

Global Well-posedness of Solutions for the p -Laplacian Hyperbolic Type Equation with Weak and Strong Damping Terms and Logarithmic Nonlinearity

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Abstract. In this paper, we consider the p -Laplacian hyperbolic type equation with weak and strong damping terms and logarithmic nonlinearity. By using the potential well method and a logarithmic Sobolev inequality, we prove global existence, infinite time blow up and asymptotic behavior of solutions in two cases $E(0) < d$ and $E(0) = d$. Furthermore, the infinite time blow up of solutions for the problem with $E(0) > 0$ ($\omega = 0$) is studied.

1. Introduction

In this paper, we study the initial boundary value problem of the p -Laplacian hyperbolic type equation with weak and strong damping terms and logarithmic nonlinearity

$$(1.1) \quad \begin{cases} u_{tt} - \Delta_p u - \omega \Delta u_t + u_t = |u|^{p-2} u \ln |u| & \text{in } \Omega \times (0, \infty), \\ u(x, t) = 0 & \text{in } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $p > 2$, $\omega \geq 0$, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the classical p -Laplacian operator.

The wave equation of p -Laplacian type has an active area of research and the most of the results are mainly concerned with the global well-posedness and stability of solutions. Messaoudi [24] considered the following quasilinear hyperbolic equations with nonlinear damping and source terms

$$(1.2) \quad u_{tt} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) - \Delta u_t + |u_t|^{q-1} u_t = |u|^{p-1} u,$$

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and studied decay of solutions by using the techniques combination of the perturbed energy and potential well methods. Wu and Xue [31] studied (1.2) and proved the uniform energy decay rates of the solutions by utilizing the multiplier method. Pei et al. [26] studied the following quasilinear wave equation of p -Laplacian type

$$u_{tt} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) - \Delta u_t = f(u),$$

and proved the existence of local and global solutions. Also they proved the blow-up result of solutions with negative initial energy. Recently, Pereira et al. [27] considered the following quasilinear wave equation of p -Laplacian type with p -Laplacian damping

$$u_{tt} - \Delta_p u - \Delta_p u_t = |u|^{r-1}u.$$

They established the global solutions by means of the Faedo–Galerkin approximations and the asymptotic behavior is obtained by Nakao’s method. For other related results, the readers may see [16, 17, 28] and the references therein.

The logarithmic nonlinearity is of much interest in physics, since it appears naturally in inflation cosmology and super symmetric field theories, quantum mechanics and nuclear physics [1, 11]. Cazenave and Haraux [4] considered

$$(1.3) \quad u_{tt} - \Delta u = u \ln |u|^k \quad \text{in } \mathbb{R}^3,$$

and established the existence and uniqueness of the solution for the Cauchy problem. Górká [12] used some compactness arguments and obtained the global existence of weak solutions for all $(u_0, u_1) \in H_0^1 \times L^2$ to the initial-boundary value problem (1.3) in the one-dimensional case. Hiramatsu et al. [14] introduced the following equation

$$(1.4) \quad u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u|^k,$$

when studying the dynamics of Q -balls in theoretical physics. In [13], Han proved the global existence of weak solutions, for all $(u_0, u_1) \in H_0^1 \times L^2$ to the initial boundary value problem (1.4) in $\Omega \subset \mathbb{R}^3$. By constructing an appropriate Lyapunov function, Zhang and Liu [33] obtained the exponential decay estimates of energy with $E(0) < d$ for all $(u_0, u_1) \in H_0^1 \times L^2$, $I(u_0) > 0$. Recently, Lian and Xu [21] considered the initial boundary value problem for the following nonlinear wave equation

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = u \ln |u|,$$

and studied the well-posedness of the solution. Chen and Xu [7] considered the following wave equation

$$u_{tt} - \Delta u + \Delta^2 u - \omega(\Delta u_{tt} + \Delta u_t) + |u_t|^{r-1}u_t = u \ln |u|,$$

and proved the global existence and infinite time blow up of solutions in cases of $E(0) < d$ and $E(0) = d$. Also they proved the infinite time blow-up of solutions at arbitrarily high initial energy level ($E(0) > 0$) under two hypothetical situations. For the related works of equations with logarithmic nonlinearity, we also refer other works [2, 5, 6, 8, 10, 15, 20, 22, 23, 32] and the references therein.

In recent years, the p -Laplacian equation with logarithmic nonlinearity have become an active area of research, see for example [3, 10, 19, 25, 29, 30] and the references therein. Nhan and Truong [25] considered the p -Laplacian pseudo-parabolic equations with logarithmic nonlinearity

$$(1.5) \quad u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) - \Delta u_t = u^{p-2} u \ln |u|,$$

and established the global existence, blow up and decay of solutions for $p > 2$. Cao and Liu [3] studied (1.5) and proved the existence and boundedness of global weak solutions, and also obtained the blow up at infinite time for $1 < p < 2$. Pişkin and Irkil [30] considered the following equation with logarithmic nonlinearity

$$u_{tt} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) - \Delta u + u_t = ku \ln |u|.$$

They established the local existence of the solutions by using Faedo–Galerkin method. Pişkin et al. [29] studied the following p -Laplacian hyperbolic type equation with logarithmic nonlinearity

$$u_{tt} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) - |u|^{p-2} u + u_t = u^{p-2} u \ln |u|.$$

They obtained the existence of global of solutions by the potential well theory and a logarithmic Sobolev inequality. Furthermore, the growth and the decay estimates of solutions for the equation are studied.

Motivated by the above mentioned papers, our purpose in this research is to investigate the global existence, asymptotic behavior and infinite time blow up of solution in cases $E(0) < d$ and $E(0) = d$ to the initial boundary value problem (1.1). Furthermore, the infinite time blow up of solutions for the problem (1.1) with $E(0) > 0$ is studied.

Our paper is organized as follows. In Section 2, we give some notations, preliminary lemmas, and introduce some potential wells. In Section 3, we prove the global existence, asymptotic behavior and infinite time blow up of solution for $E(0) < d$. In Section 4, we prove the global existence, asymptotic behavior and infinite blow up of solution for $E(0) = d$. In Section 5, we prove infinite time blow up of solutions for $E(0) > 0$ ($\omega = 0$).

2. Preliminaries and potential wells

2.1. Preliminaries

In this subsection we give some notations for function spaces and some preliminary lemmas.

Denote

$$\|u\|_p = \|u\|_{L^p(\Omega)}, \quad \|u\|_{1,p} = \|u\|_{W_0^{1,p}(\Omega)} = (\|u\|_p^p + \|\nabla u\|_p^p)^{1/p}$$

for $1 < p < \infty$. Also, we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $W^{-1,p'}$ and $W_0^{1,p}$, where p' is Hölder conjugate exponent of $p > 1$. Moreover, we also denote

$$(u, v)_\omega = \int_\Omega (\omega \nabla u \nabla v + uv) \, dx,$$

and the norm induced by the product $(u, v)_\omega$ is

$$\|u\|_\omega^2 = (u, u)_\omega.$$

Then, $\|u\|_\omega$ is an equivalent eccentric module over $H_0^1(\Omega)$ due to $\omega > 0$.

Lemma 2.1. Logarithmic Sobolev inequality, see [9] *Let $p > 1$, $\mu > 0$ and $u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}$. Then we have*

$$p \int_{\mathbb{R}^n} |u|^p \ln \left(\frac{|u|}{\|u\|_{L^p(\mathbb{R}^n)}} \right) \, dx + \frac{n}{p} \ln \left(\frac{p\mu e}{n\mathcal{L}_p} \right) \int_{\mathbb{R}^n} |u|^p \, dx \leq \mu \int_{\mathbb{R}^n} |\nabla u|^p \, dx,$$

where

$$\mathcal{L}_p = \frac{n}{p} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-p/2} \left[\frac{\Gamma(n/2 + 1)}{\Gamma(n(p-1)/p + 1)} \right]^{p/n}.$$

Remark 2.2. For $u \in W_0^{1,p}(\Omega)$, we can define $u(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$, we derive

$$(2.1) \quad p \int_\Omega |u|^p \ln \left(\frac{|u|}{\|u\|_{L^p(\Omega)}} \right) \, dx + \frac{n}{p} \ln \left(\frac{p\mu e}{n\mathcal{L}_p} \right) \int_\Omega |u|^p \, dx \leq \mu \int_\Omega |\nabla u|^p \, dx$$

for any real number $\mu > 0$.

Lemma 2.3. [18]

(a) *For any function $u \in W_0^{1,p}(\Omega)$, we have the inequality*

$$\|u\|_q \leq B_{q,p} \|\nabla u\|_q$$

for all $1 \leq q \leq p^$, where $p^* = np/(n-p)$ if $n > p$ and $p^* = \infty$ if $n \leq p$. The best constant $B_{q,p}$ depends only on Ω , n , p and q .*

(b) For any $u \in W_0^{1,p}(\Omega)$, $p \geq 1$ and $r \geq 1$, the inequality

$$\|u\|_q \leq C \|\nabla u\|_p^\theta \|u\|_r^{1-\theta}$$

is valid, where the constant C depends on n , p , q and r ,

$$\theta = \left(\frac{1}{r} - \frac{1}{q}\right) \left(\frac{1}{n} - \frac{1}{p} + \frac{1}{r}\right)^{-1},$$

and

- for $p \geq n = 1$, $r \leq q \leq \infty$;
- for $n > 1$ and $p < n$, $q \in [r, p^*]$ if $r \leq p^*$ and $q \in [p^*, r]$ if $r \geq p^*$;
- for $p = n > 1$, $r \leq q < \infty$;
- for $p > n > 1$, $r \leq q < \infty$.

Now, we define the energy associated with problem (1.1) by

$$(2.2) \quad E(t) = \frac{1}{2} \|u_t\|_2^2 + \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} \int_{\Omega} u^p \ln |u| \, dx + \frac{1}{p^2} \|u\|_p^p.$$

Lemma 2.4. *Let u be a solution of problem (1.1). Then,*

$$(2.3) \quad E'(t) = -\|u_t\|_{\omega}^2 \leq 0.$$

Proof. Multiplying the first equation of problem (1.1) by u_t and integrating from 0 to t with respect to t , we obtain

$$E(t) + \int_0^t \|u_{\tau}\|_{\omega}^2 \, d\tau = E(0),$$

which yields (2.3) by a simple calculation. □

2.2. Potential wells

In this subsection, we establish the corresponding method of potential wells which is related to the logarithmic nonlinear term $u^{p-2} \ln |u|$.

First, we define the functionals

$$(2.4) \quad \begin{aligned} J(t) &:= J(u(t)) = \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} \int_{\Omega} u^p \ln |u| \, dx + \frac{1}{p^2} \|u\|_p^p, \\ I(t) &:= I(u(t)) = \|\nabla u\|_p^p - \int_{\Omega} u^p \ln |u| \, dx. \end{aligned}$$

Then, it is obvious that

$$(2.5) \quad J(u) = \frac{1}{p} I(u) + \frac{1}{p^2} \|u\|_p^p,$$

and

$$(2.6) \quad E(t) := E(u(t)) = \frac{1}{2} \|u_t\|_2^2 + J(u).$$

Similar in [25, 29], the potential well depth is defined as

$$d = \inf \left\{ \sup_{\lambda \geq 0} J(\lambda u) \mid u \in W_0^{1,p}(\Omega), \|u\|_p^p \neq 0 \right\},$$

and

$$0 < d = \inf_{u \in \mathcal{N}} J(u),$$

where \mathcal{N} is the Nehari manifold, which is defined by

$$\mathcal{N} = \{u \in W_0^{1,p}(\Omega) \setminus \{0\} : I(u) = 0\}.$$

Next, we define

$$W = \{u : u \in W_0^{1,p}(\Omega) / I(u) > 0, J(u) < d\} \cup \{0\},$$

$$V = \{u : u \in W_0^{1,p}(\Omega) / I(u) < 0, J(u) < d\}.$$

Lemma 2.5. *For any $u \in W_0^{1,p}(\Omega)$, $\|u\|_p \neq 0$ and let $f(\lambda) = J(\lambda u)$. Then we have*

$$I(\lambda u) = \lambda f'(\lambda) \begin{cases} > 0, & 0 \leq \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < +\infty, \end{cases}$$

where

$$\lambda^* = \exp \left(\frac{\|\nabla u\|_p^p - \int_{\Omega} u^p \ln |u| \, dx}{\|u\|_p^p} \right).$$

Proof. By definition of J , we have

$$f(\lambda) = J(\lambda u) = \frac{\lambda^p}{p} \|\nabla u\|_p^p - \frac{\lambda^p}{p} \int_{\Omega} u^p \ln |u| \, dx + \lambda^p \left(\frac{1}{p^2} - \frac{\ln |\lambda|}{p} \right) \|u\|_p^p.$$

Since $\|u\|_p \neq 0$, we have $\lim_{\lambda \rightarrow 0} f(\lambda) = 0$, $\lim_{\lambda \rightarrow +\infty} f(\lambda) = -\infty$, and

$$I(\lambda u) = \lambda f'(\lambda) = \lambda^p \|\nabla u\|_p^p - \lambda^p \int_{\Omega} u^p \ln |u| \, dx - \lambda^p \ln |\lambda| \|u\|_p^p = \lambda \frac{dJ(\lambda u)}{d\lambda}.$$

So, we have

$$I(\lambda u) = \lambda f'(\lambda) \begin{cases} > 0, & 0 \leq \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < +\infty. \end{cases}$$

□

Now, we denote r the constant given by

$$r := \left(\frac{p^2 e}{n \mathcal{L}_p} \right)^{n/p^2},$$

where \mathcal{L}_p is defined as in Lemma 2.1.

Lemma 2.6. *Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. Then, we have*

- (1) *if $0 < \|u\|_p \leq r$, then $I(u) > 0$;*
- (2) *if $I(0) < 0$, then $\|u\|_p > r$;*
- (3) *if $I(0) = 0$, then $\|u\|_p \geq r$.*

Proof. By using the logarithmic Sobolev inequality (2.1), for $\mu > 0$, we have

$$\begin{aligned} I(t) &= \|\nabla u\|_p^p - \int_{\Omega} u^p \ln |u| \, dx \\ &= \|\nabla u\|_p^p - \int_{\Omega} u^p \left(\ln \frac{|u|}{\|u\|_p} + \ln \|u\|_p \right) \, dx \\ &\geq \left(1 - \frac{\mu}{p} \right) \|\nabla u\|_p^p + \left(\frac{n}{p^2} \ln \frac{p\mu e}{n \mathcal{L}_p} - \ln \|u\|_p \right) \int_{\Omega} |u|^p \, dx. \end{aligned}$$

Taking $\mu = p$, we get

$$(2.7) \quad I(t) \geq \left(\frac{n}{p^2} \ln \frac{p^2 e}{n \mathcal{L}_p} - \ln \|u\|_p \right) \|u\|_p^p.$$

- (1) If $0 < \|u\|_p \leq r$, then it follows from (2.7) that

$$I(u) > 0.$$

- (2) From (2.7) and $I(0) < 0$, we can see that

$$\left(\frac{n}{p^2} \ln \frac{p^2 e}{n \mathcal{L}_p} - \ln \|u\|_p \right) \|u\|_p^p < 0.$$

This implies

$$\|u\|_p > r.$$

- (3) This conclusion is similar to the proof of (2). □

Lemma 2.7. *The following statements hold:*

- (1) $d = \inf \left\{ \sup_{\lambda \geq 0} J(\lambda u) \mid u \in W_0^{1,p}(\Omega) \setminus \{0\}, \|u\|_p^p \neq 0 \right\}.$

(2) d has a positive lower bound, namely,

$$d \geq \frac{1}{p^2} \left(\frac{p^2 e}{n \mathcal{L}_p} \right)^{n/p} = \frac{r^p}{p^2} = M,$$

where \mathcal{L}_p is defined as in Lemma 2.1.

(3) There is a positive function $u \in \mathcal{N}$ such that $J(u) = d$.

Proof. The proof of this lemma is almost the same to that of Lemma 7 in [29]. So we omit it here. □

Lemma 2.8. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$. If $E(0) < d$ and $u_0 \in W$, then for any solution u to (1.1) we have*

$$u \in W, \quad \text{and} \quad \|u\|_p^p < p^2 d, \quad \forall t \in [0, T).$$

Lemma 2.9. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$. If $E(0) < d$ and $u_0 \in V$, then for any solution u to (1.1) we have*

$$u \in V, \quad \text{and} \quad \|u\|_p^p > p^2 d, \quad \forall t \in [0, T).$$

Lemma 2.10. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$. If $E(0) = d$ and $u_0 \in V$, then for any solution u to (1.1) we have*

$$u \in V, \quad \text{and} \quad \|u\|_p^p \geq p^2 d, \quad \forall t \in [0, T).$$

The proofs of Lemmas 2.8–2.10 are similar to those in [7, 21], we omit the details.

3. Global existence, asymptotic behavior, and blow up for $E(0) < d$

In this section we prove the global existence, asymptotic behavior, and blow up of solutions for problem (1.1) with $E(0) < d$.

3.1. Global existence for $E(0) < d$

Theorem 3.1. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$ hold. If $E(0) < d$ and $I(u_0) > 0$, then problem (1.1) admits a global weak solution $u(t) \in L^\infty(0, +\infty; W_0^{1,p}(\Omega))$, $u_t(t) \in L^\infty(0, +\infty; L^2(\Omega))$.*

Proof. Let $\{w_j\}$ be the eigenfunctions of the Laplacian operator subject to the Dirichlet boundary condition:

$$\begin{cases} -\Delta w_j = \lambda w_j, & x \in \Omega, \\ w_j = 0, & x \in \partial\Omega, \end{cases}$$

which is chosen as the complete orthogonal basis of $W_0^{1,p}(\Omega)$ and it is also a basis of $H_0^1(\Omega)$. Construct the approximate solution $u_m(x, t)$ of problem (1.1) as

$$(3.1) \quad u_m(x, t) = \sum_{j=1}^m g_{jm}(t)w_j(x), \quad m = 1, 2, \dots,$$

which satisfies

$$(3.2) \quad \begin{aligned} & (u_{mtt}, w_j) + (|\nabla u_m|^{p-2}\nabla u_m, \nabla w_j) + \omega(\nabla u_{mt}, \nabla w_j) + (u_{mt}, w_j) \\ & = (|u_m|^{p-2}u_m \ln |u_m|, w_j), \end{aligned}$$

and

$$(3.3) \quad u_m(0) = \sum_{j=1}^m b_{jm}(0)w_j(x) \rightarrow u_0 \quad \text{in } W_0^{1,p}(\Omega),$$

$$(3.4) \quad u_{mt}(0) = \sum_{j=1}^m c_{jm}(0)w_j(x) \rightarrow u_1 \quad \text{in } L^2(\Omega).$$

Multiplying (3.2) by g'_{jm} and summing for j , we get

$$(3.5) \quad \frac{d}{dt}E_m(t) = -\|u_{mt}\|_\omega^2.$$

By integrating (3.5) with respect to t over $(0, t)$, we have

$$E_m(t) + \int_0^t \|u_{m\tau}\|_\omega^2 d\tau = \frac{1}{2}\|u_{mt}\|_2^2 + J(u_m) + \int_0^t \|u_{m\tau}\|_\omega^2 d\tau = E_m(0).$$

From (3.3) and (3.4),

$$E_m(0) \rightarrow E(0) \quad \text{as } m \rightarrow \infty.$$

Hence for sufficiently large m , we have

$$(3.6) \quad \frac{1}{2}\|u_{mt}\|_2^2 + J(u_m) + \int_0^t (\omega\|\nabla u_{m\tau}\|_2^2 + \|u_{m\tau}\|_2^2) d\tau < d.$$

Further from (2.5) and (3.6), we have

$$(3.7) \quad \frac{1}{2}\|u_{mt}\|_2^2 + \frac{1}{p}I(u_m) + \frac{1}{p^2}\|u_m\|_p^p + \int_0^t (\omega\|\nabla u_{m\tau}\|_2^2 + \|u_{m\tau}\|_2^2) d\tau < d.$$

Noting that $u_m(0) \rightarrow u_0$ in $W_0^{1,p}(\Omega)$ as $m \rightarrow \infty$, we see that $u_0 \in W$ implies $u_m(0) \in W$ for sufficiently large m . Coupled with (3.7) and an argument similar to the proof of Lemma 2.8, we have $u_m(t) \in W$ for sufficiently large m and $0 \leq t < \infty$. Hence (3.7) turns to

$$(3.8) \quad \frac{1}{2}\|u_{mt}\|_2^2 + \frac{1}{p^2}\|u_m\|_p^p + \int_0^t (\omega\|\nabla u_{m\tau}\|_2^2 + \|u_{m\tau}\|_2^2) d\tau < d.$$

For a sufficiently large m , (3.8) gives

$$(3.9) \quad \begin{aligned} u_m &\text{ is bounded in } L^\infty(0, \infty; L^p(\Omega)), \\ u_{mt} &\text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)), \\ u_{mt} &\text{ is bounded in } L^2(0, \infty; H_0^1(\Omega)), \text{ if } \omega > 0. \end{aligned}$$

Next, we estimate the term $\|\nabla u_m\|_p^p$. By Lemma 2.1, we have

$$\begin{aligned} \|\nabla u_m\|_p^p &= I(u_m) + \int_\Omega |u_m|^p \ln |u_m| \, dx \\ &\leq I(u_m) + \frac{\mu}{p} \|\nabla u_m\|_p^p + \left(\log \|u_m\|_p - \frac{n}{p^2} \log \frac{p\mu e}{n\mathfrak{L}_p} \right) \|u_m\|_p^p \\ &= pJ(u_m) + \frac{\mu}{p} \|\nabla u_m\|_p^p + \left(\log \|u_m\|_p - \frac{1}{p} - \frac{n}{p^2} \log \frac{p\mu e}{n\mathfrak{L}_p} \right) \|u_m\|_p^p. \end{aligned}$$

By taking $\mu < p$, we deduce from Lemma 2.8 that

$$\|\nabla u_m\|_p^p \leq Cd.$$

On the other hand, by a direct calculation, we have

$$(3.10) \quad \begin{aligned} \int_\Omega |u_m|^p \ln |u_m| \, dx &= \int_{\Omega_1} |u_m|^p \ln |u_m| \, dx + \int_{\Omega_2} |u_m|^p \ln |u_m| \, dx \\ &\leq C \int_{\Omega_2} |u_m|^{p+\nu} \, dx \leq C \|u_m\|_{p+\nu}^{p+\nu}, \end{aligned}$$

where $\Omega_1 = \{x \in \Omega : |u_m| < 1\}$, $\Omega_2 = \{x \in \Omega : |u_m| \geq 1\}$, ν is chosen such that $p + \nu \leq p^*$ as $p < n$ and ν is positive as $p \geq n$. By using Lemma 2.3, it follows from (3.10) that

$$\begin{aligned} \int_\Omega |u_m|^p \ln |u_m| \, dx &\leq C \|u_m\|_{p+\nu}^{p+\nu} \leq C \|\nabla u_m\|_p^{\theta(p+\nu)} \|u_m\|_p^{(1-\theta)(p+\nu)} \\ &\leq \varepsilon \|\nabla u_m\|_p^p + c_\varepsilon \|u_m\|_p^{\frac{(1-\theta)p(p+\nu)}{p-\theta(p+\nu)}}, \end{aligned}$$

where

$$\theta = \left(\frac{1}{p} - \frac{1}{p+\nu} \right) \left(\frac{1}{n} - \frac{1}{p} + \frac{1}{p} \right)^{-1}.$$

At this point, we choose $\nu > 0$ such that $\theta(p + \nu) < p$, we deduce that

$$\int_\Omega |u_m|^p \ln |u_m| \, dx \leq Cd.$$

Then, we have

$$(3.11) \quad u_m \text{ is bounded in } L^\infty(0, \infty; W_0^{1,p}(\Omega)).$$

Integrating (3.2) with respect to t , we get

$$(3.12) \quad \begin{aligned} & (u_{mt}, w_j) + \int_0^t (|\nabla u_m|^{p-2} \nabla u_m, \nabla w_j) ds + \omega(\nabla u_m, \nabla w_j) + \int_0^t (u_{mt}, w_j) ds \\ &= (u_1, w_j) + (\nabla u_0, \nabla w_j) + \int_0^t (|u_m|^{p-2} u_m \ln |u_m|, w_j) ds. \end{aligned}$$

Therefore, up to a subsequence, by (3.9) and (3.11) we may pass to the limit in (3.12) and obtain a weak solution $u(x, t)$ of problem (1.1) with the corresponding regularity stated above. On the other hand, from (3.3) and (3.4) we have $u(x, 0) = u_0(x)$ in $W_0^{1,p}(\Omega)$ and $u_t(x, 0) = u_1(x)$ in $L^2(\Omega)$. \square

3.2. Asymptotic behavior for $E(0) < d$

Theorem 3.2. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$ hold. If $E(0) < d$ and $I(u_0) > 0$, then there exist two positive constants c and k such that*

$$E(t) \leq ce^{-kt}, \quad \forall t > 0.$$

Let

$$(3.13) \quad F(t) := E(t) + \varepsilon \int_{\Omega} uu_t dx + \frac{\varepsilon\omega}{2} \|\nabla u\|_2^2 + \frac{\varepsilon}{2} \|u\|_2^2,$$

where ε is a positive constant to be specified later. In order to show our result, we need the following lemma.

Lemma 3.3. *There exist two positive constants α_1 and α_2 such that*

$$(3.14) \quad \alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t), \quad t \geq 0.$$

Proof. It is easy to see that $F(t)$ and $E(t)$ are equivalent. So we omit it here. \square

Proof of Theorem 3.2. Taking a derivative of (3.13) with respect to t , using (1.1) and Lemma 2.4, we have

$$\begin{aligned} F'(t) &= E'(t) + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_p^p + \varepsilon \int_{\Omega} u^p \ln |u| \\ &= -\omega \|\nabla u_t\|_2^2 - (1 - \varepsilon) \|u_t\|_2^2 - \varepsilon \|\nabla u\|_p^p + \varepsilon \int_{\Omega} u^p \ln |u| dx. \end{aligned}$$

It follows from (2.2), for a constant $N > 0$, we see that

$$\begin{aligned} F'(t) &= -N\varepsilon E(t) - \omega \|\nabla u_t\|_2^2 + \left(\frac{\varepsilon N}{2} + \varepsilon - 1 \right) \|u_t\|_2^2 + \varepsilon \left(\frac{N}{p} - 1 \right) \|\nabla u\|_p^p \\ &\quad + \varepsilon \frac{N}{p^2} \|u\|_p^p + \varepsilon \left(1 - \frac{N}{p} \right) \int_{\Omega} u^p \ln |u| dx. \end{aligned}$$

For $N < p$, by applying logarithmic Sobolev inequality (Lemma 2.1), we have

$$(3.15) \quad \begin{aligned} F'(t) \leq & -N\varepsilon E(t) - \omega \|\nabla u_t\|_2^2 + \left(\frac{\varepsilon N}{2} + \varepsilon - 1 \right) \|u_t\|_2^2 + \varepsilon \left(1 - \frac{N}{p} \right) \left(\frac{\mu}{p} - 1 \right) \|\nabla u\|_p^p \\ & + \varepsilon \left\{ \frac{N}{p^2} + \left(1 - \frac{N}{p} \right) \left(\ln \|u\|_p - \frac{n}{p^2} \ln \frac{p\mu e}{n\mathfrak{L}_p} \right) \right\} \|u\|_p^p. \end{aligned}$$

Noting that

$$\ln \|u\|_p^p \leq \ln(p^2 J(t)) \leq \ln(p^2 E(t)) \leq \ln(p^2 E(0)) < \ln(p^2 d).$$

For any fixed N , we choose ε sufficiently small satisfying

$$\varepsilon \frac{N}{2} + \varepsilon - 1 < 0.$$

Then inequality (3.15) becomes

$$(3.16) \quad \begin{aligned} F'(t) \leq & -N\varepsilon E(t) + \varepsilon \left(1 - \frac{N}{p} \right) \left(\frac{\mu}{p} - 1 \right) \|\nabla u\|_p^p \\ & + \frac{\varepsilon}{p} \left\{ \frac{N}{p} + \frac{1}{p} \left(1 - \frac{N}{p} \right) \left(\ln(p^2 d) - \frac{n}{p} \ln \frac{p\mu e}{n\mathfrak{L}_p} \right) \right\} \|u\|_p^p. \end{aligned}$$

Taking μ satisfying

$$\left(\frac{n\mathfrak{L}_p}{pe} \right) e^{p\beta/n} < \mu < p,$$

where $\beta = Np/(p - N) + \ln(p^2 d)$, such that

$$\left(1 - \frac{N}{p} \right) \left(\frac{\mu}{p} - 1 \right) < 0, \quad \text{and} \quad \frac{N}{p} + \frac{1}{p} \left(1 - \frac{N}{p} \right) \left(\ln(p^2 d) - \frac{n}{p} \ln \frac{p\mu e}{n\mathfrak{L}_p} \right) < 0.$$

Consequently, inequality (3.16) becomes

$$F'(t) \leq -N\varepsilon E(t), \quad \forall t > 0.$$

From (3.14), we have

$$(3.17) \quad F'(t) \leq -N\varepsilon E(t) \leq -\frac{N}{\alpha_2} \varepsilon F(t), \quad \forall t > 0.$$

A simple integration of (3.17) leads to

$$F(t) \leq c_1 e^{-kt}, \quad \forall t > 0.$$

Again

$$E(t) \leq ce^{-kt}, \quad \forall t > 0.$$

This completes the proof. □

3.3. Blow-up for $E(0) < d$

Theorem 3.4. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$ hold. If $E(0) < d$ and $I(u_0) < 0$, the solution of the problem (1.1) blows up as time t goes to infinity.*

Proof. Arguing by contradiction, we suppose that the solution u is global. Then for any $T > 0$, we define the following auxiliary function

$$(3.18) \quad \theta(t) = \|u\|_2^2 + \int_0^t \|u(\tau)\|_\omega^2 d\tau + (T - t)\|u_0\|_\omega^2.$$

It is clear that $\theta(t) > 0$ for all $t \in [0, T]$. In view of the continuity of $\theta(t)$, we obtain that there is a $\kappa > 0$ such that

$$(3.19) \quad \theta(t) > \kappa \quad \text{for all } t \in [0, T],$$

where κ is independent of T . Taking a derivative of (3.18) with respect to t , using (1.1) and (2.4), we obtain

$$(3.20) \quad \begin{aligned} \theta'(t) &= 2 \int_\Omega uu_t dx + (\|u\|_\omega^2 - \|u_0\|_\omega^2) \\ &= 2 \int_\Omega uu_t dx + 2 \int_0^t (u(\tau), u_t(\tau))_\omega d\tau, \end{aligned}$$

and

$$(3.21) \quad \begin{aligned} \theta''(t) &= 2\|u_t\|_2^2 + 2\langle u_{tt}, u \rangle + 2(u, u_t)_\omega \\ &= 2\|u_t\|_2^2 - 2\|\nabla u\|_p^p + 2 \int_\Omega u^p \ln |u| dx. \end{aligned}$$

It follows from (3.18)–(3.21), we see that

$$(3.22) \quad \begin{aligned} \theta(t)\theta''(t) - (\theta'(t))^2 &= 2\theta(t) \left[\|u_t\|_2^2 - \|\nabla u\|_p^p + \int_\Omega u^p \ln |u| dx \right] \\ &\quad - 4 \left[\int_\Omega uu_t dx + \int_0^t (u(\tau), u_t(\tau))_\omega d\tau \right]^2 \\ &= 2\theta(t) [\|u_t\|_2^2 - I(t)] \\ &\quad - 4 \left[\int_\Omega uu_t dx + \int_0^t (u(\tau), u_t(\tau))_\omega d\tau \right]^2. \end{aligned}$$

In what follows, we will estimate the second term in (3.22). By using Hölder and Young inequalities, we obtain

$$(3.23) \quad \left(\int_\Omega uu_t dx \right)^2 \leq \|u\|_2^2 \|u_t\|_2^2,$$

$$(3.24) \quad \left(\int_0^t (u(\tau), u_t(\tau))_\omega d\tau \right)^2 \leq \int_0^t \|u(\tau)\|_\omega^2 d\tau \int_0^t \|u_t(\tau)\|_\omega^2 d\tau,$$

and

$$\begin{aligned}
 & 2 \int_{\Omega} uu_t dx \int_0^t (u(\tau), u_t(\tau))_{\omega} d\tau \\
 (3.25) \quad & \leq 2\|u\|_2\|u_t\|_2 \left(\int_0^t \|u(\tau)\|_{\omega}^2 d\tau \right)^{1/2} \left(\int_0^t \|u(\tau)\|_{\omega}^2 d\tau \right)^{1/2} \\
 & \leq \|u\|_2^2 \int_0^t \|u_t(\tau)\|_{\omega}^2 d\tau + \|u_t\|_2^2 \int_0^t \|u(\tau)\|_{\omega}^2 d\tau.
 \end{aligned}$$

By (3.23)–(3.25), we have

$$\begin{aligned}
 & \left[\int_{\Omega} uu_t dx + \int_0^t (u(\tau), u_t(\tau))_{\omega} d\tau \right]^2 \\
 & \leq \|u\|_2^2\|u_t\|_2^2 + \int_0^t \|u(\tau)\|_{\omega}^2 d\tau \int_0^t \|u_t(\tau)\|_{\omega}^2 d\tau \\
 (3.26) \quad & + \|u\|_2^2 \int_0^t \|u_t(\tau)\|_{\omega}^2 d\tau + \|u_t\|_2^2 \int_0^t \|u(\tau)\|_{\omega}^2 d\tau \\
 & = \|u\|_2^2 \left[\|u_t\|_2^2 + \int_0^t \|u_t(\tau)\|_{\omega}^2 d\tau \right] + \int_0^t \|u(\tau)\|_{\omega}^2 d\tau \left[\|u_t\|_2^2 + \int_0^t \|u_t(\tau)\|_{\omega}^2 d\tau \right] \\
 & = \left[\|u\|_2^2 + \int_0^t \|u(\tau)\|_{\omega}^2 d\tau \right] \left[\|u_t\|_2^2 + \int_0^t \|u_t(\tau)\|_{\omega}^2 d\tau \right] \\
 & \leq \theta(t) \left[\|u_t\|_2^2 + \int_0^t \|u_t(\tau)\|_{\omega}^2 d\tau \right].
 \end{aligned}$$

Inserting (3.26) into (3.22), we obtain that

$$\theta(t)\theta''(t) - (\theta'(t))^2 \geq \theta(t)\sigma(t), \quad t \in [0, T],$$

where

$$\sigma(t) = -2\|u_t\|_2^2 - 2I(t) - 4 \int_0^t \|u(\tau)\|_{\omega}^2 d\tau.$$

On the other hand, from Lemmas 2.4, 2.9, (2.6), $E(0) < d$, and $p > 2$, we conclude that

$$\begin{aligned}
 (3.27) \quad & \sigma(t) = 4J(t) - 4E(t) - 2I(t) - 4 \int_0^t \|u(\tau)\|_{\omega}^2 d\tau \\
 & = \frac{4}{p^2}\|u\|_p^p + \frac{2(2-p)}{p}I(t) - 4E(t) - 4 \int_0^t \|u(\tau)\|_{\omega}^2 d\tau \\
 & \geq \frac{4}{p^2}\|u\|_p^p - 4E(t) - 4 \int_0^t \|u(\tau)\|_{\omega}^2 d\tau \\
 & \geq \frac{4}{p^2}\|u\|_p^p - 4E(0) > \frac{4}{p^2}\|u\|_p^p - 4d > \nu > 0.
 \end{aligned}$$

Then, by using (3.27), we can see clearly that

$$\theta(t)\theta''(t) - (\theta'(t))^2 \geq \theta(t)\nu > 0, \quad t \in [0, T].$$

Let $Y(t) = \ln(\theta(t))$, then

$$(3.28) \quad Y'(t) = \frac{\theta'(t)}{\theta(t)},$$

and

$$Y''(t) = \frac{\theta(t)\theta''(t) - (\theta'(t))^2}{(\theta(t))^2} > 0.$$

Therefore, the function $Y'(t)$ is increasing with respect to t . A simple integration of (3.28) leads to

$$Y(t) - Y(0) = \ln \theta(t_0) = \int_0^t \frac{\theta'(\tau)}{\theta(\tau)} d\tau \geq \int_0^t \frac{\theta'(0)}{\theta(0)} d\tau = \frac{\theta'(0)}{\theta(0)}t,$$

which implies that

$$\theta(t) \geq \theta(0) \exp\left(\frac{\theta'(0)}{\theta(0)}t\right).$$

Namely,

$$\lim_{t \rightarrow +\infty} \theta(t) = +\infty,$$

which implies that the weak solution u to the problem (1.1) blows up at infinite time. \square

4. Global existence, asymptotic behavior, and blow up for $E(0) = d$

In this section we prove the global existence, asymptotic behavior, and blow up of solutions for problem (1.1) with $E(0) = d$.

Lemma 4.1. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) = d$, then there exists a $t_0 \in (0, t)$ such that*

$$\int_0^{t_0} \|u_s\|_{\omega}^2 ds > 0.$$

Proof. The proof of this lemma is almost the same to that of Lemma 4.1 in [32]. So, we omit the details. \square

4.1. Global existence for $E(0) = d$

Theorem 4.2. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$ hold. Assume that $E(0) = d$ and $u_0 \in W$, then problem (1.1) admits a global weak solution $u(t) \in L^\infty(0, +\infty; W_0^{1,p}(\Omega))$ with $u_t(t) \in L^\infty(0, +\infty; L^2(\Omega))$.*

Proof. We prove this theorem in the following two cases:

Case 1: $\|\nabla u_0\|_p \neq 0$. Let $\lambda_m = 1 - 1/m$ and $u_{0m} = \lambda_m u_0$ for $m \geq 2$. Obviously, $\lambda_m \rightarrow 1$ as $m \rightarrow \infty$. Thus, we consider the following initial boundary value problem

$$(4.1) \quad \begin{cases} u_{tt} - \Delta_p u - \omega \Delta u_t + u_t = |u|^{p-2}u \ln |u| & \text{in } \Omega \times (0, \infty), \\ u(x, 0) = u_{0m}(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega, \\ u(x, t) = 0 & \text{in } \partial\Omega \times (0, \infty). \end{cases}$$

By $u_0 \in W$ and Lemma 2.5, we have $\lambda^* = \lambda^*(u_0) > 1$. Therefore, we conclude that $u_{0m} \in W$. Thus, we have $J(u_{0m}) < J(u_0)$, i.e.,

$$0 < E_m(0) = \frac{1}{2}\|u_1\|_2^2 + J(u_{0m}) < \frac{1}{2}\|u_1\|_2^2 + J(u_0) = E(0) = d.$$

Therefore, for sufficiently large m , we claim by Theorem 3.1 that, the problem (4.1) admits a global weak solution $u_m(t) \in L^\infty([0, +\infty), W_0^{1,p}(\Omega))$, $u_{mt}(t) \in L^\infty([0, +\infty), L^2(\Omega))$ and $u_m \in W$ for $0 \leq t \leq \infty$. Similar to the proof of Theorem 3.1, we derive (3.1), (3.3) and (3.4) which implies that u is the global weak solution of problem (1.1).

Case 2: $\|\nabla u_0\|_p = 0$. Let $\lambda_m = 1 - 1/m$ and $u_{1m} = \lambda_m u_1$ for $m \geq 2$. Similar to Case 1, we consider the following problem

$$(4.2) \quad \begin{cases} u_{tt} - \Delta_p u - \omega \Delta u_t + u_t = |u|^{p-2} u \ln |u| & \text{in } \Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_{1m}(x) & \text{in } \Omega, \\ u(x, t) = 0 & \text{in } \partial\Omega \times (0, \infty). \end{cases}$$

Note that $\|\nabla u_0\|_p = 0$ implies $J(u_0) = 0$ and

$$\frac{1}{2}\|u_1\|_2^2 = E(0) = d.$$

From

$$E_m(0) = \frac{1}{2}\|u_{1m}\|_2^2 + J(u_0) = \frac{1}{2}\|\lambda_m u_1\|_2^2 = \frac{\lambda_m^2}{2}\|u_1\|_2^2 < \frac{1}{2}\|u_1\|_2^2 = E(0) = d$$

and Theorem 3.1, it follows that for sufficiently large m , problem (4.2) admits a global solution $u_m(t) \in L^\infty([0, +\infty), W_0^{1,p}(\Omega))$, $u_{mt}(t) \in L^\infty([0, +\infty), L^2(\Omega))$ and $u_m \in W$ for $0 \leq t \leq \infty$. The remainder proof is the same as Case 1. □

4.2. Asymptotic behavior for $E(0) = d$

Theorem 4.3. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$ hold. Assume that $E(0) = d$ and $u_0 \in W$, then there exist two positive constants \tilde{c}_* and k such that*

$$E(t) \leq \tilde{c}_* e^{-kt}, \quad \forall t > 0.$$

Proof. From Lemma 4.1, this implies that there exists a $t_0 > 0$ such that

$$\int_{t_0}^t \|u_s\|_\omega^2 ds > 0 \quad \text{and} \quad E(t_0) = E(0) - \int_{t_0}^t \|u_s\|_\omega^2 ds = d - \int_{t_0}^t \|u_s\|_\omega^2 ds < d.$$

By Theorem 3.2, we have

$$E(t) \leq \tilde{c} e^{-k(t-t_0)} = \tilde{c}_* e^{-kt}, \quad \forall t > 0,$$

where $\tilde{c}_* = \tilde{c} e^{kt_0}$. The proof is now complete. □

4.3. Blow up for $E(0) = d$

Theorem 4.4. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$ hold. Assume that $E(0) = d$ and $u_0 \in V$, then solutions to problem (1.1) blow up in infinite time.*

Proof. Recall the auxiliary function $\theta(t)$ defined as (3.18) and the proof of Theorem 3.4, we have

$$\sigma(t) \geq \frac{4}{p^2} \|u\|_p^p - 4E(0) \geq \frac{4}{p^2} \|u\|_p^p - 4d \geq 0,$$

and hence

$$\theta(t)\theta''(t) - (\theta'(t))^2 > 0.$$

The remainder proof is similar to that in Theorem 3.4. □

5. Blow up for $E(0) > 0$

In this section, we state and prove the blow-up result of solution to of problem (1.1) with $E(0) > 0$ and $\omega = 0$.

Theorem 5.1. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$ hold. Assume that $\omega = 0$, the initial data and initial energy satisfy*

- (1) $u_0 \in V$;
- (2) $(u_0, u_1) > 0$;
- (3) $\|u_0\|_2^2 > (p^2 E(0)/c_2^p)^{2/p} > 0$,

then the solutions to problem (1.1) blow up in infinite time.

In order to show our blow up result, we need the following lemmas.

Lemma 5.2. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$. Assume that $\omega = 0$ and the initial data satisfy*

$$(u_0, u_1) > 0,$$

then the map $\{t \mapsto \|u\|_2^2\}$ is strictly increasing when $u(t) \in V$.

Lemma 5.3. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$ and u be a weak solution to problem (1.1) with the maximum existence time interval $[0, T)$. Assume that the conditions of Theorem 5.1 hold, then the solution $u(t)$ belongs to V for problem (1.1).*

Remark 5.4. The proofs of Lemmas 5.2 and 5.3 are similar to those in [21, 32], we omit the details.

Proof of Theorem 5.1. Recalling the auxiliary function $\theta(t)$ defined as (3.18) and the proof of Theorem 3.4, when $\omega = 0$, we see that

$$\sigma(t) = -2\|u_t\|_2^2 - 2I(t) - 4 \int_0^t \|u(\tau)\|_2^2 d\tau.$$

By the same arguments as of (3.27), we can deduce

$$\sigma(t) \geq \frac{4}{p^2} \|u\|_p^p - 4E(0) \geq \frac{4c_2^p}{p^2} \|u\|_2^p - 4E(0) \geq \frac{4c_2^p}{p^2} \|u_0\|_2^p - 4E(0) > 0.$$

The remainder proof is similar to that in Theorem 3.4. □

Remark 5.5. In the next paper, we hope to consider the global existence, asymptotic behavior and blow-up of solutions to problem (1.1) with $1 < p < 2$.

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