TAIWANESE JOURNAL OF MATHEMATICS
Vol. 26, No. 6, pp. 1113–1131, December 2022
DOI: 10.11650/tjm/220701

Mean Values of Derivatives of L-functions for Cyclotomic Characters in Function Fields

Jungyun Lee and Yoonjin Lee*

Abstract. We compute the mean value of derivatives of L-functions for cyclotomic characters in function fields. We also show the non-vanishing property of derivatives of L-functions for cyclotomic characters in function fields.

1. Introduction

Many mathematicians have studied the values of L-functions at their central points. The Birch and Swinnerton–Dyer conjecture states that the rank of an abelian variety is related to the vanishing order of the central point of its related L-function. It is thus a natural question whether the central values of L-functions are nonzero or not. However, in general, it is a hard task to determine whether a given single L-value is nonzero or not. Therefore, many mathematicians consider a family of L-functions, and they prove that the mean value of their central values is nonzero.

We consider the function field case as follows. Let $\mathbb{F}_q$ be the finite field of order q, where q is a power of a prime p. Let t be an indeterminate over $\mathbb{F}_q$, and let $A = \mathbb{F}_q[t]$ and $k = \mathbb{F}_q(t)$. We denote the set of all monic elements of A by A^+. Let $\chi_{\mathfrak{m}} \colon (A/\mathfrak{m}A)^* \to \mathbb{C}^*$ be a primitive quadratic character with conductor $\mathfrak{m} \in A^+$. We define

$$L(s, \chi_{\mathfrak{m}}) = \sum_{a \in A^+} \frac{\chi_{\mathfrak{m}}(a)}{|a|^s},$$

where $|a| = q^{\deg a}$. Rosen [5,7] obtained the mean value of primitive quadratic characters:

$$\frac{1}{q^M} \sum_{\substack{\mathfrak{m} \in A^+ \\ \deg \mathfrak{m} = M}} L(1/2, \chi_{\mathfrak{m}}).$$

Later, Andrade et al. [1–4] studied the asymptotic behavior of several moments of L-values

$$\sum_{\mathfrak{m} \in S} L(1/2, \chi_{\mathfrak{m}})^k \quad \text{and} \quad \sum_{\mathfrak{m} \in S} L''(1/2, \chi_{\mathfrak{m}})^k,$$

Received May 12, 2021; Accepted July 6, 2022.
Communicated by Ming-Lun Hsieh.
2020 *Mathematics Subject Classification.* 11R29, 11R58.
Key words and phrases. function field, L-function, central value, mean value.
*Corresponding author.

where S is either a set of all square free polynomials in A^+ or a set of irreducible polynomials in A^+. The authors [6] computed the first moment of L and proved the non-vanishing property of L-functions for cyclotomic characters in function fields.

Throughout this paper, let $\mathbb{F}_q$ be a finite field of of order $q = p^r$ with a prime p, $A = \mathbb{F}_q[t]$, f a monic irreducible polynomial in A, and $\chi\colon (A/f^nA)^* \to \mathbb{C}^*$ be a primitive cyclotomic character with some p power order. Let g be a monic polynomial in A which is relatively prime to f and $\eta\colon (A/gA)^* \to \mathbb{C}^*$ be a primitive even character. Let O_{f^n} be the set of all primitive cyclotomic characters χ modulo f^n with some p power order, and log denotes $\log_{10}$. Let $h \in A$ be relatively prime to f with $\deg h \geq 1$.

In this paper we compute the mean value of derivatives of L-functions for cyclotomic characters in function fields. We also show the non-vanishing property of derivatives of L-functions for cyclotomic characters in function fields. In detail, we find the mean value of derivatives of an infinite family of L-functions for cyclotomic characters in function fields: that is, we compute the mean value

$$\frac{1}{|O_{f^n}|} \sum_{\chi \in O_{f^n}} L_i(1/2, \eta\chi)\chi(\overline{h}),$$

where $L_1 = L'$ and $L_2 = L''$. We also show the non-vanishing property of the derivatives of L-functions for cyclotomic characters in function fields. That is, we obtain that $L_i(1/2, \eta\chi)\chi(\overline{h}) \neq 0$ for some χ in O_{f^n} with $n > M_i(f, g, h)$, where $i = 1, 2$ and $M_i(f, g, h)$ is a positive real number depending on f, g and h. We point out that we explicitly determine $M_i(f, g, h)$ for $i = 1, 2$.

We explain our main results in more detail as follows. In Theorem 4.1, for $h \in A$ relatively prime to f with $\deg h \geq 1$, we find that

$$\frac{1}{|O_{f^n}|} \sum_{\chi \in O_{f^n}} L'(1/2, \eta\chi)\chi(\overline{h}) = -\eta(h) \deg h |h|^{-1/2} (\log q) + \mathcal{E}_n,$$

where for $n > (|f| - 1)\frac{\deg h}{\deg f}$,

$$|\mathcal{E}_n| \leq 9 \frac{\log q}{q-1} |f|^{2 - \frac{n}{2(|f|-1)}} |g|^3 (n \deg f + \deg g).$$

Using this result, in Theorem 4.2, we prove that if $n > 2(|f| - 1)M_1(f, g, h)$, then

$$\frac{1}{|O_{f^n}|} \sum_{\chi \in O_{f^n}} L'(1/2, \eta\chi)\chi(\overline{h}) \neq 0,$$

where $M_1(f, g, h)$ is defined as in (4.12). In Theorem 5.1, we obtain that

$$\frac{1}{|O_{f^n}|} \sum_{\chi \in O_{f^n}} L''(1/2, \eta\chi)\chi(\overline{h}) = \eta(h)(\deg h)^2 |h|^{-1/2} (\log q)^2 + \mathcal{E}_n,$$

where for $n \geq (|f|-1)\frac{\deg h}{\deg f}$,

$$|\mathcal{E}_n| \leq 9\frac{(\log q)^2}{q-1}|f|^{2-\frac{n}{2(|f|-1)}}|g|^3(n\deg f + \deg g)^2.$$

We show that for $h \in A$ with $\deg h \geq 1$ which is relatively prime to f, if $n > 2(|f|-1)M_2(f,g,h)$, then

$$\frac{1}{|O_{f^n}|}\sum_{\chi \in O_{f^n}} L''(1/2,\eta\chi)\chi(\overline{h}) \neq 0,$$

where $M_2(f,g,h)$ is defined as in (5.1).

We discuss some contributions of our results. In regards to computation of the first moments for values of derivatives of L-functions, we find an explicit upper bound of the error term while it is obtained just *asymptotically* in the number field case. Using this bound, we are able to explicitly determine a lower bound n_0 of n such that the first moment

$$\frac{1}{|O_{f^n}|}\sum_{\chi \in O_{f^n}} L'(1/2,\eta\chi)\chi(\overline{h})$$

is nonzero for $n \geq n_0$. Moreover, in the function field case, there have been some developments on computation of the mean value of L'' *only for quadratic characters*. In this work, we compute the mean value of L' and that of L'' for *cyclotomic characters* in function fields.

2. Derivatives of L-functions in function fields

Let $\mathbb{F}_q$ be a finite field of order q, where q is a power of a prime p. Let t be an indeterminate over $\mathbb{F}_q$, and we set $A = \mathbb{F}_q[t]$ and $k = \mathbb{F}_q(t)$. We denote a set of all monic elements of A by A^+. For a monic polynomial $\mathfrak{m} \in A$, let $\chi_{\mathfrak{m}} \colon (A/\mathfrak{m}A)^* \to \mathbb{C}^*$ be a primitive character of conductor $\mathfrak{m}$. Let

$$L(s,\chi_{\mathfrak{m}}) = \sum_{a \in A^+}\frac{\chi_{\mathfrak{m}}(a)}{|a|^s}, \quad \Re(s) > 1,$$

where $|a| = q^{\deg a}$ and $\Re(s)$ denotes the real part of s. Since $L(s,\chi_{\mathfrak{m}})$ is absolutely convergent on $\Re(s) > 1$, reordering of the summands, one can write $L(s,\chi_{\mathfrak{m}})$ as follows:

$$L(s,\chi_{\mathfrak{m}}) = \sum_{d=0}^{\infty}\Bigg(\sum_{\substack{a \in A^+\\ \deg a = d}}\chi_{\mathfrak{m}}(a)\Bigg)q^{-ds}.$$

For $d \geq \deg \mathfrak{m}$, as $\sum_{\substack{a \in A^+\\ \deg a = d}}\chi_{\mathfrak{m}}(a) = 0$, $L(s,\chi_{\mathfrak{m}})$ is a polynomial in q^{-s}:

$$L(s,\chi_{\mathfrak{m}}) = \sum_{d=0}^{\deg \mathfrak{m}-1}\Bigg(\sum_{\substack{a \in A^+\\ \deg a = d}}\chi_{\mathfrak{m}}(a)\Bigg)q^{-ds} = \sum_{\substack{a \in A^+\\ \deg a < \deg \mathfrak{m}}}\chi_{\mathfrak{m}}(a)q^{-ds},$$

which is defined on the whole complex plane. Hence, it is natural to consider that

$$L(1/2,\chi_{\mathfrak{m}}) = \sum_{\substack{a\in A^+\\ \deg a<\deg \mathfrak{m}}} \chi_{\mathfrak{m}}(a)|a|^{-1/2},$$
$$L^{(k)}(1/2,\chi_{\mathfrak{m}}) = (-\log q)^k \sum_{\substack{a\in A^+\\ \deg a<\deg \mathfrak{m}}} \chi_{\mathfrak{m}}(a)(\deg a)^k|a|^{-1/2},$$

where k is a positive integer and $L^{(k)}(s,\chi_{\mathfrak{m}})$ is the kth derivative of $L(s,\chi_{\mathfrak{m}})$. In particular, if $\chi_{\mathfrak{m}}$ is an even character, we have that

$$\begin{aligned} L(1/2,\chi_{\mathfrak{m}}) &= \frac{1}{q-1} \sum_{\substack{a\in A\\ \deg a<\deg \mathfrak{m}}} \chi_{\mathfrak{m}}(a)|a|^{-1/2},\\ L^{(k)}(1/2,\chi_{\mathfrak{m}}) &= \frac{(-\log q)^k}{q-1} \sum_{\substack{a\in A\\ \deg a<\deg \mathfrak{m}}} \chi_{\mathfrak{m}}(a)(\deg a)^k|a|^{-1/2}. \end{aligned} \tag{2.1}$$

3. Necessary lemmas

In this section, we keep notations of the previous section. We establish several lemmas, and we find the mean value of characters, which are very crucial for later computation of the mean value of L-functions.

Let $k=\mathbb{F}_q(t)$, $A=\mathbb{F}_q[t]$, and f be a monic irreducible polynomial in A as before. Let A_f be the completion of A at f, and let $\omega\colon (A/fA)^*\to A_f^*$ be the function field analogue of the *Teichmüller character* (see [8, page 51]) such that

$$\text{for } b\in (A/fA)^*,\ \omega(b)^{|f|-1}=1 \text{ and } \omega(b)\equiv b \pmod{f}.$$

For positive integers n and m with $1\le m<n$, let $\pi_n\colon (A_f)^*\to (A/f^nA)^*$ be a natural projection map satisfying $\pi_n(a)\equiv a \pmod{f^n}$, $\pi_{n,m}\colon (A/f^nA)^*\to (A/f^mA)^*$, and $\pi_{n,m}(a)\equiv a \pmod{f^m}$. We define that $\omega_n=\omega\circ\pi_n$.

$$\begin{tikzcd}
(A/fA)^* \arrow[r, "\omega"] \arrow[rrrr, bend left, "\omega_1"] \arrow[rr, bend right, "\omega"'] \arrow[rrr, bend right, "\omega_m"'] & A_f^* \arrow[r, "\pi_n"] & (A/f^nA)^* \arrow[r, "\pi_{n,m}"] & (A/f^m)^* \arrow[r] & (A/fA)^*
\end{tikzcd}$$

For $a\in A$, $[a]_n$ denotes the residue of a modulo f^n. For a polynomial $g\in A$, $b\in A/gA$ means that b is the (least) residue of some polynomial in A modulo g such that $\deg b<\deg g$.

Lemma 3.1. *Let f be an irreducible polynomial in A^+. Let $b \in (A/fA)^*$, $r \in A$ be relatively prime to f and n be a positive integer. Then we have the following results.*

(i) *If $b \in \mathbb{F}_q^*$, then $[\omega(b)]_n = b$, $\forall\, n \geq 1$.*

(ii) *If $b \notin \mathbb{F}_q^*$, then for $n \geq (|f|-1)\frac{\deg r}{\deg f}$, we have*

$$\deg([r\omega(b)]_n) \geq n\frac{\deg f}{|f|-1}.$$

Proof. (i) It is clear because $b^{|f|-1} = 1$ for $b \in \mathbb{F}_q^*$.

(ii) Since $(r\omega(b))^{|f|-1} = r^{|f|-1}$, we have

$$[r\omega(b)]_n^{|f|-1} \equiv r^{|f|-1} \pmod{f^n}.$$

We suppose that two elements $[r\omega(b)]_n^{|f|-1}$ and $r^{|f|-1}$ in A are equal. Then there is $c \in \mathbb{F}_q^*$ such that

$$[r\omega(b)]_n = rc,$$

which is because every element $a \in A$ such that $a^{|f|-1} = 1$ belongs to $\mathbb{F}_q^*$. Thus, we have

$$A/f^{n-\alpha}Ar\omega(b) \equiv rc \pmod{f^n}. \tag{3.1}$$

Since r is relatively prime to f, (3.1) implies that

$$\omega(b) \equiv c \pmod{f^n},$$

which is a contradiction to the fact that $\omega(b) \equiv b \pmod{f}$ for $b \notin \mathbb{F}_q^*$. Thus, there is a nonzero element $d \in A$ such that

$$[r\omega(b)]_n^{|f|-1} = r^{|f|-1} + f^n d.$$

Therefore, we have that

$$(|f|-1)\deg[r\omega(b)]_n \geq n\deg f.$$

This completes the proof. □

For $1 \leq \alpha \leq n$, we define $W_{n,\alpha} := \pi_{n,\alpha}^{-1}(\mathrm{image}(\omega_\alpha))$. Then we have that

$$W_{n,\alpha} = \{a \in (A/f^nA)^* \mid a \equiv \omega_\alpha(b) \pmod{f^\alpha}, b \in (A/fA)^*\} \tag{3.2}$$

and $W_{n,\alpha}$ is a subgroup of $(A/f^nA)^*$. Moreover we have that

$$|W_{n,\alpha}| = |(A/fA)^*||(A/f^{n-\alpha}A)| = (|f|-1)|f|^{n-\alpha}.$$

We also note that

$$W_{n,n} \subset W_{n,n-1} \subset \cdots \subset W_{n,1} = (A/f^nA)^*.$$

For $1 \leq \alpha \leq n$, let $G_{n,\alpha}$ be defined by

$$G_{n,\alpha} = \{\chi_n \colon (A/f^nA)^* \to \mu_{p^\infty} \mid W_{n,\alpha} \subset \ker \chi_n\}, \tag{3.3}$$

where μ_{p^n} is the set of all p^n-th roots of unity for a positive integer n and $\mu_{p^\infty} = \bigcup_{n=1}^{\infty} \mu_{p^n}$. Then $G_{n,\alpha}$ is a subgroup of $G_{n,\alpha+1}$, that is,

$$\{1\} = G_{n,1} \subset G_{n,2} \subset \cdots \subset G_{n,n-1} \subset G_{n,n}.$$

For positive integers n and m with $1 \leq m < n$, we have that

$$f^nA \subset f^mA.$$

There is a natural projection $\pi_{n,m} \colon (A/f^nA)^* \to (A/f^mA)^*$ given by $\pi_{n,m}(a) \equiv a \pmod{f^m}$. Hence, every character $\chi_m \colon (A/f^mA)^* \to \mathbb{C}^*$ can be considered as a character χ_n defined on $(A/f^nA)^*$ as follows:

$$\chi_n := \chi_m \circ \pi_{n,m} \colon (A/f^nA)^* \to \mathbb{C}^*.$$

In this case, we say that χ_n and χ_m are *isomorphic*. If a character χ_n is not isomorphic to χ_m for every m with $1 \leq m < n$, then we say that χ_n is a *primitive character modulo* f^n.

Lemma 3.2. *Let O_{f^n} be the set of all primitive characters modulo f^n with p power order, and $G_{n,\alpha}$ be defined as in* (3.3)*. Then we have that*

$$O_{f^n} = G_{n,n} \setminus G_{n,n-1}.$$

Proof. We easily check that

$$W_{n,m} = \{a \in (A/f^nA)^* \mid a \equiv \omega_m(b) \pmod{f^m}\} \subset \ker(\chi_m \circ \pi_{n,m}).$$

Thus, if a character $\chi_n \colon (A/f^nA)^* \to \mathbb{C}^*$ is isomorphic to some character $\chi_m \colon (A/f^mA)^* \to \mathbb{C}^*$ for $1 \leq m < n$, then

$$\chi_n = \chi_m \circ \pi_{n,m} \in G_{n,m} \subset G_{n,n-1}.$$

Therefore, $\chi_n \in G_{n,n-1}$. We thus find that O_{f^n} is equal to $G_{n,n} \setminus G_{n,n-1}$. □

Now, we obtain the following lemma.

Lemma 3.3. *For positive integers n and α with $1 \leq \alpha \leq n$, let $O_{f^n} = G_{n,n} \setminus G_{n,n-1}$ and $W_{n,\alpha}$ be defined as in* (3.2)*. Then we have that*

$$(\chi_n)_{av}(a) := \frac{1}{|O_{f^n}|} \sum_{\chi_n \in O_{f^n}} \chi_n(a) = \begin{cases} 1 & \text{if } a \in W_{n,n}, \\ -\frac{1}{|f|-1} & \text{if } a \in W_{n,n-1} \setminus W_{n,n}, \\ 0 & \text{if } a \notin W_{n,n-1}. \end{cases}$$

Proof. If $a \in W_{n,\alpha}$, then $\chi_n(a) = 1$ for $\chi_n \in G_{n,\alpha}$. Hence, one can easily obtain the first formula. Now, we suppose that $a \in (A/f^nA)^* \setminus W_{n,\alpha}$. We consider the subgroup $\langle a, W_{n,\alpha}\rangle$ of $(A/f^nA)^*$ generated by a and the elements of $W_{n,\alpha}$. Let λ be a homomorphism $\lambda\colon \langle a, W_{n,\alpha}\rangle \to \mathbb{C}^*$ sending a to a nontrivial element in $\mathbb{C}^*$ and $W_{n,\alpha}$ to $1 \in \mathbb{C}^*$. This homomorphism λ can be extended to a homomorphism $\overline{\lambda}$ from $(A/f^nA)^*$ to $\mathbb{C}^*$. By definition, we have that

$$\overline{\lambda} \in G_{n,\alpha} \quad \text{and} \quad \overline{\lambda}(a) \neq 1.$$

Thus,

$$\sum_{\chi_n \in G_{n,\alpha}} \chi_n(a) = \sum_{\chi_n \in G_{n,\alpha}} \overline{\lambda}(a)\chi_n(a).$$

Finally, we have that

$$(\overline{\lambda}(a) - 1) \sum_{\chi_n \in G_{n,\alpha}} \chi_n(a) = 0.$$

Therefore, we have that for $a \in (A/f^nA)^* \setminus W_{n,\alpha}$,

$$\sum_{\chi_n \in G_{n,\alpha}} \chi_n(a) = 0.$$

Thus, we find that for $1 \leq \alpha \leq n$,

$$\sum_{\chi_n \in G_{n,\alpha}} \chi_n(a) = \begin{cases} |G_{n,\alpha}| & \text{if } a \in W_{n,\alpha}, \\ 0 & \text{otherwise.} \end{cases}$$

Using the formula as above, we obtain that

$$\sum_{\chi_n \in G_{n,n}} \chi_n(a) - \sum_{\chi_n \in G_{n,n-1}} \chi_n(a) = \begin{cases} |G_{n,n}| - |G_{n,n-1}| & \text{if } a \in W_{n,n}, \\ -|G_{n,n-1}| & \text{if } a \in W_{n,n-1} \setminus W_{n,n}, \\ 0 & \text{if } a \notin W_{n,n-1}. \end{cases}$$

Since $|G_{n,\alpha}| = |f|^\alpha$, the result follows immediately. $\square$

The following two lemmas are necessary for computing the mean values in Sections 4 and 5.

Lemma 3.4. *Let η be a character with conductor $g \in A$. Let $a, b \in A$ with $(b, g) = 1$. For $m > \deg g$, we have*

$$\sum_{\deg \alpha = m} \eta(a + b\alpha) = 0. \tag{3.4}$$

Proof. We note that

$$\{\alpha \in A \mid \deg \alpha = m\} = \{gh + r \mid r \in A/gA \text{ and } h \in A, \deg h = m - \deg g\}.$$

Thus, the left-hand side of (3.4) is equal to

$$\sum_{r \in A/gA} \sum_{\substack{h \in A \\ \deg h = m - \deg g}} \eta(a + b(r + gh)) = \sum_{\substack{h \in A \\ \deg h = m - \deg g}} \sum_{r \in A/gA} \eta(a + br).$$

We consider the map $\phi\colon A/gA \to A/gA$ mapping r into $ar + b$. We note that for any $r \in A/gA$, if we take $r' = b^{-1}(r - a)$, then

$$a + br' \equiv r \pmod{g}.$$

Thus, the map $\phi\colon A/gA \to A/gA$ is a surjective map. Since $(b, g) = 1$, if $a + br_1 \equiv a + br_2 \pmod{g}$ for $r_1, r_2 \in A/gA$, then

$$r_1 \equiv r_2 \pmod{g},$$

which implies that $r_1 = r_2$. Hence, ϕ is a bijective map. It thus follows that

$$\sum_{r \in A/gA} \eta(a + br) = \sum_{r \in A/gA} \eta(\phi(r)) = \sum_{r \in A/gA} \eta(r) = 0.$$

This completes the proof. $\square$

Lemma 3.5. *For a nonnegative integer k and a positive integer d, let $\mathcal{F}_k(d)$ be defined by*

$$\mathcal{F}_k(d) = \sum_{\substack{\deg a \leq d \\ a \in A, a \neq 0}} (\deg a)^k |a|^{-1/2}.$$

Then we have that

$$\mathcal{F}_k(d) \leq 2\sqrt{q}^{\,d+2} d^k.$$

Proof.

$$\begin{aligned}
\mathcal{F}_k(d) &= \sum_{\substack{\deg a \leq d \\ a \in A, a \neq 0}} (\deg a)^k |a|^{-1/2} = \sum_{i=0}^{d} \#\{a \in A \mid \deg a = i, a \neq 0\} i^k \sqrt{q}^{\,-i} \\
&= \sum_{i=0}^{d} (q-1) i^k \sqrt{q}^{\,i} \leq (q-1) d^k \frac{\sqrt{q}^{\,d+1} - 1}{\sqrt{q} - 1} \\
&= (\sqrt{q}^{\,d+1} - 1)(\sqrt{q} + 1) d^k \leq 2\sqrt{q}^{\,d+2} d^k.
\end{aligned}$$

$\square$

4. Mean values of the first derivatives of L-functions

In this section, we compute the mean value of $L'(1/2, \eta\chi)\chi(\overline{h})$, where η is a general character and $h \in A$ is relatively prime to f, and thus we show that $L'(1/2, \eta\chi)$ is nonzero for sufficiently large n. We keep the notations of the previous sections in this section.

In the following theorem, we find a main term of the mean value of $L'(1/2, \eta\chi)$, and we estimate an upper bound on its error term.

Theorem 4.1. *Let f be a monic irreducible polynomial in A and $\chi\colon (A/f^nA)^* \to \mathbb{C}^*$ be a primitive character with some p power order. For $g \in A$ relatively prime to f, let $\eta\colon (A/gA)^* \to \mathbb{C}^*$ be an even primitive character. Then for $h \in A$ relatively prime to f with $\deg h \geq 1$, we have that*

$$\frac{1}{|O_{f^n}|} \sum_{\chi \in O_{f^n}} L'(1/2, \eta\chi)\chi(\overline{h}) = -\eta(h) \deg h |h|^{-1/2} (\log q) + \mathcal{E}_n, \tag{4.1}$$

where $\overline{h}$ is the multiplicative inverse modulo f^n and for $n > (|f| - 1)\frac{\deg h}{\deg f}$,

$$|\mathcal{E}_n| \leq \frac{9 \log q}{q - 1} |f|^{2 - \frac{n}{2(|f|-1)}} |g|^3 (n \deg f + \deg g).$$

Proof. Assume that $n > (|f| - 1)\frac{\deg h}{\deg f}$. From (2.1) and Lemma 3.3, we have that

$$\begin{aligned}
&- \frac{q - 1}{|O_{f^n}| \log q} \sum_{\chi \in O_{f^n}} L'(1/2, \eta\chi)\chi(\overline{h}) \\
&= \sum_{d \in A/gA} \sum_{a \in A/f^nA} \eta(a + f^n d)(\chi)_{av}(a\overline{h}) \deg(a + f^n d)|a + f^n d|^{-1/2} \\
&= \sum_{d \in A/gA} \Bigg(\sum_{a\overline{h} \in W_{n,n-1} \setminus W_{n,n}} \eta(a + f^n d)(\chi)_{av}(a\overline{h}) \deg(a + f^n d)|a + f^n d|^{-1/2} \\
&\qquad + \sum_{a\overline{h} \in W_{n,n}} \eta(a + f^n d)(\chi)_{av}(a\overline{h}) \deg(a + f^n d)|a + f^n d|^{-1/2} \Bigg) \\
&= \frac{-1}{(|f| - 1)} \sum_{d \in A/gA} \sum_{a\overline{h} \in W_{n,n-1} \setminus W_{n,n}} \Omega + \sum_{d \in A/gA} \sum_{a\overline{h} \in W_{n,n}} \Omega \\
&= \frac{-1}{(|f| - 1)} \sum_{d \in A/gA} \Bigg(\sum_{a\overline{h} \in W_{n,n-1}} \Omega - \sum_{a\overline{h} \in W_{n,n}} \Omega \Bigg) + \sum_{d \in A/gA} \sum_{a\overline{h} \in W_{n,n}} \Omega \\
&= \frac{-1}{|f| - 1} \sum_{d \in A/gA} \sum_{a\overline{h} \in W_{n,n-1}} \Omega + \frac{|f|}{|f| - 1} \sum_{d \in A/gA} \sum_{a\overline{h} \in W_{n,n}} \Omega,
\end{aligned}$$

where

$$\Omega = \eta(a + f^n d) \deg(a + f^n d)|a + f^n d|^{-1/2}.$$

From now on, let us set

$$\mathcal{A}_n := \sum_{d\in A/gA}\sum_{a\overline{h}\in W_{n,n-1}} \Omega \quad \text{and} \quad \mathcal{B}_n := \sum_{d\in A/gA}\sum_{a\overline{h}\in W_{n,n}} \Omega.$$

First, we compute $\mathcal{A}_n$. From the definition of $W_{n,\alpha}$ in (3.2), $\mathcal{A}_n$ is written as follows:

$$\mathcal{A}_n = \sum_{b\in(A/fA)^*}\sum_{d\in A/gA}\sum_{c\in(A/fA)} \eta(\delta_{n-1}(b,c,d))\deg(\delta_{n-1}(b,c,d))|\delta_{n-1}(b,c,d)|^{-1/2},$$

where $\delta_{n-1}(b,c,d) := [\omega(b)h]_{n-1} + f^{n-1}c + f^n d$. Now, the summation $\mathcal{A}_n$ is divided into four parts according to whether d or c is zero or not, and in the case where both d and c are zero, whether b is contained in $\mathbb{F}_q^*$ or not:

$$\mathcal{A}_n = \mathcal{A}_{n,1} + \mathcal{A}_{n,2} + \mathcal{A}_{n,3} + \mathcal{A}_{n,4},$$

where

$$\begin{aligned}
\mathcal{A}_{n,1} &= \sum_{b\in\mathbb{F}_q^*}\sum_{d=0,c=0} \eta(\delta_{n-1}(b,c,d))\deg(\delta_{n-1}(b,c,d))|\delta_{n-1}(b,c,d)|^{-1/2},\\
\mathcal{A}_{n,2} &= \sum_{b\in(A/fA)^*\setminus\mathbb{F}_q^*}\sum_{d=0,c=0} \eta(\delta_{n-1}(b,c,d))\deg(\delta_{n-1}(b,c,d))|\delta_{n-1}(b,c,d)|^{-1/2},\\
\mathcal{A}_{n,3} &= \sum_{b\in(A/fA)^*}\sum_{d=0,c\neq0} \eta(\delta_{n-1}(b,c,d))\deg(\delta_{n-1}(b,c,d))|\delta_{n-1}(b,c,d)|^{-1/2},\\
\mathcal{A}_{n,4} &= \sum_{b\in(A/fA)^*}\sum_{d\neq0,c\neq0} \eta(\delta_{n-1}(b,c,d))\deg(\delta_{n-1}(b,c,d))|\delta_{n-1}(b,c,d)|^{-1/2}.
\end{aligned}$$

Then by Lemma 3.1(i), it follows that

$$\begin{aligned}
\mathcal{A}_{n,1} &= \sum_{b\in\mathbb{F}_q^*} \eta([\omega(b)h]_{n-1})\deg([\omega(b)h]_{n-1})\big|[\omega(b)h]_{n-1}\big|^{-1/2}\\
&= (q-1)\eta(h)(\deg h)|h|^{-1/2},
\end{aligned} \tag{4.2}$$

and from Lemma 3.1(ii), we have that

$$\mathcal{A}_{n,2} = \sum_{b\in(A/fA)^*\setminus\mathbb{F}_q^*} \eta([\omega(b)h]_{n-1})\deg([\omega(b)h]_{n-1})\big|[\omega(b)h]_{n-1}\big|^{-1/2}$$

and

$$\begin{aligned}
|\mathcal{A}_{n,2}| &\le (n-1)\deg f|f|^{-\frac{n-1}{2(|f|-1)}}|(A/fA)^*\setminus\mathbb{F}_q^*|\\
&= (n-1)\deg f|f|^{-\frac{n-1}{2(|f|-1)}}(|f|-q).
\end{aligned} \tag{4.3}$$

Moreover, from Lemma 3.4, we have that

$$\begin{aligned}
\mathcal{A}_{n,3} &= \sum_{b\in(A/fA)^*} \sum_{\substack{c\in(A/fA)\\ c\neq 0}} \eta([\omega(b)h]_{n-1} + f^{n-1}c) \deg([\omega(b)h]_{n-1} + f^{n-1}c) \\
&\qquad\qquad \times \left|[\omega(b)h]_{n-1} + f^{n-1}c\right|^{-1/2} \\
&= |f|^{-(n-1)/2} \sum_{b\in(A/fA)^*} \sum_{\substack{c\in(A/fA)\\ c\neq 0}} \eta([\omega(b)h]_{n-1} + f^{n-1}c) \deg(f^{n-1}c)|c|^{-1/2} \\
&= |f|^{-(n-1)/2} \sum_{b\in(A/fA)^*} \sum_{\substack{c\neq 0\\ \deg c\leq \deg g}} \eta([\omega(b)h]_{n-1} + f^{n-1}c)((n-1)\deg f + \deg c)|c|^{-1/2}
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{A}_{n,3}| &\leq |f|^{-(n-1)/2} \sum_{b\in(A/fA)^*} \sum_{\substack{c\neq 0\\ \deg c\leq \deg g}} ((n-1)\deg f + \deg c)|c|^{-1/2} \\
&= |f|^{-(n-1)/2}(|f|-1)((n-1)\deg f\mathcal{F}_0(\deg g) + \mathcal{F}_1(\deg g)).
\end{aligned} \tag{4.4}$$

Furthermore, from Lemma 3.4, we also have that

$$\begin{aligned}
\mathcal{A}_{n,4} &= |f|^{-n/2} \sum_{b\in(A/fA)^*} \sum_{d\neq 0, c\neq 0} \eta([h\omega(b)]_{n-1} + f^{n-1}c + f^n d)(n\deg f + \deg d)|d|^{-1/2} \\
&= |f|^{-n/2} \sum_{b\in(A/fA)^*} \sum_{\substack{d\neq 0\\ d\in A/gA}} \sum_{\substack{c\neq 0\\ \deg c\leq \deg g}} \eta([h\omega(b)]_{n-1} + f^{n-1}c + f^n d) \\
&\qquad\qquad \times (n\deg f + \deg d)|d|^{-1/2}
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{A}_{n,4}| &\leq |f|^{-n/2} \sum_{b\in(A/fA)^*} \sum_{\substack{d\neq 0\\ d\in A/gA}} \sum_{\substack{c\neq 0\\ \deg c\leq \deg g}} (n\deg f + \deg d)|d|^{-1/2} \\
&\leq |f|^{-n/2} \sum_{b\in(A/fA)^*} \sum_{\substack{c\neq 0\\ \deg c\leq \deg g}} \sum_{\substack{d\neq 0\\ \deg d< \deg g}} (n\deg f + \deg d)|d|^{-1/2} \\
&= |f|^{-n/2}(|f|-1)(q^{\deg g+1}-1)(n\deg f\mathcal{F}_0(\deg g - 1) + \mathcal{F}_1(\deg g - 1)).
\end{aligned} \tag{4.5}$$

Now, we compute $\mathcal{B}_n$. We note that $\mathcal{B}_n$ is written as follows:

$$\begin{aligned}
\mathcal{B}_n &= \sum_{d\in A/gA} \sum_{a\bar{h}\in W_{n,n}} \eta(a + f^n d)\deg(a + f^n d)|a + f^n d|^{-1/2} \\
&= \sum_{d\in A/gA} \sum_{b\in(A/fA)^*} \eta([\omega(b)h]_n + f^n d)\deg([\omega(b)h]_n + f^n d)\left|[\omega(b)h]_n + f^n d\right|^{-1/2}.
\end{aligned}$$

The summation $\mathcal{B}_n$ can be divided into three parts according to whether d is zero or not, and in the case where $d = 0$, whether b is contained in $\mathbb{F}_q^*$ or not:

$$\mathcal{B}_n = \mathcal{B}_{n,1} + \mathcal{B}_{n,2} + \mathcal{B}_{n,3},$$

where

$$\begin{aligned}
\mathcal{B}_{n,1} &= \sum_{b\in\mathbb{F}_q^*}\sum_{d=0} \eta([\omega(b)h]_n + f^n d)\deg([\omega(b)h]_n + f^n d)\big|[\omega(b)h]_n + f^n d\big|^{-1/2},\\
\mathcal{B}_{n,2} &= \sum_{b\in(A/fA)^*\setminus\mathbb{F}_q^*}\sum_{d=0} \eta([\omega(b)h]_n + f^n d)\deg([\omega(b)h]_n + f^n d)\big|[\omega(b)h]_n + f^n d\big|^{-1/2},\\
\mathcal{B}_{n,3} &= \sum_{b\in(A/fA)^*}\sum_{\substack{d\neq 0\\ d\in(A/gA)^*}} \eta([\omega(b)h]_n + f^n d)\deg([\omega(b)h]_n + f^n d)\big|[\omega(b)h]_n + f^n d\big|^{-1/2}.
\end{aligned}$$

Then from Lemma 3.1(i), we have that

$$\begin{aligned}
\mathcal{B}_{n,1} &= \sum_{b\in\mathbb{F}_q^*} \eta([\omega(b)h]_n)\deg([\omega(b)h]_n)\big|[\omega(b)h]_n\big|^{-1/2}\\
&= (q-1)\eta(h)\deg h|h|^{-1/2},
\end{aligned}\tag{4.6}$$

and from Lemma 3.1(ii), we have that

$$\mathcal{B}_{n,2} = \sum_{b\in(A/fA)^*\setminus\mathbb{F}_q^*} \eta([\omega(b)h]_n)\deg([\omega(b)h]_n)\big|[\omega(b)h]_n\big|^{-1/2},$$

$$|\mathcal{B}_{n,2}| \le n\deg f|f|^{-\frac{n}{2(|f|-1)}}|(A/fA)^*\setminus\mathbb{F}_q^*| = n\deg f|f|^{-\frac{n}{2(|f|-1)}}(|f|-q).\tag{4.7}$$

Furthermore, we have that

$$\mathcal{B}_{n,3} = |f|^{-n/2}\sum_{\substack{d\in A/gA\\ d\neq 0}}\sum_{b\in(A/fA)^*} \eta([\omega(b)h]_n + f^n d)(n\deg f + \deg d)|d|^{-1/2}$$

and

$$|\mathcal{B}_{n,3}| \le |f|^{-n/2}(|f|-1)(|g|-1)(n\deg f\mathcal{F}_0(\deg g) + \mathcal{F}_1(\deg g)).\tag{4.8}$$

Finally, we obtain that

$$\frac{1}{|O_{f^n}|}\sum_{\chi\in O_{f^n}} L'(1/2,\eta\chi)\chi(\overline{h}) = \mathcal{M}_n + \mathcal{E}_n,$$

where

$$\begin{aligned}
\mathcal{M}_n &= \frac{\log q}{q-1}\left(\frac{1}{|f|-1}\mathcal{A}_{n,1} - \frac{|f|}{|f|-1}\mathcal{B}_{n,1}\right),\\
\mathcal{E}_n &= \frac{\log q}{q-1}\left(\frac{1}{|f|-1}\sum_{i=2}^{4}\mathcal{A}_{n,i} - \frac{|f|}{|f|-1}\sum_{i=2}^{3}\mathcal{B}_{n,i}\right).
\end{aligned}$$

From (4.2) and (4.6), we have that

$$\mathcal{M}_n = -\eta(h) \deg h |h|^{-1/2} (\log q).$$

Moreover, we obtain that

$$|\mathcal{E}_n| \leq \frac{\log q}{q-1} \left(\frac{1}{|f|-1} \sum_{i=2}^{4} |\mathcal{A}_{n,i}| + \frac{|f|}{|f|-1} \sum_{i=2}^{3} |\mathcal{B}_{n,i}| \right). \tag{4.9}$$

From (4.3) and (4.7), we get that

$$|\mathcal{B}_{n,2}|,\ |\mathcal{A}_{n,2}| \leq n \deg f |f|^{-\frac{n-1}{2(|f|-1)}} (|f| - q).$$

Thus,

$$\frac{1}{|f|-1} |\mathcal{A}_{n,2}| + \frac{|f|}{|f|-1} |\mathcal{B}_{n,2}| \leq n \deg f |f|^{-\frac{n-1}{2(|f|-1)}} (|f| - q) \left(\frac{|f|+1}{|f|-1} \right). \tag{4.10}$$

Moreover, combining (4.4), (4.5), (4.8) and Lemma 3.5 all together, we get that

$$\begin{aligned} |\mathcal{B}_{n,3}|,\ |\mathcal{A}_{n,3}| &\leq |f|^{-(n-1)/2} |f||g| (n \deg f \mathcal{F}_0(\deg g) + \mathcal{F}_1(\deg g)) \\ &\leq 2q |f|^{-(n-1)/2} |f||g|^{3/2} (n \deg f + \deg g) \end{aligned}$$

and

$$\begin{aligned} |\mathcal{A}_{n,4}| &\leq |f|^{-n/2} (|f| - 1)(|g|q - 1)(n \deg f \mathcal{F}_0(\deg g - 1) + \mathcal{F}_1(\deg g - 1)) \\ &\leq 2q |f|^{-(n-1)/2} |f||g|^{3/2} \frac{\sqrt{q}}{\sqrt{|f|}} (n \deg f + \deg g - 1) \\ &\leq 2q |f|^{-(n-1)/2} |f||g|^{3/2} (n \deg f + \deg g). \end{aligned}$$

Thus, we have

$$\begin{aligned} &\frac{1}{|f|-1} \big(|\mathcal{A}_{n,3}| + |\mathcal{A}_{n,4}| \big) + \frac{|f|}{|f|-1} |\mathcal{B}_{n,3}| \\ &\leq 2q |f|^{-(n-1)/2} |f||g|^{3/2} (n \deg f + \deg g) \left(\frac{|f|+2}{|f|-1} \right). \end{aligned} \tag{4.11}$$

Combining (4.9), (4.10) and (4.11) all together, we obtain that

$$|\mathcal{E}_n| \leq \frac{\log q}{q-1} \frac{|f|+2}{|f|-1} (n \deg f + \deg g) |f| \big(|f|^{-\frac{n-1}{2(|f|-1)}} + 2q |f|^{-(n-1)/2} |g|^{3/2} \big).$$

Since we have

$$(1 + 2q|g|^{3/2}) \leq 3|g|^3 \quad \text{and} \quad \frac{|f|+2}{|f|-1} \leq 3,$$

we finally obtain that

$$\begin{aligned}
|\mathcal{E}_n| &\le \frac{\log q}{q-1}\frac{|f|+2}{|f|-1}(n\deg f+\deg g)|f|\big(|f|^{-\frac{n-1}{2(|f|-1)}}+2q|f|^{-(n-1)/2}|g|^{3/2}\big)\\
&\le \frac{9\log q}{q-1}|f|^{1-\frac{n-1}{2(|f|-1)}}|g|^3(n\deg f+\deg g)\\
&\le \frac{9\log q}{q-1}|f|^{2-\frac{n}{2(|f|-1)}}|g|^3(n\deg f+\deg g).
\end{aligned}$$

□

In the following theorem, we find a lower bound n_0 of n such that

$$\big|\eta(h)\deg h|h|^{-1/2}(\log q)\big| > |\mathcal{E}_n|;$$

hence, we prove that for $n \ge n_0$,

$$\frac{1}{|O_{f^n}|}\sum_{\chi\in O_{f^n}} L'(1/2,\eta\chi)\chi(\overline{h}) \ne 0.$$

Theorem 4.2. *Let f be a monic irreducible polynomial in A and O_{f^n} be the set of primitive characters χ modulo f^n with p power order. For $g \in A$ relatively prime to f, let $\eta\colon (A/gA)^* \to \mathbb{C}^*$ be an even primitive character and $h \in A$ be relatively prime to f with* $\deg h \ge 1$. *If $n > 2M_1(f,g,h)(|f|-1)$, then*

$$\frac{1}{|O_{f^n}|}\sum_{\chi\in O_{f^n}} L'(1/2,\eta\chi)\chi(\overline{h}) \ne 0,$$

where $M_1(f,g,h)$ is the positive real number such that

$$\frac{|f|^{M_1(f,g,h)}}{M_1(f,g,h)} = 18|f|^4|g|^4|h|. \tag{4.12}$$

Consequently, $L'(1/2,\eta\chi)\ne 0$ for some $\chi \in O_{f^n}$ with $n > 2M_1(f,g,h)(|f|-1)$.

Proof. From Theorem 4.1, we find that if n satisfies the following two inequalities

$$\begin{aligned}
\deg(h)|h|^{-1/2} &> \frac{9}{q-1}|g|^3|f|^{2-\frac{n}{2(|f|-1)}}(n\deg f+\deg g),\\
n &> (|f|-1)\frac{\deg h}{\deg f},
\end{aligned}\tag{4.13}$$

then

$$\deg(h)|h|^{-1/2}(\log q) > 9\frac{\log q}{q-1}|g|^3|f|^{2-\frac{n}{2(|f|-1)}}(n\deg f+\deg g) \ge |\mathcal{E}_n|,$$

where $\mathcal{E}_n$ is defined as in (4.1). Hence, for n satisfying (4.13), we have that

$$\sum_{\chi\in O_{f^n}} L'(1/2,\eta\chi)\chi(\overline{h}) \ne 0.$$

From now on, we prove that if $n > 2M_1(f,g,h)(|f|-1)$, then n satisfies (4.13). We note that $\frac{|f|^x}{x}$ is an increasing function for $x > \frac{1}{(\log_e q)\deg f}$. Since

$$\frac{|f|^{M_1(f,g,h)}}{M_1(f,g,h)} > \frac{|f|^{\frac{\deg h}{2\deg f}}}{\frac{\deg h}{2\deg f}}$$

and

$$\frac{\deg h}{2\deg f} > \frac{1}{(\log_e q)\deg f}, \tag{4.14}$$

we obtain that

$$M_1(f,g,h) > \frac{\deg h}{2\deg f}. \tag{4.15}$$

Hence, for $n > 2M_1(f,g,h)(|f|-1)$, we have that

$$n > 2M_1(f,g,h)(|f|-1) > \frac{\deg h}{\deg f}(|f|-1),$$

which shows the second inequality of (4.13). From (4.14) and (4.15), we can easily check that

$$M_1(f,g,h) > \frac{1}{(\log_e q)\deg f}. \tag{4.16}$$

Since $\frac{|f|^x}{x}$ is an increasing function for $x > \frac{1}{(\log_e q)\deg f}$, from (4.16), we also find that if $\frac{n}{2(|f|-1)} > M_1(f,g,h)$, then

$$\frac{|f|^{\frac{n}{2(|f|-1)}}}{\frac{n}{2(|f|-1)}} > \frac{|f|^{M_1(f,g,h)}}{M_1(f,g,h)} = 18|f|^4|g|^4|h|. \tag{4.17}$$

Moreover, we can easily check that

$$18|f|^4|g|^4|h| > \frac{18|f|^2|g|^3|h|^{1/2}(|f|-1)\deg(fg)}{\deg h(q-1)}. \tag{4.18}$$

From (4.17) and (4.18), for $n > 2M_1(f,g,h)(|f|-1)$, we have that

$$\frac{|f|^{\frac{n}{2(|f|-1)}}}{\frac{n}{2(|f|-1)}} > \frac{18|f|^2|g|^3|h|^{1/2}(|f|-1)\deg(fg)}{\deg h(q-1)}.$$

Finally, we obtain that for $n > 2M_1(f,g,h)(|f|-1)$,

$$|f|^{\frac{n}{2(|f|-1)}} > 9\frac{|f|^2|g|^3|h|^{1/2}n\deg(fg)}{\deg h(q-1)} > 9\frac{|f|^2|g|^3|h|^{1/2}(n\deg f + \deg g)}{\deg h(q-1)}.$$

which shows the first inequality of (4.13). This completes the proof. $\square$

5. Mean values of the second derivatives of L-functions

In this section, we keep notations of previous sections. We compute the mean value of $L''(1/2,\eta\chi)\chi(\overline{h})$ for a general character η, and thus we show that $L''(1/2,\eta\chi)$ is nonzero for sufficiently large n.

Theorem 5.1. *Let f be a monic irreducible polynomial, and let g and h be monic polynomials relatively prime to f with $\deg h \geq 1$. Let $\chi\colon (A/f^nA)^* \to \mathbb{C}^*$ be a character with some p power order and $\eta\colon (A/gA)^* \to \mathbb{C}^*$ be a character. Then we get*

$$\frac{1}{|O_{f^n}|}\sum_{\chi\in O_{f^n}} L''(1/2,\eta\chi)\chi(\overline{h}) = \eta(h)(\deg h)^2|h|^{-1/2}(\log q)^2 + \mathcal{E}_n,$$

where for $n > (|f|-1)\frac{\deg h}{\deg f}$,

$$|\mathcal{E}_n| \leq 9\frac{(\log q)^2}{q-1}|f|^{2-\frac{n}{2(|f|-1)}}|g|^3(n\deg f + \deg g)^2.$$

Proof. Using Lemmas 3.1, 3.3, 3.4 and 3.5, we obtain the result by a similar computation as in the proof of Theorem 4.1. □

Theorem 5.2. *Let p be an odd prime, f be a monic irreducible polynomial, and g and h be monic polynomials relatively prime to f with $\deg h \geq 1$. Let $\chi\colon (A/f^nA)^* \to \mathbb{C}^*$ be a primitive character with p power order and $\eta\colon (A/gA)^* \to \mathbb{C}^*$ be a character. If $n > 2M_2(f,g,h)(|f|-1)$, then we have that*

$$\frac{1}{|O_{f^n}|}\sum_{\chi\in O_{f^n}} L''(1/2,\eta\chi)\chi(\overline{h}) \neq 0,$$

where $M_2(f,g,h)$ is the positive number such that

$$\frac{|f|^{M_2(f,g,h)}}{M_2(f,g,h)^2} = 36|f|^6|g|^5|h|. \tag{5.1}$$

Consequently, $L''(1/2,\eta\chi) \neq 0$ for some $\chi \in O_{f^n}$ such that $n > 2M_2(f,g,h)(|f|-1)$.

Proof. For $h \in A$ which is relatively prime to f, we recall that

$$\frac{1}{|O_{f^n}|}\sum_{\chi\in O_{f^n}} L''(1/2,\eta\chi)\chi(\overline{h}) = \eta(h)(\deg h)^2|h|^{-1/2}(\log q)^2 + \mathcal{E}_n,$$

where $n \geq (|f|-1)\frac{\deg h}{\deg f}$,

$$|\mathcal{E}_n| \leq \frac{9}{q-1}|g|^3|f|^{2-\frac{n}{2(|f|-1)}}n^2(\deg fg)^2(\log q)^2.$$

Thus, if n satisfies the following two inequalities

$$(5.2)\qquad \begin{aligned} |h|^{-1/2}(\deg h)^2(q-1) &> 9|g|^3|f|^2\frac{n^2(\deg fg)^2}{|f|^{\frac{n}{2(|f|-1)}}}, \\ n &\geq (|f|-1)\frac{\deg h}{\deg f}, \end{aligned}$$

then

$$|h|^{-1/2}(\deg h)^2(\log q)^2 > \frac{9}{q-1}|g|^3|f|^2\frac{n^2(\deg fg)^2}{|f|^{\frac{n}{2(|f|-1)}}} > |\mathcal{E}_n|.$$

Thus, if n satisfies the condition (5.2), then we have that

$$\sum_{\chi \in O_{f^n}} L''(1/2, \eta\chi)\chi(\overline{h}) \neq 0.$$

From now on, we prove that if $n > 2M_2(f,g,h)(|f|-1)$, then n satisfies (5.2).

We note that

$$(5.3)\qquad \frac{|f|^{M_2(f,g,h)}}{M_2(f,g,h)^2} > \frac{|f|^{\frac{\deg h}{2\deg f}}}{\frac{\deg h}{2\deg f}}.$$

Since $\frac{|f|^x}{x^2}$ is an increasing function for $x > \frac{2}{(\log_e q)\deg f}$ and

$$(5.4)\qquad \frac{\deg h}{2\deg f} > \frac{2}{(\log_e q)\deg f},$$

from (5.3) and (5.4), we find that

$$(5.5)\qquad M_2(f,g,h) > \frac{\deg h}{2\deg f}.$$

From (5.5), we conclude that if $n > 2M_2(f,g,h)(|f|-1)$, then

$$(5.6)\qquad n > (|f|-1)\frac{\deg h}{\deg f},$$

which is the second inequality of (5.2). From (5.4) and (5.5), we find that

$$(5.7)\qquad M_2(f,g,h) > \frac{1}{(\log_e q)\deg f}.$$

As $\frac{|f|^x}{x^2}$ is an increasing function for $x > \frac{2}{(\log_e q)\deg f}$, from (5.7), we have that for $\frac{n}{2(|f|-1)} > M_2(f,g,h)$,

$$\frac{|f|^{\frac{n}{2(|f|-1)}}}{\frac{n^2}{4(|f|-1)^2}} > \frac{|f|^{M_2(f,g,h)}}{M_2(f,g,h)^2} = 36|f|^6|g|^5|h|.$$

Moreover, we can easily check that

$$36|f|^6|g|^5|h| > \frac{36|f|^2|g|^3|h|^{1/2}(|f|-1)^2(\deg fg)^2}{(\deg h)^2(q-1)}.$$

Thus, if $\frac{n}{2(|f|-1)} > M_2(f,g,h)$, then

$$\frac{|f|^{\frac{n}{2(|f|-1)}}}{\frac{n^2}{4(|f|-1)^2}} > \frac{36|f|^2|g|^3|h|^{1/2}(|f|-1)^2(\deg fg)^2}{(\deg h)^2(q-1)}. \tag{5.8}$$

From (5.6) and (5.8), we conclude that if $\frac{n}{2(|f|-1)} > M_2(f,g,h)$, then the conditions in (5.2) are satisfied. Now the proof is completed. □

Acknowledgments

The authors would like to thank the reviewers for very helpful comments. J. Lee is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1I1A3057692). Y. Lee is a corresponding author and supported by the Ewha Womans University Research Grant of 2021, the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (NRF-2022R1A2C1003203), and Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2019R1A6A1A11051177).

References

[1] J. Andrade, *Mean values of derivatives of L-functions in function fields: II*, J. Number Theory **183** (2018), 24–39.

[2] ______, *Mean values of derivatives of L-functions in function fields: III*, Proc. Roy. Soc. Edinburgh Sect. A **149** (2019), no. 4, 905–913.

[3] J. C. Andrade and J. P. Keating, *The mean value of $L(\frac{1}{2},\chi)$ in the hyperelliptic ensemble*, J. Number Theory **132** (2012), no. 12, 2793–2816.

[4] J. Andrade and S. Rajagopal, *Mean values of derivatives of L-functions in function fields: I*, J. Math. Anal. Appl. **443** (2016), no. 1, 526–541.

[5] J. Hoffstein and M. Rosen, *Average values of L-series in function fields*, J. Reine Angew. Math. **426** (1992), 117–150.

[6] J. Lee and Y. Lee, *Non-vanishing of L-functions for cyclotomic characters in function fields*, Proc. Amer. Math. Soc. **150** (2022), no. 2, 455–468.

[7] M. Rosen, *Average value of class numbers in cyclic extensions of the rational function field*, in: *Number Theory (Halifax, NS, 1994)*, 307–323, CMS Conf. Proc. **15**, Amer. Math. Soc., Providence, RI, 1995.

[8] L. C. Washington, *Introduction to Cyclotomic Fields*, Second edition, Graduate Texts in Mathematics **83**, Springer-Verlag, New York, 1997.

Jungyun Lee
Department of Mathematics Education, Kangwon National University, Chunchen, Gangwon-do, 24341, South Korea
E-mail address: `lee9311@kangwon.ac.kr`

Yoonjin Lee
Department of Mathematics, Ewha Womans University, 52, Ewhayeodae-gil, Seodaemun-gu, Seoul, 03760, South Korea
E-mail address: `yoonjinl@ewha.ac.kr`