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On a Higher-order Reaction-diffusion Equation with a Special Medium Void via Potential Well Method

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Abstract. Let $d \in \{1, 2, 3, \ldots\}$ and $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. Consider the reaction-diffusion parabolic problem

$$\begin{cases} \frac{u_t}{|x|^4} + \Delta^2 u = k(t)|u|^{p-1}u, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $T > 0$, $p \in (1,\infty)$, $0 \neq u_0 \in H_0^2(\Omega)$ and ν is the outward normal vector to $\partial\Omega$. We investigate the existence of a global weak solution to the problem together with the decaying and blow-up properties using the potential well method.

1. Introduction

This paper concerns the existence of global solution to a class of parabolic equations whose diffusion process is determined by the bi-harmonic operator with special coefficients. The decaying rate for global solutions and blow-up estimates for local solutions are also presented. The principal method for the investigation is the potential well technique developed by Levine and Payne in 1970. This technique has been successfully employed to solve various reaction-diffusion equations, as is well-known to experts in the field. Of our particular interest here, Tan in [11] and Han in [3] investigated the weak solutions to the equation of the form

$$\begin{cases} \frac{u_t}{|x|^2} + Lu = f(u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Tan discussed the existence of a global weak solution as well as the decaying and blow-up properties when $L = \Delta_p$ and $f(u) = u^q$, whereas Han derived an upper bound on the blow-up time of weak solutions when $L = \Delta$ and $f(u) = |u|^{p-1}u$.

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Various extensions of (1.1) then follow. For examples, one may confer [14] for an investigation of (1.1) in the porous medium setting, [7] for the existence of global or finite time blow-up weak solutions of (1.1) when the initial energy is critical and most recently [12] for a fractional Laplace operator consideration.

For our problem, let $d \in \{1, 2, 3, \ldots\}$ and $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. We consider the following higher-order reaction-diffusion parabolic problem:

$$\text{(P)} \qquad \begin{cases} \frac{u_t}{|x|^4} + \Delta^2 u = k(t)|u|^{p-1}u, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $T > 0$, $p \in (1, \infty)$, $0 \neq u_0 \in H_0^2(\Omega)$ and ν is the outward normal vector to $\partial\Omega$. We will focus on sufficient conditions for the existence of a global weak solution to (P) together with the decaying and blow-up properties, the precise definitions of which are given below. The appearance of the bi-harmonic operator Δ^2 and the coefficient $k(t)$ in the equation is motivated by the work of Philippin in [9] and Han in [3] respectively. The special diffusion coefficient (or special medium void) $1/|x|^4$ is driven by the inspiring Rellich's inequality (cf. [2, Corollary 6.3.5]) and parallels the inverse-square coefficient in (1.1). The time-dependent function $k(t)$ adds an extra new feature into our investigation. In particular, all quantities and functionals used in the potential well method will also be time-dependent. The details are shown below.

It is worth mentioning that other methods for the investigation of reaction-diffusion equations exist beside the potential well technique. They include the first eigenvalue method by Kaplan in 1963, the comparison method and other methods involving integration. A recent overview of the account can be found in the monograph [4].

Back to the problem (P), the following definitions are crucial to our development.

Definition 1.1. A function $u(x,t)$ is called a weak solution to (P) if $u \in L^2(0,T; H_0^2(\Omega))$ with $u(0) = u_0$,

$$\int_0^T \int_\Omega \frac{|u_t|^2}{|x|^4}\, dxdt < \infty$$

and $u(x,t)$ satisfies

$$\left(\frac{u_t}{|x|^4}, \varphi \right) + (\Delta u, \Delta \varphi) = k(t)(|u|^{p-1}u, \varphi) \tag{1.2}$$

for all $\varphi \in H_0^2(\Omega)$ and $t \in [0,T)$.

Definition 1.2. Let $u(t)$ be a weak solution to (P). Then $u(t)$ is said to blow up at a finite time T^* if $u(t)$ exists for all $t \in [0, T^*)$ and

$$\lim_{t \to T^*} \left\| \frac{u(t)}{|x|^2} \right\|_{L^2(\Omega)}^2 = \infty. \tag{1.3}$$

Such a T^* is called the maximal existence time for $u(t)$. If (1.3) does not happen for any finite time T^*, then $u(t)$ is called a global solution and the maximal existence time of $u(t)$ is ∞.

Hereafter, for each $u \in H_0^2(\Omega)$, define the following functionals:

- Energy functional:

$$J(u,t) = \frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} - \frac{k(t)}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}.$$

- Nehari functional:

$$I(u,t) = \|\Delta u\|^2_{L^2(\Omega)} - k(t)\|u\|^{p+1}_{L^{p+1}(\Omega)}.$$

Due to Lemma 2.2 below, these two functionals are well-defined whenever $2 < p+1 < 2^*$, where

$$2^* := \begin{cases} \infty & \text{if } d \le 4, \\ \frac{2d}{d-4} & \text{if } d \ge 5. \end{cases}$$

Next for each $t \ge 0$, define the following quantities:

- Nehari's manifold:

$$\mathcal{N}(t) = \big\{u \in H_0^2(\Omega) \setminus \{0\} : I(u,t) = 0\big\}.$$

- Potential well depth:

$$d(t) = \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \sup_{\lambda>0} J(\lambda u, t) = \inf_{u \in \mathcal{N}(t)} J(u,t). \tag{1.4}$$

It is straightforward to verify that $\mathcal{N}(t)$ is non-empty for each $t \ge 0$. Furthermore, to justify the second equality in (1.4) we argue as follows. One has

$$\begin{aligned} d(t) &= \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \sup_{\lambda>0} J(\lambda u, t) \\ &= \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \sup_{\lambda>0} \left[\frac{\lambda^2}{2}\|\Delta u\|^2_{L^2(\Omega)} - \frac{k(t)}{p+1}\lambda^{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}\right] \\ &= \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \left[\frac{\lambda_0^2}{2}\|\Delta u\|^2_{L^2(\Omega)} - \frac{k(t)}{p+1}\lambda_0^{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}\right] \\ &= \inf_{u \in H_0^2(\Omega)\setminus\{0\}} J(\lambda_0 u, t) = \inf_{\lambda_0 u \in H_0^2(\Omega)\setminus\{0\}} J(\lambda_0 u, t) = \inf_{v \in \mathcal{N}(t)} J(v,t), \end{aligned}$$

where $\lambda_0 > 0$ is such that

$$\lambda_0\|\Delta u\|^2_{L^2(\Omega)} - k(t)\lambda_0^p\|u\|^{p+1}_{L^{p+1}(\Omega)} = 0 \quad \Longleftrightarrow \quad I(\lambda_0 u, t) = 0.$$

With the idea of potential well depth in mind, we are now able to define the stable and unstable sets as follows for each $t \ge 0$:

- Stable set:

$$\Sigma_1(t) = \big\{u \in H_0^2(\Omega) : J(u,t) < d_\infty \text{ and } I(u,t) > 0\big\}.$$

- Unstable set:

$$\Sigma_2(t) = \big\{u \in H_0^2(\Omega) : J(u,t) < d_\infty \text{ and } I(u,t) < 0\big\}.$$

These two sets are crucial to our study. Here

$$d_\infty := \lim_{t\to\infty} d(t).$$

It follows from Lemma 2.7 below that d_∞ is positive finite number. Observe that J, I, $\mathcal{N}$, d, Σ_1 and Σ_2 all depend on time, which is due to the presence of $k(t)$ in (P). This time-dependent feature adds extra technicality into our analysis.

Due to the presence of the inverse coefficient $1/|x|^4$, it is worth emphasizing the difference between the two cases when $0 \in \Omega$ and $0 \notin \Omega$. If $0 \in \Omega$ then $1/|x|^4$ develops a singularity. This necessitates the use of Rellich's inequality, which is valid for $d \geq 5$, in the proofs of our main results. On the other hand, if $0 \notin \Omega$ then there is no singularity and (P) can be regarded as a slight extension of the model in [3]. In this case our results are valid for all $d \in \{1, 2, 3, \ldots\}$. To deal with these two cases simultaneously, we employ the notation

$$d_\Omega := \begin{cases} 5 & \text{if } 0 \in \Omega, \\ 1 & \text{if } 0 \notin \Omega. \end{cases}$$

Our first result concerns the existence of a global weak solution to (P) when the initial datum u_0 belongs to the stable set Σ_1.

Theorem 1.3. *Let $d \geq d_\Omega$ and $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. Let $2 < p+1 < 2^*$. Let $u_0 \in \Sigma_1(0)$. Suppose $k \in C^1[0,\infty)$ satisfies $k(0) > 0$ and $k'(t) \geq 0$ for all $t \in [0,\infty)$. Furthermore suppose that $\lim_{t\to\infty} k(t) = 1$. Then there exists a global weak solution to* (P).

Such a global weak solution in Theorem 1.3 also enjoys a decaying property as given next.

Theorem 1.4. *Adopt the assumptions and notation from Theorem* 1.3*. Let u be the global solution to* (P)*. Then there exists an $\alpha > 0$ such that*

$$\|\Delta u(t)\|_{L^2(\Omega)} = O(e^{-\alpha t})$$

when $t \to \infty$.

Next we consider the blow-up behaviour of a weak solution to (P). The following result provides an upper bound on the blow-up time for a weak solution to (P) when the initial datum u_0 belongs to the unstable set Σ_2. In what follows, it is convenient to denote

$$L(t) = \frac{1}{2} \left\| \frac{u(t)}{|x|^2} \right\|^2_{L^2(\Omega)}$$

for each $t \in [0, T)$.

Theorem 1.5. *Let $d \geq d_\Omega$ and $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. Let $2 < p+1 < 2^*$. Let $u_0 \in \Sigma_2(0)$. Suppose $k \in C^1[0,\infty)$ satisfies $k(0) > 0$ and $k'(t) \geq 0$ for all $t \in [0,\infty)$. Let u be a weak solution to* (P) *with $T > 0$ being the maximal existence time. Then T is finite and u blows up at T. Moreover,*

$$T \leq \frac{4pL(0)}{(p-1)^2(p+1)(d_\infty - J(u_0,0))}.$$

The critical case when the initial energy $J(u_0, 0) = d_\infty$ is also investigated.

Theorem 1.6. *Let $d \geq d_\Omega$ and $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. Let $2 < p+1 < 2^*$. Let $u_0 \in H^2_0(\Omega)$ be such that*

$$J(u_0, 0) = d_\infty \quad \textit{and} \quad I(u_0, 0) < 0.$$

Suppose $k \in C^1[0,\infty)$ satisfies $k(0) > 0$ and $k'(t) \geq 0$ for all $t \in [0,\infty)$. Let u be a weak solution to (P) *with $T > 0$ being the maximal existence time. Then there exists a $t_0 \in (0,T)$ such that $J(u(t_0), t_0) < \infty$. Moreover, T is finite, u blows up at T and there holds*

$$T \leq t_0 + \frac{4pL(t_0)}{(p-1)^2(p+1)(d_\infty - J(u(t_0),t_0))}.$$

The paper is planned as follows. In Section 2, we collect preliminary results for proving our main theorems. Theorems 1.3 and 1.4 are proved in Section 3. The investigation of the blow-up in finite time, which is the content of Theorems 1.5 and 1.6, is done in Section 4.

2. Preliminaries

In this section we discuss some preliminary estimates to be used in the proof of the main results. Recall that we set

$$d_\Omega = \begin{cases} 5 & \text{if } 0 \in \Omega, \\ 1 & \text{if } 0 \notin \Omega \end{cases} \quad \text{and} \quad 2^* = \begin{cases} \infty & \text{if } d \leq 4, \\ \frac{2d}{d-4} = 2 + \frac{8}{d-4} & \text{if } d \geq 5. \end{cases}$$

Let us begin with the following Rellich's inequality.

Lemma 2.1. *Let $d \geq d_\Omega$ and $u \in H_0^2(\Omega)$. Then $u/|x|^2 \in L^2(\Omega)$ and there exists a constant $\mathcal{R} = \mathcal{R}(\Omega, d) > 0$ such that*

$$\int_\Omega \frac{|u|^2}{|x|^4}\,dx \leq \mathcal{R} \int_\Omega |\Delta u|^2\,dx.$$

Proof. First assume $0 \in \Omega$. Then $d \geq 5$. Let $u \in H_0^2(\Omega)$ and $\widetilde{u}$ be its zero extension to $\mathbb{R}^d$. It follows that $\widetilde{u} \in H^2(\mathbb{R}^d)$ by [1, Lemma 3.27] and

$$\int_\Omega \frac{|u|^2}{|x|^4}\,dx = \int_{\mathbb{R}^d} \frac{|\widetilde{u}|^2}{|x|^4}\,dx \leq \frac{16}{d^2(d-4)^2} \int_{\mathbb{R}^d} |\Delta \widetilde{u}|^2\,dx = \frac{16}{d^2(d-4)^2} \int_\Omega |\Delta u|^2\,dx,$$

where we used [2, Corollary 6.3.5] in the second step. Hence the claim is valid with

$$\mathcal{R} = \frac{16}{d^2(d-4)^2}.$$

Next assume $0 \notin \Omega$. Then $d \geq 1$. Set

$$\kappa_0 := \inf_{x \in \Omega} |x|^{-4} > 0.$$

We have

$$\int_\Omega \frac{|u|^2}{|x|^4}\,dx \leq \left(\inf_{x\in\Omega} |x|^{-4}\right) \int_\Omega |u|^2\,dx \leq \kappa_0 C(\Omega, d) \int_\Omega |\nabla^2 u|^2\,dx \leq \kappa_0 C(\Omega, d) \int_\Omega |\Delta u|^2\,dx,$$

where we used the Friedrichs inequality (cf. [8, Theorem 1.10]) and [10, Chapter 3, Proposition 3] in the last two steps respectively. □

The next result is the Gagliardo–Nirenberg inequality.

Lemma 2.2. *Let $d \in \{1, 2, 3, \ldots\}$ and $2 < q \leq 2^*$ with $q \neq \infty$ when $d = 4$. Then there exists an $N_0 = N_0(d, q) > 0$ such that*

$$\|u\|_{L^q(\Omega)} \leq N_0 \|\Delta u\|_{L^2(\Omega)}^{\alpha} \|u\|_{L^2(\Omega)}^{1-\alpha}$$

for all $u \in H_0^2(\Omega)$, where

$$\alpha = \frac{d(q-2)}{4q} \in (0, 1].$$

Proof. Let $u \in H_0^2(\Omega)$. It follows from the Gagliardo–Nirenberg inequality (cf. [1, Theorem 5.8 and Corollary 6.31]) that

$$\|u\|_{L^q(\Omega)}^q \leq C(d, q) \|\nabla^2 u\|_{L^2(\Omega)}^{\alpha q} \|u\|_{L^2(\Omega)}^{(1-\alpha)q}.$$

But

$$\|\nabla^2 u\|_{L^2(\Omega)} \leq C(d) \|\Delta u\|_{L^2(\Omega)}$$

by [10, Chapter 3, Proposition 3]. Combining these two inequalities together justifies the claim. □

The following result is immediate from Lemma 2.2 and the Friedrichs inequality (cf. [8, Theorem 1.10]).

Lemma 2.3. *Let* $d \in \{1, 2, 3, \ldots\}$, $u \in H_0^2(\Omega)$ *and* $2 < p+1 < 2^*$. *Then there exists a constant* $S_p = S_p(d,p) > 0$ *such that*

$$\|u\|_{L^{p+1}(\Omega)} \le S_p \|\Delta u\|_{L^2(\Omega)}.$$

Additionally, we note that the constant S_p in Lemma 2.3 can be made explicit and sharp when $d \ge 5$. In fact, one has the following statement.

Lemma 2.4. [13, Theorem 1] *Let* $d \ge 5$ *and* $u \in H^2(\Omega) \cap H_0^1(\Omega)$. *Then*

$$\|u\|_{L^{2^*}(\Omega)} \le \Lambda_p^{4/d} \|\Delta u\|_{L^2(\Omega)},$$

where

$$\Lambda_p = \left[\pi^2 d(d-4)(d^2-4)\left(\frac{\Gamma(d/2)}{\Gamma(d)}\right)\right]^{-1/2},$$

provided that $\partial\Omega$ *is sufficiently smooth. Moreover, the inequality is sharp.*

A remark is immediate.

Remark 2.5. We mainly deal with the case $u \in H_0^2(\Omega)$. As such the domain Ω with a Lipschitz boundary is sufficient to use Lemma 2.4.

Next assume $d \ge 5$. Using Lemma 2.4 and Holder's inequality, we obtain

$$S_p = |\Omega|^{\frac{1}{p+1} - \frac{1}{2^*}} \Lambda_p.$$

The roles of the energy and Nehari functionals are fundamental to our analysis. The following identities hold for them.

Lemma 2.6. *Let* $d \ge d_\Omega$ *and* $2 < p+1 < 2^*$. *Suppose* $k \in C^1[0,\infty)$ *satisfies* $k(0) > 0$ *and* $k'(t) \ge 0$ *for all* $t \in [0,\infty)$. *Let* u *be a weak solution to* (P). *Then the following identities hold.*

(i) *For a.e.* $t_0 \in [0,T)$, *one has*

$$J(u(t_0), t_0) + \int_0^{t_0} \left(\left\| \frac{u_t(s)}{|x|^2} \right\|_{L^2(\Omega)}^2 + \frac{k'(s)}{p+1} \|u(s)\|_{L^{p+1}(\Omega)}^{p+1} \right) ds = J(u_0, 0).$$

(ii) *For a.e.* $t_0 \in [0,T)$, *one has*

$$\frac{d}{dt}\left(\frac{1}{2} \left\| \frac{u(t_0)}{|x|^2} \right\|_2^2 \right) = \left(\frac{u(t_0)}{|x|^4}, u_t(t_0) \right) = -I(u(t_0), t_0).$$

Proof. Regarding (i), first suppose that $u_t \in L^2(0,T;H_0^2(\Omega))$. Then by using u_t as a test function in (1.2), we obtain

$$\left\| \frac{u_t}{|x|^2} \right\|_{L^2(\Omega)}^2 + (\Delta u, \Delta u_t) = k(t)(|u|^{p-1}u, u_t).$$

On the other hand, direct calculations give

$$\frac{d}{dt} J(u(t),t) = (\Delta u(t), \Delta u_t(t)) - k(t)(|u(t)|^{p-1}u(t), u_t(t)) - \frac{k'(t)}{p+1} \|u(t)\|_{L^{p+1}(\Omega)}^{p+1}$$

for each $t \in (0,T)$. Combining these two identities together yields that

$$\frac{d}{dt} J(u(t),t) = -\left\| \frac{u_t(t)}{|x|^2} \right\|_{L^2(\Omega)}^2 - \frac{k'(t)}{p+1} \|u(t)\|_{L^{p+1}(\Omega)}^{p+1} \tag{2.1}$$

for each $t \in (0,T)$. Now (i) follows by integrating both sides of (2.1) with respect to t over $(0,t_0)$, where $t_0 \in (0,T)$.

To finish, we observe that (2.1) holds without the assumption that $u_t \in L^2(0,T;H_0^2(\Omega))$ by an approximation argument.

The proof of (ii) follows the same line and hence is omitted. □

Next we present an explicit expression for the potential well depth.

Lemma 2.7. *Let $d \in \{1,2,3,\ldots\}$ and $2 < p+1 < 2^*$. Suppose $k \in C^1[0,\infty)$ satisfies $k(0) > 0$ and $k'(t) \geq 0$ for all $t \in [0,\infty)$. Then the following statements hold.*

(i) *One has*

$$d(t) = \frac{p-1}{2(p+1)} k(t)^{2/(1-p)} S_p^{-2(p+1)/(p-1)}$$

for all $t \in [0,\infty)$, where S_p is given in Lemma 2.3.

(ii) *The potential well depth d is decreasing as a function of t on $[0,\infty)$. Moreover,*

$$d_\infty := \lim_{t\to\infty} d(t) \in [0, d(0)].$$

Proof. (i) Let $w \in H_0^2(\Omega) \setminus \{0\}$ and $t \geq 0$. For each $\lambda \geq 0$, define

$$F(\lambda) := J(\lambda w, t) = \frac{\lambda^2}{2} \|\Delta w\|_{L^2(\Omega)}^2 - \frac{k(t)}{p+1} \lambda^{p+1} \|w\|_{L^{p+1}(\Omega)}^{p+1}.$$

It is elementary to check that F as a function of λ on $[0,\infty)$ has exactly one critical point

$$\lambda_0(w) = \left(\frac{\|\Delta w\|_{L^2(\Omega)}^2}{k(t)\|w\|_{L^{p+1}(\Omega)}^{p+1}} \right)^{1/(p-1)}$$

which is also the maximum point. With that in mind we now have

$$\begin{aligned} d(t) &= \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \sup_{\lambda>0} J(\lambda u, t) = \inf_{u\in H_0^2(\Omega)\setminus\{0\}} F(\lambda_0(u)) \\ &= \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \left\{ \frac{\lambda_0(u)^2}{2} \|\Delta u\|^2_{L^2(\Omega)} - \frac{k(t)}{p+1} \lambda_0(u)^{p+1} \|u\|^{p+1}_{L^{p+1}(\Omega)} \right\} \\ &= \frac{p-1}{2(p+1)} k(t)^{2/(1-p)} \left(\inf_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_{L^2(\Omega)}}{\|u\|_{L^{p+1}(\Omega)}} \right)^{2(p+1)/(p-1)} \\ &= \frac{p-1}{2(p+1)} k(t)^{2/(1-p)} S_p^{-2(p+1)/(p-1)}, \end{aligned}$$

where we used Lemma 2.3 in the last step.

(ii) By hypothesis k is increasing on $[0,\infty)$. This fact together with (i) justify the claim. □

The next concavity argument is classic and is used extensively in the literature for a sufficient condition of blow-up time.

Lemma 2.8. [5] *Let $\theta > 0$. Let $\psi \geq 0$ be weakly twice-differentiable on a certain interval $(0,\tau) \subset (0,\infty)$ such that $\psi(0) > 0$, $\psi'(0) > 0$ and*

$$\psi''(t)\psi(t) - (1+\theta)(\psi'(t))^2 \geq 0$$

for all $t \in (0,\tau)$. Then there exists a $T \geq \tau$ such that ψ is continuously extended to $(0,T)$ with

$$\lim_{t\to T^-} \psi(t) = \infty \quad \text{and} \quad T \leq \frac{\psi(0)}{\theta\psi'(0)}.$$

3. Existence of a global weak solution

In this section we prove the existence of a global weak solution to (P), which is Theorem 1.3. Although the proof follows the standard arguments of Faedo–Galerkin approximation, the appearance of the fourth-order operator in (P) necessitates a detailed justification. For an ease of notation, in this section we employ the dot notation

$$\dot{u}_n = (u_n)_t = \frac{\partial}{\partial t} u_n.$$

Hereafter

$$a \wedge b := \min\{a,b\} \quad \text{and} \quad a \vee b := \max\{a,b\}.$$

Recall that we set

$$d_\Omega = \begin{cases} 5 & \text{if } 0 \in \Omega, \\ 1 & \text{if } 0 \notin \Omega \end{cases} \quad \text{and} \quad 2^* = \begin{cases} \infty & \text{if } d \leq 4, \\ \frac{2d}{d-4} & \text{if } d \geq 5 \end{cases}$$

as well as

$$\Sigma_1(t) = \big\{u \in H_0^2(\Omega) : J(u,t) < d_\infty \text{ and } I(u,t) > 0\big\}$$

and

$$\Sigma_2(t) = \big\{u \in H_0^2(\Omega) : J(u,t) < d_\infty \text{ and } I(u,t) < 0\big\}$$

for each $t \geq 0$.

We start with an approximation problem.

Lemma 3.1. *Let $d \geq d_\Omega$ and $2 < p+1 < 2^*$. Let $n \in \mathbb{N}$, $T > 0$ and $u_{n0} \in C_c^\infty(\Omega)$. Then the problem*

$$(P_n) \qquad \begin{cases} \rho_n(x)\dot{u}_n + \Delta^2 u_n = \beta_n(u_n), & (x,t) \in \Omega \times (0,T], \\ u_n(x,t) = \frac{\partial u_n}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \\ u_n(x,0) = u_{n0}, & x \in \Omega, \end{cases}$$

admits a global solution $u_n \in C([0,T]; H_0^2(\Omega))$ such that $\dot{u}_n \in L^2(0,T; H_0^2(\Omega))$, where

$$\rho_n(x) = |x|^{-4} \wedge n \quad \textit{and} \quad \beta_n(u_n) = k(t)\big[(-n) \vee (|u_n|^{p-1}u_n) \wedge n\big].$$

Proof. We proceed via three steps.

Step 1: We construct and solve an approximate problem of (P_n) whose solutions possesses certain regularity. To this end, let $\{e_k\}_{k\in\mathbb{N}} \subset H_0^2(\Omega)$ be the set of all orthonormal eigenvectors of Δ in the sense that

$$-\Delta e_j = \lambda_j e_j \quad \text{and} \quad (e_i, e_j) = \delta_{ij}$$

for all $i, j \in \mathbb{N}$, where $\lambda_j \in \mathbb{R}$ and δ_{ij} is the Kronecker's delta. Then $\{e_k\}_{k\in\mathbb{N}}$ forms a complete orthogonal basis of $H_0^2(\Omega)$ and

$$\Delta^2 e_j = \lambda_j^2 e_j.$$

Set

$$W_k = \operatorname{span}\{e_1, \ldots, e_k\}.$$

Let $\{u_{n0}\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$ be such that $u_{n0} \to u_0$ in $H_0^2(\Omega)$ as $n \to \infty$. In what follows, fix $k \in \mathbb{N}$. Write

$$u_{n0} = \sum_{j=1}^{\infty} \xi_{nj} e_j \quad \text{and} \quad u_{n0k} = \sum_{j=1}^{k} \xi_{nj} e_j \in W_k.$$

Clearly,

$$\lim_{n,k\to\infty} u_{n0k} = u_0 \quad \text{in } H_0^2(\Omega). \tag{3.1}$$

We aim to look for a solution

$$u_{nk}(t,x) = \sum_{j=1}^{k} \xi_{nkj}(t)e_j(x) \in W_k, \quad \xi_{nkj} \in C^1([0,T]) \tag{3.2}$$

to the problem

$$\int_\Omega \big[\rho_n(x)\dot{u}_{nk} + \Delta^2 u_{nk} - \beta_n(u_{nk})\big]\eta\, dx = 0, \quad u_{nk}(x,0) = u_{n0k} \tag{P_{nk}}$$

for all $\eta \in W_k$. To this purpose, use $\eta = e_i$ for each $i \in \mathbb{N}$ as a test function to obtain

$$(\rho_n \dot{u}_{nk}, e_i) = \sum_{j=1}^{k} \left(\int_\Omega \rho_n(x) e_j e_i\, dx \right) \dot{\xi}_{nkj}(t) =: \sum_{j=1}^{k} a_{ij}\dot{\xi}_{nkj}(t).$$

Clearly $a_{ij} \le n$. Furthermore, one has

$$(\Delta^2 u_{nk}, e_i) = \left(\sum_{j=1}^{k} \xi_{nkj}(t)\lambda_j^2 e_j, e_i \right) = \lambda_j^2 \xi_{nki}(t),$$

and

$$\psi_{ni} = \psi_{ni}(t, \xi_{nk1}, \ldots, \xi_{nkk}) := (\beta_n(u_{nk}), e_i).$$

Hence $\{\xi_{nkj}\}_{j=1}^{k}$ is determined by the following Cauchy problem

$$\sum_{j=1}^{k} a_{ij}\dot{\xi}_{nkj}(t) + \lambda_j^2 \xi_{nkj}(t) = \psi_{ni}, \quad \xi_{nki}(0) = \int_\Omega u_{n0} e_i\, dx. \tag{C}$$

A standard result on ODE systems now confirms the existence of a unique solution $\xi_{nki} \in C^1([0,T])$ to (C). To see this, it suffices to verify that ψ_{ni} is Lipschitz continuous with respect to t on $[0,T]$ and at the same time is Lipschitz continuous with respect to $\vec{\xi} := (\xi_{nk1}, \ldots, \xi_{nkk})$ in $(C([0,T]))^k$. Since $k(t) \in C^1([0,\infty))$ by hypothesis, the former is clear. To prove the latter, first observe that for all $s \in \mathbb{R}$, one has

$$(-n) \vee (|s|^{p-1}s) \wedge n = |\mathfrak{s}|^{p-1}\mathfrak{s},$$

where $\mathfrak{s} := (-n^{1/p}) \vee s \wedge n^{1/p}$. Now let $\vec{A}, \vec{B} \in (C([0,T]))^k$ and write

$$a = \sum_{j=1}^{k} A_j e_j \quad \text{and} \quad b = \sum_{j=1}^{k} B_j e_j.$$

Correspondingly, we have

$$\mathfrak{a} = (-n^{1/p}) \vee a \wedge n^{1/p} \quad \text{and} \quad \mathfrak{b} = (-n^{1/p}) \vee b \wedge n^{1/p}.$$

Then

$$
\begin{aligned}
&\left|\psi_{ni}(t_0,\vec{A})-\psi_{ni}(t_0,\vec{B})\right| \\
&= k(t_0)\left|\left(|\mathfrak{a}|^{p-1}\mathfrak{a},e_i\right)-\left(|\mathfrak{b}|^{p-1}\mathfrak{b},e_i\right)\right| \\
&= k(t_0)\Bigg|\left([\mathfrak{a}-\mathfrak{b}]\int_0^1|\mathfrak{a}+z(\mathfrak{a}-\mathfrak{b})|^{p-1}\,dz,e_i\right) \\
&\qquad +(p-1)\left(\int_0^1|\mathfrak{a}+z(\mathfrak{a}-\mathfrak{b})|^{p-3}[\mathfrak{a}+z(\mathfrak{a}-\mathfrak{b})]^2[\mathfrak{a}-\mathfrak{b}]\,dz,e_i\right)\Bigg| \\
&\le k(t_0)p3^pn\|\mathfrak{a}-\mathfrak{b}\|_{L^2(\Omega)}\|e_i\|_{L^2(\Omega)} \le k(t_0)p3^pn\|a-b\|_{L^2(\Omega)}\|e_i\|_{L^2(\Omega)} \\
&\le k(t_0)p3^pn\|\vec{A}-\vec{B}\|_{(C([0,T]))^k}\left(\sum_{j=1}^k\|e_j\|_{L^2(\Omega)}\right)\|e_i\|_{L^2(\Omega)}
\end{aligned}
$$

for each fixed $t_0\in[0,T]$, where we used [6, (IV), p. 96] in the second step and the fact that $|\mathfrak{a}-\mathfrak{b}|\le|a-b|$ a.e. in $[0,T]\times\Omega$ in the fourth step. That is, ψ_{ni} is Lipschitz continuous on $(C([0,T]))^k$ as claimed.

Thus there exists a unique solution u_{nk} given by (3.2) to (P_{nk}) due to the prescribed boundary conditions.

Step 2: We aim to derive some a priori estimates. For convenience, each estimate is considered in a sub-step.

(i) We show that $\{\rho_n^{1/2}u_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and $\{u_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H^2(\Omega))$.

Using $\eta=u_{nk}$ as a test function in (P_{nk}), we derive

$$
\begin{aligned}
&\int_\Omega\rho_nu_{nk}^2(t)\,dx+2\int_0^t\|\Delta u_{nk}(s)\|^2_{L^2(\Omega)}\,ds \\
&=2\int_0^t\int_\Omega\beta_n(u_{nk}(s))u_{nk}(s)\,dxds+\int_\Omega\rho_n|u_{n0k}|^2\,dx.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
2\int_0^t\int_\Omega\beta_n(u_{nk}(s))u_{nk}(s)\,dxds &\le 2n\int_0^t\int_\Omega u_{nk}(s)\,dxds \\
&\le 2n\operatorname{diam}(\Omega)^2|\Omega|^{1/2}\int_0^t\left\|\frac{u_{nk}(s)}{x^2}\right\|_{L^2(\Omega)}ds \\
&\le 2n\operatorname{diam}(\Omega)^2|\Omega|^{1/2}\mathcal{R}^{1/2}\int_0^t\|\Delta u_{nk}(s)\|_{L^2(\Omega)}\,ds \\
&\le 4n^2\operatorname{diam}(\Omega)^2|\Omega|\mathcal{R}T+\int_0^t\|\Delta u_{nk}(s)\|^2_{L^2(\Omega)}\,ds \\
&=: K_1T+\int_0^t\|\Delta u_{nk}(s)\|^2_{L^2(\Omega)}\,ds,
\end{aligned}
$$

where we used Lemma 2.1 in the third step. It follows that

$$\int_\Omega \rho_n u_{nk}^2(t)\,dx + \int_0^t \|\Delta u_{nk}(s)\|^2_{L^2(\Omega)}\,ds \le K_1 T + \int_\Omega \rho_n |u_{n0k}|^2\,dx.$$

Due to (3.1) we conclude that $\{\rho_n^{1/2} u_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and $\{u_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H^2(\Omega))$.

(ii) We show that $\{\rho_n^{1/2}\dot{u}_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$ and $\{u_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H_0^2(\Omega))$.

It follows from (3.2) that $\dot{u}_{nk} \in H_0^2(\Omega)$. Therefore using $\eta = \dot{u}_{nk}$ as a test function in (P_{nk}) leads to

$$\begin{aligned} &2\int_0^t\int_\Omega \rho_n \dot{u}_{nk}^2(s)\,dxds + \|\Delta u_{nk}(t)\|^2_{L^2(\Omega)} \\ &= 2\int_0^t\int_\Omega \beta_n(u_{nk}(s))\dot{u}_{nk}(s)\,dxds + \|\Delta u_{nk}(0)\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.3}$$

But

$$\begin{aligned} 2\int_0^t\int_\Omega \beta_n(u_{nk}(s))\dot{u}_{nk}(s)\,dxds &\le \int_0^t\int_\Omega \frac{|\beta_n(u_{nk}(s))|^2}{\rho_n}\,dxds + \int_0^t\int_\Omega \rho_n \dot{u}_{nk}(s)^2\,dxds \\ &\le \int_0^t\int_\Omega \frac{n^2}{\rho_n}\,dxds + \int_0^t\int_\Omega \rho_n \dot{u}_{nk}(s)^2\,dxds. \end{aligned}$$

Define

$$\Omega_1 = \{x\in\Omega : |x|^{-4} \ge n\} \quad \text{and} \quad \Omega_2 = \{x\in\Omega : |x|^{-4} < n\}.$$

Then

$$\begin{aligned} \int_0^t\int_\Omega \frac{n^2}{\rho_n}\,dxds &= \int_0^t\int_{\Omega_1} \frac{n^2}{\rho_n}\,dxds + \int_0^t\int_{\Omega_2} \frac{n^2}{\rho_n}\,dxds \\ &= n|\Omega_1|t + \int_0^t\int_{\Omega_2} \frac{n^2}{|x|^{-4}}\,dxds \\ &\le n|\Omega_1|T + n^2\operatorname{diam}(\Omega)^4|\Omega_2|T =: K_2 T. \end{aligned}$$

Consequently,

$$2\int_0^t\int_\Omega \beta_n(u_{nk}(s))\dot{u}_{nk}(s)\,dxds \le K_2 T + \int_0^t\int_\Omega \rho_n \dot{u}_{nk}(s)^2\,dxds.$$

Combining this last display with (3.3), we arrive at

$$\int_0^t\int_\Omega \rho_n \dot{u}_{nk}^2(s)\,dxds + \|\Delta u_{nk}(t)\|^2_{L^2(\Omega)} \le K_2 T + \|\Delta u_{nk}(0)\|^2_{L^2(\Omega)}.$$

Again using (3.1) we infer that $\{\rho_n^{1/2}\dot{u}_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$ and $\{u_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H_0^2(\Omega))$.

(iii) We show that $\{\dot{u}_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$. As a by-product, we obtain $u_{nk} \in C([0,T];L^2(\Omega))$.

Notice that

$$\begin{aligned}\int_0^t\int_\Omega \dot{u}_{nk}(s)^2\,dxds &= \int_0^t\int_\Omega \frac{1}{\rho_n}\rho_n\dot{u}_{nk}(s)^2\,dxds\\ &= \int_0^t\int_{\Omega_1} \frac{1}{\rho_n}\rho_n\dot{u}_{nk}(s)^2\,dxds + \int_0^t\int_{\Omega_2} \frac{1}{\rho_n}\rho_n\dot{u}_{nk}(s)^2\,dxds\\ &\le \left(\frac{1}{n}+\mathrm{diam}(\Omega)^4\right)\int_0^t\int_\Omega \rho_n\dot{u}_{nk}(s)^2\,dxds.\end{aligned}$$

Hence $\{\dot{u}_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$.

(iv) We show that $\{\rho_n\dot{u}_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and $\{\dot{u}_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H_0^2(\Omega))$. Hence as a by-product, we also have $u_{nk} \in C([0,T];H_0^2(\Omega))$.

First we show that $\rho_n\ddot{u}_{nk} \in H^{-2}(\Omega)$. As $\dot{u}_{nk} \in H_0^2(\Omega)$, we have $\Delta^2\dot{u}_{nk} \in H^{-2}(\Omega)$. Then

$$\begin{aligned}|(\rho_n\ddot{u}_{nk},\eta)| &= \big|-(\Delta^2\dot{u}_{nk},\eta)+((\beta_n(u_{nk}))_t,\eta)\big|\\ &\le \big|-(\Delta^2\dot{u}_{nk},\eta)\big| + p\big|\big(k(t)|u_{nk}|^{p-1}\dot{u}_{nk},\eta\big)\big| + \big|\big(k'(t)|u_{nk}|^{p-1}u_{nk},\eta\big)\big|\\ &\le \|\Delta^2\dot{u}_{nk}\|_{H^{-2}(\Omega)}\|\eta\|_{H_0^2(\Omega)}\\ &\quad + pk(T)\big\||u_{nk}|^{p-1}\big\|_{L^{d/4}(\Omega)}\|\dot{u}_{nk}\|_{L^{2d/(d-4)}(\Omega)}\|\eta\|_{L^{2d/(d-4)}(\Omega)}\\ &\quad + \left(\sup_{0\le s\le T}k'(s)\right)\big\||u_{nk}|^{p-1}\big\|_{L^{d/4}(\Omega)}\|u_{nk}\|_{L^{2d/(d-4)}(\Omega)}\|\eta\|_{L^{2d/(d-4)}(\Omega)}\\ &= \|\Delta^2\dot{u}_{nk}\|_{H^{-2}(\Omega)}\|\eta\|_{H_0^2(\Omega)}\\ &\quad + pk(T)\|u_{nk}\|^{p-1}_{L^{(p-1)d/4}(\Omega)}\|\dot{u}_{nk}\|_{L^{2^*}(\Omega)}\|\eta\|_{L^{2^*}(\Omega)}\\ &\quad + \left(\sup_{0\le s\le T}k'(s)\right)\|u_{nk}\|^{p-1}_{L^{d/4}(\Omega)}\|u_{nk}\|_{L^{2d/(d-4)}(\Omega)}\|\eta\|_{L^{2d/(d-4)}(\Omega)}\end{aligned}$$

for all $\eta \in H_0^2(\Omega)$. By hypothesis, $p+1<2^*$, from which it follows that $(p-1)d/4 < 2^*$. Therefore, we further obtain

$$\begin{aligned}|(\rho_n\ddot{u}_{nk},\eta)| &\le \|\Delta^2\dot{u}_{nk}\|_{H^{-2}(\Omega)}\|\eta\|_{H_0^2(\Omega)}\\ &\quad + pk(T)|\Omega|^{\frac{4}{d}-\frac{p-1}{2^*}}\|u_{nk}\|^{p-1}_{L^{2^*}(\Omega)}\|\dot{u}_{nk}\|_{L^{2^*}(\Omega)}\|\eta\|_{L^{2^*}(\Omega)}\\ &\quad + \left(\sup_{0\le s\le T}k'(s)\right)|\Omega|^{\frac{4}{d}-\frac{p-1}{2^*}}\|u_{nk}\|^{p}_{L^{2^*}(\Omega)}\|\eta\|_{L^{2^*}(\Omega)}\\ &< \infty\end{aligned}$$

for all $\eta \in H_0^2(\Omega)$, where we use the convention that $1/\infty = 0$ in case $2^* = \infty$ hereafter. This means $\rho_n \ddot{u}_{nk} \in H^{-2}(\Omega)$.

Next choose $\eta = \dot{u}_{nk}$ to be a test function in the equation

$$\rho_n(x)(u_{nk})_{tt} + \Delta^2 \dot{u}_{nk} = (\beta_n(u_{nk}))_t.$$

Then

$$\begin{aligned}
&\int_\Omega \rho_n \dot{u}_{nk}(t)^2\,dx - \int_\Omega \rho_n \dot{u}_{nk}(0)^2\,dx + 2\int_0^t\int_\Omega |\Delta \dot{u}_{nk}|^2\,dx \\
&= \int_0^t\int_\Omega (\beta_n(u_{nk}))_t \dot{u}_{nk}\,dx \\
&= \begin{cases} 0 & \text{if } \beta_n(u_{nk}) = n, \\ 2p\int_0^t\int_\Omega k(s)|u_{nk}(s)|^{p-1}\dot{u}_{nk}(s)^2\,dxds & \\ \quad +2\int_0^t\int_\Omega k'(s)|u_{nk}(s)|^{p-1}u_{nk}(s)\dot{u}_{nk}(s)\,dxds & \text{if } \beta_n(u_{nk}) = k(t)|u_{nk}|^{p-1}u_{nk}. \end{cases}
\end{aligned} \tag{3.4}$$

Next we estimate the right-hand side of (3.4). Again notice that $p+1 < 2^*$ implies $(p-1)d/4 < 2^*$. Let $\varepsilon \in (0,1)$ be such that $(p-1)(\varepsilon + d/4) < 2^*$. Then

$$\frac{2(d+4\varepsilon)}{d+4\varepsilon-4} = 2 + \frac{8}{d+4\varepsilon-4} < 2 + \frac{8}{d-4} = 2^*$$

and

$$\begin{aligned}
\int_\Omega |u_{nk}|^{p-1}\dot{u}_{nk}^2\,dx &\le \big\| |u_{nk}|^{p-1} \big\|_{L^{\varepsilon+d/4}(\Omega)} \big\| \dot{u}_{nk}^2 \big\|_{L^{(d+4\varepsilon)/(d+4\varepsilon-4)}(\Omega)} \\
&= \|u_{nk}\|^{p-1}_{L^{(p-1)(\varepsilon+d/4)}(\Omega)} \|\dot{u}_{nk}\|^2_{L^{2(d+4\varepsilon)/(d+4\varepsilon-4)}(\Omega)} \\
&\le C|\Omega|^{\frac{4}{d+4\varepsilon} - \frac{p-1}{2^*}} \|u_{nk}\|^{p-1}_{L^{2^*}(\Omega)} \|\Delta \dot{u}_{nk}\|^{\alpha}_{L^2(\Omega)} \|\dot{u}_{nk}\|^{1-\alpha}_{L^2(\Omega)},
\end{aligned}$$

where

$$\alpha := \frac{d}{2}\left(\frac{1}{2} - \frac{d+4\varepsilon-4}{2(d+4\varepsilon)}\right) = \frac{d}{d+4\epsilon} \in (0,1)$$

and we used Hölder's inequality and Lemma 2.2 in the last step.

Thus

$$\begin{aligned}
&\int_0^t\int_\Omega k(s)|u_{nk}(s)|^{p-1}\dot{u}_{nk}(s)^2\,dxds \\
&\le Ck(T)\int_0^t \|u_{nk}(s)\|^{p-1}_{L^{2^*}(\Omega)} \|\Delta \dot{u}_{nk}(s)\|^{\alpha}_{L^2(\Omega)} \|\dot{u}_{nk}(s)\|^{1-\alpha}_{L^2(\Omega)}\,ds \\
&\le \frac{1}{4}\int_0^t \|\Delta \dot{u}_{nk}(s)\|^2_{L^2(\Omega)}\,ds + C\int_0^t \|\dot{u}_{nk}(s)\|^{2(1-\alpha)/(2-\alpha)}_{L^2(\Omega)}\,ds \\
&\le \frac{1}{4}\int_0^t \|\Delta \dot{u}_{nk}(s)\|^2_{L^2(\Omega)}\,ds + C\left(T + \sup_{k\in\mathbb{N}} \|\dot{u}_{nk}\|_{L^2(0,T;L^2(\Omega))}\right),
\end{aligned}$$

where we used the fact that $\{u_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H_0^2(\Omega))$ in the second step as well as Young's inequality and the fact that $\{\dot{u}_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$ in the third step.

Analogously,

$$\begin{aligned}\int_\Omega |u_{nk}|^{p-1}u_{nk}\dot{u}_{nk}\,dx &\le \big\||u_{nk}|^{p-1}\big\|_{L^{d/4}(\Omega)}\|u_{nk}\|_{L^{2d/(d-4)}(\Omega)}\|\dot{u}_{nk}\|_{L^{2d/(d-4)}(\Omega)}\\ &= \|u_{nk}\|^{p-1}_{L^{(p-1)d/4}(\Omega)}\|u_{nk}\|_{L^{2^*}(\Omega)}\|\dot{u}_{nk}\|_{L^{2^*}(\Omega)}\\ &\le C|\Omega|^{\frac{4}{d}-\frac{p-1}{2^*}}\|u_{nk}\|^p_{L^{2^*}(\Omega)}\|\Delta\dot{u}_{nk}\|_{L^2(\Omega)},\end{aligned}$$

where we used Hölder's inequality and Lemma 2.2 in the last step.

It follows that

$$\begin{aligned}&\int_0^t\int_\Omega k'(s)|u_{nk}(s)|^{p-1}u_{nk}(s)\dot{u}_{nk}(s)\,dxds\\ &\le C\left(\sup_{0\le s\le T}k'(s)\right)\int_0^t\|u_{nk}(s)\|^p_{L^{2^*}(\Omega)}\|\Delta\dot{u}_{nk}(s)\|_{L^2(\Omega)}\,ds\\ &\le \frac{1}{4}\int_0^t\|\Delta\dot{u}_{nk}(s)\|^2_{L^2(\Omega)}\,ds+CT,\end{aligned}$$

where we used the fact that $\{u_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H_0^2(\Omega))$ and Young's inequality in the second step.

Back to (3.4), we now obtain

$$\begin{aligned}&\int_\Omega \rho_n\dot{u}_{nk}(t)^2\,dx+\int_0^t\int_\Omega|\Delta\dot{u}_{nk}(s)|^2\,dxds\\ &\le\begin{cases}\int_\Omega\rho_n\dot{u}_{nk}(0)^2\,dx & \text{if } \beta_n(u_{nk})=n,\\ \int_\Omega\rho_n\dot{u}_{nk}(0)^2\,dx+C\left(T+\sup_{k\in\mathbb{N}}\|\dot{u}_{nk}\|_{L^2(0,T;L^2(\Omega))}\right) & \text{if } \beta_n(u_{nk})=k(t)|u_{nk}|^{p-1}u_{nk}.\end{cases}\end{aligned}$$

Since $u_{nk}(0)\in C_c^\infty(\Omega)$ satisfies

$$\rho_n\dot{u}_{nk}(0)+\Delta^2u_{nk}(0)=\beta_nu_{nk}(0),$$

we may conclude that $\{\rho_n\dot{u}_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and $\{\dot{u}_{nk}\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H_0^2(\Omega))$.

Step 3: We acquire a global weak solution to (P_n). Having achieved the a-priori estimates in Step 2, by using a subsequence when necessary, we may now let $k\to\infty$ in (P_{nk}) to obtain a weak solution u_n to (P_n) with the required regularity. □

In view of Lemma 3.1, we are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Since $u_0 \in \Sigma_1(0)$, there exists a constant $\epsilon_0 > 0$ such that

$$J(u_0, 0) + \epsilon_0 < d_\infty.$$

By Lemma 3.1 for each $n \in \mathbb{N}$ there exists a weak solution $u_n \in C([0,T]; H_0^2(\Omega))$ with $\dot{u}_n \in L^2(0,T; H_0^2(\Omega))$ to the problem (P_n), where $u_{n0} \in C_c^\infty(\Omega)$ is such that

$$\lim_{n\to\infty} u_{n0} = u_0 \quad \text{in } H_0^2(\Omega).$$

By choosing a sufficiently large $n \in \mathbb{N}$, we may assume also that

$$J(u_{n0}, 0) \le J(u_0, 0) + \epsilon_0 < d_\infty. \tag{3.5}$$

Using $\dot{u}_n$ as a test function in (P_n), we derive that

$$\begin{aligned}&\int_0^t\int_\Omega \rho_n \dot{u}_n(s)^2\, dxds + \int_0^t\int_\Omega \Delta^2 u_n(s)\dot{u}_n(s)\, dxds \\ &= \int_0^t\int_\Omega \beta_n(u_n)\dot{u}_n(s)\, dxds \le \int_0^t\int_\Omega |u_n(s)|^{p-1}u_n(s)\dot{u}_n(s)\, dxds.\end{aligned}$$

On noticing that

$$\int_\Omega \Delta^2 u_n \dot{u}_n\, dx = \frac{d}{dt}\left(\frac{1}{2}\int_\Omega \|\Delta u_n\|^2_{L^2(\Omega)}\, dx\right)$$

and

$$\int_\Omega |u_n|^{p-1}u_n\dot{u}_n\, dx = \frac{d}{dt}\left(\frac{1}{p+1}\int_\Omega \|u_n\|^{p+1}_{L^{p+1}(\Omega)}\, dx\right),$$

we can rewrite the above inequality as

$$\int_0^t\int_\Omega \rho_n\dot{u}_n(s)^2\, dxds + J(u_n(t), t) \le J(u_{n0}, 0) < d_\infty, \tag{3.6}$$

where we used (3.5) in the last step. This implies $u_n(t) \in \Sigma_1$ for each $t \in [0,T]$. Indeed, by way of contradiction we assume the opposite statement holds. Let t^* be the minimal time at which $u_n(t^*) \notin \Sigma_1$. Then using the fact that $u_n \in C([0,T]; H_0^2(\Omega))$ we infer $u_n(t^*) \in \partial\Sigma_1$. That is, either $J(u_n(t^*), t^*) = d_\infty$ or $I(u_n(t^*), t^*) = 0$. The former is impossible due to (3.6). Consequently, we must have $I(u_n(t^*), t^*) = 0$ or equivalently

$$\|\Delta u_n(t^*)\|^2_{L^2(\Omega)} = k(t^*)\|u_n(t^*)\|^{p+1}_{L^{p+1}(\Omega)},$$

whence

$$\begin{aligned}J(u_n(t^*), t^*) &= \frac{p-1}{2(p+1)}\|\Delta u_n(t^*)\|^2_{L^2(\Omega)} \ge \frac{p-1}{2(p+1)}S_p^{-2}\|u_n(t^*)\|^2_{L^{p+1}(\Omega)} \\ &= \frac{p-1}{2(p+1)}S_p^{-2}\left(\frac{k(t^*)^{-1/2}\|\Delta u_n(t^*)\|_{L^2(\Omega)}}{\|u\|_{L^{p+1}(\Omega)}}\right)^{\frac{2}{p+1}\left(\frac{1}{2}-\frac{1}{p+1}\right)^{-1}} \\ &\ge \frac{p-1}{2(p+1)}k(t^*)^{2/(1-p)}S_p^{-2(p+1)/(p-1)} = d(t^*) \ge d_\infty.\end{aligned}$$

This contradicts (3.6). Hence $u_n(t) \in \Sigma_1(t)$ for each $t \in [0,T]$ as claimed.

Let $t \in [0,T]$. Then $u_n(t) \in \Sigma_1(t)$ implies

$$\|\Delta u_n(t)\|^2_{L^2(\Omega)} > k(t)\|u_n(t)\|^{p+1}_{L^{p+1}(\Omega)}.$$

Using (3.6) we further obtain

$$\int_0^t\int_\Omega \rho_n \dot{u}_n(s)^2\,dxds + \left(\frac{1}{2} - \frac{k(t)}{p+1}\right)\|\Delta u_n(t)\|^2_{L^2(\Omega)} < J(u_{n0},0) < d_\infty. \tag{3.7}$$

In particular, one has

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{p+1}\right)\|\Delta u_n(t)\|^2_{L^2(\Omega)} &= \left(\frac{1}{2} - \frac{k_\infty}{p+1}\right)\|\Delta u_n(t)\|^2_{L^2(\Omega)}\\ &< \left(\frac{1}{2} - \frac{k(t)}{p+1}\right)\|\Delta u_n(t)\|^2_{L^2(\Omega)} < J(u_{n0},0),\end{aligned} \tag{3.8}$$

where $k_\infty := \lim_{t\to\infty} k(t) = 1$ by hypothesis. Now it follows from Lemma 2.3, (3.8) and (3.5) that

$$\begin{aligned}\int_\Omega |u_n(t)|^{p+1}\,dx &< S_p^{p+1}\big(\|\Delta u_n(t)\|^2_{L^2(\Omega)}\big)^{(p+1)/2}\\ &= S_p^{p+1}\big(\|\Delta u_n(t)\|^2_{L^2(\Omega)}\big)^{(p+1)/2-1}\|\Delta u_n(t)\|^2_{L^2(\Omega)}\\ &< S_p^{p+1}\left[\left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} J(u_{n0},0)\right]^{(p+1)/2-1}\|\Delta u_n(t)\|^2_{L^2(\Omega)}\\ &< S_p^{p+1}\left[\left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} (J(u_0,0)+\epsilon_0)\right]^{(p+1)/2-1}\|\Delta u_n(t)\|^2_{L^2(\Omega)}\\ &=: \delta\|\Delta u_n(t)\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.9}$$

Note that

$$0 < \delta < S_p^{p+1}\left[\left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} d_\infty\right]^{(p+1)/2-1} = \left[\left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} \frac{p-1}{2(p+1)}\right]^{(p-1)/2} = 1.$$

Next we use u_n as a test function in (P_n) to arrive at

$$\begin{aligned}\frac{1}{2}\int_\Omega \rho_n u_n^2\,dx + \int_0^t\int_\Omega |\Delta u_n(s)|^2\,dxds &\le \int_0^t\int_\Omega |u_n(s)|^{p+1}\,dxds + \frac{1}{2}\int_\Omega \rho_n u_{n0}^2\,dx\\ &< \delta\int_0^t\int_\Omega |\Delta u_n(s)|^2\,dxds + \frac{1}{2}\int_\Omega \rho_n u_{n0}^2\,dx,\end{aligned}$$

where we used (3.9) in the second step.

It follows that

$$\frac{1}{2}\int_\Omega \rho_n u_n^2\,dx + (1-\delta)\int_0^t\int_\Omega |\Delta u_n(s)|^2\,dxds < \frac{1}{2}\int_\Omega \rho_n u_{n0}^2\,dx < C, \tag{3.10}$$

where $C > 0$ is independent of n and T. Consequently, $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H_0^2(\Omega))$.

By (3.7) and (3.10), the following properties hold:

$$\begin{cases} u_n \to u & \text{a.e. in } (0,T)\times\Omega,\\ \rho_n^{1/2} u_n \overset{w}{\to} \frac{u_t}{|x|^2} & \text{in } L^2(0,T;L^2(\Omega)),\\ \Delta u_n \overset{w}{\to} \Delta u & \text{in } L^2(0,T;L^2(\Omega)),\\ u_n \overset{w}{\to} u & \text{in } L^2(0,T;L^{p+1}(\Omega)),\\ u_n \overset{w}{\to} u & \text{in } L^\infty(0,T;L^{p+1}(\Omega)) \end{cases}$$

for all $T > 0$. The theorem now follows by taking limits when $n \to \infty$ in (P_n). Since $T > 0$ is arbitrary, the solution is global. □

Next we prove Theorem 1.4.

Proof of Theorem 1.4. Let $t \in [0,T)$. By repeating the arguments used to obtain (3.8) in the proof of Theorem 1.3, we also have that

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \|\Delta u_n(t)\|^2_{L^2(\Omega)} < J(u_0, 0).$$

This and Lemma 2.3 together imply

$$\begin{aligned} \int_\Omega |u(t)|^{p+1}\,dx &< S_p^{p+1}\left(\|\Delta u(t)\|^2_{L^2(\Omega)}\right)^{(p+1)/2} \\ &= S_p^{p+1}\left(\|\Delta u(t)\|^2_{L^2(\Omega)}\right)^{(p+1)/2-1}\|\Delta u(t)\|^2_{L^2(\Omega)} \\ &< S_p^{p+1}\left[\left(\frac{1}{2}-\frac{1}{p+1}\right)^{-1} J(u_0,0)\right]^{(p+1)/2-1} \|\Delta u(t)\|^2_{L^2(\Omega)} \\ &=: \delta\|\Delta u(t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.11}$$

Note that

$$0 < \delta < S_p^{p+1}\left[\left(\frac{1}{2}-\frac{1}{p+1}\right)^{-1} d_\infty\right]^{(p+1)/2-1} = \left[\left(\frac{1}{2}-\frac{1}{p+1}\right)^{-1}\frac{p-1}{2(p+1)}\right]^{(p-1)/2} = 1.$$

Hence (3.11) leads to

$$\begin{aligned} I(u(t),t) &= \|\Delta u(t)\|^2_{L^2(\Omega)} - k(t)\|u\|^{p+1}_{L^{p+1}(\Omega)} \\ &\ge (1-\delta k(t))\|\Delta u(t)\|^2_{L^2(\Omega)} \ge (1-\delta)\|\Delta u(t)\|^2_{L^2(\Omega)} \end{aligned} \tag{3.12}$$

since k is increasing on $[0,\infty)$ with $\lim_{t\to\infty} k(t) = 1$ by assumptions.

Next observe that

$$J(u(t),t) = \frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} - \frac{k(t)}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)} = \frac{p-1}{2(p+1)}\|\Delta u\|^2_{L^2(\Omega)} + \frac{1}{p+1}I(u(t),t).$$

Two consequences are immediate from this equality. First using Lemma 2.6(i), we arrive at

$$J(u_0,0) > J(u(t),t) > \frac{p-1}{2(p+1)}\|\Delta u\|^2_{L^2(\Omega)} \tag{3.13}$$

as $I(u(t),t) > 0$. Secondly using (3.12), we have

$$J(u(t),t) \le \left(\frac{p-1}{2(p+1)(1-\delta)} + \frac{1}{p+1}\right) I(u(t),t).$$

Next we use Lemma 2.6(ii) to derive

$$\begin{aligned} \int_t^T I(u(s),s)\,ds &= -\int_t^T \left(\frac{u(s)}{|x|^2}, u_t(s)\right) ds = -\int_t^T L'(s)\,ds \\ &= L(t) - L(T) \le L(t) \le \frac{\mathcal{R}}{2}\int_\Omega |\Delta u(t)|^2\,dx, \end{aligned} \tag{3.14}$$

where

$$L(t) := \frac{1}{2}\left\|\frac{u(t)}{|x|^2}\right\|^2_{L^2(\Omega)}$$

for each $t \in [0,T)$ and we used Lemma 2.1 in the last step.

Combining (3.12), (3.13) and (3.14) together yields

$$\begin{aligned} \int_t^T J(u(s),s)\,ds &\le \left(\frac{p-1}{2(p+1)(1-\delta)} + \frac{1}{p+1}\right)\int_t^T I(u(s),s)\,ds \\ &\le \left(\frac{p-1}{2(p+1)(1-\delta)} + \frac{1}{p+1}\right)\frac{(p+1)\mathcal{R}}{p-1}J(u(t),t) =: AJ(u(t),t). \end{aligned}$$

Letting $T \to \infty$ in the above inequality, one has

$$\int_t^\infty J(u(s),s)\,ds \le AJ(u(t),t).$$

Set

$$M(t) = \int_t^\infty J(u(s),s)\,ds.$$

Then the above inequality can be rewritten as

$$M'(t) \le -\frac{1}{A}M(t).$$

Using Gronwall's inequality, we deduce that

$$M(t) \le M(A)\exp\left(-\frac{t-A}{A}\right) \le AJ(u(A),A)\exp\left(1 - \frac{t}{A}\right).$$

In addition, as $J(u(\cdot),\cdot)$ is decreasing on $[0,\infty)$, it holds that

$$M(t) \geq \int_t^{A+t} J(u(s),s)\,ds \geq AJ(u(A+t),A+t).$$

Hence

$$J(u(A+t),A+t) \leq J(u(A),A)\exp\left(1-\frac{t}{A}\right) \tag{3.15}$$

provided that $t > A$.

Lastly, we combine (3.15) with (3.13) to arrive at

$$\begin{aligned}\|\Delta u(A+t)\|^2_{L^2(\Omega)} &< \frac{2(p+1)}{p-1}J(u(A+t),A+t)\\ &< \frac{2(p+1)}{p-1}J(u(A),A)\exp\left(1-\frac{t}{A}\right) = Be^{-\alpha t},\end{aligned}$$

where

$$B = \frac{2(p+1)}{p-1}J(u(A),A)e \quad \text{and} \quad \alpha = \frac{1}{A}.$$

The proof is complete. □

4. Upper bound for blow-up time

In this section we work with the upper bounds for the blow-up time. These are the contents of Theorems 1.5 and 1.6. Recall that we set

$$L(t) = \frac{1}{2}\left\|\frac{u(t)}{|x|^2}\right\|^2_{L^2(\Omega)}$$

for each $t \in [0,T)$.

First we prove Theorem 1.5 which deals with the case of u_0 being in the unstable set.

Proof of Theorem 1.5. We aim to show that the maximal existence time $T < \infty$ and then to provide an upper bound for T. We divide the proof into two steps.

Step 1: We will show that

$$I(u(t),t) < 0 \quad \text{for all } t \in [0,T). \tag{4.1}$$

Since $I(u(t),t)$ is continuous as a function of t over $[0,T)$, using the fact that $I(u_0,0) < 0$ we deduce that there exists a $t_1 \in (0,T)$ such that

$$I(u(t),t) < 0$$

for all $t \in [0, t_1)$. If there is a $t_2 \in (0, T)$ such that $t_2 > t_1$, $I(u(t_2), t_2) = 0$ and $I(u(t), t) < 0$ for all $t \in [0, t_2)$, then

$$J(u(t_2), t_2) \geq \inf_{w \in \mathcal{N}(t_2)} J(w, t_2) = d(t_2) \tag{4.2}$$

by virtue of (1.4). On the other hand, Lemmas 2.6(i) and 2.7(ii) together give

$$J(u(t), t) \leq J(u_0, 0) < d_\infty < d(t)$$

for all $t \in [0, T)$. This means (4.2) is impossible, which implies (4.1).

Step 2: In view of Step 1,

$$\|\Delta u\|^2_{L^2(\Omega)} < k(t)\|\Delta u\|^{p+1}_{L^{p+1}(\Omega)} \leq k(t) S_p^{p+1} \|\Delta u\|^{p+1}_{L^2(\Omega)}$$

for all $t \in [0, T)$, where we used Lemma 2.3 in the second step. Therefore, Lemma 2.7(i) and the hypothesis that $J(u_0, 0) < d_\infty$ implies

$$\|\Delta u\|^2_{L^2(\Omega)} \geq \frac{2(p+1)}{p-1} d(t) \geq \frac{2(p+1)}{p-1} d_\infty > \frac{2(p+1)}{p-1} J(u_0, 0) \tag{4.3}$$

for all $t \in [0, T)$.

Next fix $\tau \in [0, T)$ as well as

$$\beta \in \left(0, \frac{p+1}{p}(d_\infty - J(u_0, 0))\right) \quad \text{and} \quad \sigma \in \left(\frac{L(0)}{(p-1)\beta}, \infty\right). \tag{4.4}$$

The choices of β and σ are justified below by (4.6) and (4.7) respectively. Define the nonnegative functional

$$G(h) = \int_0^h L(s)\, ds + (\tau - h)L(0) + \beta(h + \sigma)^2,$$

where $h \in [0, \tau]$. Then

$$G'(h) = L(h) - L(0) + 2\beta(h + \sigma) = 2\int_0^h \left(\frac{u(s)}{|x|^2}, u_t(s)\right) ds + 2\beta(h + \sigma)$$

and

$$\begin{aligned} G''(h) &= 2\left(\frac{u(h)}{|x|^2}, u_t(h)\right) + 2\beta = -2I(u(h), h) + 2\beta \\ &= -2(p+1)J(u(h), h) + (p-1)\|\Delta u(h)\|^2_{L^2(\Omega)} + 2\beta \\ &= -2(p+1)\left[J(u_0, 0) - \int_0^h \left(\left\|\frac{u_t(s)}{|x|^2}\right\|^2_{L^2(\Omega)} + \frac{k'(s)}{p+1}\|u(s)\|^{p+1}_{L^{p+1}(\Omega)}\right) ds\right] \\ &\quad + (p-1)\|\Delta u(h)\|^2_{L^2(\Omega)} + 2\beta \end{aligned} \tag{4.5}$$

for each $h \in [0,\tau]$, where we used Lemma 2.6 in the fourth step.

In what follows, it is convenient to denote

$$\begin{aligned}\theta(h) &= \left(2\int_0^h L(s)\,ds + \beta(h+\sigma)^2\right)\left(\int_0^h \left\|\frac{u_t(s)}{|x|^2}\right\|_{L^2(\Omega)}^2 ds + \beta\right)\\ &\quad - \left(\int_0^h \left(\frac{u(s)}{|x|^2}, u_t(s)\right) ds + \beta(h+\sigma)\right)^2 \geq 0\end{aligned}$$

for each $h \in [0,\tau]$, where we used Cauchy–Schwartz inequality to verify the last step.

In view of Lemma 2.8, consider

(4.6)

$$\begin{aligned}&G(h)G''(h) - \frac{p+1}{2}(G'(h))^2\\ &= G(h)G''(h) - 2(p+1)\left[\int_0^h \left(\frac{u(s)}{|x|^2}, u_t(s)\right) ds + \beta(h+\sigma)\right]^2\\ &= G(h)G''(h) + 2(p+1)\left[\theta(h) - (G(h) - (\tau-h)L(0))\left(\int_0^h \left\|\frac{u_t(s)}{|x|^2}\right\|_{L^2(\Omega)}^2 ds + \beta\right)\right]\\ &\geq G(h)G''(h) - 2(p+1)G(h)\left(\int_0^h \left\|\frac{u_t(s)}{|x|^2}\right\|_{L^2(\Omega)}^2 ds + \beta\right)\\ &\geq G(h)\left[G''(h) - 2(p+1)\left(\int_0^h \left\|\frac{u_t(s)}{|x|^2}\right\|_{L^2(\Omega)}^2 ds + \beta\right)\right]\\ &= G(h)\big[-2(p+1)J(u_0,0) + (p-1)\|\Delta u(h)\|_{L^2(\Omega)}^2 - 2p\beta\big]\\ &\geq G(h)\big[2(p+1)(d_\infty - J(u_0,0)) - 2p\beta\big] \geq 0\end{aligned}$$

for all $h \in [0,\tau]$, where we used (4.5), (4.3) and (4.4) in the last three steps respectively.

Next observe that

$$G(0) = \tau L(0) + \beta\sigma^2 > 0 \quad \text{and} \quad G'(0) = 2\beta\sigma > 0.$$

Consequently, Lemma 2.8 implies

$$\tau \leq \frac{2G(0)}{(p-1)G'(0)} = \frac{2(\tau L(0) + \beta\sigma^2)}{2(p-1)\beta\sigma} = \frac{L(0)}{(p-1)\beta\sigma}\tau + \frac{\sigma}{p-1}.$$

This in turn yields

$$\tau\left(1 - \frac{L(0)}{(p-1)\beta\sigma}\right) \leq \frac{\sigma}{p-1}$$

or equivalently

$$\tau \leq \frac{\sigma}{p-1}\left(1 - \frac{L(0)}{(p-1)\beta\sigma}\right)^{-1} = \frac{\beta\sigma^2}{(p-1)\beta\sigma - L(0)}. \tag{4.7}$$

Minimizing this last display over the range of σ in (4.4) leads to

$$\tau \leq \frac{4L(0)}{(p-1)^2\beta}. \tag{4.8}$$

Then we minimize (4.8) over the the range of β in (4.4) to see that

$$\tau \leq \frac{4pL(0)}{(p-1)^2(p+1)(d_\infty - J(u_0, 0))}. \tag{4.9}$$

Lastly, (4.9) holds for all $\tau \in (0, T)$, from we deduce that

$$T \leq \frac{4pL(0)}{(p-1)^2(p+1)(d_\infty - J(u_0, 0))}$$

as required. □

Now it remains to prove Theorem 1.6. To this end, it is necessary to present the following technical result.

Lemma 4.1. *Let $d \geq 1$ and $p > 1$. Suppose $k \in C^1[0, \infty)$ satisfies $k(0) > 0$ and $k'(t) \geq 0$ for all $t \in [0, \infty)$. Let u be a weak solution to* (P) *with initial datum $u_0 \in H_0^2(\Omega)$. Then*

$$\left\| \frac{u_t(0)}{|x|^2} \right\|_{L^2(\Omega)}^2 > 0$$

if u_0 is not a weak solution to the problem

$$\begin{cases} \Delta^2 u = k(0)|u|^{p-1}u & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{E}$$

Proof. First, since u is a weak solution to (P), we have

$$\left(\frac{u_t}{|x|^4}, \varphi \right) + (\Delta u, \Delta \varphi) = k(t)(|u|^{p-1}u, \varphi)$$

for all $\varphi \in H_0^2(\Omega)$ and $t \in [0, T)$. In particular,

$$\left(\frac{u_t(0)}{|x|^4}, \varphi \right) + (\Delta u_0, \Delta \varphi) = k(t)(|u_0|^{p-1}u_0, \varphi)$$

for all $\varphi \in H_0^2(\Omega)$.

Secondly, suppose that u_0 is not a weak solution to (E). Then there exists a function $0 \neq \psi \in C_c^\infty(\Omega)$ such that

$$(\Delta u_0, \Delta \psi) - k(0) \int_\Omega |u_0|^{p-1}u_0 \psi \neq 0.$$

With the above two facts in mind, one has

$$\left\| \frac{u_t(0)}{|x|^2} \right\|_{L^2(\Omega)}^2 = \left[\sup_{\varphi \in L^2(\Omega)\setminus\{0\}} \frac{\big(u_t(0)/|x|^2, \varphi\big)}{\|\varphi\|_{L^2(\Omega)}} \right]^2 \geq \left[\frac{\big(u_t(0)/|x|^2, \psi\big)}{\|\psi\|_{L^2(\Omega)}} \right]^2$$
$$= \left[\frac{(\Delta u_0, \Delta \psi) - k(0)(|u_0|^{p-1}u_0, \psi)}{\|\psi\|_{L^2(\Omega)}} \right]^2 > 0.$$

This justifies the claim. □

Now we prove Theorem 1.6.

Proof of Theorem 1.6. We aim to show that the maximal existence time $T < \infty$ and then to provide an upper bound for T. Set

$$W(t) = \big\{ w \in H_0^2(\Omega) : J(w, t) < d(t) \text{ and } I(w, t) < 0 \big\}.$$

We first show that there exists a $t_0 \in (0, T)$ such that $u(t) \in W(t)$ for all $t \in [t_0, T)$.

Since $I(u(t), t)$ is continuous as a function of t over $[0, T)$, using the fact that $I(u_0, 0) < 0$ we deduce that there exists a $t_1 \in (0, T)$ such that

$$I(u(t), t) < 0$$

for all $t \in [0, t_1)$. In addition, the fact that $I(u_0, 0) < 0$ also implies u_0 is not a weak solution to problem (E) given in Lemma 4.1. As such Lemma 4.1 confirms that

$$\left\| \frac{u_t(0)}{|x|^2} \right\|_{L^2(\Omega)}^2 > 0.$$

By continuity there exists a $t_0 \in (0, t_1)$ such that $I(u(t_0), t_0) < 0$ and

$$\left\| \frac{u_t(t)}{|x|^2} \right\|_{L^2(\Omega)}^2 > 0$$

for all $t \in [0, t_0)$. Then in view of Lemmas 2.6(i) and 2.7(ii), we deduce that

$$J(u(t), t) \leq J(u(t_0), t_0) < J(u_0, 0) < d_\infty < d(t) \tag{4.10}$$

for all $t \in [t_0, T)$. Hence it suffices to prove that $I(u(t), t) < 0$ for all $t \in [t_0, T)$. To achieve this, we proceed via proof by contradiction. Now suppose there is a $t_2 \in [t_0, T)$ such that $I(u(t_2), t_2) = 0$ and $I(u(t), t) < 0$ for all $t \in [0, t_2)$, then

$$J(u(t_2), t_2) \geq \inf_{w \in \mathcal{N}(t_2)} J(w, t_2) = d(t_2)$$

by virtue of (1.4). This contradicts (4.10), whence no such t_2 exists. That is, $u(t) \in W(t)$ for all $t \in [t_0, T)$ as required.

For the rest of the proof, we repeat the arguments used in Step 2 in the proof of Theorem 1.5, with u_0 being replaced by $u(t_0)$. This completes our proof. □

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