_{1,8}$, respectively in $\mathcal{U}_{3,8}$. We denote by $\mathcal{C}_{1,1}(n)$, respectively $\mathcal{C}_{1,2}(n)$ be subset of partitions in $\mathcal{C}_1(n)$ such that there is at least one blue 1, respectively no blue 1.

A b-modular diagram of a partition λ is a Ferrers diagram in which the ith row consists of $\lceil \lambda_i/b \rceil$ boxes with the first box filled with $\lambda_i - \lceil \lambda_i/b \rceil$ and the remaining boxes filled with b. Note that elsewhere in the literature, in modular Ferrers diagrams, it is the last box that is filled with the residue. For a positive integer b and a positive residue a modulo b, we denote by $\delta_m(a,b)$ the b-modular staircase partition with residue a and length m, i.e.,

$$\delta_m(a,b) = \big(b(m-1)+a, b(m-2)+a, \dots, b+a, a\big)$$

if $m \neq 0$, and $\delta_0(a) := \emptyset$. We have $|\delta_m(a,b)| = bm(m-1)/2 + ma$.

The injection $\chi\colon \mathcal{Q}(n-1) \to \mathcal{Q}(n)$ defined by $\chi(\lambda) = \lambda \cup (1)$ shows combinatorially that

$$Q_1(n) = Q(n) - Q(n-1). \tag{4.5}$$

In [13], Yee gave an elegant description of Wright's bijection for proving the Jacobi triple product identity. We denote Yee's bijection by $\psi_{a,b}$. For positive integers a, b with $0 < a < b$, the mapping $\psi_{a,b}$ is a bijection from $\mathcal{U}_{a,b}(n)$ to the set

$$\big\{(\delta_m(x,b),\rho) \mid x \in \{a, b-a\}, \rho \in \mathcal{U}_{0,b} \text{ and } |\delta_m(x,b)| + |\rho| = n\big\}.$$

Thus, the image of $\psi_{a,b}$ consists of pairs of partitions (of sizes adding up to n) such that the first partition is a b-modular staircase with residue a or $b-a$ and all parts of the second partition are divisible by b. Either partition in the pair may be empty. Moreover, if $\lambda \in \mathcal{U}_{a,b(n)}$, and $\ell_{a,b}(\lambda) \geq \ell_{b-a,b}(\lambda)$, then the staircase $\delta_m(x,b)$ in $\psi_{a,b}(\lambda)$ has $m = \ell_{a,b}(\lambda) - \ell_{b-a,b}(\lambda)$ and $x = a$. Otherwise, $m = \ell_{b-a,b}(\lambda) - \ell_{a,b}(\lambda)$ and $x = b-a$.

Next, we create a bijection $f\colon \mathcal{Q}(2n) \to \mathcal{C}_3(n)$ as follows. Start with $\mu \in \mathcal{Q}(2n)$. Then μ can be written uniquely as $\mu = \mu^1 \cup \mu^2$, where μ^1 has distinct parts and each part in μ^2 has even multiplicity. Here, μ^1 consists of one copy of each part of μ that occurs with odd multiplicity. Clearly $\ell(\mu^1)$ is even. Since $\mu^1 \in \mathcal{U}_{1,4}$, we have $\psi_{1,4}(\mu^1) = (\delta_m(x,4),\rho)$ with m even and $x \in \{1,3\}$. We transform μ^2, $\delta_m(x,4)$, ρ as follows.

(I) We map μ^2 to $\mu^2_{1/2}$, the partitions consisting of the parts of μ^2, each with half its multiplicity in μ^2. All parts in μ^2 are odd.

(II) We map ρ to $\rho_{1/2}$, the partition whose parts are the parts of ρ divided by 2. Since $\rho \in \mathcal{U}_{0,4}$, all parts of $\rho_{1/2}$ are even.

(III) Let $m=2k$. We map $\delta_{2k}(x,4)$ to $\delta_k(x+2,8)$. Since $x\in\{1,3\}$, we have $x+2\in\{3,5\}$. Note that

$$|\delta_k(x+2,8)|=4k(k-1)+(x+2)k=\frac{1}{2}(2(2k)(2k-1)+x(2k))=\frac{1}{2}|\delta_{2k}(x,4)|.$$

Then, $\rho_{1/2}\cup\mu^2_{1/2}\cup\delta_k(x+2,8)\vdash n$. We write $\rho_{1/2}\cup\mu^2_{1/2}:=\zeta_{1/2}\cup\gamma_{1/2}$, where $\zeta_{1/2}$ is the partition consisting of the parts of $\rho_{1/2}\cup\mu^2_{1/2}$ divisible by 8. Then $\gamma_{1/2}\in\mathcal{B}_8$.

We define

$$f(\mu):=\big(\psi^{-1}_{x+2,8}(\delta_k(x+2,8),\zeta_{1/2}),\gamma_{1/2}\big)$$

with $\psi^{-1}_{x+2,8}(\delta_k(x+2,8),\zeta_{1/2})\in\mathcal{U}_{x+2,8}$ colored red and $\gamma_{1/2}$ colored blue.

The transformation above is clearly invertible.

We create a bijection $g\colon\mathcal{Q}(2n+1)\to\mathcal{C}_1(n)$ by following the same steps as for f with a slight modification in (III). First note that in this case $\ell(\mu^1)$ is odd. Then, in (III) above we let $m=2k+1$ for some $k\geq 0$. We map $\delta_{2k+1}(1,4)$ to $\delta_k(7,8)$ and we map $\delta_{2k+1}(3,4)$ to $\delta_{k+1}(1,8)$. We have

$$|\delta_k(7,8)|=\frac{1}{2}(|\delta_{2k+1}(1,4)|-1)\quad\text{and}\quad|\delta_{k+1}(1,8)|=\frac{1}{2}(|\delta_{2k+1}(3,4)|-1).$$

Finally, we create a bijection $h\colon\mathcal{Q}(2n-1)\to\mathcal{C}_{1,1}(n)$ by starting with a partition $\mu\in\mathcal{Q}(2n-1)$, map it to $g(\mu)\in\mathcal{C}_1(n-1)$, and then we insert a part equal to 1 into the 8-regular partition $\gamma_{1/2}$.

Using the combinatorial proofs of Theorem 1.3 and identity (4.5) together with the bijections f, g, and h we obtain combinatorial proofs of (i) and (ii).

To prove (iii) combinatorially, we create a bijection $\iota\colon\mathcal{C}_{1,2}(n)\to\mathcal{Q}_1(2n)\cup\mathcal{Q}_1(2n+1)$. Then, we compose with the bijections in Theorem 1.3 to complete the proof.

Let λ be a colored partition in $\mathcal{C}_{1,2}(n)$ and denote by λ^r, respectively λ^b, the partition whose parts are the red, respectively blue parts of λ. Thus, λ^b has no part equal to 1. Since $\lambda^r\in\mathcal{U}_{1,8}$, we have $\psi_{1,8}(\lambda^r)=(\delta_k(x,8),\rho)$ with $x\in\{1,7\}$ and $\rho\in\mathcal{U}_{0,8}$. Reversing the transformation in the definition of the bijection g above, we map $\delta_k(x,8)$ to $\delta_m(y,4)$, $y\in\{1,3\}$ as follows. If $x=1$, then $m=2k-1$ and $y=3$, and if $x=7$, then $m=2k+1$ and $y=1$. We have $|\delta_m(y,4)|=2|\delta_k(x,8)|+1$. Next, in the partition $\lambda^b\cup\rho$ we double the size of each even part and we double the multiplicity of each odd part to obtain a partition $\eta=\eta^e\cup\eta^o$, where η^e, respectively η^o, consists of the even, respectively odd parts of η. Thus $\eta^e\in\mathcal{U}_{0,4}$ and all parts in η^o are odd, greater than 1 and have even multiplicity. Then, $\psi^{-1}_{1,4}(\delta_m(y,4),\eta^e)$ is a partition with distinct odd parts. If this partition contains 1 as a part, we remove it, else we leave this partition unchanged. Denote by ε the obtained partition into distinct odd parts greater than 1 and let $\iota(\lambda):=\varepsilon\cup\eta^o\in\mathcal{Q}_1(2n)\cup\mathcal{Q}_1(2n+1)$. Since all steps above can be reversed, ι is a bijection.

5. Proof of Theorem 1.9

5.1. Analytic proof

Considering Theorem 1.6 and (4.3) and (4.4), we can write

$$\begin{aligned}\sum_{n=0}^{\infty} a_e(n)q^n &= \frac{1}{(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{4n^2-n} - \sum_{n=-\infty}^{\infty} q^{4n^2-3n+1}\right)\\ &= \left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2-\lfloor n/2\rfloor}\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} a_o(n)q^n &= \frac{1}{(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{4n^2-3n} - \sum_{n=-\infty}^{\infty} q^{4n^2-n}\right)\\ &= \left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(\sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{n^2-\lceil n/2\rceil}\right).\end{aligned}$$

The first two assertions of Theorem 1.9 now follow by comparing coefficients of q^n on both sides of these identities. The next assertion follows easily considering that $a(n) = a_e(n) + a_o(n)$.

5.2. Combinatorial proof

The combinatorial proof uses ideas from the combinatorial proof of Theorem 1.6. We use the notation in that proof.

We obtain a bijection from $\mathcal{Q}(2n)$ to the set

$$\big\{(4k^2-k,\lambda) \mid k\in\mathbb{Z}, \lambda \vdash n-(4k^2-k)\big\}$$

by mapping $\mu \in \mathcal{Q}(2n)$, to

$$\big(4k^2-k, \rho_{1/2}\cup\mu^2_{1/2}\big),$$

where $2k = \ell_{1,4}(\mu^1) - \ell_{3,4}(\mu^1)$.

We can obtain similar bijections from $\mathcal{Q}(2n-1)$ to

$$\big\{(4k^2+3k+1,\lambda) \mid k\in\mathbb{Z}, \lambda \vdash n-(4k^2+3k+1)\big\}$$

and from $\mathcal{Q}(2n+1)$ to

$$\big\{(4k^2+3k,\lambda) \mid k\in\mathbb{Z}, \lambda \vdash n-(4k^2+3k)\big\},$$

where in both cases $2k+1 = \ell_{1,4}(\mu^1) - \ell_{3,4}(\mu^1)$.

To complete the proof, we combine these bijections with the combinatorial proofs of Theorem 1.3 and (4.5) and the observation that

$$j^2-\lfloor j/2\rfloor = \begin{cases} 4k^2-k & \text{if } j=2k,\\ 4k^2+3k+1 & \text{if } j=2k+1 \end{cases} \quad \text{and} \quad j^2-\lceil j/2\rceil = \begin{cases} 4k^2-k & \text{if } j=2k,\\ 4k^2+3k & \text{if } j=2k+1. \end{cases}$$

6. Proof of Theorem 1.12

In [2], Andrews and Merca considered Euler's pentagonal number theorem

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2},$$

and proved the following truncated form: For any $k \geq 1$,

$$\frac{(-1)^{k-1}}{(q;q)_\infty} \sum_{n=1-k}^{k} (-1)^n q^{n(3n-1)/2} = (-1)^{k-1} + \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \tag{6.1}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0 \leq k \leq n,\\ 0 & \text{otherwise.} \end{cases}$$

We note that the series on the right hand side of (6.1) is the generating function for $M_k(n)$, i.e.,

$$\sum_{n=0}^{\infty} M_k(n) q^n = \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Multiplying both sides of (6.1) by

$$(-q^3,-q^5,q^8;q^8)_\infty - q(-q,-q^7,q^8;q^8)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2-\lfloor n/2\rfloor},$$

we obtain

$$\begin{aligned} &(-1)^{k-1} \left(\left(\sum_{n=0}^{\infty} a_e(n) q^n \right) \left(\sum_{n=1-k}^{k} (-1)^n q^{n(3n-1)/2} \right) - \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2-\lfloor n/2\rfloor} \right) \\ &= \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2-\lfloor n/2\rfloor} \right) \left(\sum_{n=0}^{\infty} M_k(n) q^n \right). \end{aligned}$$

The first identity of Theorem 1.12 follows easily considering Cauchy's multiplication of two power series.

In a similar way, multiplying both sides of (6.1) by

$$(-q,-q^7,q^8;q^8)_\infty - (-q^3,-q^7,q^8;q^8)_\infty = -\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2-\lceil n/2\rceil},$$

we obtain the second identity of Theorem 1.12. We omit the details.

7. A theta identity of Gauss

The following theta identity is often attributed to Gauss [1, p. 23, (2.2.12)]:

$$1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}. \tag{7.1}$$

In this section, we use identity (7.1) in order to obtain new relations involving the partition functions $a_e(n)$ and $a_o(n)$. We define $\varrho_e(n)$ and $\varrho_o(n)$ by

$$\sum_{n=0}^{\infty}\varrho_e(n)q^n = \sum_{n=-\infty}^{\infty}(-1)^n q^{\lceil n(3n-1)/4\rceil} \quad \text{and} \quad \sum_{n=0}^{\infty}\varrho_o(n)q^n = \sum_{n=-\infty}^{\infty}(-1)^{n-1} q^{\lfloor n(3n-1)/4\rfloor}.$$

The numbers $n(3n-1)/4$ for $n\in\mathbb{Z}$ are the half generalized pentagonal numbers. We remark that $\varrho_e(1)=-2$ and all other coefficients $\varrho_e(n)$, $\varrho_o(n)$ are from the set $\{-1,0,1\}$.

The following result shows that our functions $a_e(n)$, $a_o(n)$ and $a(n)$ share two common linear recurrence relations for most values of n.

Theorem 7.1. *Let n be a nonnegative integer. Then*

(i) $a_e(n)+2\sum_{j=1}^{\infty}(-1)^j a_e(n-j^2) = \varrho_e(n)$;

(ii) $a_o(n)+2\sum_{j=1}^{\infty}(-1)^j a_o(n-j^2) = \varrho_o(n)$.

Proof. (i) Considering the generating function for $a_e(n)$ from Corollary 1.7(i), we have

$$\begin{aligned}
&\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}a_e(n)q^n\\
&= \frac{(-q^3,-q^5,q^8;q^8)_\infty}{(-q;q)_\infty} - q\,\frac{(-q,-q^7,q^8;q^8)_\infty}{(-q;q)_\infty}\\
&= \frac{(q^6,q^{10};q^{16})_\infty(q^8;q^8)_\infty(q;q^2)_\infty}{(q^3,q^5;q^8)_\infty} - q\,\frac{(q^2,q^{14};q^{16})_\infty(q^8;q^8)_\infty(q;q^2)_\infty}{(q,q^7;q^8)_\infty}\\
&= (q^6,q^{10};q^{16})_\infty(q,q^7,q^8;q^8)_\infty - q\,(q^2,q^{14};q^{16})_\infty(q^3,q^5;q^8)_\infty.
\end{aligned}$$

The Watson quintuple product identity [6] states that

$$(q/z^2,qz^2;q^2)_\infty(z,q/z,q;q)_\infty = \sum_{n=-\infty}^{\infty}q^{n(3n+1)/2}(z^{-3n}-z^{3n+1}). \tag{7.2}$$

Letting $q\to q^8$ and successively setting $z=q$, $z=q^3$ in (7.2), we obtain

$$\begin{aligned}
(q^6,q^{10};q^{16})_\infty(q,q^7,q^8;q^8)_\infty &= \sum_{n=-\infty}^{\infty}q^{4n(3n+1)}(q^{-3n}-q^{3n+1})\\
&= \sum_{n=-\infty}^{\infty}\left(q^{G(4n)/2}-q^{G(4n+3)/2}\right)
\end{aligned}$$

and

$$
\begin{aligned}
q\,(q^2,q^{14};q^{16})_\infty(q^3,q^5,q^8;q^8)_\infty &= \sum_{n=-\infty}^{\infty} q^{4n(3n+1)+1}(q^{-9n}-q^{9n+3})\\
&= \sum_{n=-\infty}^{\infty} \big(q^{G(4n+1)/2+1/2}-q^{G(4n+2)/2+1/2}\big),
\end{aligned}
$$

where $G(n)=n(3n-1)/2$ is the nth generalized pentagonal number. Thus, we obtain

$$
\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty} a_e(n)q^n = \sum_{n=-\infty}^{\infty}(-1)^n q^{\lceil n(3n-1)/4\rceil}.
$$

On the other hand, we have

$$
\begin{aligned}
\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty} a_e(n)q^n &= \left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\right)\left(\sum_{n=0}^{\infty} a_e(n)q^n\right)\\
&= \sum_{n=0}^{\infty}\left(a_e(n)+2\sum_{k=1}^{\infty}(-1)^k a_e(n-k^2)\right)q^n
\end{aligned}
$$

and the first assertion of the theorem is proved.

(ii) The proof of the second identity is quite similar, so we omit the details. □

Next, we remark that the reciprocal of the infinite product in (7.1) is the generating function for $\overline{p}(n)$, the number of overpartitions of n (cf. [7]):

$$
\sum_{n=0}^{\infty}\overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.
$$

Overpartitions are ordinary partitions with the added condition that the first appearance of any part may be overlined or not. Thus, there are eight overpartitions of 3:

$$
(3),(\overline{3},2,1),(\overline{2},1),(2,\overline{1}),(\overline{2},\overline{1}),(1,1,1),(\overline{1},1,1).
$$

The functions $a_e(n)$, $a_o(n)$ and $a(n)$ are closely related to the overpartition function $\overline{p}(n)$ as seen in the next theorem.

Theorem 7.2. *Let n be a nonnegative integer. Then*

(i) $\displaystyle a_e(n)=\sum_{k=-\infty}^{\infty}(-1)^k\overline{p}\big(n-\lceil k(3k-1)/4\rceil\big);$

(ii) $\displaystyle a_o(n)=\sum_{k=-\infty}^{\infty}(-1)^{k-1}\overline{p}\big(n-\lfloor k(3k-1)/4\rfloor\big);$

(iii) $a(n) = \sum_{k=-\infty}^{\infty} (-1)^k \Big(\overline{p}\big(n - \lceil k(3k-1)/4 \rceil\big) - \overline{p}\big(n - \lfloor k(3k-1)/4 \rfloor\big)\Big).$

Proof. The proof follows easily considering that

$$\begin{aligned}\sum_{n=0}^{\infty} a_e(n) q^n &= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\lceil n(3n-1)/4 \rceil} \\ &= \left(\sum_{n=0}^{\infty} \overline{p}(n) q^n\right) \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{\lceil n(3n-1)/4 \rceil}\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} a_o(n) q^n &= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^{n-1} q^{\lfloor n(3n-1)/4 \rfloor} \\ &= \left(\sum_{n=0}^{\infty} \overline{p}(n) q^n\right) \left(\sum_{n=-\infty}^{\infty} (-1)^{n-1} q^{\lfloor n(3n-1)/4 \rfloor}\right).\end{aligned}$$

□

In [2], Andrews and Merca introduced the overpartition function $\overline{M}_k(n)$, which counts the number of overpartitions of n in which the first part larger than k appears at least $k+1$ time. For example, $\overline{M}_2(12) = 16$, and the partitions in question are $(4,4,4)$, $(\overline{4},4,4)$, $(3,3,3,3)$, $(\overline{3},3,3,3)$, $(3,3,3,2,1)$, $(3,3,3,\overline{2},1)$, $(3,3,3,2,\overline{1})$, $(3,3,3,\overline{2},\overline{1})$, $(\overline{3},3,3,2,1)$, $(\overline{3},3,3,\overline{2},1)$, $(\overline{3},3,3,2,\overline{1})$, $(\overline{3},3,3,\overline{2},\overline{1})$, $(3,3,3,1,1,1)$, $(3,3,3,\overline{1},1,1)$, $(\overline{3},3,3,1,1,1)$, $(\overline{3},3,3,\overline{1},1,1)$. We remark that there is a more general result where Theorem 7.1 is the limiting case $k \to \infty$.

Theorem 7.3. *For $n, k > 0$,*

(i) $(-1)^k \left(a_e(n) + 2\sum_{j=1}^{k} (-1)^j a_e(n-j^2) - \varrho_e(n)\right) = \sum_{j=-\infty}^{\infty} (-1)^j \overline{M}_k\big(n - \lceil j(3j-1)/4 \rceil\big);$

(ii) $(-1)^k \left(a_o(n) + 2\sum_{j=1}^{k} (-1)^j a_o(n-j^2) - \varrho_o(n)\right) = \sum_{j=-\infty}^{\infty} (-1)^{j-1} \overline{M}_k\big(n - \lfloor j(3j-1)/4 \rfloor\big).$

Proof. In [2], Andrews and Merca proved the following truncated form of (7.1): For any $k \geq 1$,

$$\begin{aligned}&\frac{(-q;q)_\infty}{(q;q)_\infty} \left(1 + 2\sum_{j=1}^{k} (-1)^j q^{j^2}\right) - 1 \\ &= 2(-1)^k \frac{(-q;q)_k}{(q;q)_k} \sum_{j=0}^{\infty} \frac{q^{(k+1)(k+j+1)}(-q^{k+j+2};q)_\infty}{(1-q^{k+j+1})(q^{k+j+2};q)_\infty}.\end{aligned} \tag{7.3}$$

The series on the right hand side of (7.3) is the generating function for $\overline{M}_k(n)$, i.e.,

$$\sum_{n=0}^{\infty} \overline{M}_k(n)q^n = 2\,\frac{(-q;q)_k}{(q;q)_k} \sum_{j=0}^{\infty} \frac{q^{(k+1)(k+j+1)}(-q^{k+j+2};q)_\infty}{(1-q^{k+j+1})(q^{k+j+2};q)_\infty}.$$

Multiplying both sides of (7.3) by

$$\frac{(q;q)_\infty}{(-q;q)_\infty} \sum_{n=0}^{\infty} a_e(n)q^n = \sum_{n=-\infty}^{\infty} (-1)^n q^{\lceil n(3n-1)/4 \rceil},$$

we obtain

$$\left(\sum_{n=0}^{\infty} a_e(n)q^n\right)\left(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\right) - \sum_{n=-\infty}^{\infty} (-1)^n q^{\lceil n(3n-1)/4 \rceil}$$
$$= (-1)^k \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{\lceil n(3n-1)/4 \rceil}\right)\left(\sum_{n=0}^{\infty} \overline{M}_k(n)q^n\right).$$

The first identity of theorem follows easily considering Cauchy's multiplication of two power series.

In a similar way, multiplying both sides of (7.3) by

$$\frac{(q;q)_\infty}{(-q;q)_\infty} \sum_{n=0}^{\infty} a_o(n)q^n = \sum_{n=-\infty}^{\infty} (-1)^{n-1} q^{\lfloor n(3n-1)/4 \rfloor},$$

we obtain the second identity. We omit the details. □

Finally, we remark that there is a substantial amount of numerical evidence to state the following conjectures related to Theorems 7.2 and 7.3.

Conjecture 7.4. *Let n be a nonnegative integer. For $k>0$, we have*

(i) $(-1)^k \left(a_e(n) - \sum_{j=1-k}^{k} (-1)^j \overline{p}\big(n-\lceil j(3j-1)/4 \rceil\big)\right) \geq 0;$

(ii) $(-1)^{k-1} \left(a_o(n) - \sum_{j=1-k}^{k} (-1)^{j-1} \overline{p}\big(n-\lfloor j(3j-1)/4 \rfloor\big)\right) \geq 0;$

(iii) $(-1)^{\lceil k/2 \rceil} \left(a(n) - \sum_{j=1-k}^{k} (-1)^j \Big(\overline{p}\big(n-\lceil j(3j-1)/4 \rceil\big) - \overline{p}\big(n-\lfloor j(3j-1)/4 \rfloor\big)\Big)\right) \geq 0.$

Conjecture 7.5. *Let n be a nonnegative integer. For $k>0$, we have*

(i) $(-1)^k \left(a_e(n) + 2\sum_{j=1}^{k}(-1)^j a_e(n-j^2) - \varrho_e(n)\right) \geq 0;$

(ii) $(-1)^k \left(a_o(n) + 2\sum_{j=1}^{k}(-1)^j a_o(n-j^2) - \varrho_o(n)\right) \geq 0.$

8. Concluding remarks

In this article we gave a new combinatorial interpretation for the function $Q_1(n)$ by introducing the functions $a_e(n)$, $a_o(n)$, and $a(n) = a_e(n) + a_o(n)$. We studied the connection between these functions and 8-regular partitions in two colors satisfying certain conditions. We found connections between the respective functions $a_e(n)$, $a_o(n)$, and $a(n)$ and the partition function $p(n)$, and we also gave a truncated pentagonal number theorem for each of these functions. The generating functions for $a_e(n)$, $a_o(n)$, and $a(n)$ allow us to easily deduce an interesting identity

$$\sum_{n=0}^{\infty} \frac{q^{\lfloor 3n/2 \rfloor}}{(q;q)_n} = \frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q)_\infty}.$$

To conclude, we consider the function $d(n) := a_o(n) - a_e(n)$, which has the following generating function

$$\sum_{n=0}^{\infty} d(n)q^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{\lfloor 3n/2 \rfloor}}{(q;q)_n}.$$

First note that $Q_1(n)$ is also equal to the number of partitions of n into distinct parts such that the first two parts differ by 1. To see this, recall that $Q(n)$ is also the number of partitions of n into distinct parts. We abuse notation and write $\mathcal{Q}(n)$ for the set of distinct partitions of n. Then we have an injection $\tau\colon \mathcal{Q}(n-1) \to \mathcal{Q}(n)$ defined by $\tau(\lambda_1, \lambda_2, \ldots, \lambda_k) = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_k)$. Then $Q_1(n) = Q(n) - Q(n-1) = |\mathcal{Q}(n) \setminus \tau(\mathcal{Q}(n-1))|$ and $\mathcal{Q}(n) \setminus \tau(\mathcal{Q}(n-1))$ is the set of partitions of n into distinct parts such that the first two parts differ by 1. We abuse notation and refer to this set as $\mathcal{Q}_1(n)$.

We denote by $\mathcal{Q}_2(n)$ the set of partitions λ of n into distinct parts greater than 1 such that $\lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = 1$, and set $Q_2(n) := |\mathcal{Q}_2(n)|$. Then we have the following theorem.

Theorem 8.1. *For $n \geq 3$ we have $d(n) = Q_2(2n+1)$.*

Proof. To prove the theorem, we need to show that $Q_2(2n+1) = Q_1(2n+1) - Q_1(2n)$. We will show the more general result that $Q_2(n) = Q_1(n) - Q_1(n-1)$ for $n \geq 5$. We define an injection $\sigma\colon \mathcal{Q}_1(n-1) \to \mathcal{Q}_1(n)$ as follows. Let $\lambda \vdash n-1$ be a partition into distinct part such that $\lambda_1 - \lambda_2 = 1$. If 1 is not a part of λ, we define $\sigma(\lambda) = \lambda \cup (1)$. If 1 is a part of λ, we define $\sigma(\lambda)$ to be the partition obtained from λ by removing the part equal to 1 and increasing each of λ_1 and λ_2 by 1. Clearly, this is an injection and $\mathcal{Q}_1(n) \setminus \sigma(\mathcal{Q}_1(n-1)) = \mathcal{Q}_2(n)$. $\square$

It would be interesting to describe $\mathcal{Q}_2$ as a subset of partitions into odd parts greater than 1.

Finally, we remark that the proof of Theorem 8.1 implies that, starting with $n = 4$, the sequence $Q_1(n)$ is non-decreasing. Thus, $a_e(n) \leq a_o(n) \leq a_e(n+1)$ for $n \geq 2$. This also implies the function $Q(n)$ is convex for $n \geq 4$, i.e.,

$$2Q(n) \leq Q(n-1) + Q(n+1).$$

Of course, the convexity of $Q(n)$ follows from the Bateman–Erdős Theorem [5] but the argument above provides a very easy combinatorial proof.

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