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Novel Results on Persistence and Attractivity of Delayed Nicholson's Blowflies System with Patch Structure

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Abstract. This paper is concerned with the dynamic characteristics of a class of Nicholson's blowflies system with patch structure and multiple pairs of distinct time-varying delays. We aim to find the influence of the distinct time-varying delays in the same reproductive function on its asymptotic behavior. First, we derive the global existence, positiveness and uniform persistence of solutions for the addressed system. Then, by employing the theory of functional differential equations, the fluctuation lemma and the technique of differential inequalities, we build up some new delay-dependent criteria for the global attractivity of the positive equilibrium point vector, which does not possess the same components. In addition, we exam the effectiveness and feasibility of the theoretical achievements by some numerical simulations.

1. Introduction

In the real world, considering that logical self limiting control can occur at any stage of the population life cycle, it is indispensable to introduce maturity delay and feedback delay corresponding to maturity period and feedback time in the same time-dependent reproductive function of population dynamics model, which are often different [7, 8]. In particular, Berezansky and Braverman [1] established the following Nicholson's blowflies model with different mature delay and feedback delay:

$$x'(t) = \beta(t)[-\delta x(t) + \rho x(t-\tau(t))e^{-hx(t-\sigma(t))}], \tag{1.1}$$

which in the case $\beta(t) \equiv 1$ and $\tau(t) \equiv \sigma(t)$ is in accord with the classical scalar Nicholson's blowflies equation [2, 24] and has been widely studied, including the existence, persistence, oscillation, periodicity, almost periodicity and stability [2, 5, 11, 17, 18, 21, 23, 24, 29, 31]. It should be noted that Berezansky and Braverman showed by examples in [1] that distinct delays in the same reproductive function may lead to chaotic oscillation. In this case, the feedback term $\rho x(t-\tau(t))e^{-hx(t-\sigma(t))}$ is actually a binary function, which greatly

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improves the difficulty of studying the dynamics on the model, resulting in the almost stagnation of the research on the model (1.1). So far, we only found that the authors of [9, 13, 22, 25] explored the stability of Nicholson's blowflies (1.1) with different maturity delay and feedback delay in the same reproductive function. Particularly, the authors in [9] investigated a scalar Nicholson's equation incorporating multiple pairs of time-varying delays, and obtained several sufficient criterion for the permanence, local stability and global attractivity of the positive equilibrium.

On the other hand, considering that living environment of many species is fragmented in the process of reproduction, and the natural division of spatial regions is discrete (each region is usually described as a patch), we naturally extend the model (1.1) to the following Nicholson's blowflies system with patch structure:

$$x_i'(t) = \beta_i(t)\Big[- \overline{\delta}_i x_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij} x_j(t) + \sum_{j=1}^{m} \rho_{ij} x_i(t - \tau_{ij}(t)) e^{-h_{ij} x_i(t-\sigma_{ij}(t))}\Big], \tag{1.2}$$

where $t \geq t_0$, $i \in \Lambda := \{1, 2, \dots, n\}$, x_i describes the number of the density of the ith-population at time t, a_{ij} $(i \neq j)$ is refer to the proportion of the population moving from patch j to patch i at time t, $\overline{\delta}_i$ designates the coefficient of instantaneous loss for class i at time t (which integrates both the death proportion and the dispersal proportion of the population in class i moving to the other classes), $\rho_{ij} x_i(t-\tau_{ij}(t)) e^{-h_{ij} x_i(t-\sigma_{ij}(t))}$ represents the reproductive function for class i at time t, ρ_{ij} is the birth rate for the species, $\tau_{ij}(t)$ and $\sigma_{ij}(t)$ stand for maturity delay and feedback delay respectively. For more detailed biological significance, one can refer to [14, 30, 33] and their references cited therein.

Introducing the change of variable

$$\overline{\delta}_i = \delta_i - a_{ii} \quad \text{with } a_{ii} < 0,$$

we have the equivalent expression of (1.2) as

$$x_i'(t) = \beta_i(t)\Big[- \delta_i x_i(t) + \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{m} \rho_{ij} x_i(t - \tau_{ij}(t)) e^{-h_{ij} x_i(t-\sigma_{ij}(t))}\Big], \tag{1.3}$$

where $\delta_i > 0$, $\rho_{ij} > 0$, $h_{ij} > 0$, $a_{ij} \geq 0$ $(i \neq j)$, $\beta_i, \tau_{ij}, \sigma_{ij} \colon \mathbb{R} \to (0, +\infty)$ $(i \in \Lambda,\ j \in \Omega := \{1, 2, \dots, m\})$ are bounded and continuous functions, and $(a_{ij})_{n\times n}$ is a cooperative and irreducible matrix satisfying

$$\sum_{j=1, j\neq i}^{n} a_{ij} = -a_{ii} \quad \text{for all } i \in \Lambda. \tag{1.4}$$

For the case of $\tau_{ij}(t) \equiv \sigma_{ij}(t)$ $(i \in \Lambda,\ j \in \Omega)$, the dynamic behavior for model (1.3) has been extensively studied in recent years [3, 4, 10, 15, 16, 19, 20, 32]. As pointed out in [6], the positive equilibrium point $(N_1, N_2, \ldots, N_n)$ with different components can be transformed into the positive equilibrium point with the same components, i.e.,

$$N_1 = N_2 = \cdots = N_n = N^*. \tag{1.5}$$

This has been considered as fundamental for the obtained research in [4,19,20]. Obviously, (1.5) is not consistent with the biological background in the considered system [3,10]. However, for the case of $\tau_{ij}(t) \neq \sigma_{ij}(t)$, there are relatively few studies devoted to model (1.3), we only find that the stability of its zero equilibrium has been discussed in [22, 33], and few attempts have been made to reveal the asymptotic behavior of the positive equilibrium [30]. Thus, without adopting the technical conditions (1.4) and (1.5), the attractivity analysis on the system (1.3) has not been involved, which needs further research.

Based on the above observations, we are committed to establish the global attractivity conditions of the unique positive equilibrium point for the system (1.3) with $\tau_{ij}(t) \neq \sigma_{ij}(t)$ ($i \in \Lambda$, $j \in \Omega$). In short, the contributions of this article can be summarized as below. (1) With the help of some novel differential inequality techniques, we establish the global positiveness and uniform persistence on the solutions of system (1.3); (2) Under certain assumptions, we drive some new sufficient criteria guaranteeing the attractivity of the positive equilibrium point of system (1.3) for the first time, which improve and generalize some recent existing ones; (3) We carry out some numerical examples including comparison analyses to validate the correctness and feasibility of the obtained theoretical results.

The remaining of this paper is systematized as below. The positiveness and uniform persistence are presented in Section 2. In Section 3, we substantiate the global attractivity of the unique positive equilibrium point for the addressed system (1.3). Moreover, numerical simulations in Section 4 indicate that our theoretical findings are correct, and a concise conclusion is offered in Section 5.

2. Global existence, positiveness and uniform persistence

Throughout this paper, we label the collection of all n-dimensional real vectors by $\mathbb{R}^n$ ($\mathbb{R}^1 = \mathbb{R}$) and the set of all positive integers by $\mathbb{N}^+$. For a bounded real function ϕ, let

$$\phi^H = \sup_{\vartheta \in [H, +\infty)} \phi(\vartheta), \quad \phi^+ = \sup_{\vartheta \in \mathbb{R}} \phi(\vartheta), \quad \phi^- = \inf_{\vartheta \in \mathbb{R}} \phi(\vartheta).$$

Denote

$$h = \max_{i \in \Lambda} \max_{j \in \Omega} h_{ij}, \quad r_i^H = \max\left\{ \max_{j \in \Omega} \tau_{ij}^H, \max_{j \in \Omega} \sigma_{ij}^H \right\},$$
$$r_i = \max\left\{ \max_{j \in \Omega} \tau_{ij}^+, \max_{j \in \Omega} \sigma_{ij}^+ \right\}, \quad r = \max_{i \in \Lambda}\{r_i\}.$$

Furthermore, assume that $\delta_i > 0$, $\rho_{ij} > 0$, $\sigma_{ij} > 0$, $\beta_i^- > 0$, $\tau_{ij}, \sigma_{ij}, \beta_i \colon \mathbb{R} \to (0, +\infty)$ $(i \in \Lambda,\, j \in \Omega)$ are bounded and continuous functions, $A = (a_{ij})_{n\times n}$ is a cooperative matrix with $a_{ij} \geq 0$ $(i \neq j)$, and

$$\sum_{j=1, j\neq i}^{n} a_{ij} \leq -a_{ii} \quad \text{for all } i \in \Lambda, \tag{2.1}$$

and there exists $\widetilde{t_0} \in [t_0, +\infty)$ such that

$$\sigma_{ij}(t) \geq \tau_{ij}(t) \quad \text{for arbitrary } t \in [\widetilde{t_0}, +\infty),\ i \in \Lambda,\ j \in \Omega. \tag{2.2}$$

Clearly, (2.1) is a weaker assumption than (1.4), and (2.2) suggests that the feedback delay is not less than maturity delay.

Because we will prove the persistence of the system (1.3), it is not difficult by using the argument method in the literatures [3, 10] to find a positive vector $(N_1^*, N_2^*, \ldots, N_n^*)$ such that

$$-\delta_i N_i^* + \sum_{j=1}^{n} a_{ij} N_j^* + \sum_{j=1}^{m} \rho_{ij} N_i^* e^{-h_{ij} N_i^*} = 0 \quad \text{for all } i \in \Lambda, \tag{2.3}$$

which entails that $(N_1^*, N_2^*, \ldots, N_n^*)$ is a positive equilibrium point of model (1.3).

Let $C = \prod_{i=1}^{n} C([-r_i, 0], \mathbb{R})$ be a Banach space accompanying the supremum norm $\|\cdot\|$, and $C_+ = \prod_{i=1}^{n} C([-r_i, 0], [0, +\infty))$. In addition, we label $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ as an admissible solution of (1.3) involving the initial conditions:

$$x_{t_0} = \varphi, \quad \varphi \in C_+ \quad \text{and} \quad \varphi_i(0) > 0, \quad i \in \Lambda, \tag{2.4}$$

and $[t_0, \eta(\varphi))$ as the maximal right-interval of existence of $x_t(t_0, \varphi)$.

We next give three key lemmas which will be used to prove our main results.

Lemma 2.1. *$x(t) = x(t; t_0, \varphi)$ is positive on $[t_0, +\infty)$.*

Proof. According to Theorem 5.2.1 in [26, p. 81], one has $x_t(t_0, \varphi) \in C_+$ for all $t \in [t_0, \eta(\varphi))$. Thus, owing to (1.3) and (2.4), we drive

$$\begin{aligned}
&x_i(t) \\
&= \varphi_i(0) e^{-\int_{t_0}^{t} (\delta_i - a_{ii}) \beta_i(s)\, ds} \\
&\quad + e^{-\int_{t_0}^{t} (\delta_i - a_{ii}) \beta_i(s)\, ds} \int_{t_0}^{t} \beta_i(s) \Bigg[\sum_{j=1, j\neq i}^{n} a_{ij} x_j(s) + \sum_{j=1}^{m} \rho_{ij} x_i(s - \tau_{ij}(s)) e^{-h_{ij} x_i(s - \sigma_{ij}(s))} \Bigg] \\
&\qquad \times e^{\int_{t_0}^{s} (\delta_i - a_{ii}) \beta_i(v)\, dv}\, ds \\
&> 0 \quad \text{for all } t \in [t_0, \eta(\varphi)) \text{ and } i \in \Lambda.
\end{aligned}$$

It remains to substantiate that $\eta(\varphi) = +\infty$. For $t \in [t_0, \eta(\varphi))$, $i \in \Lambda$, set

$$X_i(t) = \max_{t_0 - r_i \le s \le t} x_i(s), \quad \Gamma(t) = \max_{i \in \Lambda} X_i(t).$$

Then

$$x_i'(s) \le \beta_i(s)\Big(\sum_{j=1, j\ne i}^{n} a_{ij} + \sum_{j=1}^{m} \rho_{ij}\Big)\Gamma(s), \quad \forall\, s \in [t_0, t],\ i \in \Lambda,$$

and

$$\begin{aligned} x_i(s) &\le x_i(t_0) + \int_{t_0}^{s} \beta_i(v)\Big(\sum_{j=1, j\ne i}^{n} a_{ij} + \sum_{j=1}^{m} \rho_{ij}\Big)\Gamma(v)\, dv, \\ &\le \|\varphi\| + \int_{t_0}^{t} \max_{i \in \Lambda}\Big\{\beta_i^+\Big(\sum_{j=1, j\ne i}^{n} a_{ij} + \sum_{j=1}^{m} \rho_{ij}\Big)\Big\}\Gamma(v)\, dv, \quad \forall\, s \in [t_0, t],\ i \in \Lambda. \end{aligned}$$

This, together with the definition of $\Gamma(t)$, implies

$$\Gamma(t) \le \|\varphi\| + \int_{t_0}^{t} \max_{i \in \Lambda}\Big\{\beta_i^+\Big(\sum_{j=1, j\ne i}^{n} a_{ij} + \sum_{j=1}^{m} \rho_{ij}\Big)\Big\}\Gamma(v)\, dv \quad \text{for all } t \in [t_0, \eta(\varphi)).$$

In view of the Gronwall–Bellman inequality, we have

$$\begin{aligned} 0 &< x_i(t) \le X_i(t) \le \Gamma(t) \\ &\le \|\varphi\| e^{\int_{t_0}^{t} \max_{i \in \Lambda}\left\{\beta_i^+\left(\sum_{j=1, j\ne i}^{n} a_{ij} + \sum_{j=1}^{m} \rho_{ij}\right)\right\} dv}, \quad \forall\, t \in [t_0, \eta(\varphi)),\ i \in \Lambda, \end{aligned}$$

which, together with Theorem 2.3.1 in [12], indicates that $\eta(\varphi) = +\infty$ and completes the proof. □

Lemma 2.2. *Assume that*

(2.5)
$$\delta_i - \sum_{j=1}^{m} \rho_{ij} \limsup_{t \to +\infty}(\sigma_{ij}(t) - \tau_{ij}(t)) \times \limsup_{t \to +\infty} \beta_i(t)\Big(\sum_{j=1, j\ne i}^{n} a_{ij} + \sum_{j=1}^{m} \rho_{ij}\Big) > 0 \quad \textit{for all } i \in \Lambda$$

holds. Then $x(t)$ is bounded on $[t_0, +\infty)$.

Proof. For $t > t_0$, let $i_0 \in \Lambda$ and $M_{i_0}(t) \in [t_0 - r_{i_0}, t]$ agree with

$$x_{i_0}(M_{i_0}(t)) = \max_{t_0 - r_{i_0} \le s \le t} x_{i_0}(s) = \max_{i \in \Lambda}\Big\{\max_{t_0 - r_i \le s \le t} x_i(s)\Big\}.$$

Now, we validate that $x(t)$ is bounded on $[t_0, +\infty)$. Assume on the contrary that

$$\lim_{t \to +\infty} x_{i_0}(M_{i_0}(t)) = +\infty, \quad \lim_{t \to +\infty} M_{i_0}(t) = +\infty.$$

On account of (2.2) and (2.5), there exist $\varepsilon > 0$ and $T_0 > \widetilde{t}_0$ such that for all $t \geq T_0$, $i \in \Lambda$, $j \in \Omega$,

$$\sigma_{ij}(t) - \tau_{ij}(t) < \limsup_{t\to+\infty}(\sigma_{ij}(t) - \tau_{ij}(t)) + \varepsilon, \quad \beta_i(t) < \limsup_{t\to+\infty}\beta_i(t) + \varepsilon, \tag{2.6}$$

and

(2.7)
$$\delta_i - \sum_{j=1}^{m}\rho_{ij}\Big(\limsup_{t\to+\infty}(\sigma_{ij}(t) - \tau_{ij}(t)) + \varepsilon\Big)\Big(\limsup_{t\to+\infty}\beta_i(t) + \varepsilon\Big)\Big(\sum_{j=1,j\neq i}^{n} a_{ij} + \sum_{j=1}^{m}\rho_{ij}\Big) > 0.$$

Consequently, there must be a $\widetilde{T}_0 > T_0$ obeying

$$M_{i_0}(t) > T_0 + r \quad \text{for all } t \geq \widetilde{T}_0. \tag{2.8}$$

Apparently, for all $t \in [t_0, +\infty)$, (1.3) and (2.1) lead to

$$\begin{aligned}
x'_{i_0}(s) &= \beta_{i_0}(s)\Big[-\delta_{i_0}x_{i_0}(s) + \sum_{j=1}^{n}a_{i_0j}x_j(s) + \sum_{j=1}^{m}\rho_{i_0j}x_{i_0}(s-\tau_{i_0j}(s))e^{-h_{i_0j}x_{i_0}(s-\sigma_{i_0j}(s))}\Big]\\
&\leq \beta_{i_0}(s)\Big[-\delta_{i_0}x_{i_0}(s) + \sum_{j=1}^{n}a_{i_0j}x_j(s) + \sum_{j=1}^{m}\rho_{i_0j}x_{i_0}(s-\tau_{i_0j}(s))\Big]\\
&\leq \beta_{i_0}(s)\Big[\sum_{j=1,j\neq i_0}^{n}a_{i_0j}x_{i_0}(M_{i_0}(t)) + \sum_{j=1}^{m}\rho_{i_0j}x_{i_0}(M_{i_0}(t))\Big]\\
&= \beta_{i_0}(s)\Big[\sum_{j=1,j\neq i_0}^{n}a_{i_0j} + \sum_{j=1}^{m}\rho_{i_0j}\Big]x_{i_0}(M_{i_0}(t)) \quad \text{for all } s \in [t_0, t],
\end{aligned}$$

which, together with (1.3), (2.1), (2.6), (2.7), (2.8) and the fact $\sup_{x\geq 0} xe^{-x} = \frac{1}{e}$, yields

$$\begin{aligned}
0 &\leq x'_{i_0}(M_{i_0}(t))\\
&= \beta_{i_0}(M_{i_0}(t))\Big[-\delta_{i_0}x_{i_0}(M_{i_0}(t)) + \sum_{j=1}^{n}a_{i_0j}x_j(M_{i_0}(t))\\
&\qquad + \sum_{j=1}^{m}\rho_{i_0j}x_{i_0}(M_{i_0}(t)-\tau_{i_0j}(M_{i_0}(t)))e^{-h_{i_0j}x_{i_0}(M_{i_0}(t)-\sigma_{i_0j}(M_{i_0}(t)))}\Big]\\
&\leq \beta_{i_0}(M_{i_0}(t))\Big\{-\delta_{i_0}x_{i_0}(M_{i_0}(t)) + \sum_{j=1}^{n}a_{i_0j}x_{i_0}(M_{i_0}(t))\\
&\qquad + \sum_{j=1}^{m}\frac{\rho_{i_0j}}{h_{i_0j}}h_{i_0j}x_{i_0}(M_{i_0}(t)-\sigma_{i_0j}(M_{i_0}(t)))e^{-h_{i_0j}x_{i_0}(M_{i_0}(t)-\sigma_{i_0j}(M_{i_0}(t)))}\\
&\qquad + \sum_{j=1}^{m}\rho_{i_0j}\big[x_{i_0}(M_{i_0}(t)-\tau_{i_0j}(M_{i_0}(t))) - x_{i_0}(M_{i_0}(t)-\sigma_{i_0j}(M_{i_0}(t)))\big]
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times e^{-h_{i_0j}x_{i_0}(M_{i_0}(t)-\sigma_{i_0j}(M_{i_0}(t)))}\Big\}\\
&\le \beta_{i_0}(M_{i_0}(t))\Big[-\delta_{i_0}x_{i_0}(M_{i_0}(t))+\sum_{j=1}^{n}a_{i_0j}x_{i_0}(M_{i_0}(t))+\sum_{j=1}^{m}\frac{\rho_{i_0j}}{h_{i_0j}e}\\
&\qquad +\sum_{j=1}^{m}\rho_{i_0j}\int_{M_{i_0}(t)-\sigma_{i_0j}(M_{i_0}(t))}^{M_{i_0}(t)-\tau_{i_0j}(M_{i_0}(t))}x'_{i_0}(s)\,ds\Big]\\
&\le \beta_{i_0}(M_{i_0}(t))\Big[-\delta_{i_0}x_{i_0}(M_{i_0}(t))+\sum_{j=1}^{m}\frac{\rho_{i_0j}}{h_{i_0j}e}+\sum_{j=1}^{m}\rho_{i_0j}\Big(\limsup_{t\to+\infty}(\sigma_{i_0j}(t)-\tau_{i_0j}(t))+\varepsilon\Big)\\
&\qquad \times\Big(\limsup_{t\to+\infty}\beta_{i_0}(t)+\varepsilon\Big)\Big(\sum_{j=1,j\ne i_0}^{n}a_{i_0j}+\sum_{j=1}^{m}\rho_{i_0j}\Big)x_{i_0}(M_{i_0}(t))\Big]
\end{aligned}
$$

for all $t \ge \widetilde{T}_0$, and then

$$
\begin{aligned}
&0 < x_{i_0}(M_{i_0}(t))\\
&\le \frac{\displaystyle\sum_{j=1}^{m}\frac{\rho_{i_0j}}{h_{i_0j}e}}{\delta_{i_0}-\displaystyle\sum_{j=1}^{m}\rho_{i_0j}\Big(\limsup_{t\to+\infty}(\sigma_{i_0j}(t)-\tau_{i_0j}(t))+\varepsilon\Big)\Big(\limsup_{t\to+\infty}\beta_{i_0}(t)+\varepsilon\Big)\Big(\sum_{j=1,j\ne i_0}^{n}a_{i_0j}+\sum_{j=1}^{m}\rho_{i_0j}\Big)}
\end{aligned}
$$

for arbitrary $t \ge \widetilde{T}_0$. This is contrary to $\lim_{t\to+\infty} x_{i_0}(M_{i_0}(t)) = +\infty$ and terminates the proof of Lemma 2.2. □

Lemma 2.3. *If*

$$
\delta_i - a_{ii} < \sum_{j=1}^{m}\rho_{ij} \quad \textit{and} \quad \lim_{t\to+\infty}[\sigma_{ij}(t)-\tau_{ij}(t)] = 0 \quad \textit{for all } i\in\Lambda,\ j\in\Omega. \tag{2.9}
$$

Then $\min_{i\in\Lambda}\liminf_{t\to+\infty} x_i(t) > 0$.

Proof. To derive a contradiction, we assume that $l = \min_{i\in\Lambda}\liminf_{t\to+\infty} x_i(t) = 0$. Define

$$
v(t) = \max\Big\{\xi : \xi\le t \;\Big|\; \text{there exists } \widehat{i}\in\Lambda \text{ obeying } x_{\widehat{i}}(\xi) = \min_{i\in\Lambda}\Big\{\min_{t_0\le s\le t} x_i(s)\Big\}\Big\}.
$$

Then, $\lim_{t\to+\infty} v(t) = +\infty$. Meanwhile, for a strictly monotone increasing infinite sequence $\{t_p\}_{p\ge1}$, one can find $\widehat{i}\in\Lambda$ and a subsequence $\{t_{p_k}\}_{k\ge1}\subseteq\{t_p\}_{p\ge1}$ such that

$$
x_{\widehat{i}}(v(t_{p_k})) = \min_{t_0\le s\le t_{p_k}} x_{\widehat{i}}(s) = \min_{i\in\Lambda}\Big\{\min_{t_0\le s\le t_{p_k}} x_i(s)\Big\} \quad \text{and} \quad \lim_{k\to+\infty} x_{\widehat{i}}(v(t_{p_k})) = 0. \tag{2.10}
$$

In view of (1.3), (2.1), (2.9) and (2.10), we acquire

$$\begin{aligned}
0 &\geq x'_{\widehat{i}}(v(t_{p_k}))\\
&= \beta_{\widehat{i}}(v(t_{p_k}))\Big[-\delta_{\widehat{i}}x_{\widehat{i}}(v(t_{p_k})) + \sum_{j=1}^{n} a_{\widehat{i}j}x_j(v(t_{p_k}))\\
&\qquad + \sum_{j=1}^{m} \rho_{\widehat{i}j}x_{\widehat{i}}(v(t_{p_k}) - \tau_{\widehat{i}j}(v(t_{p_k})))e^{-h_{\widehat{i}j}x_{\widehat{i}}(v(t_{p_k})-\sigma_{\widehat{i}j}(v(t_{p_k})))}\Big]\\
&\geq \beta_{\widehat{i}}(v(t_{p_k}))\Big[-(\delta_{\widehat{i}} - a_{\widehat{i}\widehat{i}})x_{\widehat{i}}(v(t_{p_k})) + x_{\widehat{i}}(v(t_{p_k}))\sum_{j=1, j\neq\widehat{i}}^{n} a_{\widehat{i}j}\\
&\qquad + \sum_{j=1}^{m} \rho_{\widehat{i}j}x_{\widehat{i}}(v(t_{p_k}) - \tau_{\widehat{i}j}(v(t_{p_k})))e^{-h_{\widehat{i}j}x_{\widehat{i}}(v(t_{p_k})-\sigma_{\widehat{i}j}(v(t_{p_k})))}\Big]\\
&\geq \beta_{\widehat{i}}(v(t_{p_k}))\Big[-(\delta_{\widehat{i}} - a_{\widehat{i}\widehat{i}})x_{\widehat{i}}(v(t_{p_k}))\\
&\qquad + \sum_{j=1}^{m} \rho_{\widehat{i}j}x_{\widehat{i}}(v(t_{p_k}) - \tau_{\widehat{i}j}(v(t_{p_k})))e^{-h_{\widehat{i}j}x_{\widehat{i}}(v(t_{p_k})-\sigma_{\widehat{i}j}(v(t_{p_k})))}\Big]\\
&\geq \beta_{\widehat{i}}(v(t_{p_k}))\Big[-(\delta_{\widehat{i}} - a_{\widehat{i}\widehat{i}})x_{\widehat{i}}(v(t_{p_k}))\\
&\qquad + \sum_{j=1}^{m} \rho_{\widehat{i}j}x_{\widehat{i}}(v(t_{p_k}) - \tau_{\widehat{i}j}(v(t_{p_k})))e^{-h_{\widehat{i}j}\sup_{s\in[-r_{\widehat{i}},+\infty)} x_{\widehat{i}}(s)}\Big]
\end{aligned}$$

for all $v(t_{p_k}) > t_0 + r$. Consequently,

$$\begin{aligned}
(\delta_{\widehat{i}} - a_{\widehat{i}\widehat{i}}) &\geq \sum_{j=1}^{m} \rho_{\widehat{i}j}\frac{x_{\widehat{i}}(v(t_{p_k}) - \tau_{\widehat{i}j}(v(t_{p_k})))}{x_{\widehat{i}}(v(t_{p_k}))}e^{-h_{\widehat{i}j}x_{\widehat{i}}(v(t_{p_k})-\sigma_{\widehat{i}j}(v(t_{p_k})))}\\
&\geq \sum_{j=1}^{m} \rho_{\widehat{i}j}e^{-h_{\widehat{i}j}x_{\widehat{i}}(v(t_{p_k})-\sigma_{\widehat{i}j}(v(t_{p_k})))} \quad \text{for all } v(t_{p_k}) > t_0 + r,
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&(\delta_{\widehat{i}} - a_{\widehat{i}\widehat{i}})x_{\widehat{i}}(v(t_{p_k}))\\
&\geq \sum_{j=1}^{m} \rho_{\widehat{i}j}x_{\widehat{i}}(v(t_{p_k}) - \tau_{\widehat{i}j}(v(t_{p_k})))e^{-h_{\widehat{i}j}\sup_{s\in[-r_{\widehat{i}},+\infty)} x_{\widehat{i}}(s)} \quad \text{for all } v(t_{p_k}) > t_0 + r.
\end{aligned} \tag{2.12}$$

By taking limits of (2.12), we gain

$$\lim_{k\to+\infty} x_{\widehat{i}}(v(t_{p_k}) - \tau_{\widehat{i}j}(v(t_{p_k}))) = \lim_{k\to+\infty} x_{\widehat{i}}(v(t_{p_k}) - \sigma_{\widehat{i}j}(v(t_{p_k}))) = \lim_{k\to+\infty} x_{\widehat{i}}(v(t_{p_k})) = 0,$$

which, together with (2.9) and (2.11), suggests that $\delta_{\widehat{i}} - a_{\widehat{i}\widehat{i}} \geq \sum_{j=1}^{m} \rho_{\widehat{i}j}$. This yields a contradiction and finishes the proof. $\square$

Theorem 2.4. *Let* (2.9) *be satisfied, and define*

$$M^* = \max_{i\in\Lambda} \frac{\sum_{j=1}^m \frac{\rho_{ij}}{h_{ij}}\frac{1}{e}}{\delta_i}, \quad \chi_i = \sup\left\{\chi \;\middle|\; \chi \in (0,+\infty), \delta_i - a_{ii} < \sum_{j=1}^m \rho_{ij} e^{-\chi}\right\}, \tag{2.13}$$

$$k_{ij} \in (0,1] \quad \text{with} \quad k_{ij}e^{-k_{ij}} = h_{ij}M^* e^{-h_{ij}M^*}, \tag{2.14}$$

and

$$k^{\min} = \min\left\{\min_{i\in\Lambda,\, j\in\Omega} \frac{k_{ij}}{h_{ij}}, \min_{i\in\Lambda,\, j\in\Omega} \frac{\chi_i}{h_{ij}}\right\}, \tag{2.15}$$

where $i \in \Lambda$, $j \in \Omega$. *Then*

$$k^{\min} \le \min_{i\in\Lambda} \liminf_{t\to+\infty} x_i(t) \le \max_{i\in\Lambda} \limsup_{t\to+\infty} x_i(t) \le M^*. \tag{2.16}$$

Proof. First, one can choose $i_{**} \in \Lambda$ satisfying

$$L^{\sup} = \limsup_{t\to+\infty} x_{i_{**}}(t) = \max_{i\in\Lambda} \limsup_{t\to+\infty} x_i(t).$$

Owing to the fluctuation lemma [27, Lemma A.1], it is an easy matter to find a sequence $\{t_k^*\}_{k\ge1}$ obeying

$$\lim_{k\to+\infty} t_k^* = +\infty, \quad \lim_{k\to+\infty} x_{i_{**}}(t_k^*) = \limsup_{t\to+\infty} x_{i_{**}}(t) = L^{\sup}, \quad \lim_{k\to+\infty} x'_{i_{**}}(t_k^*) = 0. \tag{2.17}$$

For simplicity but without loss of generality, we also assume that $\lim_{k\to+\infty} x_l(t_k^*)$ ($l \in \Lambda \setminus \{i_{**}\}$), $\lim_{k\to+\infty} \beta_{i_{**}}(t_k^*)$, $\lim_{k\to+\infty} x_{i_{**}}(t_k^* - \tau_{i_{**}j}(t_k^*))$ and $\lim_{k\to+\infty} x_{i_{**}}(t_k^* - \sigma_{i_{**}j}(t_k^*))$ exist for all $j \in \Omega$. Due to (1.3), (2.1) and (2.17), we gain

$$\begin{aligned}
0 &= \lim_{k\to+\infty} x'_{i_{**}}(t_k^*) \\
&= \lim_{k\to+\infty} \beta_{i_{**}}(t_k^*)\Big[-\delta_{i_{**}} \lim_{k\to+\infty} x_{i_{**}}(t_k^*) + \sum_{j=1}^n a_{i_{**}j} \lim_{k\to+\infty} x_j(t_k^*) \\
&\qquad + \sum_{j=1}^m \frac{\rho_{i_{**}j}}{h_{i_{**}j}} h_{i_{**}j} \lim_{k\to+\infty} x_{i_{**}}(t_k^* - \tau_{i_{**}j}(t_k^*)) e^{-h_{i_{**}j}\lim_{k\to+\infty} x_{i_{**}}(t_k^* - \sigma_{i_{**}j}(t_k^*))}\Big] \\
&\le \lim_{k\to+\infty} \beta_{i_{**}}(t_k^*)\Big[-\delta_{i_{**}} L^{\sup} + \sum_{j=1}^n a_{i_{**}j} L^{\sup} + \sum_{j=1}^m \frac{\rho_{i_{**}j}}{h_{i_{**}j}}\frac{1}{e}\Big] \\
&\le \lim_{k\to+\infty} \beta_{i_{**}}(t_k^*)\Big[-\delta_{i_{**}} L^{\sup} + \sum_{j=1}^m \frac{\rho_{i_{**}j}}{h_{i_{**}j}}\frac{1}{e}\Big],
\end{aligned}$$

which yields

$$L^{\sup} \le \frac{\sum_{j=1}^m \frac{\rho_{i_{**}j}}{h_{i_{**}j}}\frac{1}{e}}{\delta_{i_{**}}} \le \max_{i\in\Lambda} \frac{\sum_{j=1}^m \frac{\rho_{ij}}{h_{ij}}\frac{1}{e}}{\delta_i} = M^*.$$

Next, define $l_{**} \in \Lambda$ with

$$L^{\inf} = \liminf_{t\to+\infty} x_{l_{**}}(t) = \min_{i\in\Lambda} \liminf_{t\to+\infty} x_i(t).$$

Again from the fluctuation lemma [27, Lemma A.1], one can find a sequence $\{t_k^{**}\}_{k\geq 1}$ obeying

$$\lim_{k\to+\infty} t_k^{**} = +\infty, \quad \lim_{k\to+\infty} x_{l_{**}}(t_k^{**}) = \liminf_{t\to+\infty} x_{l_{**}}(t) = L^{\inf}, \quad \lim_{k\to+\infty} x'_{l_{**}}(t_k^{**}) = 0. \tag{2.18}$$

In particular, $\lim_{k\to+\infty} x_l(t_k^{**})$ $(l \in \Lambda \setminus \{l_{**}\})$, $\lim_{k\to+\infty} \beta_{l_{**}}(t_k^{**})$, $\lim_{k\to+\infty} x_{l_{**}}(t_k^{**} - \tau_{l_{**}j}(t_k^{**}))$ and $\lim_{k\to+\infty} x_{l_{**}}(t_k^{**} - \sigma_{l_{**}j}(t_k^{**}))$ exist for all $j \in \Omega$. Now, we claim that

$$L^{\inf} \geq k^{\min}.$$

Otherwise,

$$L^{\inf} < k^{\min}, \quad h_{ij}L^{\inf} < k_{ij} \quad \text{and} \quad \min_{U\in[h_{ij}L^{\inf}, h_{ij}L^{\sup}]} Ue^{-U} = h_{ij}L^{\inf}e^{-h_{ij}L^{\inf}}.$$

Clearly, (2.13)–(2.15) lead to

$$h_{ij}L^{\inf} \leq h_{ij} \lim_{k\to+\infty} x_{l_{**}}(t_k^{**} - \tau_{l_{**}j}(t_k^{**})) = h_{ij} \lim_{k\to+\infty} x_{l_{**}}(t_k^{**} - \sigma_{l_{**}j}(t_k^{**})) \leq h_{ij}L^{\sup}$$

and

$$h_{ij} \lim_{k\to+\infty} x_{l_{**}}(t_k^{**} - \tau_{l_{**}j}(t_k^{**}))e^{-h_{ij}\lim_{k\to+\infty} x_{l_{**}}(t_k^{**}-\tau_{l_{**}j}(t_k^{**}))} \geq h_{ij}L^{\inf}e^{-h_{ij}L^{\inf}}. \tag{2.19}$$

Due to (1.3), (2.1), (2.18) and (2.19), we gain

$$\begin{aligned}
0 &= \lim_{k\to+\infty} x'_{l_{**}}(t_k^{**}) \\
&= \lim_{k\to+\infty} \beta_{l_{**}}(t_k^{**}) \Big[-\delta_{l_{**}} \lim_{k\to+\infty} x_{l_{**}}(t_k^{**}) + \sum_{j=1}^{n} a_{l_{**}j} \lim_{k\to+\infty} x_j(t_k^{**}) \\
&\qquad + \sum_{j=1}^{m} \frac{\rho_{l_{**}j}}{h_{l_{**}j}} h_{l_{**}j} \lim_{k\to+\infty} x_{l_{**}}(t_k^{**} - \tau_{l_{**}j}(t_k^{**}))e^{-h_{l_{**}j}\lim_{k\to+\infty} x_{l_{**}}(t_k^{**}-\sigma_{l_{**}j}(t_k^{**}))} \Big] \\
&= \lim_{k\to+\infty} \beta_{l_{**}}(t_k^{**}) \Big[-\delta_{l_{**}} \lim_{k\to+\infty} x_{l_{**}}(t_k^{**}) + \sum_{j=1}^{n} a_{l_{**}j} \lim_{k\to+\infty} x_j(t_k^{**}) \\
&\qquad + \sum_{j=1}^{m} \frac{\rho_{l_{**}j}}{h_{l_{**}j}} h_{l_{**}j} \lim_{k\to+\infty} x_{l_{**}}(t_k^{**} - \tau_{l_{**}j}(t_k^{**}))e^{-h_{l_{**}j}\lim_{k\to+\infty} x_{l_{**}}(t_k^{**}-\tau_{l_{**}j}(t_k^{**}))} \Big] \\
&\geq \lim_{k\to+\infty} \beta_{l_{**}}(t_k^{**}) \Big[-(\delta_{l_{**}} - a_{l_{**}l_{**}})L^{\inf} \\
&\qquad + \sum_{j=1, j\neq l_{**}}^{n} a_{l_{**}j}L^{\inf} + \sum_{j=1}^{m} \frac{\rho_{l_{**}j}}{h_{l_{**}j}} h_{l_{**}j}L^{\inf}e^{-h_{l_{**}j}L^{\inf}} \Big]
\end{aligned}$$

$$\begin{aligned}
&\geq \lim_{k\to+\infty} \beta_{l_{**}}(t_k^{**})L^{\inf}\Big[-(\delta_{l_{**}} - a_{l_{**}l_{**}}) + \sum_{j=1}^{m} \rho_{l_{**}j} e^{-h_{l_{**}j}L^{\inf}}\Big] \\
&> \lim_{k\to+\infty} \beta_{l_{**}}(t_k^{**})L^{\inf}\Big[-(\delta_{l_{**}} - a_{l_{**}l_{**}}) + \sum_{j=1}^{m} \rho_{l_{**}j} e^{-h_{l_{**}j}k^{\min}}\Big] \\
&\geq \lim_{k\to+\infty} \beta_{l_{**}}(t_k^{**})L^{\inf}\Big[-(\delta_{l_{**}} - a_{l_{**}l_{**}}) + \sum_{j=1}^{m} \rho_{l_{**}j} e^{-\chi_{l_{**}}}\Big] \\
&= 0,
\end{aligned}$$

which yields a contradiction and proves the above claim. Hence, (2.16) holds, which verifies Theorem 2.4. □

Remark 2.5. Theorem 2.4 indicates that, under the assumption (2.9), system (1.3) has uniform persistence, and then $(0, 0, \ldots, 0)$ is unstable.

3. Global attractivity analysis

In this section, the strategies of the proof of the present paper follow from those used in some earlier papers [8, 19, 28], but some modifications are nontrivial.

First, we establish the attractivity conditions for non-oscillatory solutions of model (1.3).

Proposition 3.1. *Provided that* (2.9) *is satisfied, and*

$$\liminf_{t\to+\infty} x_i(t) \geq N_i^* \quad \textit{for all } i \in \Lambda \tag{3.1}$$

hold. Moreover, assume that

$$\sum_{j=1}^{n} a_{ij} N_j^* = 0 \quad \textit{for all } i \in \Lambda. \tag{3.2}$$

Then $\lim_{t\to+\infty} x_i(t) = N_i^*$ *for arbitrary* $i \in \Lambda$.

Proof. Denote $z_i(t) = x_i(t) - N_i^*$ $(i \in \Lambda)$, it can be deduced from Lemma 2.2 that

$$0 \leq \limsup_{t\to+\infty} z_i(t) < +\infty \quad \text{for all } i \in \Lambda.$$

Let $i^* \in \Lambda$ be such an index as $\limsup_{t\to+\infty} z_{i^*}(t) = \max_{i\in\Lambda} \limsup_{t\to+\infty} z_i(t)$. In order to verify Proposition 3.1, it suffices to state that

$$\limsup_{t\to+\infty} z_{i^*}(t) = 0.$$

Otherwise, $\limsup_{t\to+\infty} z_{i^*}(t) > 0$. Owing to the fluctuation lemma [27, Lemma A.1], it is an easy matter to find a sequence $\{t_k\}_{k\geq 1}$ obeying

$$\lim_{k\to+\infty} t_k = +\infty, \quad \lim_{k\to+\infty} z_{i^*}(t_k) = \limsup_{t\to+\infty} z_{i^*}(t), \quad \lim_{k\to+\infty} z'_{i^*}(t_k) = 0. \tag{3.3}$$

Due to (1.3) and (2.3), we gain

$$\begin{aligned} z'_{i^*}(t_k) = \beta_{i^*}(t_k)\Bigg[&- \delta_{i^*} x_{i^*}(t_k) + \sum_{j=1}^n a_{i^*j} z_j(t_k) + \sum_{j=1}^n a_{i^*j} N_j^* \\ &+ \sum_{j=1}^m \rho_{i^*j} x(t_k - \tau_{i^*j}(t_k)) e^{-h_{i^*j} x_{i^*}(t_k - \sigma_{i^*j}(t_k))}\Bigg]. \end{aligned} \tag{3.4}$$

Because $\beta_{i^*}(t)$, $x_{i^*}(t-\tau_{i^*j}(t))$ and $x_{i^*}(t-\sigma_{i^*j}(t))$ are bounded on $[t_0, +\infty)$, we can select a subsequence of $\{t_k\}$ (for convenience of exposition, we still label by $\{t_k\}$) satisfying that $\lim_{k\to+\infty} z_l(t_k)$ $(l \in \Omega \setminus \{i^*\})$, $\lim_{k\to+\infty} \beta_{i^*}(t_k)$, $\lim_{k\to+\infty} x_{i^*}(t_k - \tau_{i^*j}(t_k))$ and $\lim_{k\to+\infty} x_{i^*}(t_k - \sigma_{i^*j}(t_k))$ exist for all $j \in \Omega$. Moreover, (2.9) and (3.1) yield

$$N_{i^*}^* \leq \lim_{k\to+\infty} x_{i^*}(t_k - \tau_{i^*j}(t_k)) = \lim_{k\to+\infty} x_{i^*}(t_k - \sigma_{i^*j}(t_k)) \leq N_{i^*}^* + \lim_{k\to+\infty} z_{i^*}(t_k) \tag{3.5}$$

for all $j \in \Omega$.

Hereafter, with the help of (3.3), we regard two cases as follows.

Case 1. If $\lim_{k\to+\infty} x_{i^*}(t_k - \tau_{i^*j}(t_k)) = \lim_{k\to+\infty} x_{i^*}(t_k - \sigma_{i^*j}(t_k)) = N_{i^*}^*$ for all $j \in \Omega$, by taking limits, (2.1)–(2.3) and (3.3)–(3.5) give us

$$\begin{aligned} 0 &= \lim_{k\to+\infty} z'_{i^*}(t_k) \\ &= \lim_{k\to+\infty} \beta_{i^*}(t_k)\Bigg[- \delta_{i^*} \lim_{k\to+\infty} x_{i^*}(t_k) + \sum_{j=1}^n a_{i^*j} \lim_{k\to+\infty} z_j(t_k) + \sum_{j=1}^n a_{i^*j} N_j^* \\ &\qquad + \sum_{j=1}^m \rho_{i^*j} N_{i^*}^* e^{-h_{i^*j} N_{i^*}^*}\Bigg] \\ &\leq \lim_{k\to+\infty} \beta_{i^*}(t_k)\Bigg[- \delta_{i^*}\left(\limsup_{t\to+\infty} z_{i^*}(t) + N_{i^*}^*\right) + \limsup_{t\to+\infty} z_{i^*}(t) \sum_{j=1}^n a_{i^*j} \\ &\qquad + \sum_{j=1}^n a_{i^*j} N_j^* + \sum_{j=1}^m \rho_{i^*j} N_{i^*}^* e^{-h_{i^*j} N_{i^*}^*}\Bigg] \\ &< \lim_{k\to+\infty} \beta_{i^*}(t_k)\Bigg[- \delta_{i^*} N_{i^*}^* + \sum_{j=1}^n a_{i^*j} N_j^* + \sum_{j=1}^m \rho_{i^*j} N_{i^*}^* e^{-h_{i^*j} N_{i^*}^*}\Bigg] \\ &= 0, \end{aligned}$$

which leads to a contradiction, and suggests that $\limsup_{t\to+\infty} z_{i^*}(t) = 0$.

Case 2. If there are some $j \in \Omega$ such that $N_{i^*}^* < \lim_{k\to+\infty} x_{i^*}(t_k - \tau_{i^*j}(t_k)) = \lim_{k\to+\infty} x_{i^*}(t_k - \sigma_{i^*j}(t_k))$, it follows from (2.1)–(2.3), (3.2), (3.4) and (3.5) that

$$\begin{aligned}
0 &= \lim_{k\to+\infty} z'_{i^*}(t_k) \\
&< \lim_{k\to+\infty} \beta_{i^*}(t_k)\Bigg[-\delta_{i^*} \lim_{k\to+\infty} x_{i^*}(t_k) + \sum_{j=1}^{n} a_{i^*j} \lim_{k\to+\infty} z_j(t_k) + \sum_{j=1}^{n} a_{i^*j} N_j^* \\
&\qquad + \sum_{j=1}^{m} \rho_{i^*j}\Big(\lim_{k\to+\infty} x_{i^*}(t_k - \tau_{i^*j}(t_k))\Big) e^{-h_{i^*j}N_{i^*}^*}\Bigg] \\
&\le \lim_{k\to+\infty} \beta_{i^*}(t_k)\Big(\limsup_{t\to+\infty} z_{i^*}(t) + N_{i^*}^*\Big)\Bigg[-\delta_{i^*} + \sum_{j=1}^{m} \rho_{i^*j} e^{-h_{i^*j}N_{i^*}^*}\Bigg] \\
&= 0,
\end{aligned}$$

which is also a contradiction and proves the above statement. This accomplishes the proof of Proposition 3.1. □

Proposition 3.2. *Assume that* (2.9) *and* (3.2) *hold, and*

$$\limsup_{t\to+\infty} x_i(t) \le N_i^* \quad \textit{for all } i \in \Lambda \tag{3.6}$$

is satisfied. Then, $\lim_{t\to+\infty} x_i(t) = N_i^*$ *for arbitrary* $i \in \Lambda$.

Proof. Denote $z_i(t) = x_i(t) - N_i^*$ $(i \in \Lambda)$, according to Lemma 2.1, one can see that

$$-\infty < \limsup_{t\to+\infty} z_i(t) \le 0 \quad \text{for all } i \in \Lambda.$$

Label $i^{**} \in \Lambda$ be such an index as $\liminf_{t\to+\infty} z_{i^{**}}(t) = \min_{i\in J} \liminf_{t\to+\infty} z_i(t)$. So as to confirm Proposition 3.2, we only need to show that

$$\liminf_{t\to+\infty} z_{i^{**}}(t) = 0.$$

Conversely, suppose that $\liminf_{t\to+\infty} z_{i^{**}}(t) < 0$. On account of the fluctuation lemma [27, Lemma A.1], there exists a sequence $\{t_k\}_{k\ge1}$ agreeing with

$$\lim_{k\to+\infty} t_k = +\infty, \quad \lim_{k\to+\infty} z_{i^{**}}(t_k) = \liminf_{t\to+\infty} z_{i^{**}}(t), \quad \lim_{k\to+\infty} z'_{i^{**}}(t_k) = 0. \tag{3.7}$$

It follows from (1.4), (1.5) and (2.3) that

$$\begin{aligned}
z'_{i^{**}}(t_k) = \beta_{i^{**}}(t_k)\Bigg[&-\delta_{i^{**}} x_{i^{**}}(t_k) + \sum_{j=1}^{n} a_{i^{**}j} z_j(t_k) + \sum_{j=1}^{n} a_{i^{**}j} N_j^* \\
&+ \sum_{j=1}^{m} \rho_{i^{**}j} x(t_k - \tau_{i^{**}j}(t_k)) e^{-h_{i^{**}j} x_{i^{**}}(t_k - \sigma_{i^{**}j}(t_k))}\Bigg].
\end{aligned} \tag{3.8}$$

Since $\beta_{i^{**}}(t)$, $x_{i^{**}}(t-\tau_{i^{**}j}(t))$ and $x_{i^{**}}(t-\sigma_{i^{**}j}(t))$ are bounded on $[t_0,+\infty)$, one can choose a subsequence of $\{t_k\}$ (still labeled by $\{t_k\}$) such that $\lim_{k\to+\infty} z_l(t_k)$ $(l\in\Omega\setminus\{i^{**}\})$, $\lim_{k\to+\infty}\beta_{i^{**}}(t_k)$, $\lim_{k\to+\infty} x_{i^{**}}(t_k-\tau_{i^{**}j}(t_k))$ and $\lim_{k\to+\infty} x_{i^{**}}(t_k-\sigma_{i^{**}j}(t_k))$ exist for all $j\in\Omega$. Moreover, (2.3) and (3.6) lead to

$$N_{i^{**}}^* + \lim_{k\to+\infty} z_{i^{**}}(t_k) \le \lim_{k\to+\infty} x_{i^{**}}(t_k-\tau_{i^{**}j}(t_k)) = \lim_{k\to+\infty} x_{i^{**}}(t_k-\sigma_{i^{**}j}(t_k)) \le N_{i^{**}}^*, \tag{3.9}$$

for all $j\in\Omega$, and one of the following cases must occur:

Case 1. If $\lim_{k\to+\infty} x_{i^{**}}(t_k-\tau_{i^{**}j}(t_k)) = \lim_{k\to+\infty} x_{i^{**}}(t_k-\sigma_{i^{**}j}(t_k)) = N_{i^{**}}^*$ for all $j\in\Omega$, by taking limits, (2.1)–(2.3) and (3.7)–(3.9) reveal that

$$\begin{aligned}
0 &= \lim_{k\to+\infty} z_{i^{**}}'(t_k)\\
&= \lim_{k\to+\infty}\beta_{i^{**}}(t_k)\Big[-\delta_{i^{**}}\lim_{k\to+\infty} x_{i^{**}}(t_k) + \sum_{j=1}^n a_{i^{**}j}\lim_{k\to+\infty} z_j(t_k) + \sum_{j=1}^n a_{i^{**}j}N_j^*\\
&\qquad\qquad + \sum_{j=1}^m \rho_{i^{**}j}N_{i^{**}}^* e^{-h_{i^{**}j}N_{i^{**}}^*}\Big]\\
&\ge \lim_{k\to+\infty}\beta_{i^{**}}(t_k)\Big[-\delta_{i^{**}}\Big(\liminf_{t\to+\infty} z_{i^{**}}(t) + N_{i^{**}}^*\Big) + \liminf_{t\to+\infty} z_{i^{**}}(t)\sum_{j=1}^n a_{i^{**}j}\\
&\qquad\qquad + \sum_{j=1}^n a_{i^{**}j}N_j^* + \sum_{j=1}^m \rho_{i^{**}j}N_{i^{**}}^* e^{-h_{i^{**}j}N_{i^{**}}^*}\Big]\\
&> \lim_{k\to+\infty}\beta_{i^*}(t_k)\Big[-\delta_{i^{**}}N_{i^{**}}^* + \sum_{j=1}^n a_{i^{**}j}N_j^* + \sum_{j=1}^m \rho_{i^{**}j}N_{i^{**}}^* e^{-h_{i^{**}j}N_{i^{**}}^*}\Big]\\
&= 0,
\end{aligned}$$

which is impossible and implies that $\liminf_{t\to+\infty} z_{i^{**}}(t) = 0$.

Case 2. If $\lim_{k\to+\infty} x_{i^{**}}(t_k-\tau_{i^{**}j}(t_k)) = \lim_{k\to+\infty} x_{i^{**}}(t_k-\sigma_{i^{**}j}(t_k)) < N_{i^{**}}^*$ holds for some $j\in\Omega$, it follows from (2.1)–(2.3), (3.2), (3.7) and (3.8) that

$$\begin{aligned}
0 &= \lim_{k\to+\infty} z_{i^{**}}'(t_k)\\
&> \lim_{k\to+\infty}\beta_{i^{**}}(t_k)\Big[-\delta_{i^{**}}\lim_{k\to+\infty} x_{i^{**}}(t_k) + \sum_{j=1}^n a_{i^{**}j}\lim_{k\to+\infty} z_j(t_k) + \sum_{j=1}^n a_{i^{**}j}N_j^*\\
&\qquad\qquad + \sum_{j=1}^m \rho_{i^{**}j}\Big(\lim_{k\to+\infty} x_{i^{**}}(t_k-\tau_{i^{**}j}(t_k))\Big)e^{-h_{i^{**}j}N_{i^{**}}^*}\Big]\\
&\ge \lim_{k\to+\infty}\beta_{i^{**}}(t_k)\Big(\liminf_{k\to+\infty} z_{i^{**}}(t) + N_{i^{**}}^*\Big)\Big[-\delta_{i^{**}} + \sum_{j=1}^m \rho_{i^{**}j}e^{-h_{i^{**}j}N_{i^{**}}^*}\Big]\\
&= 0,
\end{aligned}$$

leads to a clear contradiction and the above statement is proved. This completes the proof of Proposition 3.2. $\square$

Corollary 3.3. *Assume that* (2.9) *and* (3.2) *hold, and for each* $i \in \Lambda$, $x_i(t)$ *is eventually nonoscillatory about* N_i^*, *i.e., there exists* $T^* \geq t_0$ *agreeing with*

$$x_i(t) \geq N_i^* \quad (\text{or } x_i(t) \leq N_i^*) \quad \text{for all } t \geq T^* \text{ and } i \in \Lambda. \tag{3.10}$$

Then, $\lim_{t\to+\infty} x_i(t) = N_i^*$ *for all* $i \in \Lambda$.

Remark 3.4. It can be concluded from Corollary 3.3 that, under the assumptions (2.9), (3.2) and (3.10), the convergence criterion for non-oscillatory solutions of the system (1.3) is independent of the size of the delays, and all conclusions of Propositions 2.1 and 2.2 in [28] and the results of Theorem 4.1 in [19] are special ones of Corollary 3.3.

Theorem 3.5. *Let* (2.9) *and* (3.2) *be satisfied, and there exists* $H \geq t_0$ *such that for all* $i \in \Lambda$,

$$\frac{(e^{(\delta_i - a_{ii})\beta_i^H r_i^H} - 1)h\delta_i N_i^*}{\delta_i - a_{ii}} \leq 1 \tag{3.11}$$

and

$$0 < \frac{h\delta_i N_i^*(1 - e^{-r_i^H(\delta_i - a_{ii})\beta_i^H})}{\delta_i - \delta_i e(1 - e^{-r_i^H(\delta_i - a_{ii})\beta_i^H}) - a_{ii}e^{-r_i^H(\delta_i - a_{ii})\beta_i^H}} \leq 1 \tag{3.12}$$

hold. Then $\lim_{t\to+\infty} x_i(t) = N_i^*$ *for all* $i \in \Lambda$.

Proof. Label

$$\zeta_i(t) = h(x_i(t) - N_i^*), \quad i \in \Lambda.$$

In light of (1.3), we drive

$$\begin{aligned}
&\zeta_i'(t) + h\delta_i\beta_i(t)N_i^* + \delta_i\beta_i(t)\zeta_i(t) \\
&= \beta_i(t)\sum_{j=1}^n a_{ij}(t)\zeta_j(t) + h\beta_i(t)\sum_{j=1}^n a_{ij}N_j^* \\
&\quad + h\beta_i(t)\sum_{j=1}^m \rho_{ij}\left[\frac{\zeta_i(t-\tau_{ij}(t))}{h} + N_i^*\right]e^{-\frac{h_{ij}\zeta_i(t-\sigma_{ij}(t))}{h} - h_{ij}N_i^*},
\end{aligned}$$

and

$$\begin{aligned}
&(\zeta_i(t)e^{\int_{t_0}^t(\delta_i - a_{ii})\beta_i(v)\,dv})' \\
&= \Bigg[\sum_{j=1, j\neq i}^n a_{ij}\beta_i(t)\zeta_j(t) + h\beta_i(t)\sum_{j=1}^n a_{ij}N_j^* \\
&\quad + h\beta_i(t)\sum_{j=1}^m \rho_{ij}\left(\frac{\zeta_i(t-\tau_{ij}(t))}{h} + N_i^*\right)e^{-\frac{h_{ij}\zeta_i(t-\sigma_{ij}(t))}{h} - h_{ij}N_i^*} - h\beta_i(t)\delta_i N_i^*\Bigg] \\
&\quad \times e^{\int_{t_0}^t(\delta_i - a_{ii})\beta_i(v)\,dv}, \quad t \geq t_0,\ i \in \Lambda.
\end{aligned} \tag{3.13}$$

To accomplish the verification, we shall reveal that

$$\min_{i\in\Lambda}\liminf_{t\to+\infty}\zeta_i(t)=\max_{i\in\Lambda}\limsup_{t\to+\infty}\zeta_i(t)=0.$$

Owing to Corollary 3.3, it suffices to treat the case that for each $T^*>t_0$, there are $t^*,t^{**}\in(T^*,+\infty)$ agreeing with

$$\min_{i\in\Lambda}\zeta_i(t^*)<0 \quad \text{and} \quad \max_{i\in\Lambda}\zeta_i(t^{**})>0. \tag{3.14}$$

Denote

$$\mu=\limsup_{t\to+\infty}\zeta_{i_1}(t)=\max_{i\in\Lambda}\limsup_{t\to+\infty}\zeta_i(t), \quad \lambda=\liminf_{t\to+\infty}\zeta_{i_2}(t)=\min_{i\in\Lambda}\liminf_{t\to+\infty}\zeta_i(t). \tag{3.15}$$

In view of (3.14), we acquire

$$\lambda\le 0\le\mu.$$

Now, our problem reduces to proof that $\lambda=\mu=0$. If the conclusion was false, then $\mu>0$ or $\lambda<0$.

We only consider the case of $\mu>0$. ($\lambda<0$ can be handled similarly).

If $\lambda=0$, i.e., $\lambda=\min_{i\in\Lambda}\liminf_{t\to+\infty}\zeta_i(t)=0$. It follows from Proposition 3.1 that $\mu=\limsup_{t\to+\infty}\zeta_{i_1}(t)=0$, and $\lim_{t\to+\infty}x_i(t)=N_i^*$ for all $i\in\Lambda$.

When $\mu>0$ and $\lambda<0$, with the aid of the fluctuation lemma [27, Lemma A.1], one can pick two strictly monotone increasing infinite sequences $\{l_q\}_{q\ge1}$, $\{s_q\}_{q\ge1}$ obeying that, for all $q\in\mathbb{N}^+$,

$$l_q>\widetilde{t}_0+r, \quad \zeta_{i_1}(l_q)>0, \;\; l_q\to+\infty, \quad \zeta_{i_1}(l_q)\to\mu, \;\; \zeta'_{i_1}(l_q)\to0 \quad \text{as } q\to+\infty, \tag{3.16}$$

and

$$s_q>\widetilde{t}_0+r, \quad \zeta_{i_2}(s_q)<0, \;\; s_q\to+\infty, \quad \zeta_{i_2}(s_q)\to\lambda, \;\; \zeta'_{i_2}(s_q)\to0 \quad \text{as } q\to+\infty.$$

Since a bounded sequence has a convergent subsequence, it can be assumed that for all $j\in\Omega$,

$$\lim_{q\to+\infty}\beta_{i_1}(l_q)=\beta^*_{i_1}, \quad \lim_{q\to+\infty}\zeta_{i_1}(l_q-\tau_{i_1j}(l_q))=\zeta^j_{i_1}, \quad \lim_{q\to+\infty}\zeta_i(l_q)=\zeta^l_i, \quad i\in\Lambda\setminus\{i_1\}, \tag{3.17}$$

and

$$\lim_{q\to+\infty}\beta_{i_2}(s_q)=\beta^{**}_{i_2}, \quad \lim_{q\to+\infty}\zeta_{i_2}(s_q-\tau_{i_2j}(s_q))=\zeta^j_{i_2}, \quad \lim_{q\to+\infty}\zeta_i(s_q)=\zeta^s_i, \quad i\in\Lambda\setminus\{i_2\}.$$

To drive a contradiction, the proof falls naturally into three parts.

First, we state that there is an $H_1 > 0$ such that, for any $q \geq H_1$, there exists $L_q \in [l_q - r_{i_1}, l_q)$ obeying

$$\zeta_{i_1}(L_q) = 0, \quad \text{and} \quad \zeta_{i_1}(t) > 0 \quad \text{for all } t \in (L_q, l_q). \tag{3.18}$$

Otherwise, we can choose a subsequence of $\{l_q\}$ (do not relabel) such that

$$\zeta_{i_1}(t) > 0 \quad \text{for all } t \in [l_q - r_{i_1}, l_q),\ q = 1, 2, \ldots.$$

Subsequently,

$$0 \leq \lim_{q \to +\infty} \zeta_{i_1}(l_q - \tau_{i_1 j}(l_q)) \leq \mu \quad \text{for all } j \in \Omega,$$

and

$$\begin{aligned}
\zeta'_{i_1}(l_q) &= \beta_{i_1}(l_q) \sum_{j=1}^{n} a_{i_1 j} \zeta_j(l_q) + h\beta_{i_1}(l_q) \sum_{j=1}^{n} a_{i_1 j} N_j^* \\
&\quad + h\beta_{i_1}(l_q) \sum_{j=1}^{m} \rho_{i_1 j} \left[\frac{\zeta_{i_1}(l_q - \tau_{i_1 j}(l_q))}{h} + N_{i_1}^* \right] e^{-\frac{h_{i_1 j} \zeta_{i_1}(l_q - \sigma_{i_1 j}(l_q))}{h} - h_{i_1 j} N_{i_1}^*} \\
&\quad - h\delta_{i_1} \beta_{i_1}(l_q) N_{i_1}^* - \delta_{i_1} \beta_{i_1}(l_q) \zeta_{i_1}(l_q) \\
&< \beta_{i_1}(l_q) \sum_{j=1}^{n} a_{i_1 j} \zeta_j(l_q) + h\beta_{i_1}(l_q) \sum_{j=1}^{n} a_{i_1 j} N_j^* \\
&\quad + h\beta_{i_1}(l_q) \sum_{j=1}^{m} \rho_{i_1 j} \left[\frac{\zeta_{i_1}(l_q - \tau_{i_1 j}(l_q))}{h} + N_{i_1}^* \right] e^{-h_{i_1 j} N_{i_1}^*} \\
&\quad - h\delta_{i_1} \beta_{i_1}(l_q) N_{i_1}^* - \delta_{i_1} \beta_{i_1}(l_q) \zeta_{i_1}(l_q).
\end{aligned} \tag{3.19}$$

In what follows, we assert that

$$\lim_{q \to +\infty} \zeta_{i_1}(l_q - \tau_{i_1 j}(l_q)) = \lim_{q \to +\infty} \zeta_{i_1}(l_q - \sigma_{i_1 j}(l_q)) = \zeta_{i_1}^{j} = \mu \quad \text{for all } j \in \Omega. \tag{3.20}$$

If the assertion would not hold, then there exists $\widehat{j} \in \Omega$ such that

$$\lim_{q \to +\infty} \zeta_{i_1}(l_q - \tau_{i_1 \widehat{j}}(l_q)) = \lim_{q \to +\infty} \zeta_{i_1}(l_q - \sigma_{i_1 \widehat{j}}(l_q)) = \zeta_{i_1}^{\widehat{j}} < \mu,$$

which, together with (2.9), (3.2), (3.15), (3.16), (3.17) and (3.19), leads to

$$\begin{aligned}
0 &\leq a_{i_1 i_1} \beta_{i_1}^* \lim_{q \to +\infty} \zeta_{i_1}(l_q) + \beta_{i_1}^* \sum_{j=1, j \neq i_1}^{n} a_{i_1 j} \lim_{q \to +\infty} \zeta_j(l_q) + h\beta_{i_1}^* \sum_{j=1}^{n} a_{i_1 j} N_j^* \\
&\quad + h\beta_{i_1}^* \sum_{j=1}^{m} \rho_{i_1 j} \left[\frac{\lim_{q \to +\infty} \zeta_{i_1}(l_q - \tau_{i_1 j}(l_q))}{h} + N_{i_1}^* \right] e^{-h_{i_1 j} N_{i_1}^*} \\
&\quad - h\delta_{i_1} \beta_{i_1}^* N_{i_1}^* - \delta_{i_1} \beta_{i_1}^* \lim_{q \to +\infty} \zeta_{i_1}(l_q)
\end{aligned}$$

$$
\begin{aligned}
&< h\beta_{i_1}^* \sum_{j=1}^{m} \rho_{i_1 j} \left(N_{i_1}^* + \frac{\mu}{h}\right) e^{-h_{i_1 j} N_{i_1}^*} - h\beta_{i_1}^* \delta_{i_1} \left(N_{i_1}^* + \frac{\mu}{h}\right) \\
&= 0,
\end{aligned}
$$

which derives a contradiction and verifies (3.20). Furthermore, (2.3), (3.2), (3.17) and (3.20) yield

$$
\begin{aligned}
0 &= a_{i_1 i_1} \beta_{i_1}^* \lim_{q\to+\infty} \zeta_{i_1}(l_q) + \beta_{i_1}^* \sum_{j=1, j\neq i_1}^{n} a_{i_1 j} \lim_{q\to+\infty} \zeta_j(l_q) + h\beta_{i_1}^* \sum_{j=1}^{n} a_{i_1 j} N_j^* \\
&\quad + h\beta_{i_1}^* \sum_{j=1}^{m} \rho_{i_1 j} \left(N_{i_1}^* + \frac{\mu}{h}\right) e^{-h_{i_1 j}\left(N_{i_1}^* + \frac{\mu}{h}\right)} \\
&\quad - h\delta_{i_1} \beta_{i_1}^* N_{i_1}^* - \delta_{i_1} \beta_{i_1}^* \lim_{q\to+\infty} \zeta_{i_1}(l_q) \\
&< h\beta_{i_1}^* \sum_{j=1}^{m} \rho_{i_1 j} \left(N_{i_1}^* + \frac{\mu}{h}\right) e^{-h_{i_1 j} N_{i_1}^*} - h\beta_{i_1}^* \delta_{i_1} \left(N_{i_1}^* + \frac{\mu}{h}\right) \\
&= 0,
\end{aligned}
$$

which is a clear contradiction and proves the assertion (3.18).

Likewise, it is not difficult to find $H_1^* > 0$ such that for any $q \geq H_1^*$, there exists $S_q \in [s_q - r_{i_2}, s_q)$ agreeing with

$$
\zeta_{i_2}(S_q) = 0 \quad \text{and} \quad \zeta_{i_2}(t) < 0 \quad \text{for all } t \in (S_q, s_q). \tag{3.21}
$$

Secondly, we certificate that

$$
e^{-\mu} - 1 \leq \lambda \leq 0 \leq \mu \leq e^{-\lambda} - 1. \tag{3.22}
$$

For any $\varepsilon > 0$, (3.15) indicates that one can select a positive integer $q^* > H_1 + H_1^*$ obeying

$$
\lambda - \varepsilon < \zeta_i(t) < \mu + \varepsilon \quad \text{for arbitrary } t > \min\{l_{q^*}, s_{q^*}\} - 2r \text{ and } i \in \Lambda. \tag{3.23}
$$

In view of (1.3), (2.3), (3.2), (3.13), (3.21) and (3.23), we acquire

$$
\begin{aligned}
&\zeta_{i_2}(s_q) e^{\int_{t_0}^{s_q} (\delta_{i_2} - a_{i_2 i_2}) \beta_{i_2}(v)\, dv} \\
&= -h\delta_{i_2} N_{i_2}^* \frac{e^{\int_{t_0}^{s_q} (\delta_{i_2} - a_{i_2 i_2}) \beta_{i_2}(v)\, dv} - e^{\int_{t_0}^{S_q} (\delta_{i_2} - a_{i_2 i_2}) \beta_{i_2}(v)\, dv}}{\delta_{i_2} - a_{i_2 i_2}} \\
&\quad + \sum_{j=1, j\neq i_2}^{n} a_{i_2 j} \int_{S_q}^{s_q} \zeta_j(t) \beta_{i_2}(t) e^{\int_{t_0}^{t} (\delta_{i_2} - a_{i_2 i_2}) \beta_{i_2}(v)\, dv}\, dt \\
&\quad + h \sum_{j=1}^{m} \rho_{i_2 j} \int_{S_q}^{s_q} \left[N_{i_2}^* + \frac{\zeta_{i_2}(t - \tau_{i_2 j}(t))}{h}\right] e^{-h_{i_2 j} N_{i_2}^* - \frac{h_{i_2 j}}{h} \zeta_{i_2}(t - \sigma_{i_2 j}(t))}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad \times \beta_{i_2}(t) e^{\int_{t_0}^{t}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv}\,dt \\
&> -h\delta_{i_2} N_{i_2}^{*} \frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv} - e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv}}{\delta_{i_2}-a_{i_2 i_2}} \\
&\quad + (\lambda-\varepsilon) \frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv} - e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv}}{\delta_{i_2}-a_{i_2 i_2}} \sum_{j=1, j\neq i_2}^{n} a_{i_2 j} \\
&\quad + h \sum_{j=1}^{m} \rho_{i_2 j} \int_{S_q}^{s_q} \left(N_{i_2}^{*} + \frac{\lambda-\varepsilon}{h} \right) e^{-h_{i_2 j} N_{i_2}^{*} - \frac{h_{i_2 j}}{h}(\mu+\varepsilon)} \beta_{i_2}(t) e^{\int_{t_0}^{t}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv}\,dt \\
&> -h\delta_{i_2} N_{i_2}^{*} \frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv} - e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv}}{\delta_{i_2}-a_{i_2 i_2}} \\
&\quad + (\lambda-\varepsilon) \frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv} - e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv}}{\delta_{i_2}-a_{i_2 i_2}} (-a_{i_2 i_2}) \\
&\quad + \frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv} - e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv}}{\delta_{i_2}-a_{i_2 i_2}} \\
&\qquad \times \left[h \sum_{j=1}^{m} \rho_{i_2 j} N_{i_2}^{*} e^{-h_{i_2 j} N_{i_2}^{*}} e^{-(\mu+\varepsilon)} + (\lambda-\varepsilon) \sum_{j=1}^{m} \rho_{i_2 j} e^{-h_{i_2 j} N_{i_2}^{*}} \right] \\
&= h\delta_{i_2} N_{i_2}^{*} \frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv} - e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv}}{\delta_{i_2}-a_{i_2 i_2}} [e^{-(\mu+\varepsilon)} - 1] \\
&\quad + (\lambda-\varepsilon)\big(e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv} - e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv} \big), \quad q > q^{*}
\end{aligned}
$$

and

$$
\begin{aligned}
&z_{i_2}(s_q) + (\lambda-\varepsilon)\big(e^{-(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}^{H} r_{i_2}^{H}} - 1 \big) \\
&\geq z_{i_2}(s_q) + (\lambda-\varepsilon)\big(e^{-\int_{S_q}^{s_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv} - 1 \big) \\
&> \big(1 - e^{-\int_{S_q}^{s_q}(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}(v)\,dv} \big) \frac{h\delta_{i_2} N_{i_2}^{*}}{\delta_{i_2}-a_{i_2 i_2}} [e^{-(\mu+\varepsilon)} - 1] \\
&\geq \big(1 - e^{-(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}^{H} r_{i_2}^{H}} \big) \frac{h\delta_{i_2} N_{i_2}^{*}}{\delta_{i_2}-a_{i_2 i_2}} [e^{-(\mu+\varepsilon)} - 1], \quad q > q^{*}.
\end{aligned} \tag{3.24}
$$

Letting $q \to \infty$ and $\varepsilon \to 0$, (3.11) and (3.24) reveal that

$$
\begin{aligned}
\lambda &\geq \frac{1 - e^{-(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}^{H} r_{i_2}^{H}}}{(e^{-(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}^{H} r_{i_2}^{H}} - 1) + 1} \times \frac{h\delta_{i_2} N_{i_2}^{*}}{\delta_{i_2}-a_{i_2 i_2}} (e^{-\mu} - 1) \\
&= \frac{(e^{(\delta_{i_2}-a_{i_2 i_2})\beta_{i_2}^{H} r_{i_2}^{H}} - 1) h\delta_{i_2} N_{i_2}^{*}}{\delta_{i_2}-a_{i_2 i_2}} (e^{-\mu} - 1) \geq e^{-\mu} - 1 \geq -1.
\end{aligned} \tag{3.25}
$$

In the light of (1.3), (2.3), (3.2), (3.13), (3.17), (3.18), (3.23) and (3.25), we obtain

$$\begin{aligned}
&\zeta_{i_1}(l_q)e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}\\
&= -h\delta_{i_1}N_{i_1}^*\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}}{\delta_{i_1}-a_{i_1i_1}}\\
&\quad+\sum_{j=1,j\neq i_1}^{n}a_{i_1j}\int_{L_q}^{l_q}\zeta_j(t)\beta_{i_1}(t)e^{\int_{t_0}^{t}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}\,dt\\
&\quad+h\sum_{j=1}^{m}\rho_{i_1j}\int_{L_q}^{l_q}\left[N_{i_1}^*+\frac{\zeta_{i_1}(t-\tau_{i_1j}(t))}{h}\right]e^{-h_{i_1j}N_{i_1}^*-\frac{h_{i_1j}}{h}\zeta_{i_1}(t-\sigma_{i_1j}(t))}\\
&\qquad\qquad\times\beta_{i_1}(t)e^{\int_{t_0}^{t}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}\,dt\\
&< -h\delta_{i_1}N_{i_1}^*\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}}{\delta_{i_1}-a_{i_1i_1}}\\
&\quad+(\mu+\varepsilon)\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}}{\delta_{i_1}-a_{i_1i_1}}\sum_{j=1,j\neq i_1}^{n}a_{i_1j}\\
&\quad+h\sum_{j=1}^{m}\rho_{i_1j}\int_{L_q}^{l_q}\left(N_{i_1}^*+\frac{\mu+\varepsilon}{h}\right)e^{-h_{i_1j}N_{i_1}^*-\frac{h_{i_1j}}{h}(\lambda-\varepsilon)}\beta_{i_1}(t)e^{\int_{t_0}^{t}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}\,dt\\
&< -h\delta_{i_1}N_{i_1}^*\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}}{\delta_{i_1}-a_{i_1i_1}}\\
&\quad+(\mu+\varepsilon)\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}}{\delta_{i_1}-a_{i_1i_1}}(-a_{i_1i_1})\\
&\quad+\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)\,dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)\,dv}}{\delta_{i_1}-a_{i_1i_1}}\\
&\qquad\times\left[h\sum_{j=1}^{m}\rho_{i_1j}N_{i_1}^*e^{-h_{i_1j}N_{i_1}^*}e^{-(\lambda-\varepsilon)}+(\mu+\varepsilon)e^{1+\varepsilon}\sum_{j=1}^{m}\rho_{i_1j}e^{-h_{i_1j}N_{i_1}^*}\right]\\
&= h\delta_{i_1}N_{i_1}^*\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}}{\delta_{i_1}-a_{i_1i_1}}[e^{-(\lambda-\varepsilon)}-1]\\
&\quad+\frac{\delta_{i_1}e^{1+\varepsilon}-a_{i_1i_1}}{\delta_{i_1}-a_{i_1i_1}}(\mu+\varepsilon)e^{1+\varepsilon}\big(e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}\big),
\end{aligned}$$

for $q>q^*$, and

$$\begin{aligned}
z_{i_1}(l_q) &< h\delta_{i_1}N_{i_1}^*\frac{1-e^{\int_{l_q}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}}{\delta_{i_1}-a_{i_1i_1}}[e^{-(\lambda-\varepsilon)}-1]\\
&\quad+\frac{\delta_{i_1}e^{1+\varepsilon}-a_{i_1i_1}}{\delta_{i_1}-a_{i_1i_1}}(\mu+\varepsilon)\big(1-e^{\int_{l_q}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta_{i_1}(v)\,dv}\big)
\end{aligned}$$

$$\begin{aligned}(3.26)\qquad &\le h\delta_{i_1}N^*_{i_1}\frac{1-e^{-r^H_{i_1}(\delta_{i_1}-a_{i_1i_1})\beta^H_{i_1}}}{\delta_{i_1}-a_{i_1i_1}}[e^{-(\lambda-\varepsilon)}-1]\\ &\quad+\frac{\delta_{i_1}e^{1+\varepsilon}-a_{i_1i_1}}{\delta_{i_1}-a_{i_1i_1}}(\mu+\varepsilon)\big(1-e^{-r^H_{i_1}(\delta_{i_1}-a_{i_1i_1})\beta^H_{i_1}}\big),\quad q>q^*.\end{aligned}$$

Letting $q\to\infty$ and $\varepsilon\to 0$, (3.12) and (3.26) entail that

$$\begin{aligned}\mu &\le \frac{h\delta_{i_1}N^*_{i_1}\big(1-e^{-r^H_{i_1}(\delta_{i_1}-a_{i_1i_1})\beta^H_{i_1}}\big)}{\delta_{i_1}-\delta_{i_1}e\big(1-e^{-r^H_{i_1}(\delta_{i_1}-a_{i_1i_1})\beta^H_{i_1}}\big)-a_{i_1i_1}e^{-r^H_{i_1}(\delta_{i_1}-a_{i_1i_1})\beta^H_{i_1}}}(e^{-\lambda}-1)\\ &\le e^{-\lambda}-1,\end{aligned}$$

which, together with (3.25), entails that (3.22) holds.

Finally, using the same argument as in the proof of Theorem 4.1 of [28], we can conclude from (3.22) that $\lambda=\mu=0$, which contradicts the fact that $\mu>0$. This ends the proof. □

Remark 3.6. In fact, $\lim_{r^H_i\to 0^+} e^{r^H_i(\delta_i-a_{ii})\beta^H_i}=1$ means that the conditions of (3.11) and (3.12) naturally hold, but $\lim_{r^H_i\to+\infty} e^{r^H_i(\delta_i-a_{ii})\beta^H_i}=+\infty$ shows that (3.11) and (3.12) are obviously not satisfied. Thus, one can find that the sufficiently small pairs of timing-varying delays have little effect on the global attractivity of the positive equilibrium for system (1.3), but the system (1.3) may produce chaotic oscillation when the time-varying delays are sufficiently large. This will be verified through some numerical simulations in the next section.

Remark 3.7. It should be pointed out that, for $i\in\Lambda$, $j\in\Omega$ and $i\neq j$, if $a_{ij}=0$ and without patch structure in (1.3), then it is simplified to the scalar Nicholson's equation in [28], and one can easily discovery that our results cover the corresponding results in [28]. In addition, since the actual biological background often leads to the positive equilibrium point possesses different components, then all the results in literature [4, 19, 20] are the special case of this paper when $N_1=N_2=\cdots=N_n=N^*$ and $\tau_{ij}(t)\equiv\sigma_{ij}(t)$ ($i\in\Lambda$, $j\in\Omega$), which has been adopted as a basic assumption for the considered attractivity of (1.3) in previous work.

Remark 3.8. In Theorem 5.1 of [10] and Corollary 3.4 of [3], one can find that

$$(3.27)\qquad \frac{\sum_{j=1}^n\rho_{ij}}{\delta_i-\sum_{j=1}^n a_{ij}}<e^2,\quad i\in\Lambda$$

has been considered as a technical assumption for the permanence and stability for Nicholson's blowflies system with patch structure. Evidently, the assumption (3.27) is no longer required in Theorems 2.4 and 3.5 of our manuscript. This implies that the theoretical results of this present paper improve and complement some existing ones in the above mentioned papers.

4. Numerical simulations

In this section, we carry out some numerical simulations to verify the theoretical findings of this paper.

Example 4.1. Consider the following patch structure Nicholson's blowflies system incorporating multiple pairs of time-varying delays:

$$
\begin{aligned}
x_1'(t) = (10+\cos^2(t))\Big[&-\frac{1}{5}x_1(t) + \Big(-\frac{1}{20}x_1(t) + \frac{1}{20}x_2(t)\Big) \\
&+ \frac{11}{100}e^{\frac{2}{5}}x_1(t-\tau_{11}(t))e^{-\frac{1}{5}x_1(t-\sigma_{11}(t))} \\
&+ \frac{9}{100}e^{\frac{4}{5}}x_1(t-\tau_{12}(t))e^{-\frac{2}{5}x_1(t-\sigma_{12}(t))}\Big], \\
x_2'(t) = (12+\sin^2(2t))\Big[&-\frac{1}{20}x_2(t) + \Big(\frac{1}{40}x_1(t) - \frac{1}{40}x_2(t)\Big) \\
&+ \frac{3}{100}e^{\frac{2}{3}}x_2(t-\tau_{21}(t))e^{-\frac{1}{3}x_2(t-\sigma_{21}(t))} \\
&+ \frac{1}{50}e^{\frac{1}{2}}x_2(t-\tau_{22}(t))e^{-\frac{1}{4}x_2(t-\sigma_{22}(t))}\Big], \\
x_3'(t) = (10+\sin^2(3t))\Big[&-\frac{\sqrt{2}}{4}x_3(t) + \Big(\frac{1}{40}x_1(t) + \frac{1}{40}x_2(t) - \frac{\sqrt{2}}{20}x_3(t)\Big) \\
&+ \frac{13}{80}\sqrt{2}e^{\frac{\sqrt{2}}{3}}x_3(t-\tau_{31}(t))e^{-\frac{1}{3}x_3(t-\sigma_{31}(t))} \\
&+ \frac{7}{80}\sqrt{2}e^{\frac{\sqrt{2}}{2}}x_3(t-\tau_{32}(t))e^{-\frac{1}{2}x_3(t-\sigma_{32}(t))}\Big],
\end{aligned} \tag{4.1}
$$

which takes on a unique positive equilibrium point $(N_1^*, N_2^*, N_3^*) = (2, 2, \sqrt{2})$.

Now, an easy computation shows that

$$
\tau_{ij}(t) = \frac{1}{20}|\cos 2(i+j)t|, \quad \sigma_{ij}(t) = \frac{1}{20}|\sin 2(i+j)t| + \frac{1}{1+t^2}, \quad i,j = 1,2,3. \tag{4.2}
$$

satisfy (2.9), (3.2), (3.11) and (3.12). Thus, we deduce from Theorem 3.5 that the positive equilibrium point $(2, 2, \sqrt{2})$ is a global attractor of (4.1) incorporating delays (4.2). The numerical simulation in Figure 4.1 is consistent with this assertion.

In addition, take

$$
\tau_{ij}(t) = 40j, \quad \sigma_{ij}(t) = 40j + \frac{1}{1+t^2}, \quad i,j = 1,2,3, \tag{4.3}
$$

we check at once that (3.11) and (3.12) do not hold for system (4.1) involving delays (4.3). From Figure 4.2, we can find that $(2, 2, \sqrt{2})$ is unstable and maybe not the global attractor of (4.1) with delays (4.3). This confirms the conclusions reached in Remark 3.6.

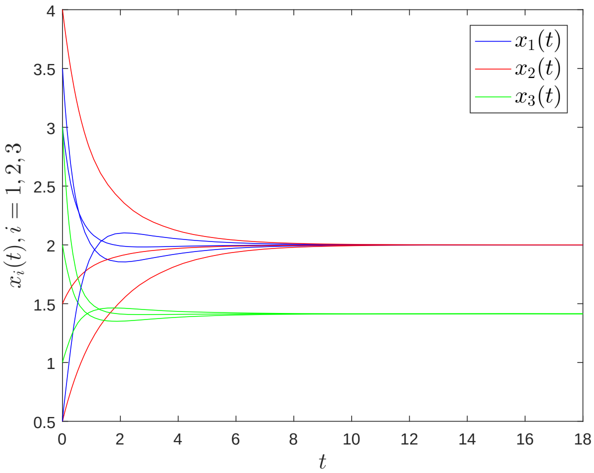

Figure 4.1: Numerical solutions to example (4.1) with delays (4.2) and initial values:$(3|\cos(t)|, 1.5 + |\sin(t)|, 3 + |\sin(t)|), (0.5 + 5|\sin(t)|, 1 + 3|\cos(t)|, 0.5 + 0.5|\cos(t)|), (3.5|\cos(t)|, 0.5 + 2|\sin(t)|, 2 + 2|\sin(t)|)$.

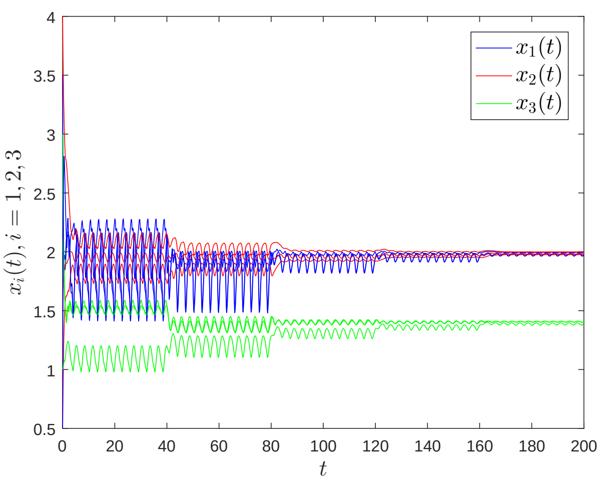

Figure 4.2: Numerical solutions to example (4.1) with delays (4.3) and initial values:$(3|\cos(t)|, 1.5 + |\sin(t)|, 3 + |\sin(t)|), (0.5 + 5|\sin(t)|, 1 + 3|\cos(t)|, 0.5 + 0.5|\cos(t)|), (3.5|\cos(t)|, 0.5 + 2|\sin(t)|, 2 + 2|\sin(t)|)$.

Remark 4.2. Because $\tau_{ij}(t) \not\equiv \sigma_{ij}(t)$ $(i \in \Lambda,\ j \in \Omega)$, and the compositions of the positive equilibrium point vector of system (4.1) are unequal, and hence the results in literature [3, 4, 6, 9, 10, 13–16, 19, 20, 28, 30, 32, 33] cannot be used to reveal the global attractiveness of system (4.1). This implies that our results are novel and generalize all the ones in the above-mentioned references. On the other hand, the positive equilibrium point of system (4.1) is globally attractive when the time-varying delays are sufficiently small, but when the time-varying delays are sufficiently large, the system (4.1) may yield complex dynamic behavior, which confirms our findings.

5. Conclusions

This paper explores the effect of delays on the attractivity of Nicholson's blowflies model with patch structure and multiple pairs of distinct time-varying delays. Without assuming that the equilibrium vector possesses the same components, by applying some novel differential inequality analyses and the fluctuation lemma, the uniform persistence on the positive solutions, as well as the global attractivity on the positive equilibrium point have firstly been established for the addressed system. The obtained results substantiate that, by controlling the death rate, the dispersal rates and the related parameters in the reproduction function, the attractivity of the positive equilibrium point can be attained if the time-varying delays are sufficiently small in the development process. The results of this manuscript are verified by some numerical simulations, and supplement some early publications to a certain extent. The adopted strategies in this present study could be taken into consideration in the area of dynamics problems on other patch structure population systems incorporating two or more distinct delays in the same reproduction function. This is our future research direction.

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